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A METHOD FOR INVESTIGATING GEOMETRIC PROPERTIES OF SUPPORT POINTS AND APPLICATIONS

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ABSTRACT. A normalized univalent function f is a support point of S if there exists a continuous linear functional L (which is nonconstant on S) for which f maximizes $\operatorname{Re} L(g)$, $g \in S$. For such functions it is known that $\Gamma = \mathbf{C} - f(U)$ is a single analytic arc that is part of a trajectory of a certain quadratic differential $Q(w)\,dw^2$. A method is developed which is used to study geometric properties of support points. This method depends on consideration of $\operatorname{Im}\{w^2 Q(w)\}$ rather than the usual $\operatorname{Re}\{w^2 Q(w)\}$. Qualitative, as well as quantitative, applications are obtained. Results related to the Bieberbach conjecture when the extremal functions have initial real coefficients are also obtained.

1. Introduction. Let $\mathcal{H}(U)$ denote the space of all functions analytic in the unit disk $U = \{z\colon |z| < 1\}$. Given the topology of uniform convergence on compact subsets of U, the space $\mathcal{H}(U)$ becomes a locally convex topological vector space. A particular subset of $\mathcal{H}(U)$ is the class S. This class consists of all functions f which are univalent in U and normalized so that $f(0) = 0$ and $f'(0) = 1$. We call $f \in S$ a support point if there exists a continuous linear functional L defined on $\mathcal{H}(U)$ which is nonconstant on S and

$$\max_{g \in S} \operatorname{Re} L(g) = \operatorname{Re} L(f).$$

It is well known that all rotations of the Koebe function $k_\theta(z) = z/(1 + e^{i\theta} z)^2$ are support points of S as well as extreme points of the closed convex hull of S [**1**, **14**]. These functions map the unit disk onto the complement of a radial slit from $e^{-i\theta}/4$ to infinity. A natural question to ask is which of the geometric properties of the functions k_θ are typical of those of arbitrary support points of S. It is known that if f is a support point of S then $\Gamma = \mathbf{C} - f(U)$ is a single analytic arc which tends to infinity with increasing modulus and Γ possesses the $\pi/4$-property: the angle between the radius and tangent vectors never exceeds $\pi/4$ in absolute value [**9**, **3**, **6**].

The principal tool used in the study of support points is the Schiffer variational method [**12**]. It implies that the arc Γ satisfies a differential equation of the form

$$w^{-2} L(f^2/(f - w))\, dw^2 > 0. \tag{1}$$

In the past it was consideration of $\operatorname{Re}\{L(f^2/(f - w))\}$ which led to geometric properties of Γ (in particular, the $\pi/4$-property is obtained in this way). The basic

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reason stems from the fact that for each $w \in \Gamma$, the competing function defined by $f_w(z) = wf(z)/(w - f(z))$ gives $L(f - f_w) = L(f^2/(f - w))$. The purpose of this paper is to present a method whereupon consideration of $\operatorname{Im}\{L(f^2/(f - w))\}$ will also lead to geometric properties of Γ. We apply this method to obtain qualitative, as well as quantitative, results about various support points. It is believed that the omitted arc Γ of a support point has monotonic argument. We prove that this is indeed the case for the functional $L(g) = \alpha a_2 + \beta a_3$ ($\alpha, \beta \in \mathbf{C}$). This generalizes the result in [**4**]. For this functional we also show that Γ must lie in a certain half-plane that can be determined. A result is presented which implies the Bieberbach conjecture under a certain hypothesis.

2. A geometric method. The method presented in this section is implicit in work of Charzynski and Schiffer [**5**] and later in Bombieri [**2**]. We shall present the method in a form applicable to support points of S. This method is based on the behavior of trajectories of certain quadratic differentials. These properties can be found in [**7** and **10**].

LEMMA 1. *Let Ω be a simply-connected region not containing the origin and let Ω be bounded by a trajectory arc γ of $\psi(\omega)\,d\omega^2/\omega^2$ and a simple arc C. Let $\gamma \cap C = \{p_0, p_1\}$ and suppose that ψ/ω^2 is analytic on $\overline{\Omega}\backslash\{p_0, p_1\}$. Suppose further that p_0 and p_1 are not poles of order larger than one for $\psi(\omega)\,d\omega^2/\omega^2$. If $\psi \neq 0$ on $\partial\Omega\backslash\{p_0, p_1\}$ then there exists a simply-connected region $\Omega^* \subset \Omega$, bounded by a trajectory arc γ^* of $\psi(\omega)\,d\omega^2/\omega^2$ and a connected subarc $C^* \subset C$, such that $\psi \neq 0$ on $\overline{\Omega^*}$.*

PROOF. It is well known that there are exactly $n + 2$ trajectories issuing from each zero of order n of a quadratic differential and a single trajectory issuing at each simple pole [**10**]. By our hypotheses we see that $\psi(\omega)\,d\omega^2/\omega^2$ has no poles in $\overline{\Omega}\backslash\{p_0, p_1\}$, and the points p_0 and p_1 are at worst simple poles. Hence, there are no trajectories in $\overline{\Omega}$ homotopic to a point since these occur only for poles of order larger than one [**10**].

Case 1. $\psi \neq 0$ in Ω.

There are only finitely many trajectories issuing from p_0 and p_1. If z^* is any fixed point of Ω not on these trajectories then there is a unique trajectory arc γ^* in Ω passing through z^*. Now $\overline{\gamma}^*$ does not contain p_0 and p_1, and clearly, $\overline{\gamma}^* \cap C$ consists of exactly two points ς_1 and ς_2 (since no trajectory is homotopic to a point, and trajectories do not cross). Let C^* be the connected subarc of C from ς_1 to ς_2 and Ω^* the resulting simply-connected region with $\partial\Omega^* = \gamma^* \cup C^*$. This region satisfies the conclusion of the lemma.

Case 2. $\psi = 0$ in Ω.

There are at least three trajectories issuing from each of the finite number of zeros of ψ in Ω and a finite number of trajectories from p_0 and p_1. Let $\mathcal{T}$ denote the union of all such trajectories in Ω, together with γ. Since there are no trajectories in $\overline{\Omega}$ homotopic to a point, each $\tilde{\gamma} \in \mathcal{T}$ has two endpoints which must be either p_0, p_1, a zero of ψ, or a point of C. Let $I = \overline{\mathcal{T}} \cap C = \{\varsigma_0, \varsigma_1, \ldots, \varsigma_n\}$. This set is nonempty since $p_0, p_1 \in I$. For convenience we let $\varsigma_0 = p_0$, $\varsigma_n = p_1$, and we reindex if necessary so that as we traverse C from p_0 to p_1 we follow $\varsigma_0, \varsigma_1, \ldots, \varsigma_n$ in this order. (See Figure 1.)

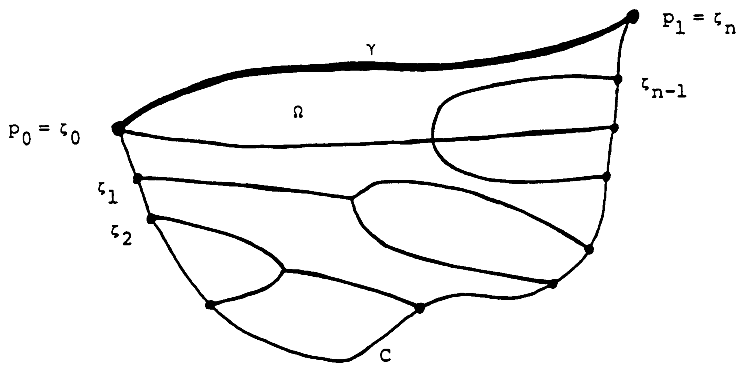

FIGURE 1

We let I_0 denote the set of all points of I that can be joined to ς_0 by a union of trajectories in $\mathcal{T}$. Let ς_{m_0} be the point of I_0 with m_0 minimal. Let $\Omega_0 \subset \Omega$ be the resulting region bounded by the subarc $C_0 \subset C$ from ς_0 to ς_{m_0} and the corresponding (unique) union of trajectories joining ς_0 to ς_{m_0}. If $\psi \neq 0$ in Ω_0, we proceed as in Case 1 and we are done. If $\psi = 0$ in Ω_0, we let I_1 be the set of all points in I that can be joined to ς_1 by a union of trajectories in $\mathcal{T}$. Let ς_{m_1} be the point of I_1 with m_1 minimal (clearly $m_1 < m_0$). Let $\Omega_1 \subset \Omega_0$ be the resulting region bounded by the subarc $C_1 \subset C_0$ from ς_1 to ς_{m_1} and the corresponding union of trajectories in $\mathcal{T}$ joining ς_1 to ς_{m_1}. If $\psi \neq 0$ in Ω_1, we proceed as in Case 1. If not, we continue this process, which terminates since ψ has only a finite number of zeros in Ω. The proof of the lemma is complete.

THEOREM 1. *Let $\psi(\omega)\, d\omega^2/\omega^2$ be a quadratic differential which has a simple pole at $\omega = 0$ and no other poles in $|\omega| \leq \rho$. Let Γ_0 be the unique trajectory which terminates at $\omega = 0$. Suppose ψ is nonzero on Γ_0 except at $\omega = 0$.*

(a) *If $\operatorname{Im}\psi(\omega) \neq 0$ on the radial segment J: $\omega = te^{i\theta}, 0 < t < \rho$, then $\Gamma_0 \cap J = \emptyset$.*

(b) *If $\operatorname{Im}\psi(\omega) \neq 0$ on the radial segment J': $\omega = te^{i\theta}, 0 \leq \rho_0 < t < \rho_1 < \rho$, and Γ_0 lies in a sector of opening less than 2π, then Γ_0 intersects $\overline{J'}$ (closure of J') at most once.*

The condition $\operatorname{Im}\psi(\omega) \neq 0$ on J says geometrically that no trajectory or orthogonal trajectory is ever tangent to J. In particular, if Γ_0 intersects J it must actually cross J. Analytically the condition implies that $\operatorname{Im}\{\sqrt{\psi(\omega)}\} \neq 0$ on J; i.e., $\operatorname{Im}\{\sqrt{\psi(\omega)}\}$ retains its sign along J.

PROOF. (a) Assume that $\Gamma_0 \cap J \neq \emptyset$. We would like to be able to apply Lemma 1, so we first prove the existence of a simply-connected region Ω as in the lemma. Suppose first that there exists a point $\omega_0 \in \Gamma_0 \cap J$ nearest the origin. Let C be that part of J from 0 to ω_0, and let γ be that part of Γ_0 from 0 to ω_0. Let Ω be the corresponding simply-connected region bounded by γ and C. Assume now that Γ_0 crosses J an infinite number of times near $\omega = 0$. Since $\psi(\omega)\, d\omega^2/\omega^2$ has a simple pole at $\omega = 0$, we know that Γ_0 is asymptotic to a line at $\omega = 0$. Thus, the part of Γ_0 in a sufficiently small disk $|\omega| < \rho^* < \rho$ lies in a half-plane. In this disk we choose two consecutive points ς_1, ς_2 of $\Gamma_0 \cap J$. Let γ be the subarc of Γ_0 from ς_1

to ς_2, C the part of J from ς_1 to ς_2, and Ω the region bounded by γ and C. Thus, in either case, we have found a region Ω as asserted.

For the choice of Ω as above we let $\gamma \cap C = \{p_0, p_1\}$ and observe that ψ is analytic on $\overline{\Omega}\backslash\{p_0, p_1\}$, $\psi \neq 0$ on $\partial\Omega\backslash\{p_0, p_1\}$, and p_0 and p_1 are at worst simple poles of $\psi(\omega)\,d\omega^2/\omega^2$. We can thus apply Lemma 1 to conclude that there exists a region $\Omega^* \subset \Omega$ bounded by an arc γ^* of a trajectory and a connected arc C^* of J with ψ nonzero on $\overline{\Omega}^*$. Suppose C^* is the segment of J from ω_1 to ω_2. Apply Cauchy's Theorem to conclude that $\int_{\partial\Omega^*} \sqrt{\psi(\omega)}\,d\omega/\omega = 0$. Now since $\sqrt{\psi(\omega)}d\omega/\omega$ is real along γ^*, this implies that

$$0 = \operatorname{Im} \int_{\partial\Omega^*} \sqrt{\psi(\omega)}\frac{d\omega}{\omega} = \operatorname{Im} \int_{\omega_1}^{\omega_2} \sqrt{\psi(\omega)}\frac{d\omega}{\omega} = \int_{\omega_1}^{\omega_2} \operatorname{Im}\left\{\sqrt{\psi(\omega)}\frac{d\omega}{\omega}\right\}.$$

However, as noted earlier, $\operatorname{Im}\sqrt{\psi(\omega)} \neq 0$ on J and so $\int_{\omega_1}^{\omega_2} \operatorname{Im}\{\sqrt{\psi(\omega)}d\omega/\omega\} \neq 0$. This gives a contradiction and, hence, $\Gamma_0 \cap J = \emptyset$.

(b) Assume Γ_0 meets $\overline{J'}$ at least twice, say at ω_1 and ω_2. Now since Γ_0 lies in a sector of opening less than 2π, we let Ω be the region bounded by the subarc γ of Γ_0 from ω_1 to ω_2 and by C, the part of $\overline{J'}$ from ω_1 to ω_2. We apply the same argument as in (a) to arrive at a contradiction. This completes the proof of the theorem.

In view of the known properties of support points, this theorem is easily seen to be applicable. Let $f \in S$ be a support point for L and let Γ be its omitted arc. Then, by inverting $\omega = 1/w$, we see from (1) that $\Gamma_0 = 1/\Gamma$ is a trajectory of the quadratic differential $\psi(\omega)\,d\omega^2/\omega^2$, where

$$\psi(\omega) = L(\omega f^2/(\omega f - 1)). \tag{2}$$

Schiffer [**13**] proved that $L(f^2) \neq 0$. Thus we see that $\psi(\omega)\,d\omega^2/\omega^2$ has a simple pole at $\omega = 0$. Brickman and Wilken [**3**] have shown that ψ is analytic on Γ_0. It is known that if ψ vanishes on Γ_0, then Γ_0 must be radial and, hence, by the subordination principle, $f = k_\theta$ [**14**]. Thus, we may assume ψ is nonzero on Γ_0. We now turn our attention to applications.

3. Applications. It is well known that if $\omega = 0$ is a simple pole of $\psi(\omega)\,d\omega^2/\omega^2$, then precisely one trajectory and one orthogonal trajectory will terminate there [**10**, p. 216]. Let Ω_0 be a single trajectory arc in $|\omega| \leq \rho$ terminating at $\omega = 0$. Since $d\omega^2/\omega^2 > 0$ holds for all radial lines, we see that if

$$\operatorname{Im}\psi(te^{i\theta_1}) \equiv \operatorname{Im}\psi(te^{i\theta_2}) \equiv 0, \qquad 0 \leq t \leq \rho,$$

for distinct $\theta_1, \theta_2 \in [0, 2\pi)$, then one of the radial segments $\omega = te^{i\theta_1}$ or $\omega = te^{i\theta_2}$, $0 \leq t \leq \rho$, must be a trajectory terminating at $\omega = 0$. Suppose R: $\omega = te^{i\theta_1}$, $0 \leq t \leq \rho$, is a trajectory terminating at $\omega = 0$. If ψ is analytic on Γ_0 and nonzero on $\Gamma_0\backslash\{0\}$, then, because Γ_0 is a single analytic arc also terminating at $\omega = 0$, we must have $\Gamma_0 = R$.

THEOREM 2. *If $f(z) = z + \sum_{n=2}^{\infty} A_n z^n$ is a support point for the functional $L(g) = \alpha a_2 + \beta a_3$ $(\alpha, \beta \in \mathbf{C})$, and if Γ is the arc omitted by f, then Γ lies entirely in a half-plane and has monotonic argument.*

PROOF. Clearly, if $\beta = 0$ the only support points are k_θ, so we may assume $\beta \neq 0$. We also invert by $\omega = 1/w$ and let $\Gamma_0 = 1/\Gamma$. Thus, Γ_0 lies in $|\omega| \leq 4$ by the Koebe $\frac{1}{4}$-Theorem. It follows from (2) that Γ_0 is the trajectory (terminating at $\omega = 0$) of $\psi(\omega)\,d\omega^2/\omega^2$, where

$$\psi(\omega) = -C\omega(1 + D\omega),$$

with $C = 2A_2\beta + \alpha$ and $D = \beta/C$. (Note that $L(f^2) = C \neq 0$.)

We first show that Γ_0 lies in a half-plane. Suppose $\operatorname{Im}\{D\overline{C}\} \neq 0$; then it is clear that $\operatorname{Im}\psi(\overline{C}t) = -|C|^2t^2\operatorname{Im}\{D\overline{C}\} \neq 0$ for $t \neq 0$. It is easy to check that all the hypotheses of Theorem 1 are satisfied, and from (a) we can conclude that Γ_0 lies entirely in a half-plane. In the case $\operatorname{Im}\{D\overline{C}\} = 0$, we simply note that $\operatorname{Im}\psi(i\overline{C}t) = -|C|^2t \neq 0$ for $t \neq 0$. We apply Theorem 1(a) again to conclude that Γ_0 lies in a half-plane.

For each real θ we consider the radial segments in $|\omega| \leq 4$ defined by

$$J_\theta\colon\ \omega = te^{i\theta}/D, \qquad 0 < t \leq 4|D|.$$

Then by putting $\theta_0 = \arg(C/D)$ we see that

$$\operatorname{Im}\psi(te^{i\theta}/D) = -|Ct/D|[\sin(\theta_0 + \theta) + t\sin(2\theta_0 + \theta)]. \tag{3}$$

If $\operatorname{Im}\psi(te^{i\theta'}/D) \equiv 0$, $0 \leq t \leq 4|D|$, for some θ' we see from (3) that

$$\operatorname{Im}\psi(te^{i(\theta'+\pi)}/D) \equiv 0, \qquad 0 \leq t \leq 4|D|,$$

also holds. Now since $\omega = 0$ is a simple pole of $\psi(\omega)\,d\omega^2/\omega^2$, by the above remarks we can conclude that Γ_0 is a radial segment. The subordination principle then yields $f = k_{\theta^*}$ for some real θ^*. Hence, suppose $\operatorname{Im}\psi(te^{i\theta}/D) \not\equiv 0$, $0 \leq t \leq 4|D|$, for all $\theta \in [0, 2\pi)$. Partition each J_θ at the zero of $\operatorname{Im}\psi(te^{i\theta}/D)$. That is, let $J_\theta = L_\theta \cup \overline{l}_\theta$, where L_θ is the open segment of J_θ such that $\overline{L}_\theta$ contains the origin. (If $\operatorname{Im}\psi(te^{i\theta}/D) \neq 0$ for $0 < t \leq 4|D|$, we set $l_\theta = \{4e^{i\theta}|D|/D\}$ and $L_\theta = J_\theta \backslash l_\theta$.) By construction, $\operatorname{Im}\psi(\omega)$ is nonzero on L_θ and l_θ. We apply Theorem 1(a) to each L_θ to conclude that $\Gamma_0 \cap L_\theta = \emptyset$. Applying the second part of the theorem to each $\overline{l}_\theta$, we see that Γ_0 intersects $\overline{l}_\theta$ at most once. Hence, Γ_0 can intersect each radial segment in $|\omega| \leq 4$ at most once. This says that Γ_0, hence Γ, has monotonic argument. The proof of the theorem is complete.

Let us suppose that $f(z) = z + \sum_{n=2}^\infty A_n z^n$ belongs to S and is a support point for $L(g) = a_n$ $(n \geq 2)$. If we set $f(z)^k = \sum_{n=k}^\infty A_n^{(k)} z^n$, then the omitted arc $\Gamma = \mathbf{C} - f(U)$ satisfies

$$-P_n\left(\frac{1}{w}\right)\left(\frac{dw}{w}\right)^2 > 0, \tag{4}$$

where

$$P_n\left(\frac{1}{w}\right) = \sum_{k=1}^{n-1} \frac{A_n^{(k+1)}}{w^k}. \tag{5}$$

In [**8**] it is shown that $A_2 \neq 0$. If $A_2, \ldots, A_{n-1}$ are all real, then from (5) we see that P_n is real on the real axis. The quadratic differential $-P_n(\omega)\,d\omega^2/\omega^2$ has a simple pole at $\omega = 0$. Hence, the remark preceding Theorem 2 implies that $1/\Gamma$ lies either on the positive or negative real axis. Hence, we must have $f(z) = z(1 + z)^2$ or $f(z) = z/(1 - z)^2$. This result can be improved.

THEOREM 3. *If $f(z) = z + \sum_{n=2}^{\infty} A_n z^n \in S$ is a support point for $L(g) = a_n$ $(n \geq 4)$ and $A_2, \ldots, A_{n-2}$ are real, then $f(z) = z/(1 \pm z)^2$, with $A_n = n$.*

We shall make use of the following lemma.

LEMMA 2. *If $f(z) = z + \sum_{n=2}^{\infty} A_n z^n \in S$ is a support point for $L(g) = a_n$ $(n \geq 3)$ and $A_n^{(3)}, \ldots, A_n^{(n-1)}$ are all real, then $f(z) = z/(1 \pm z)^2$, with $A_n = n$.*

PROOF. Let $\Gamma = \mathbf{C} - f(U)$ be the omitted arc of f and let $\Gamma_0 = 1/\Gamma$. Thus, by (4), the arc Γ_0 satisfies $-P_n(\omega)\, d\omega^2/\omega^2 > 0$ in $|\omega| < 4$. Since $A_n^{(3)}, \ldots, A_n^{(n-1)}$ are all real, it follows from (5) that

$$\operatorname{Im}\{P_n(t)\} = t \operatorname{Im}\{A_n^{(2)}\}, \qquad t \in \mathbf{R}. \tag{6}$$

Assume that $\operatorname{Im}\{A_n^{(2)}\} \neq 0$. In this case we see that $\operatorname{Im}\{P_n(t)\} \neq 0$ along the real axis except at the origin. We apply Theorem 1(a) to conclude that Γ_0 meets the real axis only at $\omega = 0$. In particular, $\Gamma_0 \backslash \{0\}$ lies entirely in the upper or lower half-plane. We also know that

$$-A_2 = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{f(e^{i\theta})}\, d\theta.$$

In other words, $-A_2$ lies in the closed convex hull of the point set Γ_0. Hence, A_2 lies in the upper or lower half-plane. However $A_2 = A_n^{(n-1)}/(n-1)$ is real (and nonzero) and we arrive at a contradiction.

Thus, we must have $A_n^{(2)}$ real, and so P_n is real on the real axis. We can then conclude from (4) that Γ_0 lies on the positive or negative real axis. Hence, $f(z) = z/(1 \pm z)^2$, with $A_n = n$.

PROOF OF THEOREM 3. We first note that the formula

$$A_n^{(m)} = \sum_{k=1}^{n-(m-1)} A_k A_{n-k}^{(m-1)}, \qquad 2 \leq m \leq n,$$

implies that $A_n^{(2)} = F_2(A_2, A_3, \ldots, A_{n-1})$, where F_2 is a nonlinear function (with real coefficients) of the $n-2$ variables indicated. Next we see that

$$A_n^{(3)} = \sum_{k=1}^{n-2} A_k A_{n-k}^{(2)} = F_3(A_2, \ldots, A_{n-2}),$$

where F_3 is a nonlinear function (with real coefficients) of the $n-3$ variables shown. Thus, in general, for $m = 3, 4, \ldots, n-1$ we see that

$$A_n^{(m)} = F_m(A_2, \ldots, A_{n-m+1}),$$

where F_m has real coefficients and is a nonlinear function of the $n-m$ variables shown. In particular, the highest coefficient of A_k appearing in $A_n^{(3)}, A_n^{(4)}, \ldots, A_n^{(n-1)}$ is clearly A_{n-2}. Hence, if $A_2, \ldots, A_{n-2}$ are real then $A_n^{(m)}$ is real for $m = 3, \ldots, n-1$. Now apply Lemma 2.

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