TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 287, Number 2, February 1985

# BANACH SPACES WITH THE $L^1$-BANACH-STONE PROPERTY

BY
PETER GREIM

ABSTRACT. It has previously been shown that separable Banach spaces $V$ with trivial $L$-structure have the $L^1$-Banach-Stone property, i.e. every surjective isometry between two Bochner spaces $L^1(\mu_i, V)$ induces an isomorphism of the two measure algebras. We remove the separability restriction, employing the topology of the measure algebra's Stonean space.

The result is achieved via a complete description of the $L$-structure of $L^1(\mu, V)$.

**1. Introduction.** In analogy to the Banach-Stone property [**2**, p. 142] a Banach space $V$ is said to have the $L^p$-Banach-Stone property if for any pair of positive $\sigma$-finite measure spaces $(\Omega_i, \Sigma_i, \mu_i)$ the existence of a surjective linear isometry between the Bochner spaces $L^p(\mu_i, V)$ implies that the measure algebras $\Sigma_i/\mu_i$ are isomorphic. (See [**7**] for the definition and properties of $L^p(\mu_i, V)$.) By Lamperti's extension [**15**] of Banach's classical result [**1**], the scalar field $\mathbf{K}$ has the $L^p$-Banach-Stone property for $p \neq 2$. The first vector-valued generalization is due to Cambern, who showed that separable Hilbert spaces have this property for $1 \leq p < \infty$, $p \neq 2$ [**4**] and for $p = \infty$ [**5**]. See also [**9**]. Sourour's paper [**18**] showed that, rather than the Hilbert space property, the "triviality of $V$'s $L^p$-structure", i.e. the nondecomposability of $V$ into an $l^p$-direct sum of nontrivial subspaces, was responsible for $V$'s $L^p$-Banach-Stone property. Although his proof used Hermitian operators and thus required complex scalars, his result turned out to be true also for real scalars [**10, 13**]. However, as is often the case in the investigation of Bochner $L^p$-spaces, the separability of $V$ was essential for the proofs for measure-theoretic reasons. A few "nonseparable" results have been achieved in [**11** and **12**]: all Hilbert spaces have the $L^p$-Banach-Stone property ($p \neq 2$), and all duals with trivial $L^\infty$-structure and all $CK$-spaces with connected $K$ have the $L^\infty$-Banach-Stone property.

In this paper we prove the following theorem, doing away with the previous restrictions.

1.1. THEOREM. *Banach spaces with trivial $L$-structure have the $L^1$-Banach-Stone property.*

As in [**13**], Theorem 1.1 is an immediate consequence of the description of the $L$-structure of $L^1(\mu, V)$ (see Theorems 4.2 and 4.4 below). As to our method, instead of using vector-valued liftings as in [**11**], we employ the topology of the measure algebra's Stonean space. Although this has the defect that we may have to pass

Received by the editors February 13, 1984.
1980 *Mathematics Subject Classification*. Primary 46E40; Secondary 46B20.
*Key words and phrases*. Banach-Stone theorem, Bochner $L^p$-space, centralizer, hyperstonean, integral module, $L^p$-structure.

from a very familiar measure space, e.g. Lebesgue measure, to a fairly abstract one, we feel that it is more convenient to use continuity arguments and the strong topological properties of hyperstonean spaces. (We need this latter notion anyway in order to describe the $L$-structure of $L^1(\mu, V)$ for general $V$.)

The paper is organized as follows. In §2 we present the tools we need: category measures on hyperstonean spaces, the isometric representation of $L^p(\mu, V)$ on such a measure space, the notions of $L^p$-projection, Cunningham $p$-algebra, centralizer and $M$-bounded opeators, and the representation of $V$ as an "integral module" or "function module". In §3 we describe a natural candidate for the Cunningham $p$-algebra of $L^p(\mu, V)$ $(p \neq 2)$, and in §4 we confirm this candidate for $p = 1$. We omit some partial results for the case $p = \infty$, e.g. an $L^\infty$-Banach-Stone theorem for range spaces $V$ admitting only trivial $M$-bounded operators into their biduals.

A remark concerning our notation. $B(X, Y)$ and $B(X)$ are the spaces of bounded linear operators from $X$ into $Y$ and into itself, respectively, $\langle\cdot,\cdot\rangle$ denotes the evaluation on $X \times X^*$, and $\|x\|_p$ is the $L^p$-norm of $t \mapsto \|x(t)\|$ whenever it exists $(1 \leq p \leq \infty$, $x$ not necessarily Bochner measurable). $\chi_A$ is the characteristic function of the set $A$ and $\mathbf{v}$ is the constant function taking $v$ as value, where the domain is understood. In order to avoid trivial considerations, all our Banach spaces will be at least one dimensional.

The notion of strong measurability should not be confused with Bochner measurability. A function $h$ taking values in $B(X, Y)$ is called *strongly measurable* if for every $x \in X$ the $Y$-valued function $h(\cdot)x$ is Bochner measurable.

## 2. Prerequisites.

A. *Perfect measures.* From now on let $(\Omega, \Sigma, \mu)$ be a $\sigma$-finite positive measure space—w.l.o.g. complete, since the notion of (Bochner) measurability depends only on the completion—, and $1 \leq p \leq \infty$. $V$ is a real or complex Banach space. Let $\Omega^\frown$ denote the Stonean space of $\mu$'s measure algebra.

2.1. It is well known [**16, 3**] that $\mu^\frown(C) := \mu(M)$, where $C$ is the clopen set corresponding to $M$, extends to a Borel measure $\mu^\frown$ on the (extremally disconnected compact) space $\Omega^\frown$ with the following properties:

(i) The $\mu^\frown$-null sets are exactly the nowhere dense Borel sets. (In particular, $\mu^\frown(C) > 0$ for each nonvoid clopen $C$.)

(ii) Each nonvoid clopen set contains a nonvoid clopen set with finite measure.

As a consequence from (i) and (ii) we have (compare the proof of [**3**, Corollary 3.15]):

(iii) There is a disjoint partition $\Omega^\frown = (\dot{\bigcup}_{i\in I} K_i)^-$ into clopen subsets $K_i$ with finite measure such that $\mu^\frown = \sum_{i\in I} \mu^\frown|_{K_i}$.

2.2. The Boolean isomorphism representing $\Sigma/\mu$ induces an isometry $^\frown$ of $L^p(\mu, V)$ onto $L^p(\mu^\frown, V)$ via simple functions $(p < \infty)$, respectively countably valued functions $(p = \infty)$. The inverse of this isometry is given by $x^\frown \mapsto x^\frown \circ \psi$, where $\psi \colon \Omega \to \Omega^\frown$ is a suitable inverse measure preserving mapping.

The definition of the measurable mapping $\psi$ is given in [**17**, Proposition 5.1a)] via a lifting of $L^\infty(\mu)$; the rest is routine.

We call a (not necessarily $\sigma$-finite) Borel measure $m$ on an extremally disconnected compact $K$ a *category measure* if it satisfies (i) and (ii), and consequently, (iii) above. $K$ is called *hyperstonean* if it admits a category measure. These measures have the following nice property.

2.3. LEMMA. *Each m-measurable function $x\colon K \to V$ is norm-continuous on an open dense subset $U$ of $K$ (whose complement is a null set because of* (i)).

The proof for finite $m$ is a simple (iterated) application of Egorov's theorem to a sequence of (continuous) simple functions converging a.e. to $x$. See also [**17**]. For infinite $m$ look at a partition as in (iii) and observe that $m(K \setminus \dot{\bigcup}_{i\in I} U_i) = \sum_{i\in I} m(K_i \setminus U_i) = 0$ implies that $K \setminus \dot{\bigcup}_{i\in I} U_i$ has void interior.

Since $K$ is the Stone-Čech compactification of each open dense subset $U$ we may conclude parts (a) and (b) of the following proposition.

2.4. PROPOSITION. (a) *Each scalar-valued $L^\infty(m)$-function on $K$ is a.e. equal to (exactly) one continuous function.*

(b) *For a dual $V^*$, each $V^*$-valued $L^\infty(m)$-function is a.e. equal to (exactly) one weak $*$-continuous function.*

(c) *Consequently, the aforementioned isometry $\hat{}$ induces an isometry $\tilde{}$ of $L^\infty(\mu, V^*)$ onto $C_{\mathrm{sep}}(\Omega\hat{}, (V^*, \sigma^*))$, the subspace of $C(\Omega\hat{}, (V^*, \sigma^*))$ consisting of the Bochner measurable (i.e., essentially separably valued) functions.*

(d) $C(\Omega\hat{}, (V^*, \sigma^*))$ *is isometrically isomorphic to $L^1(\mu, V)^*$ via*

$$\langle x, g\rangle := \int g(t)x\hat{}(t)\,d\mu\hat{}(t) \qquad (x \in L^1(\mu, V)).$$

For the proofs of (a) and (b) we have only to observe that, by Lemma 2.3, each $x$ in $L^\infty(m, V^*)$ has a representative that is norm (and consequently weak $*$-) continuous on an open dense $U$ and takes values in the weak $*$-compact ball with radius $\|x\|_\infty$. The uniqueness is due to the fact that for each weak $*$-continuous $z\colon K \to V^*$ the norm function $t \mapsto \|z(t)\| = \sup\{|z(t)v| \mid v \in V,\ \|v\| \le 1\}$ is lower semicontinuous, so that its essential and actual suprema coincide (nonvoid open sets have positive measure). In other words, the mapping in (c) is, in fact, an isometry. The essentially separably valued weak $*$-continuous functions are measurable by [**7**, Corollary II.4]. (d) is well known ([**6**], e.g.). □

2.5. We shall frequently make use of the following property of Bochner measurable functions: If $x\colon \Omega \to V$ is measurable and $\varepsilon > 0$, then by an application of Egorov's theorem (similar to 2.3) we can find a countably valued function $y = \sum_{i=1}^\infty \chi_{\Omega_i} v_i$ such that $\|y - x\|_\infty < \varepsilon$. The vectors $v_i$ may be chosen as $x(t_i)$ for suitable $t_i \in \Omega_i$.

B. *$L^p$-structure.* References for the following are [**2** and **3**]. An *$L^p$-projection* of $V$ is a projection $P\colon V \to V$ satisfying $\|v\|^p = \|Pv\|^p + \|v - Pv\|^p$ ($p < \infty$) or, respectively, $\|v\| = \max\{\|Pv\|, \|v - Pv\|\}$ ($p = \infty$), for all $v \in V$. *From now on let* $p \neq 2$.

The set $\mathbf{P}_p(V)$ of all $L^p$-projections is a Boolean algebra whose Stonean space $K_p$ is hyperstonean if $p < \infty$ or $V$ is a dual.

The *Cunningham p-algebra* $C_p(V)$ is the operator algebra generated by $\mathbf{P}_p(V)$. It is isomorphic to $CK_p$ as a Banach algebra; thus $V$ is a $CK_p$-module.

We shall need the following representation of $V$ as a $CK_p$-module of vector-valued functions, whereat for $p = \infty$ we assume that $V$ is a dual.

2.6. PROPOSITION. *Let $m$ be any category measure on $K := K_p$. There is a system $(V_k)_{k\in K}$ of Banach spaces and an embedding $v \mapsto \langle v\rangle$ of $V$ into the cartesian product $\prod_{k\in K}(V_k \dot\cup \{\infty\})$ such that:*

(i) *for each $v \in V$ the norm function $k \mapsto [v](k) := \|\langle v\rangle(k)\|$ ($\|\infty\| := \infty$) is a continuous numerical $L^p(m)$-function with $L^p$-norm equal to $\|v\|$; and*

(ii) *addition and $CK$-multiplication on $V$, when embedded into the cartesian product, coincide with the $m$-a.e. pointwise operations.*

This representation of $V$ as a so-called "integral module" (or "function module" for $p = \infty$) is established in [**3**] in the case $p < \infty$ for real scalars; however, the theorem holds for complex scalars as well. This is essentially due to the fact that the $L^p$-projections of the underlying real Banach space $V_r$ are complex linear. (The proof in [**14**] for $p = 1$ applies to all $p \neq 2$.) The proof for $p = \infty$ is a combination of Theorems 4.14, 5.9, 5.7(ii) and 5.13 in [**2**].

2.7. An operator $T\colon W \to V$ defined on a subspace $W$ of $V$ is called *$M$-bounded* if there is a $\lambda > 0$ such that $\|x - Ty\| \leq 1$ whenever $\|x + \lambda y\| \leq 1$ and $\|x - \lambda y\| \leq 1$. $\lambda$ can be chosen to be $\|T\|$. The real subspace of $B(W, V)$ consisting of all $M$-bounded operators is denoted by $\mathrm{Mb}(V, W)$ (and by $\mathrm{Mb}(V)$ if $W = V$).

If $V$ is a complex space, then for the underlying real space $V_r$ we have $\mathrm{Mb}(V_r) = \mathrm{Mb}(V)$, i.e. every real linear $M$-bounded $T$ is complex linear.

2.8. The *centralizer* $Z(V)$ of $V$ is the $\mathbf{K}$-linear hull of $\mathrm{Mb}(V)$; it is a Banach algebra of the form $CK$. In the complex case the isometric isomorphism $Z(V) \to CK$ maps $Z(V_r)$ onto the real part of $CK$. Analogously the isometry $C_p(V) \to CK_p$ maps $C_p(V_r)$ onto the real part of $CK_p$.

2.9. The algebra isomorphism "adjoint" from $B(V)$ into $B(V^*)$ maps $C_p(V)$ onto $C_q(V^*)$, where $q$ is the conjugate index to $p$ ($p < \infty$). $C_\infty(V)$ is a subalgebra of $Z(V)$; for duals $V^*$ we have $(C_1(V) \cong)\ C_\infty(V^*) = Z(V^*)$.

2.10. An *$L^p$-summand* is the range of an $L^p$-projection. We shall also speak of "$L$"- and "$M$-" (instead of "$L^1$-" and "$L^\infty$-") projections or summands.

**3. A candidate for $C_p(L^p(\mu, V))$.** *Throughout this section let $1 \leq p \leq \infty$, $p \neq 2$, unless otherwise stated.* Paralleling a theorem for spaces of continuous functions [**2**, Proposition 10.3], it has been shown [**10**] for separable $V$ that the centralizer of $L^\infty(\mu, V)$ is isomorphic to $L^\infty_{\mathrm{st}}(\mu, Z(V))$, the space of all strongly measurable, essentially bounded $Z(V)$-valued functions modulo $\mu$-equivalence, with the essential supremum as norm. This turned out to be the strong closure of a naturally embedded copy of $L^\infty(\mu, Z(V))$ in $B(L^\infty(\mu, V))$. If the norm and strong topologies on $Z(V)$ coincide, then $Z(L^\infty(\mu, V))$ is $L^\infty(\mu, Z(V))$ itself, in which case it depends only on $Z(V)$ and not on $V$. We are aiming at a similar characterization of $C_p(L^p(\mu, V))$, where $V$ is arbitrary. However, for nonseparable $V$, the norm function $t \mapsto \|h(t)\|$ ($h\colon \Omega \to C_p(V)$) need no longer be measurable, so that we need an alternate definition for the norm of a bounded strongly measurable function—unless (as we shall do in due course) we pass to the Stonean transform, for which the norm function *is* measurable. Furthermore, our situation is different from the one in [**10**] also insofar as the norm and strong topologies on $C_p(V)$ coincide only if this algebra is finite-dimensional ($p < \infty$). Nevertheless we shall be able to show that $L^\infty_{\mathrm{st}}(\mu, C_p(V))$ does not depend on $V$ but only on $C_p(V)$.

A. *Strongly measurable functions.*

3.1. LEMMA. *Let $Z$ be a subspace of $B(V)$ and let $h\colon \Omega \to Z$ be strongly measurable and bounded. Then $M_h x(t) := h(t)x(t)$ defines a bounded operator $M_h$ on $L^p(\mu, V)$ with $\|M_h\| \le \sup\{\|h(t)\| \mid t \in \Omega\}$.*

We omit the easy proof (compare [**10**, p. 464]). Since the countably valued functions are dense in $L^p(\mu, V)$, we have $M_h = M_g$ if and only if $h$ and $g$ are "strongly equivalent", i.e. $h(\cdot)v = g(\cdot)v$ a.e. for every $v \in V$. Thus, if we denote by $L^\infty_{\rm st}(\mu, Z)$ the vector space of all bounded strongly measurable $Z$-valued functions modulo strong equivalence, and define $\|h\| := \|M_h\|$, then $L^\infty_{\rm st}(\mu, Z)$ *becomes a subspace of* $B(L^p(\mu, V))$. Since for $h \in L^\infty(\mu, Z)$ we have $\|M_h\| \le \|h\|$, and obviously $\|M_h\| = \|h\|$ for all countably valued functions $h$, the embedding of $L^\infty(\mu, Z)$ into $L^\infty_{\rm st}(\mu, Z)$ is an isometry:

$$L^\infty(\mu, Z) \subset L^\infty_{\rm st}(\mu, Z) \subset B(L^p(\mu, V)).$$

If $Z$ is an algebra, then the above isometries are multiplicative.

3.2. LEMMA.

$$L^\infty(\mu, Z) \subset L^\infty_{\rm st}(\mu, Z) \subset L^\infty(\mu, Z)^{-\rm st},$$

*where the closure is taken in the strong operator topology.*

PROOF. This lemma is a generalization of Lemma 2 in [**10**]. Since the argument there is not quite complete, we want to give the proof here. Let $h\colon \Omega \to Z$ be strongly measurable and bounded, say $\|h(t)\| \le 1$ for all $t \in \Omega$, $x^{(1)}, \dots, x^{(N)}$ finitely many vectors in $L^p(\mu, V)$, and $\varepsilon > 0$. We are seeking a $g \in L^\infty(\mu, Z)$ with $\|g\| \le 1$ such that $\|M_h x^{(n)} - M_g x^{(n)}\|_p \le \varepsilon$ for $1 \le n \le N$. Since the simple (resp. countably valued) functions are dense in $L^p(\mu, V)$ we may assume that $x^{(n)} = \sum_{i=1}^\infty \chi_{\Omega_i} v_i^{(n)}$, $\Omega_i$ pairwise disjoint (looking at a common refinement of finitely many partitions of $\Omega$). We may even assume that the $x^{(n)}$ are constant functions. Namely, if $x^{(n)}$ is as above, $g_i \in L^\infty(\mu, Z)$ and $\|(M_h - M_{g_i})\chi_{\Omega_i} v_i^{(n)}\|_p \le 2^{-i}\varepsilon$ $(i \in \mathbf{N})$, then $g := \sum_{i=1}^\infty \chi_{\Omega_i} \cdot g_i$ is measurable and $\|(M_h - M_g)x^{(n)}\|_p \le \varepsilon$.

Thus let $x^{(n)} = \mathbf{v^{(n)}}$. Observing 2.5 and again forming a common refinement of finitely many partitions, we find a partition $\Omega = \dot\bigcup_{j=1}^\infty \Omega_j$ and $t_j \in \Omega_j$ such that $\|h(t)v^{(n)} - h(t_j)v^{(n)}\| \le \varepsilon$ a.e. on $\Omega_j$ $(j \in \mathbf{N},\ 1 \le n \le N)$. Then the Bochner measurable function $g := \sum_{j=1}^\infty h(t_j) \cdot \chi_{\Omega_j}$ has the desired properties $\|g\|_\infty \le 1$ and $\|M_h \mathbf{v^{(n)}} - M_g \mathbf{v^{(n)}}\|_p \le \varepsilon \cdot \|\mathbf{1}\|_p$. □

*Now we specialize to the case $Z = C_p(V)$.*

3.3. LEMMA. $L^\infty(\mu, C_p(V)) \subset C_p(L^p(\mu, V))$.

PROOF. Since the functions taking only countably many values in the dense subspace $\operatorname{lin} \mathbf{P}_p(V)$ of $C_p(V)$ are dense in $L^\infty(\mu, C_p(V))$, it is enough to show that they belong to $C_p(L^p(\mu, V))$. So let $h = \sum_{i=1}^\infty \chi_{\Omega_i} \cdot P_i$, where $\Omega = \dot\bigcup_{j=1}^\infty \Omega_i$ and $P_i \in \operatorname{lin} \mathbf{P}_p(V)$. Obviously, for each $i \in N$,

$$\chi_{\Omega_i} \cdot P_i \in \operatorname{lin} \mathbf{P}_p(L^p(\mu|_{\Omega_i}, V)) \subset C_p(L^p(\mu|_{\Omega_i}, V)).$$

Since $L^p(\mu, V)$ is the $l^p$-direct sum of the subspaces $L^p(\mu|_{\Omega_i}, V)$, Proposition 2.7 in [**3**] shows that $h \in C_p(L^p(\mu, V))$. □

For $p < \infty$, $C_p(X)$ is strongly closed since it is generated by $\mathbf{P}_p(X)$, a complete Boolean algebra of projections [**3**, Proposition 1.6 and **8**, Corollary XVII. 3.17]. So we have the following corollary.

3.4. PROPOSITION. *Let $p < \infty$. Then*

$$L^\infty_{\mathrm{st}}(\mu, C_p(V)) \subset L^\infty(\mu, C_p(V))^{-\mathrm{st}} \subset C_p(L^p(\mu, V)).$$

B. *$L^\infty_{\mathrm{st}}(\mu, C_p(V))$ is independent of $V$.* Formally our candidate for $C_p(L^p(\mu, V))$, namely $L^\infty_{\mathrm{st}}(\mu, C_p(V))$, depends on $V$. In this subsection we show that it actually depends only on $V$'s $L^p$-structure. *For the rest of this section we assume that $V$ is represented as in* 2.6. (In particular, if $p = \infty$, we assume that $V$ is a dual.) Observe then that

$$C_p(V) \cong CK \cong C_p(L^p(m)) \cong C_1(L^1(m)).$$

3.5. LEMMA. (a) *A function $h\colon \Omega \to C_p(V)$ is strongly measurable w.r.t. $V$ if and only if it is so w.r.t. $L^p(m)$.*

(b) *A bounded function $h\colon \Omega \to C_p(V)$ is strongly measurable w.r.t. $L^p(m)$ if and only if it is so w.r.t. $L^1(m)$ $(p < \infty)$. Thus the notion of strong measurability does not depend on $V$ or $p < \infty$ but only on $CK$.*

(c) *A function $h\colon \Omega \to C_\infty(V)$ is strongly measurable if and only if it is Bochner measurable.*

PROOF. (a) Since the mapping $v \mapsto [v]$ is a surjection of $V$ onto the positive cone of $L^p(m)$ (which is a consequence of [**3**, Lemma 3.13] $(p < \infty)$ and [**2**, Theorem 5.13(ii)] $(p = \infty)$), it suffices to show that, for every $v \in V$, $h(\cdot)v$ is measurable if and only if $h(\cdot)[v]$ is. The necessity is obvious since $v \mapsto [v]$ is a contraction. Assume now that $h(\cdot)[v]$ is measurable. Then it can be approximated pointwise a.e. by functions of the form $\sum_{i=1}^\infty h(t_i)[v] \cdot \chi_{\Omega_i}$ $(\Omega = \dot{\bigcup}_{i=1}^\infty \Omega_i,\ t_i \in \Omega_i)$, that is, by functions $g(\cdot)[v]$, with $g$ a countably valued function. The equality

$$\|(g(t) - h(t))v\| = \|\,|g(t) - h(t)| \cdot [v]\|_p = \|(g(t) - h(t)) \cdot [v]\|_p$$

shows that the measurable functions $g(\cdot)v$ approximate $h(\cdot)v$ pointwise a.e., i.e. that $h(\cdot)v$ is measurable.

(b) Since $h$ is bounded, its strong measurability is equivalent to the $\|\ \|_p$-measurability of the functions $t \mapsto h(t) \cdot \chi_C$, $C$ a clopen subset of $K$ with finite measure. Assume such a function is $\|\ \|_p$-measurable and let $1 \leq q < \infty$. We want to show that it is $\|\ \|_q$-measurable. Observe that $h(t) \cdot \chi_C \in L^q(m)$. Since the linear hull of the functions $\chi_B$, $m(B) < \infty$, is a norming subset of $L^q(m)^*$ in the sense of [**7**, p. 43], and the scalar functions $t \mapsto \int h(t) \cdot \chi_C \cdot \chi_B \, dm$ are measurable, it remains to show that $h(\cdot) \cdot \chi_C$ is essentially $\|\ \|_q$-separably valued. This, however, is a property independent of $q$, since a subspace of $L^q(m)$ is separable if and only if it is contained in a subspace of the form $L^q(m|_{\mathcal{A}})$, where $\mathcal{A}$ is a separable subalgebra of $K$'s Borel algebra. Finally we observe that the notion of strong measurability w.r.t. $L^p(m)$ does not depend on the choice of the category measure, since pointwise multiplication with a density $dm/dn$ of $m$ w.r.t. another category measure $n$ is an isometric $CK$-module isomorphism onto $L^p(n)$.

(c) Look at $t \mapsto h(t) \cdot \mathbf{1} = h(t)$. $\square$

3.6. REMARK. In the case $p = \infty$ strong measurability may depend on $V$ if we admit nonduals $V$. For example, let $\mu$ be the Lebesgue measure on $[0, 1]$, and define $V = c_0(\mathbf{Q})$, $W = l^\infty(\mathbf{Q})$. Then $C_\infty(V) \cong l^\infty(\mathbf{Q}) \cong C_\infty(W)$. The function $h\colon [0,1] \to l^\infty(\mathbf{Q})$ defined by $h(t) = \chi_{U_t}$, with $U_t := \mathbf{Q} \cap (t - \frac{1}{2}, t + \frac{1}{2})$, is strongly measurable w.r.t. $V$, but not strongly measurable w.r.t. $W$ ($h(\cdot)\mathbf{1}$ is not essentially separably valued).

C. *The Stonean representation.* In order to verify the surjectivity of the embeddings in Proposition 3.4, we pass to the hyperstonean measure space $(\Omega^\frown, \Sigma^\frown, \mu^\frown)$ introduced in §2. We show that for each element of $L^\infty_{\text{st}}(\mu^\frown, C_p(V))$ there is exactly one representative that is continuous w.r.t. the weak $*$-topology on $CK \cong L^1(m)^*$. In other words, similarly to the situation in Proposition 2.4, not only shall we be able to calculate with functions instead of equivalence classes, we can even profit from the nice topological properties of hyperstonean spaces. That will pay off in §4.

Let $C_{\text{st}}(\Omega^\frown, CK)$ denote the Banach space of all $CK$-valued $\sigma(CK, L^1(m))$-continuous functions on $\Omega^\frown$ that are strongly measurable w.r.t. $L^1(m)$, supplied with the supremum norm. Of course, $C_{\text{sep}}(\Omega^\frown, CK)$ is a subspace of $C_{\text{st}}(\Omega^\frown, CK)$.

3.7. LEMMA. *Under the canonical mappings we have*

$$L^\infty_{\text{st}}(\mu^\frown, C_p(V)) \cong C_{\text{st}}(\Omega^\frown, CK) \qquad (p < \infty)$$

*and*

$$L^\infty_{\text{st}}(\mu^\frown, C_\infty(V)) \cong C_{\text{sep}}(\Omega^\frown, CK).$$

PROOF. In view of Proposition 2.4(c) and Lemma 3.5(c) we have nothing to show for $p = \infty$. Let $p < \infty$. Firstly the computation in the proof of Lemma 2.4(c) shows that the canonical linear mapping

$$C_{\text{st}}(\Omega^\frown, CK) \to L^\infty_{\text{st}}(\mu^\frown, C_1(L^1(m))) \cong L^\infty_{\text{st}}(\mu^\frown, C_p(V))$$

is an isometric embedding. Now let $h\colon \Omega^\frown \to CK$ be strongly measurable and bounded, say $\|h(\cdot)\| \leq 1$. We show that $h$ is strongly equivalent to a $\sigma(CK, L^1(m))$-continuous $h^\sim\colon \Omega^\frown \to CK$. Assume first that $m(K) < \infty$. Then $\mathbf{1} \in L^1(m)$ and $t \mapsto h(t) \cdot \mathbf{1} = h(t)$ is an $L^1(m)$-valued Bochner-measurable function: thus it is $\|\ \|_1$-continuous on an open dense subset $U$ of $\Omega^\frown$ (Lemma 2.3). We want to show that $h|_U$ is $\sigma(CK, L^1(m))$-continuous. Since $h(U)$ is bounded in $CK$, it suffices to show that $t \mapsto \int h(t) \cdot f\, dm$ is continuous on $U$ for all characteristic functions $f$. This, however, is a consequence of the $\|\ \|_1$-continuity of $h|_U$. Thus we may extend $h|_U$ to a $\sigma(CK, L^1(m))$-continuous function $h^\sim$ on $K$. In particular we have $h^\sim = h$ a.e.

If $m$ is infinite, look at a partition $K = (\dot{\bigcup}_{i\in I} K_i)^-$ into clopen subsets $K_i$ with finite measure. For $h_i(t) := h(t) \cdot \chi_{K_i}$ define the weak $*$-continuous functions $h_i^\sim$ on $\Omega^\frown$ as in the preceding paragraph. For fixed $t \in \Omega^\frown$ consider the function $\sum_{i\in I} h_i^\sim(t)$ on $\dot{\bigcup}_{i\in I} K_i$ and denote its continuous extension to $K$ by $h^\sim(t)$. Using the boundedness of $h^\sim(\cdot)$ and the fact that $L^1$-functions are supported by only countably many $K$, it is routine to show that for each $f$ in $L^1(m)$ the mapping $t \mapsto h^\sim(t) \cdot f\, dm$ is continuous and $t \mapsto h^\sim(t) \cdot f$ is equivalent to $t \mapsto h(t) \cdot f$. $\square$

**4. The $L$-structure of $L^1(\mu, V)$.** We are going to show that, for $p = 1$, the embeddings in Proposition 3.4 are onto. We may assume w.l.o.g. that $\mu = \mu^\frown$. For, if $T \in C_1(L^1(\mu, V))$, then by assumption the Cunningham operator $T^\frown$ defined by

$T^\frown x^\frown := (Tx)^\frown$ has the form $T^\frown x^\frown = h^\frown x^\frown$, where $h^\frown$ is strongly measurable. But then, with $\psi$ as in 2.2,

$$Tx = (Tx)^\frown \circ \psi = (h^\frown x^\frown) \circ \psi = (h^\frown \circ \psi)(x^\frown \circ \psi)x = (h^\frown \circ \psi)x$$

$(x \in L^1(\mu, V))$, and $h^\frown \circ \psi \in L^\infty_{\mathrm{st}}(\mu, C_1(V))$.

We need the following lemma. Remember that $L^\infty(\mu^\frown, V^*)$ is a subspace of $C(\Omega^\frown, (V^*, \sigma^*))$ via the embedding $\sim$ in Lemma 2.4(c).

4.1. LEMMA. *Let $W$ be a closed subspace of $V^*$ and*

$$T\colon L^\infty(\mu^\frown, W) \to C(\Omega^\frown, (V^*, \sigma^*))$$

*be $M$-bounded (compare 2.7). Assume that $T$ commutes with the characteristic projections $\chi_C$ for clopen $C$. Then there is a family of $M$-bounded operators $T_t\colon W \to V^*$ such that for every $x \in L^\infty(\mu^\frown, W)$,*

$$(Tx)(t) = T_t x(t) \quad a.e.$$

PROOF. Assume w.l.o.g. $\|T\| = 1$ and define $T_t w := (T\mathbf{w})(t)$. Since the norm in the range space is the supremum norm, the linear operator $T_t$ is $M$-bounded and $\|T_t\| \leq 1$ $(\|w \pm u\| \leq 1 \Rightarrow \|\mathbf{w} \pm \mathbf{u}\|_\infty \leq 1 \Rightarrow \|w - (T\mathbf{u})(t)\| \leq \|\mathbf{w} - T\mathbf{u}\| \leq 1)$. Let $x \in L^\infty(\mu^\frown, W)$ and choose a sequence $(y_n)$ of countably valued functions converging uniformly to $x$ on an open dense set. Then $T_t y_n(t) \underset{n}{\rightarrow} T_t x(t)$ on this set. On the other hand, $Ty_n$ converges to $Tx$ uniformly. It remains to show that $Ty_n(t) = T_t y_n(t)$ a.e. $(n \in \mathbf{N})$. So let $y = \sum_{i \in I} \chi_{\Omega_i} w_i$, $\Omega^\frown = (\bigcup_{i \in I} \Omega_i)^-$ with clopen $\Omega_i$. Now

$$\chi_{\Omega_i} Ty = T\chi_{\Omega_i} y = \chi_{\Omega_i} T\mathbf{w_i} = \chi_{\Omega_i} T.y(\cdot),$$

so that we have the required equality on the dense open union of the $\Omega_i$'s. □

4.2. THEOREM. *Under the embedding of Proposition* 3.4,

$$L^\infty_{\mathrm{st}}(\mu, C_1(V)) \cong L^\infty(\mu, C_1(V))^{-\mathrm{st}} \cong C_1(L^1(\mu, V))$$

*(and this algebra depends only on $CK \cong C_1(V)$, not on $V$ itself).*

PROOF. According to the remark preceding Lemma 4.1 it is enough to show that each $S \in C_1(L^1(\mu^\frown, V))$ is of the form $M_{h^\frown}$ with a strongly measurable $h^\frown\colon \Omega^\frown \to C_1(V)$. For the rest of this proof we may, and do, assume that the scalars are real, for if $X$ is a complex Banach space then we have $C_1(X) = C_1(X_r) + i \cdot C_1(X_r)$, where $X_r$ denotes the underlying real Banach space. We also assume w.l.o.g. that $\mu^\frown$ is finite. $T := S^*$ is $M$-bounded (2.9). Thus Lemma 4.1 shows the existence of a suitable family $T_t \in Z(V^*)$ such that for all $w \in V^*$, $T\mathbf{w}(t) = T_t w$. Again because of 2.9 we have $T_t = S_t^*$ with $S_t \in C_1(V)$. We want to show that for all $v \in V$, $S\mathbf{v}(t) = S_t v$ a.e. (then $t \mapsto h^\frown(t) := S_t$ is strongly measurable and, since both $S$ and $M_{h^\frown}$ commute with characteristic projections, $S = M_{h^\frown}$).

Choose a dense open set $U \subset \Omega^\frown$ where $S\mathbf{v}$ (more precisely: a representative) is norm continuous. Let $w \in V^*$ and $C$ be a clopen subset of $U$. Then

$$\int_C \langle S\mathbf{v}(\cdot), w\rangle \, d\mu^\frown = \langle S\chi_C \mathbf{v}, \mathbf{w}\rangle = \langle \chi_C \mathbf{v}, T\mathbf{w}\rangle = \int_C \langle v, T.w\rangle \, d\mu^\frown = \int_C \langle S.v, w\rangle \, d\mu^\frown.$$

Both integrands are continuous on $U$ and, since $C$ is arbitrary, they coincide on $U$. Consequently, $S\mathbf{v}(t) = S_t v$ for all $t \in U$, that is, a.e. □

4.3. COROLLARY. *Let $C_1(V)$ be finite dimensional. Then*

$$C_1(L^1(\mu, V)) \cong L^\infty(\mu, C_1(V)).$$

*In particular, if $V$ has trivial $L$-structure, then*

$$C_1(L^1(\mu, V)) \cong L^\infty(\mu).$$

Observe that every separable $C_1(V)$ is necessarily finite dimensional since it is a dual $CK$-space. *For nonseparable $C_1(V)$ the above equality is false.* In fact, if $\mu$ is not purely atomic and $C_1(V)$ is not separable, then $L^\infty(m)$ contains an isometric copy of $l^\infty(\mathbf{Q})$ (via a disjoint sequence of clopen sets with finite measure), $\mu$ contains a direct summand $\lambda^\alpha$ ($\lambda =$ Lebesgue measure on $[0,1]$, $\alpha$ a suitable cardinal), and the projection from $[0,1]^\alpha$ onto $[0,1]$, followed by the function in Remark 3.6 and the embedding of $l^\infty(\mathbf{Q})$ into $L^\infty(m)$, is an $L^1(m)$-strongly measurable function that is not essentially separably valued.

There is an easy description of the idempotents of $C_1(L^1(\mu, V))$, i.e. the $L$-projections of $L^1(\mu, V)$. Let $K$ and $m$ be as in 2.6, $\mathcal{B}$ the Borel algebra of $K$. Since $m$ is a direct sum of finite measures $m|_{K_i}$ (2.1(iii)), we can define a product measure $\mu \otimes m$ on $\Sigma \otimes \mathcal{B}$ as a direct sum of the products $\mu \otimes (m|_{K_i})$. It is routine to show that it does not depend on the choice of the partition. In view of Lemma 3.5(a) we have

$$C_1(L^1(\mu, V)) \cong L^\infty_{\text{st}}(\mu \hat{,} C_1(V)) \cong C_1(L^1(\mu, L^1(m))) \cong C_1(L^1(\mu \otimes m)).$$

In particular, these algebras have the same idempotents. Using Evans' description of the $L$-projections in an $L^1$-space [**3**] we conclude:

4.4. THEOREM. (a) $\mathbf{P}_1(L^1(\mu, V)) \cong \mathrm{PCF}(\mu \otimes m)$, *where* $\mathrm{PCF}(\mu \otimes m)$ *denotes the Boolean algebra of all "pseudo-characteristic functions" in the sense of* [**3**, *Definition* 4.8].

(b) $\mathbf{P}_1(L^1(\mu, V)) \cong \Sigma \otimes \mathcal{B}/\mu \otimes m$ *if $m$ is $\sigma$-finite or, equivalently, if $\mathbf{P}_1(V)$ has the countable chain property.*

(c) $\mathbf{P}_1(L^1(\mu, V)) \cong \Sigma/\mu$ *if $V$ has trivial $L$-structure.*

PROOF. (c) follows trivially from (b). The countable chain property is necessary for $\sigma$-finiteness of $m$ because of 2.1(i), and 2.1(iii) shows that it is sufficient. As for $\sigma$-finite measures the pseudo-characteristic functions are just the characteristic ones, (b) follows from (a). To complete the proof, the remark preceding the theorem shows that $\mathbf{P}_1(L^1(\mu, V))$ is isomorphic to $\mathbf{P}_1(L^1(\mu \otimes m))$, and the latter is $\mathrm{PCF}(\mu \otimes m)$ according to [**3**, Proposition 4.9]. □

From Theorem 4.4 we immediately derive Theorem 1.1. In fact we have the following stronger result:

4.5. COROLLARY. *Let $V_1$ and $V_2$ have trivial $L$-structures. If there is an isometry $T$ of $L^1(\mu_1, V_1)$ onto $L^1(\mu_2, V_2)$, then $L^1(\mu_1) \cong L^1(\mu_2)$. In fact, there is* *a Boolean isomorphism* $\Phi\colon \Sigma_1/\mu_1 \leftrightarrow \Sigma_2/\mu_2$ *reducing* $T$ *in the sense that for all* $M$ *in* $\Sigma_1$,

$$T(\chi_M L^1(\mu_1, V_1)) = \chi_{\Phi M} L^1(\mu_2, V_2).$$

PROOF. $T$ induces a Boolean isomorphism $\Phi$ between the $\mathbf{P}_1(L^1(\mu_i, V_i))$ ($\cong \Sigma_i/\mu_i$) by $\Phi P := T \circ P \circ T^{-1}$. □

## REFERENCES


1. S. Banach, *Théorie des opérations linéaires*, Monogr. Mat. PWN, Warsaw, 1932.
2. E. Behrends, *M-structure and the Banach-Stone theorem*, Lecture Notes in Math., vol. 736, Springer-Verlag, Berlin and New York, 1979.
3. E. Behrends, et al., *$L^p$-structure in real Banach spaces*, Lecture Notes in Math., vol. 613, Springer-Verlag, Berlin and New York, 1977.
4. M. Cambern, *The isometries of $L^p(X, K)$*, Pacific J. Math. **55** (1974), 9–17.
5. ——, *Isometries of measurable functions*, Bull. Austral. Math. Soc. **24** (1981), 13–26.
6. M. Cambern and P. Greim, *The dual of a space of vector measures*, Math. Z. **180** (1982), 373–378.
7. J. Diestel and J. J. Uhl, Jr., *Vector measures*, Math. Surveys, no. 15, Amer. Math. Soc., Providence, R.I., 1977.
8. N. Dunford and J. T. Schwartz, *Linear operators.* Part III, Wiley, 1971.
9. R. J. Fleming and J. E. Jamison, *Classes of operators on vector-valued integration spaces*, J. Austral. Math. Soc. Ser. A **24** (1977), 129–138.
10. P. Greim, *The centralizer of Bochner $L^\infty$-spaces*, Math. Ann. **260** (1982), 463–468.
11. ——, *Banach-Stone theorems for non-separably valued Bochner $L^\infty$-spaces*, Rend. Circ. Mat. Palermo (2), Suppl. **2** (1982), 123–129.
12. ——, *Hilbert spaces have the Banach-Stone property for Bochner spaces*, Bull. Austral. Math. Soc. **27** (1983), 121–128.
13. ——, *Isometries and $L^p$-structure of separably valued Bochner $L^p$-spaces*, Measure Theory and Its Applications (Proc. Conf. Sherbrooke, 1982). Lecture Notes in Math., vol. 1033, Springer-Verlag, Berlin and New York, 1983, pp. 209–218.
14. B. Hirsberg, *M-ideals in complex function spaces and algebras*, Israel J. Math. **12** (1972), 133–146.
15. J. Lamperti, *On the isometries of certain function spaces*, Pacific J. Math. **8** (1958), 459–466.
16. Y. Mibu, *Relations between measure and topology in some Boolean spaces*, Proc. Imp. Acad. Tokyo **20** (1944), 454–458.
17. D. Sentilles, *Decomposition of weakly measurable functions*, Indiana Univ. Math. J. **32** (1983), 425–437.
18. A. R. Sourour, *The isometries of $L^p(\Omega, X)$*, J. Funct. Anal. **30** (1978), 276–285.


DEPARTMENT OF MATHEMATICS, FREE UNIVERSITY OF BERLIN, ARNIMALLEE 2-6, (D1000) BERLIN (33), FEDERAL REPUBLIC OF GERMANY

*Current address*: Department of Mathematics, University of California, Santa Barbara, California 93106