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CONWAY'S FIELD OF SURREAL NUMBERS

BY

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ABSTRACT. Conway introduced the Field **No** of numbers, which Knuth has called the *surreal numbers*. **No** is a proper class and a real-closed field, with a very high level of density, which can be described by extending Hausdorff's η_ξ condition. In this paper the author applies a century of research on ordered sets, groups, and fields to the study of **No**. In the process, a tower of subfields, ξ**No**, is defined, each of which is a real-closed subfield of **No** that is an η_ξ-set. These fields all have Conway partitions. This structure allows the author to prove that every pseudo-convergent sequence in **No** has a unique limit in **No**.

0. Introduction.

0.0. In the zeroth part of J. H. Conway's book, *On numbers and games* [**6**], a proper class of numbers, **No**, is defined and investigated. D. E. Knuth wrote an elementary didactic novella, *Surreal numbers* [**15**], on this subject. Combining the notation of the first author with the terminology of the second, we will call **No** the Field of *surreal numbers*. Following Conway [**6**, p. 4], a proper class that is a field, group,... will be called a Field, Group,.... We investigate this Field using some of the methods developed in the study of ordered sets, groups, and fields over the last 100 years or so. (A short, partial bibliography on this subject will be found at the end of the paper.)[1]

0.1. Let T be a partially-ordered class. It will be called *totally-ordered* (= *linearly-ordered* = *simply-ordered*) if for all x and y in T then $x \leqslant y$ or $y \leqslant x$. Assume T is a totally-ordered class and an additive group. It will be called a *totally-ordered group* if $x \leqslant y$ in T implies $x + z \leqslant y + z$ for all $z \in T$. Assume that T is a totally-ordered group and, in addition, that T is a field. It will be called a *totally-ordered field* if $x \geqslant 0$ and $y \geqslant 0$ in T imply $xy \geqslant 0$. T will be called *Dedekind complete* if, given any bounded subset B of T, it has a l.u.b. in T. It is well known that, up to isomorphism,

(1) the only Dedekind complete totally-ordered field is the field $\mathbb{R}$ of all real numbers.

A totally-ordered group G is called *Archimedean* if given any $g, h \in G$ with $g, h > 0$ there exists $n \in N$ (the set of all natural numbers) such that $ng > h$ and $nh > g$. Let K be a non-Archimedean totally-ordered field.

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[1] Thanks are due the referee for conjecturing a stronger version of a result that appeared in the original version of §2.3. This conjecture is now proved in §2.3.

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From (1) we see that

(2) K is not Dedekind complete.

Using the open intervals of K as a basis of the open classes in K gives K a topology called the *order topology*. One can easily see that, under this topology,

(3) K is not connected, not compact, not locally connected, and, not locally compact.

Let L and R be subclasses of a totally-ordered class T. We write $L < R$ if $x^{\mathrm{L}} \in L$ and $x^{\mathrm{R}} \in R$ implies $x^{\mathrm{L}} < x^{\mathrm{R}}$. Note that $\varnothing < R$ and $L < \varnothing$ for all L and R. Conway constructs the elements of **No** as follows [**6**, p. 4]:

(4) If L and R are two subsets of **No** with $L < R$, then there exists a number $\{L \mid R\} \equiv x \in$ **No**. All elements of **No** are constructed in this way.

In general, if $x = \{L \mid R\}$, then x^{L} will denote a typical element of L, and x^{R} a typical element of R. Conway defines

(5) $x \geqslant y$ iff $x^{\mathrm{R}} > y$ and $x > y^{\mathrm{L}}$ for all x^{R} and y^{L}.

It is well to note that equality between these numbers is an equivalence relation, namely,

(6) $x = y$ iff $x \geqslant y$ and $y \geqslant x$,

and not the following: $\{L \mid R\} = \{L' \mid R'\}$ iff $L = L'$ and $R = R'$, or some such expression. Conway shows that **No** is a proper class (i.e., a Class), defines an addition and multiplication in **No**, and shows that it is a real-closed Field. Note that if $x = \{L \mid R\}$, then $L < \{x\} < R$; thus, whenever a gap exists in the numbers defined thus far, an element is created to fill that gap. This process of creation stops only when L or R is a proper class of numbers.

There is a large body of literature on totally-ordered sets, groups, and fields that goes back to work of Cantor, Hahn, and Hausdorff. *It is the purpose of this paper to apply some of these now classical results to the Field* **No**. For the convenience of the reader, the author will make a few introductory remarks on these topics as each is introduced. References to the literature will also be given.

0.2. It has proved convenient to the author, and he hopes it will also be useful to the reader, to use one principal reference to a set theory which is embedded in a theory of classes. Although there are several variants of such a theory (see, e.g., Gödel [**10**]), the author has chosen to use *Introduction to set theory* by J. Donald Monk [**19**], a text which presents set theory from axioms that go back to Skolem and A. P. Morse, as given by Kelley [**14**]. (See also [**6**, pp. 64–67].) In general, we follow the notation of Monk with one notable exception: we use $\subset$ to denote containment, not proper containment; thus $N \subset N$. As usual, we let Z denote the ring of all integers and $\mathbf{Q}$ the field of all rational numbers. We also assume that all ordinal numbers we use are sets.

0.3. Let T be a totally-ordered class and T' a subclass of T. T' is said to be *cofinal* (resp. *coinitial*) *in* T if for all $t \in T$ there exists $t' \in T'$ such that $t \leqslant t'$ (resp. $t' \leqslant t$).

Thus Z is both cofinal and coinitial in $\mathbb{R}$, whereas N is only cofinal in $\mathbb{R}$.

(1) T is called an η-Class if, given any subsets L and R of T with $L < R$, there exists $t \in T$ such that $L < \{t\} < R$.

Assume that T is an η-Class. Since L and R may be taken to be the empty set (0.1), we see that $T \neq \varnothing$. Next note that one can construct an isomorphic copy of the Class **Ord**, of all ordinal numbers, in T; thus T is a propler class. It follows almost immediately from Conway's definitions (0.1: 4, 5, and 6) (i.e. from (4), (5), and (6) in §0.1) that

(2) **No** *is an* η-*Class*.

0.4. The author is very grateful to Norman Stein for bringing the work of Conway to his attention. Thanks are also due A. H. Stone for suggesting the book by Monk.

1. The Hahn-Krull theory of valuations on ordered algebras.

1.0. Let G be a totally-ordered additive Group (i.e., G is a group that may be a proper class). Given $g \in G$, let $|g| \equiv \max g, \ -g$; thus $g = |g|$ iff $g \geqslant 0$, and $|g + h| \leqslant |g| + |h|$ for all $g, h \in G$. A subclass C of G will be called *convex* if, for each $c_0, c_1 \in C$ and $g \in G$, with $c_0 < g < c_1$, g is in C. A convex sub-Group H of G will be called *principal* if there exists $g \in H$ such that $H = (g) \equiv \{h \in G\colon$ there exists $n \in N$ such that $|h| \leqslant n|g|\}$. Such an element g will be called a *generator* of H. g and g' in G will be called *commensurate* to one another, written $g \sim g'$, if $(g) = (g')$. Thus $g \sim g'$ iff there exists $n \in N$ such that $|g| \leqslant n|g'|$ and $|g'| \leqslant n|g|$. We write $g \ll g'$ if, for all $n \in N$, $n|g| < |g'|$. For $g \neq 0$ let $(g)^- \equiv \{h \in G\colon h \ll g\}$.

1.1. Let G be an Abelian totally-ordered group. Let Σ be the set of all nonzero convex subgroups of G. It is easily seen that Σ is totally-ordered under inclusion. Let S be the set of all nonzero principal convex subgroups of G; then it is easily seen that each element in Σ is a union of elements in S. Given $g \in G^*$ ($\equiv G - \{0\}$) let

(1) $$V(g) \equiv (g);$$

then V maps G^* onto S. In this context it is natural to order S by inclusion; however, it is more convenient, when referring to the main body of literature on the valuation theory of fields, to define

(2) $$V(g) \geqslant V(g') \quad \text{iff} \quad V(g) \subset V(g').$$

It is also convenient to define $V(0)$ to be ∞, an ideal element greater than each element of S. For g and g' in G, then

(3) $$V(g \pm g') \geqslant \min V(g), V(g'),$$

and

(4) *if* $V(g) \neq V(g')$, *equality holds in* (3).

Proof. Since $V(-g') = V(g')$ we need prove (3) and (4) only for the case in which the plus sign appears. Without loss of generality, we may assume $V(g') \leqslant V(g)$, i.e., $(g) \subset (g')$. Clearly, $g + g' \in (g')$, and thus $(g + g') \subset (g')$, which implies $V(g') \leqslant V(g + g')$, establishing (3). Assume now that $V(g') < V(g)$, i.e., $(g) \subsetneqq (g')$. Since $V(-g') = V(g')$, we may assume, without loss of generality, that

$g' > 0$. If $g > 0$, then $0 < g << g' < g + g' < 2g'$, showing that $(g') = (g + g')$ and, hence, that $V(g + g') = V(g')$. Now assume that $g < 0 < g'$. Since $g << g'$, $2g > -g'$; thus $g' = -g' + 2g' < 2g + 2g' = 2(g + g') < 2g'$, showing that $V(g + g') = V(g')$, establishing (3) and (4).

V is called *the order-valuation* on the totally-ordered group G, and S is called its *value set*. For $g \in G^*$, let $V^-(g) \equiv \{h \in G: h << g\}$; then $V^-(g)$ is the largest proper convex subgroup of $V(g)$, and

$$(5) \qquad V(g)/V^-(g) \equiv F(g)$$

is defined to be the *factor group* of $V(g)$, or merely a *factor* of G. It is well known that

(6) any Archimedean totally-ordered group (i.e., one whose value set is a single point) is isomorphic to a subgroup of $(\mathbb{R}, +)$, the additive subgroup of $\mathbb{R}$;

thus

(7) each factor of G is isomorphic to a subgroup of $(\mathbb{R}, +)$.

The value set S together with its associated factors is sometimes referred to as the *Hahn skeleton* of G.

1.2. Assume now that G is a totally-ordered Group which is not a set. How much of the Hahn-Krull theory given in §1.1 can be carried over for G? Given $g \in G^*$, (g) is a convex sub-Group of G, but it may well be a proper class and, thus, not an element of any other class [**19**, p. 14]; thus we may not be able to form S as we did in §1.1. We certainly can define $(g)^-$ to be $\{h \in G: h << g\}$ and show that $(g)^-$ is the largest proper convex sub-Group of (g). $(g)^-$ may very well be a proper class. If it is, how can we go about constructing $(g)/(g)^-$? Since each of the cosets of (g) mod $(g)^-$ is a proper class, they are elements of no class. Can we use the axiom of choice to choose coset representatives? The version of the axiom of choice at our disposal [**19**, Axiom 1.36] is not strong enough to do this, just as Gödel's very strong version of choice [**10**, Axiom E] is not.

That there problems can easily be avoided will be seen in what follows.

1.3. Let G be a totally-ordered Group, S a totally-ordered Class, ∞ an element greater than each $s \in S$, and V a mapping of G onto $S \cup \{\infty\}$ such that $V(0) = \infty$ and V maps G^* onto S. The pair (V, S) will be called a *valuation* of G if (1.1 : 3 and 4) hold. If (V, S) is a valuation of G then S is called its *value class*. A pair (V, S) will be called *an order-valuation* of G if, for all $g, g' \in G^*$,

$$(1) \qquad \begin{array}{ll} \text{(i)} & V(g) = V(g') \text{ iff } g \sim g', \text{ and} \\ \text{(ii)} & V(g') < V(g) \text{ iff } g \ll g'. \end{array}$$

Clearly (V, S) defined in §1.1 is a valuation as well as an order-valuation. Clearly, we wish to show that

(2) *every order-valuation* (V, S) *is a valuation*.

In order to facilitate this, note that $g << g'$ iff $(g) \subsetneqq (g')$. Next note that, assuming (V, S) is an order-valuation, the following hold:

(3)
$$\begin{array}{ll} \text{(i)} & V(g) = V(g') \text{ iff } (g) = (g'), \\ \text{(ii)} & V(g') < V(g) \text{ iff } (g) \subsetneqq (g'), \\ \text{(iii)} & V(g') \leqslant V(g) \text{ iff } (g) \subset (g'), \text{ and} \\ \text{(iv)} & V(g) = V(-g) \text{ for all } g \in G. \end{array}$$

Using (3), together with the proof of (1.1 : 3 and 4), one can prove (2). Let $g > 0$ be in G. For $h \in (g)$ let

(4)
$$\begin{aligned} L(h) &\equiv \{m/n : mg \leqslant nh, \text{ with } m \in Z \text{ and } n \in N\}, \\ R(h) &\equiv \{m/n : mg > nh, \text{ with } m \in Z \text{ and } n \in N\}. \end{aligned}$$

Clearly, $L(h) \cup R(h) = \mathbf{Q}$, $L(h) \neq \varnothing \neq R(h)$, and $L(h) < R(h)$. Let $\mathrm{Ddm}_g(h) \equiv \sup L(h)\ (= \inf R(h)) \in \mathbf{R}$ and let $h \in (g) \to \mathrm{Ddm}_g(h) \in \mathbf{R}$ be called the *Dedekind divisor map with base g*. Clearly $\mathrm{Ddm}_g(g) = 1$.

(5) *Ddm_g is a homomorphism of (g) into $\mathbf{R}$ having kernel $(g)^-$. Further, $h \leqslant h'$ in (g) implies $\mathrm{Ddm}_g h \leqslant \mathrm{Ddm}_g h'$.*

PROOF. Let $h, h' \in (g)$, $m/n \in L(h)$, and $m'/n' \in L(h')$; then $mg \leqslant nh$ and $m'g \leqslant n'h'$. Hence, $mn'g \leqslant nn'h$, $m'ng \leqslant nn'h'$, and, thus,

$$(mn' + m'n)g \leqslant nn'(h + h'),$$

showing that $L(h) + L(h') \subset L(h + h')$. Similarly $R(h) + R(h') \subset R(h + h')$, proving that the Dedekind divisor map is a homomorphism. Let $h \in (g)^-$, $n \in N$, and $m \in Z$. $mg \leqslant nh$ implies $m \leqslant 0$. $mg > nh$ implies $m > 0$; thus $\mathrm{Ddm}_g(h) = 0$. Conversely, let h be a positive element in $\ker \mathrm{Ddm}_g$; then for all $n \in N$, $nh \leqslant g$, proving that $h \in (g)^-$, and hence that $\ker \mathrm{Ddm}_g = (g)^-$. Now let $m/n \in L(h)$ with $n > 0$, and let $h \leqslant h'$ be in (g); then $mg \leqslant nh \leqslant nh'$, showing that $m/n \in L(h')$, and thus proving that $\mathrm{Ddm}_g h \leqslant \mathrm{Ddm}_g h'$, establishing (5).

1.4. Assume that F is a totally-ordered Field and V is an order-valuation of the additive Group $(F, +)$ of F. Let G be the value Class of V. Let us define an addition $+$ on G as follows:

(1)
$$V(x) + V(y) \equiv V(xy) \quad \text{for all } x, y \in F^*.$$

It is easy to check that the operation of $+$ on G is well defined. V is a homomorphism of F^* onto the additive Group G which we call the *value Group* of V. Let us also introduce the convention that $g + \infty = \infty + g = \infty + \infty = \infty$, for all $g \in G$; then the formula in (1) holds for all $x, y \in F$.

(2) *G is a totally-ordered Group.*

PROOF. Let $x, y, z \in F^*$. The following statements are equivalent: $V(x) < V(y)$, $y << x$, $n|y| < |x|$ for all $n \in N$, $n|yz| < |xz|$ for all $n \in N$, $yz << xz$, $V(xz) < V(yz)$, and $V(x) + V(z) < V(y) + V(z)$, proving (2).

Thus we have proved that

(3) *V is a valuation of the Field F and has G as its value group.*

Let $\mathbf{O} \equiv (1)$ and $\mathbf{M} \equiv (1)^-$. It is very easy to see that $\mathbf{O} = \{f \in F: V(f) \geqslant 0\}$ and $\mathbf{M} = \{f \in F: V(f) > 0\}$; thus

(4) *$\mathbf{O}$ is a valuation sub-Ring of F (i.e., a proper sub-Ring such that $f \in F - \mathbf{O}$ implies $1/f \in \mathbf{O}$), and $\mathbf{M}$ is its maximal ideal.*

Then $\mathbf{O} - \mathbf{M} \equiv \mathbf{U}$, the Group of units of $\mathbf{O}$, is $\{f \in F: f \sim 1\}$. As usual in valuation theory, $\mathbf{U}$ is the kernel of the homomorphism V of $(F^*, \times)$ onto G.

See, e.g., [**27**, **24**, **7** or **5**] for details on valuation theory.

1.5. Let F be a field with valuation V. Let $\mathbf{O} \equiv \{f \in F: V(f) \geqslant 0\}$ and $\mathbf{M} \equiv \{f \in F: V(f) > 0\}$; then $\mathbf{O}$ is the *valuation ring* of V and $\mathbf{M}$ is the maximal ideal of $\mathbf{O}$. The canonical homomorphism p of $\mathbf{O}$ onto $\mathbf{O}/\mathbf{M} \equiv K$ is called the *place* of V and K is called the *residue class field* of V.

Now let F be a totally-ordered Field, $\mathbf{O} \equiv (1)$, and $\mathbf{M} \equiv (1)^-$ §1; then, as we have seen, each is a convex sub-Group of $(F, +)$ and $\mathbf{M}$ is the largest proper convex sub-Group of $\mathbf{O}$. Of course, F is Archimedean if and only if $\mathbf{O} = F$. Assume that F is non-Archimedean. For each $f \in \mathbf{O}$ let $p(f) \equiv \mathrm{Ddm}_1(f)$; then by (1.3: 5) p is a homomorphism of $(\mathbf{O}, +)$ into $(\mathbb{R}, +)$ having $\mathbf{M}$ as kernel. Since F is a totally-ordered Field, its characteristic is zero; thus it has a copy of $\mathbb{Q}$ in it, which we will identify with $\mathbb{Q}$. Let $h \in \mathbf{O}$ and $g = 1$; then

(1)
$$L(h) = \{m/n : m/n \leqslant h, \text{ with } m \in Z \text{ and } n \in N\},$$
$$R(h) = \{m/n : m/n > h, \text{ with } m \in Z \text{ and } n \in N\}.$$

(2) *p is a homomorphism of the ring $\mathbf{O}$ into $\mathbb{R}$ having kernel $\mathbf{M}$; thus $\mathbf{M}$ is a prime ideal of $\mathbf{O}$. Further, $\mathbf{M}$ is the maximal ideal of $\mathbf{O}$.*

PROOF. Let h, h' be positive elements in $\mathbf{O}$, $m/n \in L(h)$, and $m'/n' \in L(h')$, with $m, m' \geqslant 0$. Thus $m \leqslant nh$, $m' \leqslant n'h'$, $mm' \leqslant nm'h \leqslant nn'hh'$, showing that $mm'/nn' \in L(hh')$. Similarly, $m/n \in R(h)$ and $m'/n' \in R(h')$ imply $mm'/nn' \in R(hh')$; thus $p(hh') = p(h)p(h')$. Since p is a homomorphism of $(\mathbf{O}, +)$ into $(\mathbb{R}, +)$, $p(-h) = -p(h)$, establishing the first sentence in (2). An element is in $\mathbf{O} - \mathbf{M}$ iff it is bounded by nonzero rational numbers of the same sign. Given such an element, its inverse has the same property, proving that $\mathbf{M}$ is the maximal ideal of $\mathbf{O}$.

(3) *$\mathbf{O}$ is a valuation sub-Ring of F.*

PROOF. Let $f \in F - \mathbf{O}$; then, by definition, $1 \ll f$. Hence $n < |f|$ for all $n \in N$. Without loss of generality, let $f > 0$; thus $1/f < 1/n$ for all $n \in N$, showing that $1/f \in \mathbf{M} \subset \mathbf{O}$, establishing (3).

From these observations, we see that

(4) *p is a place for the valuation ring $\mathbf{O}$.*

1.6. *Application of the Hahn-Krull theory to* **No**. Given a totally-ordered Field F, let $F^+ \equiv \{x \in F: x > 0\}$: then $(F^+, \times)$ is a subgroup of $(F^*, \times)$ of index 2. Conway defines a map $x \in \mathbf{No} \mapsto \omega^x \in \mathbf{No}^+$ such that

(1) each $y \in \mathbf{No}^+$ is commensurate to ω^x, for some $x \in \mathbf{No}$,

and

(2)
$$\omega^{x+y} = \omega^x \cdot \omega^y \quad \text{for all } x, y \in \mathbf{No} \text{ [6, pp. 31–32]}.$$

(3) $$x \in \mathbf{No}^{+} \quad \textit{implies} \quad \omega^{x} \gg 1.$$

PROOF. The simplest element in $\mathbf{No}^+$ (i.e., the element in $\mathbf{No}^+$ with the earliest birthday) is 1 and, by definition, $\omega^1 = \omega \gg 1$. Proceeding by induction on the simplicity of x in $\mathbf{No}^+$, recall that

(4) $$\omega^{x} \equiv \left\{0, r\omega^{x^{\mathrm{L}}} \mid s\omega^{x^{\mathrm{R}}}\right\}, \quad \text{where } r, s \in \mathbb{R}^{+} \text{ with } x = \left\{x^{\mathrm{L}} \mid x^{\mathrm{R}}\right\} \text{ and } 0 < x^{\mathrm{L}} \text{ [6, pp. 31–32]}.$$

Thus, by induction, (3) is proved.

(5) *Let $x < y$ in* **No**; *then* $\omega^x \ll \omega^y$.

PROOF. Let $z \equiv y - x$. Then $y = x + z$ and $z > 0$. Hence $\omega^y = \omega^x \cdot \omega^z \gg \omega^x$, using (2) and (3), establishing (5).

(6) *Given $y \in$* $\mathbf{No}^+$ *there exists a unique $x \in$* **No** *such that* $y \sim \omega^x$.

PROOF. By (1) there exists such an x. By (5) it is unique, establishing (6).

Let $E \equiv \{\omega^x \colon x \in \mathbf{No}\}$; then E is a multiplicative sub-Group of $\mathbf{No}^+$ and $x \in \mathbf{No} \mapsto \omega^x$ is an order-preserving isomorphism of $(\mathbf{No}, +)$ onto E. Given $z \in E$, let $\log_\omega z$ be the element $x \in \mathbf{No}$ such that $z = \omega^x$. Clearly $z \in E \mapsto \log_\omega z \in \mathbf{No}$ is an order-preserving isomorphism of E onto $(\mathbf{No}, +)$.

For $y \in \mathbf{No}^*$ ($\equiv \mathbf{No} - \{0\}$) let $V(y)$ be the element $x \in \mathbf{No}$ such that $|y| \sim \omega^{-x}$ (6). Further, let $V(0) \equiv \infty$, an element greater than each $x \in \mathbf{No}$.

(7) *V is an order-valuation for* $(\mathbf{No}, +)$ *with value-Class* **No**.

PROOF. The following are easily seen to be equivalent: $y \sim y'$, $|y| \sim |y'|$, $V(y) = V(y')$, verifying condition (1.3: 1(i)). Let $x \equiv V(y)$ and $x' \equiv V(y')$; then, by definition, $|y| \sim \omega^{-x}$ and $|y'| \sim \omega^{-x'}$. The following statements are equivalent: $y << y'$, $|y| << |y'|$, $\omega^{-x} \ll \omega^{-x'}$, $-x < -x'$, $x' < x$, and $V(y') < V(y)$; verifying condition (1.3: 1(ii)) and thus proving (7).

(8) *For all $y, y' \in$* $\mathbf{No}^*$, $V(yy') = V(y) + V(y')$; *thus V is a valuation of* **No** *whose value-Group is* $(\mathbf{No}, +)$.

PROOF. Let $x \equiv V(y)$ and $x' \equiv V(y')$. It is easy to see that each statement implies the next:

$|y| \sim \omega^{-x}$ and $|y'| \sim \omega^{-x'}$;

there exist n and m in N such that $\omega^{-x}/n < |y| < n\omega^{-x}$ and $\omega^{-x'}/m < |y'| < m\omega^{-x'}$;

there exist n and m in N such that $\omega^{-(x+x')}/nm < |yy'| < nm\omega^{-(x+x')}$;

$V(yy') = x + x' = V(y) + V(y')$,

proving (8).

1.7. Let **O** be (1) in **No**. As we saw in §1.4, **O** is a valuation Ring in **No** whose maximal ideal $\mathbf{M} = (1)^-$. The Dedekind divisor map $p \equiv \mathrm{Ddm}_1$ of **O** into $\mathbb{R}$ is a place of **O** (1.5: 4). Since $\mathbb{R}$ is a subfield of **No** [6, pp. 24–25], we see that

(1) $$p(\mathbf{O}) = \mathbb{R};$$

thus p is an $\mathbb{R}$-valued place. Given $x \in \mathbf{O}$, Conway has shown [6, pp. 32–33] that there exists a unique $r \in \mathbb{R}$ such that $x = r + x_1$, with $x_1 \in \mathbf{M}$. Of course, $r = p(x)$;

thus

(2) $$x = p(x) + x_1 \quad \text{with } x_1 \text{ in } \mathbf{M}.$$

Note. Given this result, one could define p by (2) and show very easily that it defines a place map of $\mathbf{O}$ onto $\mathbb{R}$.

1.8. Let us now consider the Group ($\mathbf{No}$, +). We have seen (1.6: 7) that V is an order-valuation for ($\mathbf{No}$, +) with value Class $\mathbf{No}$. Let $x \in \mathbf{No}^+$. By (1.3: 5), the Dedekind divisor map Ddm_g is a homomorphism of (g) into $\mathbb{R}$ with kernel $(g)^-$ such that $h \leqslant h'$ in (g) implies $\mathrm{Ddm}_g h \leqslant \mathrm{Ddm}_g h'$. Let $G \equiv \mathrm{im}\,\mathrm{Ddm}_g$. Since $\mathbb{R}$ is a subfield of $\mathbf{No}$, (g) and $(g)^-$ are vector spaces over $\mathbb{R}$ as well as over $\mathbb{Q}$; thus G is a nonzero subgroup of ($\mathbb{R}$, +) that is divisible. Since $\mathbf{No}$ is an η-Class (0.3: 2),

(1) $$\mathrm{im}\,\mathrm{Ddm}_g = \mathbb{R} \quad (\text{cf. } [\mathbf{2}, \text{p. } 712]).$$

Further, one easily sees that

(2) Ddm_g *is an* $\mathbb{R}$*-linear map of* (g) *onto* $\mathbb{R}$.

Since the topology on $\mathbf{No}$ is the order topology (0.1), and since Ddm_g preserves $\leqslant$, we see that

(3) Ddm_g *is a continuous* $\mathbb{R}$*-linear functional on* (g).

2. The valuation theory of Krull, Ostrowski and Kaplansky.

2.0. In the category of fields with valuation, an extension is called an *immediate extension* if neither the residue class field nor the value group is enlarged by the extension. A field with valuation is called *maximal* if it has no proper immediate extensions. This idea, due apparently to F. K. Schmidt, first seems to have appeared in Krull's celebrated paper of 1931 [**16**]. In Kaplansky's Harvard Dissertation of 1941 [**13**] such extensions were considered at length. Kaplansky used Ostrowski's idea of pseudo-convergent sequences [**23**] (c. 1935) to great effect in [**13**]. We will recall this theory very briefly and then apply it to $\mathbf{No}$ in this section.

2.1. Let K be a Field with valuation V and value Group G. Let λ be a (nonzero) limit ordinal. A sequence $A \equiv (a_\alpha)_{\alpha<\lambda}$ of elements in K will be called a *pseudo-convergent sequence of length* λ if

(1) $$V(a_\alpha - a_\beta) < V(a_\beta - a_\gamma) \quad \text{for all } \alpha < \beta < \gamma < \lambda.$$

(See, e.g., Kaplansky [**13**, p. 303 ff.] for details.) Assume that A is pseudo-convergent. Using the triangle inequality (1.1: 3 and 4), one can easily see that either

(2) (i) $V(a_\alpha) < V(a_\beta)$ *for all* $\alpha < \beta < \lambda$; *or*
(ii) *there exists* $\alpha < \lambda$ *such that, for all* β *with* $\alpha < \beta < \lambda$, $V(a_\beta) = V(a_\alpha)$.

It is also easily seen that

(3) $$V(a_\beta - a_\alpha) = V(a_{\alpha+1} - a_\alpha) \quad \text{for all } \alpha < \beta < \lambda.$$

Let $g_\alpha \equiv V(a_{\alpha+1} - a_\alpha) \in G$ for all $\alpha < \lambda$. By definition (1) $(g_\alpha)_{\alpha<\lambda}$ is a strictly increasing sequence of points in G, which may or may not be cofinal in G. Let B, the *breadth* of A, be $\{y \in K\colon V(y) > g_\alpha \text{ for all } \alpha < \lambda\}$. The breadth is clearly a subgroup of $(K, +)$, which may or may not be zero. $x \in K$ will be called a *pseudo-limit* of A if

(4) $$V(x - a_\alpha) = g_\alpha \quad \text{for all } \alpha < \lambda.$$

(*Note*. Kaplansky uses "limit" for the term we here define as "pseudo-limit" [**13**, p. 304].) If a pseudo-limit to A exists, it is uniquely determined modulo B. The following holds:

(5) *Assume that K is a set. K is maximal if and only if every pseudo-convergent sequence in K has a pseudo-limit in K* [**13**, *p*. 309].

Whether or not K is a set, it will be called *pseudo-complete* if every pseudo-convergent sequence in K has a pseudo-limit in K.

Note that the notions of pseudo-convergent sequences, breadth, and pseudo-limits do not depend on the multiplication structure of K or the additive structure of G; thus all these notions can be defined for a totally-ordered additive group G. Let ξ be an ordinal with $\xi > 0$. G will be called *ξ-pseudo-complete* if every pseudo-convergent sequence $(a_\alpha)_{\alpha<\lambda}$ of length $\lambda < \omega_\xi$ has a pseudo-limit in G. (The term ξ-maximal was used in [**2, 3**] for this idea.)

2.2. *Application of this theory to* **No**.

(1) **No** *is pseudo-complete.*

Proof. Let λ be a (nonzero) limit ordinal and let $A \equiv (a_\alpha)_{\alpha<\lambda}$ be a pseudo-convergent sequence in **No**. Let $g_\alpha \equiv V(a_{\alpha+1} - a_\alpha)$ for all $\alpha < \lambda$. As noted in §2.1, $(g_\alpha)_{\alpha<\lambda}$ is a strictly increasing sequence in the value group of V, namely in (**No**, +) (1.6: 8). For each $\alpha < \lambda$ let $b_\alpha \in \mathbf{No}^+$ such that $V(b_\alpha) = g_\alpha$. The following inequality holds:

(2) $$a_{\alpha+1} - b_\alpha < a_{\beta+1} - b_\beta < a_{\beta+1} + b_\beta < a_{\alpha+1} + b_\alpha \quad \text{for all } \alpha < \beta < \lambda.$$

To establish (2) note that the first quantity is less than the last, and the second inequality holds simply because b_α and b_β are positive. Let us establish the last inequality in (2). Note first that it is equivalent to

(3) $$a_{\beta+1} - a_{\alpha+1} < b_\alpha - b_\beta.$$

Since $g_\alpha < g_\beta$, $b_\beta \ll b_\alpha$, and hence $b_\alpha - b_\beta > 0$; thus (3) is implied by

(4) $$|a_{\beta+1} - a_{\alpha+1}| < b_\alpha - b_\beta.$$

To establish (4) note that by (2.1: 3), $V(a_{\beta+1} - a_{\alpha+1}) = V(a_{\alpha+2} - a_{\alpha+1}) = g_{\alpha+1}$. Using the triangle equality (1.1: 4), we see that $V(b_\alpha - b_\beta) = \min V(b_\alpha), V(b_\beta) = g_\alpha$. Since $g_{\alpha+1} > g_\alpha$ (2.1), we see that

(5) $$V(|a_{\beta+1} - a_{\alpha+1}|) > V(b_\alpha - b_\beta);$$

which proves (4), (3) and, hence, the last inequality of (2). The first inequality of (2) also follows from (4), proving (2). Let

(6) $$L \equiv \{a_{\alpha+1} - b_\alpha : \text{for all } \alpha < \lambda\},$$
$$R \equiv \{a_{\alpha+1} + b_\alpha : \text{for all } \alpha < \lambda\}.$$

By (2) $L < R$. Let $x \equiv \{L \mid R\}$; then $L < \{x\} < R$. Thus $a_{\alpha+2} - b_{\alpha+1} < x < a_{\alpha+2} + b_{\alpha+1}$ for all $\alpha < \lambda$; i.e., $-b_{\alpha+1} < x - a_{\alpha+2} < b_{\alpha+1}$. As a consequence, $|x - a_{\alpha+2}| < b_{\alpha+1}$ and, thus, $V(x - a_{\alpha+2}) \geqslant V(b_{\alpha+1}) = g_{\alpha+1}$ for all $\alpha < \lambda$. Using the triangle equality (1.1: 4) again, we see that

$$V(x - a_\alpha) = V(x - a_{\alpha+2} + a_{\alpha+2} - a_\alpha) = \min V(x - a_{\alpha+2}), V(a_{\alpha+2} - a_\alpha)$$
$$= \min V(x - a_{\alpha+2}), g_\alpha = g_\alpha,$$

proving (1).

2.3. Let (L, R) denote the class of all y in **No** for which $L < \{y\} < R$. The argument above shows that

(0) *Any y in (L, R) is a pseudo-limit of A. Further, x is the simplest such element.*

It is well known [**13**, p. 304] that

(1) z is a pseudo-limit of A iff z is in $B + \{x\}$.

Let $L' = \{x - b_\alpha : \alpha < \lambda\}$ and $R' = \{x + b_\alpha : \alpha < \lambda\}$. Then

(2) $$B + \{x\} = (L', R').$$

Indeed, let z be in $B + \{x\}$. Then $V(x - z) > g_\alpha$ for all $\alpha < \lambda$; i.e., $|x - z| < b_\alpha$ for all $\alpha < \lambda$, showing that z is in (L', R'). Now let z be in (L', R'). Then $x - b_\alpha < z < x + b_\alpha$ for all $\alpha < \lambda$; i.e., $|x - z| < b_\alpha$ for all $\alpha < \lambda$. Hence, $V(x - z) \geqslant g_\alpha$ for all $\alpha < \lambda$. Since $(g_\alpha)_{\alpha<\lambda}$ is strictly-increasing (2.1), and since λ is a nonzero limit ordinal, $x - z$ is in B, and thus z is in $B + \{x\}$, establishing (2).

Combining (0), (1) and (2) we see that (L, R) is a subclass of (L', R'). The following also holds:

(3) $$(L, R) = (L', R').$$

Indeed, let z be in (L', R'). By (2) z is a pseudo-limit of A, hence $V(z - a_{\alpha+1}) = g_{\alpha+1}$ for all $\alpha < \lambda$ (2.1: 4). As a consequence, $|z - a_{\alpha+1}| < b_\alpha$. Hence $-b_\alpha < z - a_{\alpha+1} < b_\alpha$ and, thus, $a_{\alpha+1} - b_\alpha < z < a_{\alpha+1} + b_\alpha$. As a result we see that z is in (L, R) establishing (3). Thus

(4) *x is the simplest pseudo-limit of A.*

Let such a pseudo-limit of A be called *the limit* of A. Clearly it is unique. As a consequence, we see that

(5) *every pseudo-convergent sequence in* **No** *has a unique limit in* **No.**

3. Hahn groups and formal power series fields.

3.0. Both of these topics are treated in Hahn's paper of 1907 [**11**]. Many variations have subsequently been written on these two classic themes.

3.1. Let T be a nonempty totally-ordered set and $(G_t)_{t\in T}$ a family of totally-ordered additive groups. Let P be the full Cartesian product $\prod_{t\in T} G_t$. Under pointwise operations P is an additive group. Given $x \in P$ let the *support* of x, $\operatorname{supp}(x)$, be defined to be $\{t \in T: x(t)$ is a nonzero element in $G_t\}$; let

(1) $$H \equiv \{x \in P : \operatorname{supp}(x) \text{ is a well-ordered subset of } T\}.$$

It is very easy to see that H is a subgroup of P which contais the direct sum of the G_t's. For $x \in H^*$ let $W(x)$ denote the least element in supp(x). Let $W(0) \equiv \infty$, an element ordered so that $t < \infty$ for all $t \in T$. One easily sees that, for all $x, y \in H$,

$$W(x \pm y) \geqslant \min W(x), W(y), \tag{2}$$

and

$$\textit{if } W(x) \neq W(y) \textit{ then equality holds in } (2). \tag{3}$$

(Cf. (1.1: 3 and 4).) H can be made into a totally-ordered group by declaring $x \in H^*$ to be positive if $x(W(x)) > 0$. (This is the so called *lexicographic order* on H.) Under this order, H is called the *Hahn group* of $(G_t)_{t \in T}$.

Assume now, in addition, that each G_t is a nonzero Archimedean group. When the results of §1.1 are applied to H, one finds that the value set of H is naturally isomorphic to T and the factors of H are naturally isomorphic to the G_t's.

The Hahn Imbedding Theorem tells us that given any totally-ordered group G with value set T and factors $(G_t)_{t \in T}$, then G can be isomorphically imbedded in H, its corresponding Hahn group.

3.2. Formal power series fields are constructed in much the same way. Let K be a field, G a totally-ordered additive group, and $P \equiv K^G$; then under pointwise operations P is a vector space over K. For $f \in P$ let supp$(f) \equiv \{g \in G: f(g) \neq 0\}$ and let $F \equiv \{f \in P$: supp(f) is a well-ordered subset of $G\}$. Then F is a sub-K-space of P. For $u, v \in F$ and $z \in G$ let

$$uv(z) \equiv \sum_{x+y=z} u(x)v(y). \tag{1}$$

A priori the sum on the right-hand side of (1) may involve an infinite number of nonzero terms. Fix z in G. First we may confine our attention to $x \in$ supp(u). Next we need only be concerned about $y \in$ supp$(v) \cap \{z - x : x \in$ supp$(u)\}$; but this set is both well-ordered and anti-well-ordered; i.e., it is finite. As a consequence, the sum on the right side of (1) has only a finite number of nonzero terms in it and thus uv is in P. It can be shown that $uv \in F$ and F is a commutative ring with identity. It is deeper to see that

$$F \textit{ is a field}. \tag{2}$$

Hahn proved this directly by letting $u \in F^*$ and constructing $v \in F^*$, such that $uv = 1$, by transfinite induction. Other proofs of (2) have been given; one of the most general and revealing was given by B. H. Neumann **[20]**.

Given $f \in F^*$ let $V(f)$ be the least element in supp(f); then

(3) *V is a valuation of the field F whose value group is G.*

Let K be a totally-ordered Archimedean field and let F be given the lexicographic order; then

(4) *F is a totally-ordered field and V is equivalent to the order-valuation on F.*

We will refer to F as *the field of formal power series with coefficients in K and exponents in G.*

Finally it should be noted that

(5) *the field F is maximal* [**13**].

A subfield F' of F having the same residue class field and value group as F will be called *a field of formal power series with coefficients in K and exponents in G*. Note F is an immediate extension of such a field F'.

3.3. Conway remarks [**6**, p. 33] that his theorem on the normal form for $x \in \mathbf{No}$ [**6**, Theorem 21, p. 33] "...can be interpreted as showing that the structure of **No** as a Field can be obtained from its structure as an additive Group by means of the Malcev-Neumann transfinite power-series construction."

One of the goals of the research we are now reporting on was to construct something very much like a Field F of formal power series with coefficients in $\mathbb{R}$ and exponents in a totally-ordered additive Group G, constructed using some elementary process other than that used to construct **No**, such that F and **No** are isomorphic. There are two kinds of possible obstructions to this program: set theoretic and algebraic. We saw in §1.6 that G must be a proper class. Were we to try to proceed as we did in §3.2, the first object constructed would be $P \equiv \mathbb{R}^G$, but, since G is a proper class, $P = \varnothing$ [**19**, p. 55]. Thus some variation on this method must be sought. Fortunately, much of the ordered algebra required is known. It will be recalled and synthesized for the reader in the next section.

4. η_ξ-structures of Cantor, Hausdorff, Sierpiński et al.

4.0. In the first part of Cantor's great 1895 monograph [**4**], he gives the following characterization of the order type η of the set $R \equiv \{x \in \mathbb{Q}: 0 < x < 1\}$: (i) R is countably infinite; (ii) R has no least and no greatest element; and (iii) R is everywhere dense [**4**, pp. 504–508].

In 1914, Hausdorff [**12**, pp. 180–185], generalized Cantor's order type η as follows. Let ξ be an ordinal number. A totally-ordered set E is called an η_ξ-set if, given any two subclasses H and K of E such that if

(1) $$H < K$$

(0.1), and

(2) $$|H| + |K| < \aleph_\xi,$$

there exists $e \in E$ such that $H < \{e\} < K$. (Cf. Conway's condition (0.1:4).) Clearly a countable η_0-set is one that satisfies Cantor's conditions (i)–(iii) above, and conversely. Clearly our definition (0.3: 1) of an η-Class is a variant of Hausdorff's in which (2) is replaced with the condition that H and K be sets. Hausdorff proved the following [**12**, pp. 180–185]:

(3) *Given any totally-ordered set T with $|T| \leqslant \aleph_\xi$, it may be mapped into an η_ξ-set E by means of an order-preserving map.*

(4) *Any two η_ξ-sets of power $\aleph_\xi$ are order-isomorphic.*

(5) *If $\aleph_\xi$ is singular, any η_ξ-set is an $\eta_{\xi+1}$-set.*

Assume, henceforth, that $\xi > 0$.

The simplest and most elegant construction of an η_ξ-set which the author knows of is the following, due to Sierpiński [25]:

(6) *Let* $S_\xi \equiv \{f \in \{0,1\}^{\omega_\xi}$: *there exists* $\alpha < \omega_\xi$ *such that* $f(\alpha) = 1$, *and for all* β, *with* $\alpha < \beta < \omega_\xi$, *then* $f(\beta) = 0\}$.

(*Note*. Since the ordinals in Conway [6] and Mark [19, p. 68] are von Neumann ordinals [21], any ordinal is $\{\beta: \beta < \alpha\}$ for some ordinal α. ω_ξ denotes, as usual, the least ordinal of power $\aleph_\xi$. These are sometimes called *initial ordinals*. Thus, for example, ω_0 is the set of all finite ordinals, ω_1 is the set of all countable ordinals, etc.)

(7) *If* $\aleph_\xi$ *is regular, then* S_ξ *is an* η_ξ*-set.*

(See, e.g., [17, pp. 336–338] for a proof.)

It is well known that

(8) *an* η_ξ*-set of power* $\aleph_\xi$ *exists*

if and only if

(9) $\aleph_\xi$ *is regular and* $\sum_{\alpha<\xi} 2^{\aleph_\alpha} \leqslant \aleph_\xi$.

Note that if $\xi = \alpha + 1$ then $\aleph_\xi$ is regular; thus in this case (9) is equivalent to

(10) $2^{\aleph_\alpha} = \aleph_{\alpha+1}$,

which is part of the Generalized Continuum Hypothesis. Note also that

(11) *if* $\aleph_\xi$ *is strongly inaccessible, then* (9) *holds.*

(12) *If an* $\eta_{\alpha+1}$*-set exists, it must be of power at least* $2^{\aleph_\alpha}$.

(See, e.g., [17, p. 338].)

For $f \in S_\xi$, let $\operatorname{supp}(f) \equiv \{\beta \in \omega_\xi: f(\beta) \neq 0\}$; then this set is a nonempty subset of ω_ξ which has a greatest element.

4.1. Let Γ be a divisible totally-ordered additive group which is an η_ξ-set. Then:

(1) *Given any totally-ordered additive group* H, *with* $|H| \leqslant \aleph_\xi$, *there exists an order-preserving (group) isomorphism of* H *into* Γ [1].

(2) *Any two divisible totally-ordered additive groups that are* η_ξ*-sets of power* $\aleph_\xi$ *are (order and group theoretically) isomorphic* [1].

(Cf. (4.0: 3 and 4).) Let G be a divisible totally-ordered group and S its value set (1.1).

(3) *G is an η_ξ-set if and only if* (i) *S is an η_ξ-set,* (ii) *its factors are isomorphic to* $\mathbb{R}$, *and* (iii) *it is* ξ*-pseudo-complete* (2.1) [2, 3].

Let E be an η_ξ-set. For all $e \in E$ let $G_e \equiv \mathbb{R}$ and let H be the Hahn group of $(G_e)_{e\in E}$. Let $G \equiv \{x \in H: |\operatorname{supp}(x)| < \aleph_\xi\}$; then using (3) we see that

(4) *G is a divisible totally-ordered additive group which is an η_ξ-set,*

showing that such groups exist. Further,

(5) *if* $|E| = \aleph_\xi$ *then* $|G| = \aleph_\xi$.

Concerning (5) the reader should note that in (4.0) we assumed $\xi > 0$. Hausdorff noted [**12**, pp. 180–185] (see also [**9**, p. 177]) that an η_1-set must have power at least the continuum; thus E and G have the same power.

4.2. Let G be a divisible totally-ordered additive group which is an η_ξ-set and let

(1) $F \equiv \left\{ f \in \mathbb{R}^G\text{: supp}(f) \textit{ is a well-ordered subset of } G \textit{ of power less than } \aleph_\xi \right\}.$

F is then a formal power series field with coefficients in $\mathbb{R}$ and exponents in G. Under the lexicographic order, F is a totally-ordered field.

(2) *F is a real-closed field which is an η_ξ-set* [**2**, **3**].

Further, theorems similar to (4.1: 1 and 2), but for the category of ordered fields, hold. See [**8**, or **9**, pp. 180–183, 193] for details.

4.3. With this in hand, we can synthesize these three constructions as follows: Let S_ξ be Sierpiński's set of (4.0: 6); then it is an η_ξ-set (4.0: 7). Let G_ξ be the group defined in 4.1 with $E \equiv S_\xi$; then it is a divisible totally-ordered additive group which is an η_ξ-set (4.1: 4). Let F_ξ be given by (4.2: 1) with $G \equiv G_\xi$; then it is a real-closed field which is an η_ξ-set. Further, if $|S_\xi| = \aleph_\xi$ then $|F_\xi| = \aleph_\xi$ (4.1: 5).

It is these fields F_ξ that will be our basic building blocks used to construct a field isomorphic to **No**. The first of these construction occupies the next section.

5. A construction of Conway's numbers within a universe within set theory.

5.0. Recall that a cardinal number $\mathbf{M}$ is called *strongly inaccessible* if $\mathbf{M} > \aleph_0$, $\mathbf{M}$ is regular, and if $\mathbf{N} < \mathbf{M}$ implies $2^{\mathbf{N}} < \mathbf{M}$. It is not possible to prove the existence of strongly inaccessible cardinals within set theory. Nevertheless, it has seemed both natural and useful to many mathematicians to assume that such numbers exist. For example, using Tarski's Axiom (see, e.g., [**17**, p. 326]), given any cardinal $\mathbf{N}$ there exists a strongly inaccessible cardinal $\mathbf{M}$ with $\mathbf{N} < \mathbf{M}$.

(1) Assume there exists a strongly inaccessible number $\aleph_\xi$.

One of the greatest uses of (1) is that it allows one to construct a set A of power $\aleph_\xi$, called a universe [**19**, p. 160], such that within A are enough sets to do all of the standard constructions of set theory for sets of power less than $\aleph_\xi$. (See, e.g., [**19**, pp. 112–114, 159–163] for details.) Note that each $x \in A$ is a set of power less than $\aleph_\xi$.

Let $\xi\mathbf{No}$ be the class of all numbers constructed using Conway's construction (0.1: 4) subject only to the constraint that each L and R must be elements of the universe A. Since A is a universe for set theory within set theory, Conway's proofs shows that

(2) *$\xi\mathbf{No}$ is a real-closed Field.*

Our theorem (0.3: 2), which holds for $\xi\mathbf{No}$, within the universe A now takes the following form:

(3) *$\xi\mathbf{No}$ is an η_ξ-set of power $\aleph_\xi$.*

Proof. Let L and R be subsets of $\xi\mathbf{No}$ such that $L < R$ and $|L| + |R| < \aleph_\xi$; then L and R are in A [**19**, 23.12(v), p. 160]. Hence $\{ L \mid R \} \equiv x \in \xi\mathbf{No}$ and $L < \{x\} < R$, proving that $\xi\mathbf{No}$ is an η_ξ-class. Since A is a set and $\xi\mathbf{No} \subset A$, $\xi\mathbf{No}$ is a set. Since $|A| = \aleph_\xi$, $|\xi\mathbf{No}| \leqslant \aleph_\xi$. Since $\xi\mathbf{No}$ is an η_ξ-set, its power is at least $\aleph_\xi$, proving (3).

5.1. Consider the field F_ξ constructed in §4.3. It is a real-closed field which is an η_ξ-set. Since $\aleph_\xi$ is strongly inaccessible (5.0: 1), (4.0: 9) holds; thus $|F_\xi| = \aleph_\xi$. Hence

(1) ξ**No** *and* F_ξ *are isomorphic fields* [**8**].

5.2. Monk notes that all universes within set theory are associated with a strongly inaccessible cardinal number as A is [**19**, p. 161]. We have exploited the usual advantage of working within a universe. The construction of ξ**No**, along the lines Conway gives, is done within A; however, ξ**No** is a set, even though it is not in the universe A. Thus we can use set theory to construct F_ξ, which is isomorphic to ξ**No**. On the other hand, Conway's Field **No** is a proper class, and the calculus of proper classes is by necessity much more restricted than that of sets.

We will try to use what insight we may have obtained in this section in what follows, even though we will

(1) drop assumption (5.0: 1).

6. A construction of a Field isomorphic to No.

6.0. We make our construction by taking direct limits in various categories over the index Class **Ord**, the Class of all ordinal numbers. Since **Ord** is a proper Class, care will be taken to show that the procedures we use are permissible within the set theory we have chosen to work in [**19**].

6.1. It is well known that for all $\alpha \in$ **Ord**, $\aleph_{\alpha+1}$ is regular. (See, e.g., [**17**, p. 309].) By (4.0: 7)

(1) *for each* $\alpha \in$ **Ord**, $S_{\alpha+1}$ *is an* $\eta_{\alpha+1}$*-set.*

Recall that elements of $S_{\alpha+1}$ are maps from $\omega_{\alpha+1}$ to $\{0,1\}$ (4.0: 6). Let β be an ordinal such that $\beta \geqslant \alpha$.

(2) Let i^α_β be the inclusion map of $\omega_{\alpha+1}$ into $\omega_{\beta+1}$.

Note that i^α_β is order-preserving, i^α_α is the identity map of $\omega_{\alpha+1}$, and, if $\gamma \geqslant \beta$, then $i^\beta_\gamma i^\alpha_\beta = i^\alpha_\gamma$.

(3) For $f \in S_{\alpha+1}$ let $m^\alpha_\beta(f)$ be the map g of $\omega_{\beta+1}$ into $\{0,1\}$ such that for all $\delta \in \omega_{\beta+1} - i^\alpha_\beta(\mathrm{supp}(f))$, $g(\delta) = 0$, and for all $\delta \in i^\alpha_\beta(\mathrm{supp}(f))$, $g(\delta) = f(\delta)$.

Then g is in $S_{\beta+1}$ and m^α_β is order-preserving.

Since the $\omega_{\alpha+1}$'s are distinct sets, the $S_{\alpha+1}$'s are disjoint sets. Let $\Sigma \equiv \bigcup_{\alpha \in \mathbf{Ord}} S_{\alpha+1}$. (*Note.* This is a well-defined proper Class [**19**, p. 51].) Let $f, g \in \Sigma$. There exist unique $\alpha, \beta \in$ **Ord** such that $f \in S_{\alpha+1}$ and $g \in S_{\beta+1}$. Without loss of generality we may assume $\beta \geqslant \alpha$. We say that

(4) f and g are *equivalent*, $f \sim g$, if $m^\alpha_\beta(f) = g$.

Clearly, this is an equivalence relation on Σ. Given $g \in \Sigma$ let α be the least ordinal such that there exists $f_0 \in \Sigma$ such that $m^\alpha_\beta(f_0) = g$; then f_0 is unique and $\alpha \equiv \mathrm{index}(g)$. f_0 will be called *the representative* of the equivalence Class to which g belongs.

(5) Let S be the Class of all representatives of all equivalence classes of Σ mod $\sim$.

For $f \in S_{\alpha+1}$ let $m^\alpha(f)$ be the representative of the equivalence class to which f belongs. m^α is a monomorphism of $S_{\alpha+1}$ into S and the following holds:

(6) $$m^\beta m^\alpha_\beta = m^\alpha.$$

Let f_0 and g_0 be in S with $f_0 \in S_{\alpha+1}$, $g_0 \in S_{\beta+1}$ and $\beta \geqslant \alpha$. We define

(7) $$f_0 \leqslant g_0 \quad \text{iff} \quad m^\alpha_\beta(f_0) \leqslant g_0.$$

Since each $S_{\alpha+1}$ is a totally-ordered set, S is a totally ordered Class. Thus S is the direct limit, $\mathrm{Lim}_\rightarrow\, S_{\alpha+1}$. The reason for considering S is that

(8) *S is an η-Class.*

Proof. Let L_0 and R_0 be subsets of S such that $L_0 < R_0$. Let $U_0 \equiv L_0 \cup R_0$. Since U_0 is a set, there exists $\alpha \in \mathbf{Ord}$ such that

(9) $$\{\mathrm{index}\,(f_0) : f_0 \in U_0\} \subset \omega_{\alpha+1}, \quad \text{and} \quad |U_0| < \aleph_{\alpha+1}.$$

For each $f_0 \in U_0$ there exists a unique $f \in S_{\alpha+1}$ such that $f \sim f_0$. Let $L \equiv \{f \in S_{\alpha+1}: f \sim f_0 \text{ for some } f_0 \in L_0\}$, and let R be similarly defined. Clearly $L < R$ and $|L| + |R| < \aleph_{\alpha+1}$. Since $S_{\alpha+1}$ is an $\eta_{\alpha+1}$-set (1), there exists $h \in S_{\alpha+1}$ such that $L < \{h\} < R$. Clearly $L_0 < \{h\} < R_0$, establishing (8).

Let TOC denote the category whose objects are totally-ordered Classes and whose morphisms are maps from one such object to another that preserve $\leqslant$. We have then shown that the direct limit of the $S_{\alpha+1}$'s, taken together with the m^α_β's, exists in this category.

6.2. For $\alpha \in \mathbf{Ord}$, let $G_{\alpha+1}$ be the totally-ordered divisible group that was defined in §4.3. We saw there that it is an $\eta_{\alpha+1}$-set. Let TOG denote the category whose objects are all totally-ordered additive Groups and whose morphisms are all homomorphisms between those objects that preserve $\leqslant$; then the $G_{\alpha+1}$'s are objects in TOG.

Let $\beta \geqslant \alpha$ and $f \in G_{\alpha+1}$. f is then a map of $S_{\alpha+1}$ into $\mathbb{R}$.

(1) Let $h^\alpha_\beta(f)$ be the map g of $S_{\beta+1}$ into $\mathbb{R}$ such that, for all $x \in S_{\beta+1} - m^\alpha_\beta(\mathrm{supp}(f))$, $g(x) = 0$, and for all $x \in m^\alpha_\beta(\mathrm{supp}(f))$, $g(x) = f(x)$.

Since m^α_β is order-preserving and $\beta \geqslant \alpha$, h^α_β is an order-preserving homomorphism of $G_{\alpha+1}$ into $G_{\beta+1}$; thus it is a monomorphism in TOG. Clearly, if $\gamma \geqslant \beta$ then $h^\beta_\gamma h^\alpha_\beta = h^\alpha_\gamma$. We want to show that $\mathrm{Lim}_\rightarrow\, G_{\alpha+1}$ exists in TOG.

Since the $S_{\alpha+1}$'s are disjoint, so are the $G_{\alpha+1}$'s. Let $\Gamma \equiv \bigcup_{\alpha \in \mathrm{Ord}} G_{\alpha+1}$. Let *equivalence*, $\sim$, be defined in Γ as it was in (6.1: 4), and *the representative* of some $g \in \Gamma$ as it was in §6.1. Let G be defined as S was (6.1: 5). For $f \in G_{\alpha+1}$ let $h^\alpha(f)$ be the representative of the equivalence class to which f belongs; then h^α maps $G_{\alpha+1}$ into G and $h^\alpha = h^\beta h^\alpha_\beta$ for all $\beta \geqslant \alpha$. Let the order and group structure of the $G_{\alpha+1}$'s induce the structure of an object in TOG on G; then

(2) $$G = \mathrm{Lim}_\rightarrow\, G_{\alpha+1} \quad \textit{and} \quad G \textit{ is divisible and an } \eta\textit{-Class},$$

by (6.1: 6).

6.3. For $\alpha \in \mathbf{Ord}$ let $F_{\alpha+1}$ be the totally-ordered field defined in §4.3. We noted in §4.3 that it is an $\eta_{\alpha+1}$-set that is a real-closed field. Let TOF denote the category whose objects are all totally-ordered Fields and whose morphisms are all field monomorphisms between those objects that are order-preserving. Then each $F_{\alpha+1}$ is an object in TOF. We can proceed as we did in §6.2 to show that $\mathrm{Lim}_{\rightarrow}\ F_{\alpha+1} \equiv F$ exists, in our set theory, and is an object in TOF.

(1) *F is a real-closed Field which is an η-Class.*

6.4.

(1) *F and* **No** *are isomorphic Fields.*

Let K be a totally-ordered Field. Conway [**6**, p. 42] defines K as having the *universal embedding property* if, given any subfield k (which we require to be a set) of K and any extension g of k in TOF (g being a set), there exists a subfield $\bar{g}$ of K that contains k and a k-isomorphism of g onto $\bar{g}$. Conway proves that **No** has this property [**6**, Theorem 28, p. 42]. The same proof shows that

(2) *any real-closed totally-ordered Field that is an η-Class has the universal embedding property.*

Conway then states and proves Theorem 29 [**6**, p. 43]: *any object K in* TOF *that has the universal embedding property is isomorphic to* **No**. This result leads us to:

(3) *F is isomorphic to* **No**.

Since the set theory we are using is a variant on that used by Conway, we will add a little to his discussion. Conway defines N_α to be the set of all numbers born on day α, where $\alpha \in \mathbf{Ord}$ [**6**, p. 29], and shows that

(4) $(N_\alpha)_{\alpha\in\mathbf{Ord}}$ *is a partition of* **No** [**6**, *p*. 30];

thus **No** can be given a well-ordering $(x_\alpha)_{\alpha\in\mathbf{Ord}}$. Having constructed F as a direct limit of the $F_{\alpha+1}$'s we can identify each $F_{\alpha+1}$ with its image in F and thus regard F as the union of $(F_{\alpha+1})_{\alpha\in\mathbf{Ord}}$. Since each $F_{\alpha+1}$ is a set, we can also give F a well-ordering $(y_\alpha)_{\alpha\in\mathbf{Ord}}$. We then can follow Conway's proof of Theorem 29 [**6**, p. 43], which combines elements of the Artin-Schreier Theory [**18**], with Cantor's proof of the characterization of the set R (4.0) [**4**, pp. 504–508], establishing (3), the main result of this section.

7. Some distinguished subfields of No.

7.0. The construction of F_ξ in §4.3 suggests that **No** may have some analogous subfields. That this is indeed the case will be seen in this section.

(1) Let ξ be in **Ord** such that $\xi > 0$ and with ω_ξ (or, equivalently, $\aleph_\xi$) regular.

Such ordinals are very plentiful. For example, if $\xi = \alpha + 1$ for some $\alpha \in \mathbf{Ord}$, then ξ satisfies (1).

(2) Let $\xi\mathbf{No}$ be the class of all $x \in \mathbf{No}$ such that there exist subsets L and R of $\xi\mathbf{No}$ for which $L < R$, $|L| + |R| < \aleph_\xi$, and $x = \{L \mid R\}$.

Note that if $\aleph_\xi$ is strongly inaccessible, then $\xi\mathbf{No}$, defined in (2), is the same as $\xi\mathbf{No}$ defined in §5.0; thus $\xi\mathbf{No}$ as defined in (2) is a generalization of the Field considered in §5.

7.1.

(1) $\xi\mathbf{No}$ *is a sub-Field of* $\mathbf{No}$.

PROOF. Certainly $\varnothing \subset \xi\mathbf{No}$; thus $0\ (\equiv \{\ |\ \})$ and $1\ (\equiv \{0|\ \})$ are in $\xi\mathbf{No}$. On referring to the definitions of addition and multiplication in **No** [**6**, p. 5], one sees that $\xi\mathbf{No}$ is a sub-Ring of **No**. Let $x > 0$ be in $\xi\mathbf{No}$, with $x = \{L \mid R\}$, for L and R as in (7.0: 2). The construction of $1/x$ by Conway [**6**, pp. 21–22] involves not only the usual induction but also an induction over ω. Since $\xi > 0$, $1/x \in \xi\mathbf{No}$, establishing (1).

A little further reflection yields the following:

(2) $\mathbb{R}$ *is a subfield of* $\xi\mathbf{No}$. *Each ordinal* $\alpha < \omega_\xi$ *is in* $\xi\mathbf{No}$.

From the discussion of ω^x [**6**, pp. 31–32] and the fact that $\xi > 0$ (7.0: 1), one easily sees that

(3) *for each* $y \in \xi\mathbf{No}$, ω^y *is in* $\xi\mathbf{No}$; *for each* $x > 0$ *in* $\xi\mathbf{No}$ *there exists* $y \in \xi\mathbf{No}$ *such that* x *is commensurate with* ω^y.

On the other hand, since ω_ξ is regular, it has no cofinal set L of power less than $\aleph_\xi$; thus

(4) ω_ξ *is in* $\mathbf{No}$ *but not in* $\xi\mathbf{No}$.

From (4) we see that $\xi\mathbf{No}$ is a proper sub-Field of **No**. The following is much stronger:

(5) $$\xi\mathbf{No} \subset O_{\omega_\xi}.$$

(See [**6**, p. 29] for definition.)

PROOF. Let x be in $\xi\mathbf{No}$ with $x \equiv \{L \mid R\}$, (i) $|L| + |R| < \aleph_\xi$, and (ii) $L \cup R \subset O_{\omega_\xi}$. Each $y \in L \cup R \equiv U$ is in O_α, for some $\alpha < \omega_\xi$. Since ω_ξ is regular, we may use (i) and (ii) to show that there exists $\beta < \omega_\xi$ such that $U \subset O_\beta$. Hence, $x \in M_\beta = O_{\beta+1}$ [**6**, p. 29], which is a subset of O_{ω_ξ}, establishing (5).

One corollary of (5) is that

(6) $\xi\mathbf{No}$ *is a set.*

Once we know this it is natural to try to compute the cardinal number of $\xi\mathbf{No}$. Using Conway's notion of the sign-expansion of an element in N_α [**6**, p. 30], we see that

(7) $$|N_\alpha| = 2^{|\alpha|} \quad \text{for all } \alpha \in \mathbf{Ord}.$$

From (7) and the definition of O_α and M_α [**6**, p. 29], we see that

(8) $$|O_\alpha| = \sum_{\beta<\alpha} 2^{|\beta|} \quad \text{and} \quad |M_\alpha| = \sum_{\beta\leqslant\alpha} 2^{|\beta|}.$$

From this we can see that, for any $\gamma \in \mathbf{Ord}$,

(9) $$|O_{\omega_\gamma}| = \aleph_0 + \sum_{\alpha<\gamma} 2^{\aleph_\alpha}$$

and, hence,

(10) $$|O_{\omega_{\gamma+1}}| = 2^{\aleph_\gamma}.$$

PROOF. To prove (9) note that as β runs through ω, $|\beta|$ runs through the nonnegative integers. The resulting contribution to the cardinal on the left side of (9) is thus $\aleph_0$. For $\omega \leqslant \beta < \omega_1$, $|\beta| = \aleph_0$, and thus the contribution to the cardinal number on the left side of (9) is $\aleph_1 \cdot 2^{\aleph_0}$. Continuing in this way, we see that (8) implies (9). (10) follows from (9), establishing these results.

Combining (2), (5), (9) and (10), we see that

(11) $$\aleph_\xi \leqslant |\xi\mathbf{No}| \leqslant \sum_{\alpha<\xi} 2^{\aleph_\alpha},$$

and

(12) $$\aleph_{\alpha+1} \leqslant |\xi\mathbf{No}| \leqslant 2^{\aleph_\alpha} \quad \text{if } \xi = \alpha + 1.$$

7.2. Virtually by definition (7.0: 2), we see that

(1) ξ**No** *is an* η_ξ*-set.*

Using this, (4.0: 12) and (7.1: 12), we see that

(2) $$|(\alpha + 1)\mathbf{No}| = 2^{\aleph_\alpha} \quad \text{for all } \alpha \in \mathbf{Ord}.$$

We have seen (1.6: 7 and 8) that V is an order-valuation for the Field **No** whose value-Group is (**No**, +). Using (7.1: 3), we see that V restricted to ξ**No**, which we denote by

(3) ξV, is a valuation of the ordered field ξ**No** whose value group is (ξ**No**, +).

Utilizing the main theorem of [**2**, p. 712], we know that

(4) ξ**No** *is* ξ*-pseudo-complete* (2.1).

Applying the methods in §2.3, we obtain a stronger version of (4), namely:

(5) *Every pseudo-convergent sequence in* ξ**No** *of length* $\lambda < \omega_\xi$ *has a unique limit in* ξ**No**.

Since $\mathbb{R}$ is a subfield of ξ**No** (7.1: 2) and $\xi > 0$ (7.0: 1), we may apply the argument in Conway [**6**, pp. 40–42] and thus conclude that

(6) ξ**No** *is a real-closed field.*

(7) *If* (i) $2^{\aleph_\alpha} = \aleph_{\alpha+1}$, *then* (ii) $(\alpha + 1)$**No** *and* $F_{\alpha+1}$ *are isomorphic.*

PROOF. Assume that (i) holds; then $(\alpha + 1)$**No** and $F_{\alpha+1}$ are each real-closed fields that are $\eta_{\alpha+1}$-sets of power $\aleph_{\alpha+1}$. It is well known that any two such fields are isomorphic (see [**8**, or **9**, p. 193]).

In any event we know that

(8) $$\mathbf{No} = \bigcup_{\alpha \in \mathbf{Ord}} (\alpha + 1)\mathbf{No}.$$

Since **No** is real-closed,

(9) ξ**No** *is relatively algebraically closed in* **No**.

Using (2) we know that

(10) *the transcendence degree of* $(\alpha + 2)$**No** *over* $(\alpha + 1)$**No** *is* $2^{\aleph_{\alpha+1}}$.

7.3. The results in this section show how close the structure of the very naturally defined subfields $\xi\mathbf{No}$ are to the fields F_α used to define F in §6. This similarity can be made even closer under additional set-theoretic hypotheses.

(1) *Assume the Generalized Continuum Hypotheses: then, for all* $\alpha \in \mathbf{Ord}$, $(\alpha + 1)\mathbf{No}$ *and* F_α *are isomorphic.*

PROOF. Since each field is an $\eta_{\alpha+1}$-field of power $\aleph_{\alpha+1}$ that is real-closed, they are isomorphic. (See [**8** or **9**].)

Another hypothesis that simplifies matters is Tarski's Axiom [**17**, p. 326], which implies that

(2) *given any cardinal number* $\mathbf{N}$ *there exists a strongly inaccessible cardinal* $\mathbf{M}$ *such that* $\mathbf{M} > \mathbf{N}$.

Assume that (2) holds and let $\mathbf{Ord}_0 \equiv \{\xi \in \mathbf{Ord}: \aleph_\xi \text{ is strongly inaccessible}\}$. We can then modify the construction in §6 to employ only $\xi \in \mathbf{Ord}_0$, since $\mathbf{Ord}_0$ is cofinal in $\mathbf{Ord}$; thus we would define F to be $\mathrm{Lim}_{\to \xi \in \mathbf{Ord}_0} F_\xi$ and then note that

(3) $\xi \in \mathbf{Ord}_0$ *implies that* F_ξ *and* $\xi\mathbf{No}$ *are isomorphic.*

Of course, (2) is equivalent to the statement that $\mathbf{Ord}_0$ is cofinal in $\mathbf{Ord}$. Without some assumption we cannot even prove that $\mathbf{Ord}_0 \neq \varnothing$. For this reason, in §6 we chose to take limits along the class of all nonlimit ordinal numbers.

8. Conway partitions.

8.0. Conway made the following observation [**6**, p. 43]: "As an abstract Field, **No** is the unique universally embedding totally ordered Field.

"We repeat that **No** has plenty of additional strucrture which would not emerge from this 'definition'."

The purpose of this section is to describe a way of defining this additional structure on **No** as an abstract Field.

8.1. Recall [**6**, pp. 29–30] that Conway defines N_α as the set of all numbers "born first on day α". Further, he showed that

(1) $(N_\alpha)_{\alpha \in \mathbf{Ord}}$ *is a partition of* $\mathbf{No}$ *and* $N_0 = \{0\}$.

One can define M_α to be $\bigcup_{\beta \leqslant \alpha} N_\alpha$ and $O_\alpha \equiv \bigcup_{\beta < \alpha} N_\beta$. Clearly, $\alpha \leqslant \beta$ implies $M_\alpha \subset M_\beta$ and $O_\alpha \subset O_\beta$. Let L and R be subsets of **No** with $L < R$ and let $x \equiv \{L \mid R\}$.

(2) *If* $L \cup R \subset O_\alpha$, *then* $x \in M_\alpha$ [**6**, p. 29].

Further, as we have noted before,

(3) $$L < \{x\} < R.$$

Indeed, since $L < R$, each x^{L} is less than each x^{R}. Since $x = x$, $x \geqslant x$; thus no $x^{\mathrm{R}} \leqslant x$ and $x \leqslant$ no x^{L} [**6**, p. 4]. As a consequence, $x^{\mathrm{L}} < x < x^{\mathrm{R}}$, establishing (3).

Assume now that $x \in N_\alpha$; i.e., that α is the birthday of x, or in symbols, $\alpha = b(x)$.

(4) *Given* $y \in M_\alpha$ *such that* $L < \{y\} < R$, *then* $y = x$ [**6**, p. 23].

8.2. Let T be a totally-ordered η-Class. Let $P \equiv (N(\alpha))_{\alpha \in \mathbf{Ord}}$ be a partition of T for which $N(0) \equiv \{t(0)\}$.

For each $x \in T$ there exists a unique $\alpha \in \mathbf{Ord}$, called the *birthday of* x and denoted by $b(x)$, such that $x \in N(\alpha)$. It is convenient to define $M(\alpha)$ to be $\bigcup_{\beta \leqslant \alpha} N(\beta)$ and $O(\alpha)$ to be $\bigcup_{\beta < \alpha} N(\beta)$. Clearly, $\alpha \leqslant \beta$ implies $M(\alpha) \subset M(\beta)$ and $O(\alpha) \subset O(\beta)$. Let L and R be subsets of T such that $L < R$. Since T is an η-Class, $S \equiv \{x \in T\colon L < \{x\} < R\}$ is a nonempty Class. Let α be the least element in $b(S)$ ($\equiv \{b(x)\colon x \in S\}$), and let $x \in S$ have birthday α.

The partition P will be called a *Conway partition of* T if the following hold:

(1) given $y \in S$ with birthday α, then $y = x$,

and

(2) if $L \cup R \subset O(\alpha)$ then $x \in M(\alpha)$.

EXAMPLE. $(N_\alpha)_{\alpha \in \mathbf{Ord}}$ is a Conway partition of **No** (§8.1).

Assume that P is a Conway partition of T.

(3) *There exists a unique order-preserving map t of* **No** *onto T such that for all $x \in$* **No**, *$b(t(x)) = b(x)$, i.e., such that t preserves birth order.*

PROOF. Let t take 0 in M_0 to $t(0)$ in $M(0)$. For some $\alpha \in \mathbf{Ord}$, with $\alpha > 0$, assume that t, as defined above on M_0, admits a unique order-preserving extension to O_α that maps O_α onto $O(\alpha)$ and preserves birth order. Let $x \in N_\alpha$. $x = \{L \mid R\}$ for some subsets L and R of O_α [**6**, p. 29]. By assumption, $t(L) \cup t(R) \subset O(\alpha)$ and $t(L) < t(R)$. Since T is an η-Class, $S \equiv \{u \in T\colon t(L) < \{u\} < t(R)\}$ is a nonempty Class. Let $t(x)$ be an element in S such that $b(t(x))$ is the least element β in $b(S)$. By (1), $t(x)$ is unique. By (2), $\beta \leqslant \alpha$. Assume, for a moment, that $\beta < \alpha$; then $t(x) \in O(\alpha)$. Since t is an order-preserving map of O_α onto $O(\alpha)$ which preserves birth order, there exists $y \in O_\alpha$ such that $t(y) = t(x)$. Hence $L < \{y\} < R$. By (8.1: 4) $x = y$; thus $a = b(x) = b(y) = b(t(y)) = b(t(x)) = \beta$, which is absurd, proving that $\beta = \alpha$. Thus (3) is proved by induction.

Using the Conway partition P on T, we can define addition and multiplication on T as Conway did; then the map t (3) of **No** onto T preserves these operations: thus T is a Field and

(4) *t is an isomorphism of the Field* **No** *onto the Field T.*

8.3. A glance at the proof of Conway's Theorem 29 [**6**, p. 43] suggests that **No** has a vast number of automorphisms. A little further reflection will suggest that, given any subfield k of **No**, **No** also has a vast number of k-automorphisms. Next note that an automorphism of a real-closed field must be order-preserving, since it must preserve squares and the squares are exactly the nonnegative elements of the field. From (8.2: 3) we see that

(1) *the only automorphism of* **No** *which preserves birth order is the identity map.*

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