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AN APPLICATION OF FLOWS TO TIME SHIFT AND TIME REVERSAL IN STOCHASTIC PROCESSES[1]

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ABSTRACT. A simple proposition (Theorem 1) on flows allows the investigation of random time shift and time reversal in Markov processes without assuming any regularity of paths. Theorem 5 is a generalization of Nagasawa's time reversal theorem and Theorem 4 generalizes a recent result of Getoor and Glover.

1. Flows.

1.1. We consider a flow in a measurable space $(\Omega, \mathscr{F})$ that is a family of transformations θ_t, $t \in \mathbf{R}$, such that $\theta_s\theta_t = \theta_{s+t}$ for all $s, t \in \mathbf{R}$ and $\{(t, \omega): \theta_t\omega \in A\} \in \mathscr{B}_{\mathbf{R}} \times \mathscr{B}$ for all $A \in \mathscr{F}$($\mathscr{B}_R$ is the Borel σ-algebra in $\mathbf{R}$).

We put $(\theta_t Y)(\omega) = Y(\theta_t\omega)$, for every function Y and

$$(P\theta_t)(A) = P(\theta_t^{-1}A), \qquad A \in \mathscr{F},$$

$$\overline{P} = \int_{\mathbf{R}} P\theta_t\, dt,$$

for every measure P. We denote by $\mathscr{A}$ the collection of all sets $A \in \mathscr{F}$ such that $\theta_t^{-1}A = A$ for all $t \in \mathbf{R}$. A function Y is measurable with respect to the σ-algebra $\mathscr{A}$ if and only if it is $\mathscr{F}$-measurable and invariant under all transformations θ_t. Obviously $\overline{P}(A) = 0$ or $+\infty$ for all $A \in \mathscr{A}$. Nevertheless we prove

THEOREM 1. *If $\overline{P}$ is σ-finite (over $\mathscr{F}$), it determines uniquely the values $P(A)$ for all $A \in \mathscr{A}$.*

1.2. As a tool we use stationary times. A *stationary time* τ is a measurable mapping from Ω to the extended real line $[-\infty, +\infty]$ such that $\theta_t\tau = \tau - t$ for all $t \in \mathbf{R}$.

Suppose that $\mathscr{F}$ coincides with its completion with respect to the class of all probability measures. Then the first hitting time of a set $A \in \mathscr{F}$,

$$\tau_A^+ = \inf\{t: \theta_t\omega \in A\},$$

and the last exit time from A,

$$\tau_A^- = \sup\{t: \theta_t\omega \in A\}$$

are stationary times[2] (the measurability follows from [**5**, Chapter 3, §1]).

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[2] We put $\inf \phi = +\infty$, $\sup \phi = -\infty$.

In general, every stationary time τ is the first hitting time of $\{\tau < 0\}$ and it is the last exit time from $\{\tau > 0\}$.

We denote the indicator function for the set $\{\tau \in \mathbf{R}\}$ by κ_τ. Let ρ be a positive Borel function on $\mathbf{R}$ such that $\int \rho(t)\,dt = 1$. We put $\rho(-\infty) = \rho(+\infty) = 0$. We note that $\theta_t \rho(\tau) = \rho(\tau - t)$. Hence

$$\kappa_\tau = \int \rho(\tau - t)\,dt = \int \theta_t \rho(\tau)\,dt$$

and, for every $\mathscr{A}$-measurable Y,

$$P\kappa_\tau Y = PY\int \theta_t\rho(\tau)\,dt = \int P\theta_t(Y\rho(\tau)) = \overline{P}\rho(\tau)Y. \tag{1.1}$$

1.3. Suppose that there exists a strictly positive function F such that

$$F_s = \int_{-\infty}^{s} \theta_t F\,dt \quad \text{is } P\text{-integrable for some } s \in \mathbf{R}. \tag{1.2}$$

Put

$$\tau_a = \inf\{s : F_s \geqslant a\}, \qquad a > 0.$$

Note that $P\{\tau_a = -\infty\} = 0$ and $\{\tau_a = +\infty\} = \{F_\infty \leqslant a\}$. Hence $\kappa_{\tau_a} = 1_{F_\infty > a} \uparrow 1$ P-a.s. as $a \downarrow 0$. By (1.1),

$$PY = \lim_{a \downarrow 0} \overline{P} Y \rho(\tau_a). \tag{1.3}$$

The restriction of measure P to $\mathscr{A}$ can be recovered from $\overline{P}$ by formula (1.3). An analogous formula can be written if

$$F^s = \int_{s}^{+\infty} \theta_t F\,dt \quad \text{is } P\text{-integrable for some } s \in \mathbf{R}. \tag{1.4}$$

Both conditions (1.2) and (1.4) are satisfied if $\overline{P}$ is σ-finite.

1.4. Let τ be a stationary time. For every function $Z(\omega)$ we denote by $\theta_\tau Z$ the function $Y(\omega)$ defined by the formula

$$Y(\omega) = \begin{cases} Z(\theta_{\tau(\omega)}\omega) & \text{if } \tau(\omega) \in \mathbf{R}, \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that Y is invariant and therefore formulas (1.1) and (1.3) are applicable to Y. We conclude

If P_1 and P_2 are two measures on $(\Omega, \mathscr{F})$ and if $\overline{P}_1 = \overline{P}_2$ is a σ-finite measure, then $P_1\theta_\tau = P_2\theta_\tau$ for all stationary times τ.

1.5. The definition of the measure $\overline{P}$ makes sense if an arbitrary locally compact group G acts on $(\Omega, \mathscr{F})$ (the Lebesgue measure on $\mathbf{R}$ should be replaced by a left- or right-invariant measure on G). Formula (1.1) and its proof remain valid if τ is a G-valued function defined on an invariant set Ω_τ such that $\theta_g\tau(\omega) = \tau(\omega)g^{-1}$ for all $g \in G$. (We put $\kappa_\tau = 1_{\Omega_\tau}$ and we set $\rho(\tau) = 0$ if $\tau(\omega)$ is not defined.)

However the construction in subsection 1.3 is not applicable if $G \neq \mathbf{R}$. It remains an open problem for which groups G Theorem 1 is true.

2. Random time shift in Markov processes.

2.1. Let $p(s, x; t, B)$ be a Markov transition function in $(E, \mathscr{E})$. We put

$$\mu T_t^s(B) = \int \mu(dx) p(s, x; t, B); \qquad T_t^s f(x) = \int p(s, x; t, dy) f(y).$$

Suppose that a σ-finite measure ν_t is given for every $t \in \mathbf{R}$. We say that $\nu = \{\nu_t\}$ is an *entrance rule* if, for all $B \in \mathscr{E}$,

$$\nu_s T_t^s(B) \leqslant \nu_t(B) \quad \text{and} \quad \nu_s T_t^s(B) \uparrow \nu_t(B) \quad \text{as } s \uparrow t.$$

ν is *an entrance rule at time* r if $\nu_t = 0$ for $t \leqslant r$ and $\nu_s T_t^s = \nu_t$ for $r < s < t$.[3] In an analogous way, we define *an entrance rule at time* $-\infty$. Every entrance rule has a representation

$$\nu = \int_{[-\infty, +\infty]} \nu^r \sigma(dr) \tag{2.1}$$

where ν^r is an entrance rule at time r. We note that $\sigma\{-\infty\} = 0$ if and only if $\nu_s T_t^s \downarrow 0$ as $t \uparrow \infty$ for some s (see [**1**, Formula (5.5)]).

A family $h = \{h^t, t \in \mathbf{R}\}$ of positive $\mathscr{E}$-measurable functions is called an *exit rule* if

$$T_t^s h^t \leqslant h^s \quad \text{and} \quad T_t^s h^t \uparrow h^s \quad \text{as } t \downarrow s.$$

All concepts related to entrance rules have natural analogs for exit rules.

The following result is due to Kuznetsov [**4**]. Suppose that $(E, \mathscr{E})$ is a standard Borel space and let an entrance rule μ and an exit rule h satisfy the condition

$$\mu_t(h^t = +\infty) = 0 \quad \text{for all } t. \tag{2.2}$$

Then there exists a stochastic process (X_t, P_μ^h) on a random time interval (α, β) such that, for every $t \in \mathbf{R}$,

$$P_\mu^h(\alpha < t, X_t \in dy, t < \beta) = \mu_t(dy) h^t(y), \tag{2.3}$$

and for all $t_1 < \cdots < t_n \in \mathbf{R}$,

$$\begin{aligned} &P_\mu^h(\alpha < t_1, X_{t_1} \in dy_1, \dots, X_{t_n} \in dy_n, t_n < \beta) \\ &\quad = \mu_{t_1}(dy_1) p(t_1, y_1; t_2, dy_2) \cdots p(t_{n-1}, y_{n-1}; t_n, dy_n) h^{t_n}(y_n). \end{aligned} \tag{2.4}$$

The measure P_μ^h is σ-finite.

2.2. Suppose that a transition function p is stationary. Excessive measures and functions can be characterized as entrance and exit rules independent of t. Let μ be an entrance rule. If the measure

$$\bar{\mu} = \int_{\mathbf{R}} \mu_t \, dt$$

is σ-finite, it is excessive. To every exit rule h there corresponds an excessive function

$$\bar{h} = \int_{\mathbf{R}} h^t \, dt.$$

[3] In the literature, entrance rules at time 0 are called entrance laws.

We assume that the process $X_t(\omega)$ can be chosen in such a way that there exists a flow θ_t, $t \in \mathbf{R}$, with the properties: $\theta_t\alpha = \alpha - t$, $\theta_t\beta = \beta - t$ and $\theta_t X_u = X_{u+t}$.[4]

A random time shift of the process X_t is defined on the space $\Omega_\tau = \{\omega\colon -\infty < \tau < +\infty\}$ by the formula

$$\tilde{X}_t(\omega) = X_{\tau(\omega)+t}(\omega). \tag{2.5}$$

Suppose that an entrance rule μ and an exit rule h satisfy the conditions

$$\bar{\mu} \text{ is } \sigma\text{-finite and } \bar{h} < \infty \quad \bar{\mu}\text{-a.e.} \tag{2.6}$$

Then the conditions of Kuznetsov's theorem are satisfied for three pairs $(\bar{\mu}, \bar{h})$, $(\bar{\mu}, h)$ and $(\mu, \bar{h})$ and therefore, the measures $P_{\bar{\mu}}^{\bar{h}}$, $P_{\bar{\mu}}^{h}$ and $P_{\mu}^{\bar{h}}$ are defined and σ-finite. We note that

$$\overline{P_{\bar{\mu}}^{h}} = \overline{P_{\mu}^{\bar{h}}} = P_{\bar{\mu}}^{\bar{h}}.$$

By applying Theorem 1 and subsection 1.4, we get the following result:

THEOREM 2. *Measures $P_{\bar{\mu}}^{h}$ and $P_{\mu}^{\bar{h}}$ coincide on all invariant sets. If τ is a stationary time, then*

$$P_{\bar{\mu}}^{h}\theta_\tau = P_{\mu}^{\bar{h}}\theta_\tau. \tag{2.7}$$

The shift $\tilde{X}_t$ of X_t defined by formula (2.5) *has identical laws under $P_{\bar{\mu}}^{h}$ and $P_{\mu}^{\bar{h}}$.*

3. Backward transition functions and duality. Theorem of Nagasawa and theorem of Getoor and Glover.

3.1. Along with forward transition functions $p(s, x; t, dy)$ we consider *backward transition functions* $q(s, dx; t, y)$. We say that a forward transition function p and a backward transition function q are *m-related* if

$$m_s(dx)p(s, x; t, dy) = q(s, dx; t, y)m_t(dy) \quad \text{for all } s < t. \tag{3.1}$$

It follows from this relation that m is an entrance rule for both p and q.

If $g = \{g_t\}$ is an exit rule for q, then $(g \circ m)_t(dx) = g_t(x)m_t(dx)$ is an entrance rule for p, and all entrance rules for p which are absolutely continuous with respect to m can be represented in this form.

To every statement on forward transition functions there corresponds a statement on backward transition functions. The "backward" version of Kuznetsov's theorem is as follows. Let $\nu = \{\nu^t\}$ be an entrance rule and $g = \{g_t\}$ be an exit rule for a backward transition function q. If

$$\nu^t(g_t = +\infty) = 0 \quad \text{for all } t, \tag{3.2}$$

then there exists a stochastic process (X_t, Q_g^ν) on a random time interval (α, β) such that, for every $t \in \mathbf{R}$,

$$Q_g^\nu(\alpha < t, X_t \in dy, t < \beta) = g_t(y)\nu^t(dy) \tag{3.3}$$

[4] (The σ-algebra $\mathscr{F}$ in Ω is generated by the sets $\{X_t \in B\}$, $t \in \mathbf{R}$, $B \in \mathscr{E}$.) Counterexamples show that Theorems 2 through 5 are not true without this assumption.

and, for all $t_1 < \cdots < t_n \in \mathbf{R}$,

$$Q_g^\nu(\alpha < t_1, X_{t_1} \in dy_1, \ldots, X_{t_n} \in dy_n, t_n < \beta) \\ = g_{t_1}(y_1) q(t_1, dy_1; t_2, y_2) \cdots q(t_{n-1}, dy_{n-1}; t_n, y_n) \nu^{t_n}(dy_n). \tag{3.4}$$

The measure Q_g^ν is σ-finite.

Suppose that p and q are m-related and that h is an exit rule for p and g is an exit rule for q. It follows from (3.3) and (3.4) that

$$P_{g \circ m}^h = Q_g^{h \circ m}. \tag{3.5}$$

3.2. If a backward transition function q is stationary, then we write

$$q_t(dx, y) = q(t, dx; 0, y), \qquad t < 0.$$

If p and q are both stationary and if $m_t = m$ for all t, then the condition (3.1) can be rewritten as follows

$$m(dx) p_t(x, dy) = q_{-t}(dx, y) m(dy). \tag{3.6}$$

Suppose that μ is an entrance rule for p, ν is an entrance rule for q and let $\bar{\mu}$ and $\bar{\nu}$ be σ-finite and absolutely continuous with respect to m. The densities $u = d\bar{\mu}/dm$ and $v = d\bar{\nu}/dm$ can be chosen to be q-excessive and p-excessive respectively. By (3.5),

$$\overline{P_\mu^v} = P_{\bar{\mu}}^v = P_{u \circ m}^v = Q_u^{v \circ m} = Q_u^{\bar{\nu}} = \overline{Q_u^\nu},$$

and Theorem 1 and subsection 1.4 imply

THEOREM 3. *Measures P_μ^v and Q_u^ν coincide on all invariant sets. For every stationary time τ, $P_\mu^v \theta_\tau = Q_u^\nu \theta_\tau$. The shifted process $\tilde{X}_t = X_{\tau + t}$ has identical laws under P_μ^v and Q_u^ν.*

We write $(X_t, P) \cong (X'_t, P')$ if stochastic processes (X_t, P) and (X'_t, P') have identical laws (i.e., if they have the same finite-dimensional distributions).

3.3. Formula

$$\hat{p}_t(x, dy) = q_{-t}(dy, x) \tag{3.7}$$

determines a stationary forward transition function which is in weak duality with $p_t(x, dy)$ in the sense of Getoor and Sharpe [3]. We note that $(\tilde{X}_{-t}, Q_{u^*}^{\nu^*}) \cong (X_t, \hat{P}_\nu^u)$ where an entrance rule ν for $\hat{p}$ and an entrance rule ν^* for q are related by the formula $(\nu^*)^t = \nu_{-t}$. Analogously, $(u^*)_t = u^{-t}$. We can assume without any substantial loss of generality that there exists a measurable transformation r of the space $(\Omega, \mathscr{F})$ such that $X_t(r\omega) = X_{-t}(\omega)$. If τ is a stationary time, then so is $\tau^*(\omega) = -\tau(\omega)$. Theorem 3 implies

THEOREM 4. *Let stationary transition functions p and $\hat{p}$ be in weak duality relative to m. Let μ be an entrance rule for p and ν be an entrance rule for $\hat{p}$. Suppose that $\bar{\mu}$ and $\bar{\nu}$ are σ-finite and absolutely continuous with respect to m. Let u and v be excessive functions (for p and $\hat{p}$ respectively) such that $\bar{\mu} = u \circ m$, $\bar{\nu} = v \circ m$. Then, for every stationary time τ, $(X_{\tau - t}, P_\mu^v) \cong (X_{\tau^* + t}, \hat{P}_\nu^u)$.*

This is a generalization of a theorem of Getoor and Glover (see [**2**, Theorem (6.5)]) who have considered the situation when $\tau = \beta$,

$$\begin{aligned} \mu_t &= \mu_0 P_t && \text{for } t > 0, \\ \nu_t &= \nu_0 \hat{P}_t && \text{for } t > 0, \\ \mu_t &= \nu_t = 0 && \text{for } t \leqslant 0. \end{aligned}$$

In this situation $\beta^* = \alpha = 0$ $\hat{P}^u_\nu$-a.s. and $(X_{\beta - t}, P^v_\mu) \cong (X_t, \hat{P}^u_\nu)$. Moreover

$$u = d(\mu_0 G)/dm, \qquad v = d(\nu_0 \hat{G})/dm$$

where

$$G = \int T_t\, dt, \qquad \hat{G} = \int \hat{T}_t\, dt.$$

(Actually, in [**2**] the process $X_{(\beta - t)+}$ rather than $X_{\beta - t}$ is considered and therefore certain regularity conditions for paths are required.)

3.4. We say that a random time τ is a *renewal time* for a Markov process (X_t, P) if $(\tilde{X}_t, P)$ where $\tilde{X}_t$ is defined by formula (2.5) has the same transition function as (X_t, P). If (X_t, P) is a strong Markov process with a stationary transition function, then every optional time is a renewal time.

Markov process (X_t, P^u_ν) corresponding by Kuznetsov's theorem to a stationary transition function p, an entrance rule ν and an excessive function u has the stationary transition function

$$\begin{aligned} &p^u_t(x, dy) = u(x)^{-1} p_t(x, dy) u(y) \quad \text{for } 0 < u(x) < \infty, \\ &p^u_t(x, B) = 1_B(x) \quad \text{for } u(x) = 0 \text{ or } \infty. \end{aligned}$$

We call p^u the u-transform of p.

The following result is an immediate implication of Theorem 4:

THEOREM 5. *Let p, $\hat{p}$, μ, ν, u, v have the same meaning as in Theorem* 4. *If a stationary time τ is a renewal time for $(X_t, \hat{P}^u_\nu)$ then $(X_{\tau - t}, P^v_\mu)$ is a Markov process with a stationary transition function $\hat{p}^v$ (the v-transform of $\hat{p}$).*

Theorem 5 is a generalization of Nagasawa's theorem [**6**].

4. A property of invariant measures.

4.1. Let P be an invariant measure for a flow θ_t. For every positive measurable Y we put

$$\overline{Y} = \int_{\mathbf{R}} \theta_t Y\, dt.$$

We note that if τ is a stationary time, then

$$P\kappa_\tau Y = PY \int \theta_t \rho(\tau)\, dt = \int P\theta_t(\theta_{-t} Y \rho(\tau))\, dt \tag{4.1}$$

$$= P \int \theta_{-t} Y \rho(\tau)\, dt = P\overline{Y}\rho(\tau)$$

(cf. (1.1)).

4.2. An invariant measure P is called *dissipative* if $\overline{Y} < \infty$ P-a.e. for every positive P-integrable Y, and it is called *conservative* if $\overline{Y} = 0$ or ∞ P-a.e. for every positive measurable Y. It is well known (see, e.g., [7, §V.5]) that every σ-finite invariant measure P can be represented as the sum of a dissipative measure P_d and a conservative measure P_c which are mutually singular.

THEOREM 6. *Let P be a dissipative invariant measure. If $\overline{Y}_1 = \overline{Y}_2$ P-a.e., then $PY_1 = PY_2$.*

Indeed, let F be a strictly positive P-integrable function. Since $\overline{F} < \infty$ P-a.e., the condition (1.2) is satisfied and there exist stationary times τ_a such that $\kappa_{\tau_a} \uparrow 1$ as $a \downarrow 0$. By (4.1),

$$PY_i = \lim_{a \downarrow 0} P\overline{Y}_i\rho(\tau_a), \qquad i = 1, 2.$$

4.3. Let P be an arbitrary σ-finite invariant measure. There exists a partition of Ω into disjoint sets Ω_c and Ω_d such that $P_c(\Omega_d) = P_d(\Omega_c) = 0$. Let F be a P-integrable function which is strictly positive on Ω_c and equal to 0 on Ω_d. We have $\overline{F} = 0$ or ∞ P_c-a.e., $\overline{F} = 0$ P_d-a.e. and therefore $\overline{F} = 2\overline{F}$ P-a.e. On the other hand, $P(F) \neq P(2F)$ if $P_c \neq 0$. Therefore the statement of Theorem 6 is true *only* if P is dissipative.

REMARK. It follows from subsections 4.2 and 4.3 that a σ-finite invariant measure P is dissipative if and only if it satisfies condition (1.2) (or (1.4)).

4.4. A measure P_m^1 corresponding, by Kuznetsov's theorem, to a stationary transition function p, an excessive measure m and the exit rule $h = 1$ is invariant with respect to the flow θ_t. It is dissipative if and only if $Gf < \infty$, m-a.e. where G is defined in subsection 3.3.

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REFERENCES

1. E. B. Dynkin, *Initial and final behaviour of the trajectories of Markov processes*, Uspehi Mat. Nauk **26** (160) 1971, 153–172; English transl., *Markov processes and related problems of analysis*, London Math. Soc. Lecture Notes Ser., vol. 54, Cambridge Univ. Press, London, 1982, pp. 123–144.

2. R. K. Getoor and J. Glover, *Riesz decomposition in Markov process theory*, Trans. Amer. Math. Soc. **285** (1984), 107–132.

3. R. K. Getoor and M. J. Sharpe, *Naturality, standardness, and weak duality for Markov processes*, Z. Wahrsch. Verw. Gebiete (to appear).

4. S. E. Kuznetsov, *Construction of Markov processes with random birth and death times*, Teor. Veroyatnost. i Primenen **18** (1973), 596–601. (English translation: Theory Probab. Appl.)

5. P.-A. Meyer, *Probability and potential*, Blaisdell, Waltham, Mass., 1966.

6. M. Nagasawa, *Time reversions of Markov processes*, Nagoya Math. J. **24** (1964), 177–204.

7. J. Neveu, *Bases mathématiques du calcul des probabilités*, Masson, Paris, 1964.

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