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BAIRE SETS OF k-PARAMETER WORDS ARE RAMSEY

BY

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ABSTRACT. We show that Baire sets of k-parameter words are Ramsey. This extends a result of Carlson and Simpson, *A dual form of Ramsey's theorem*, Adv. in Math. **53** (1984), 265–290.

Employing the method established therefore, we derive analogous results for Dowling lattices and for ascending k-parameter words.

1. Introduction and preliminaries. In [**GR71**], Graham and Rothschild established a Ramsey type theorem for partitioning k-parameter subsets of an n-dimensional cube A^n, where A is a finite set. As a special case, the Graham-Rothschild result implies Ramsey's theorem about partitions of k-element subsets of an n-element set. However, in contrast to Ramsey's theorem, the Graham-Rothschild result does not extend immediately to partitions of k-parameter subsets of infinite dimensional cubes. Using the axiom of choice, there exist subsets $\mathscr{B} \subseteq A^\omega$ such that every ω-parameter subcube of A^ω meets both $\mathscr{B}$ and its complement (provided, of course, that A contains at least two elements).

Applying a Baire category argument, Carlson and Simpson [**CS84**] showed that for every *Baire set* $\mathscr{B} \subseteq A^\omega$ (where A^ω is endowed with the Tychonoff product topology, with A being discrete) there exists an ω-dimensional subcube $S \subseteq A^\omega$ with $S \subseteq \mathscr{B}$ or $S \subseteq A^\omega \setminus \mathscr{B}$. In this sense, Baire sets of 0-parameter words are Ramsey. For $k > 0$, Carlson and Simpson prove that *Borel sets* of k-parameter words are Ramsey.

As a matter of fact, Pierre Matet observed that the Carlson-Simpson proof for $k > 0$ works for $\mathscr{C}$-sets, whenever $\mathscr{C}$ is a σ-algebra which is closed under continuous preimages and such that every member of $\mathscr{C}$ has the property of Baire, But, using the axiom of choice, Baire sets are not closed under continuous preimages.

In this paper we show that, also for $k > 0$, all Baire sets of k-parameter words are Ramsey. Our proof relies on an infinite *-version of the Graham-Rothschild theorem which has been established in [**Voixx**].

In §2 we define the notion of parameter words and state the infinite *-version of the Graham-Rothschild theorem (Theorem A). In §3 we then prove that *Baire sets of k-parameter words are Ramsey* (Theorem B).

Dowling [**Dow73**] introduced a class of geometric lattices which is based on finite groups $\mathscr{G}$. These *Dowling lattices* are closely related to partition lattices and to the original Graham-Rothschild concept of parameter sets. In fact, our methods from §3

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can also be applied here, yielding that *Baire sets of partial $\mathcal{G}$-partitions are Ramsey* (Theorem C). This is explained in §4.

As a tool, we need an infinite *-version for partial $\mathcal{G}$-partitions. This infinite *-version is established in §5.

In §6 we define *ascending parameter words*. E.g., Hindman's theorem on finite sums and unions **[Hin74]** as well as Milliken's topological generalization **[Mil75]** of it are particular results about ascending parameter words with respect to a singleton alphabet. Recently, Carlson **[Carxx]** (cf. **[Pri82]**) extended Milliken's Ellentuck-type theorem about ascending ω-parameter words to arbitrary finite alphabets.

We show that *Baire sets of ascending k-parameter words are Ramsey* (Theorem E). Again, we need a *-version (Theorem F). This is deduced from Carlson's result.

Finally, in §7 we conclude with a result about the general structure of Baire mappings from k-parameter words into metric spaces. We also mention some related questions.

Preliminaries. 1. Small latin letters i, j, k, l, m, n, r, t denote finite ordinals (nonnegative integers), as usual, $k = \{0, \ldots, k-1\}$.

2. ω is the smallest infinite ordinal, the set of nonnegative integers.

3. Small greek letters α, β, γ denote ordinals less or equal to ω.

4. Let $\mathcal{X}$ be a topological space. A subset $B \subseteq \mathcal{X}$ is a Borel set if it belongs to the σ-algebra generated by all open subsets of $\mathcal{X}$. A subset $B \subseteq \mathcal{X}$ is a Baire set if B is open modulo a meager set, i.e., there exists $M \subseteq \mathcal{X}$, where M is meager, such that $(B \setminus M) \cup (M \setminus B)$ is open.

5. For topological spaces $\mathcal{X}$ and $\mathcal{Y}$, a mapping $\Delta\colon \mathcal{Y} \to \mathcal{X}$ is Borel (resp. Baire) if for all open subsets $\mathcal{O} \subseteq X$ the preimage $\Delta^{-1}(\mathcal{O})$ is Borel (resp. Baire). Every Borel mapping is Baire.

6. For detailed explanations of the topological facts used in this paper see, e.g., **[Kur66]**.

2. Surjections and parameter words.

DEFINITION. Let t be a positive integer and let $\alpha \leqslant \beta \leqslant \omega$ be ordinals. By $\mathcal{S}_t\binom{\beta}{\alpha}$ we denote the set of all surjective mappings $F\colon t + \beta \to t + \alpha$ satisfying

(1) $F(i) = i$ for every $i < t$,

(2) $\min F^{-1}(i) < \min F^{-1}(j)$ for all $i < j < t + \alpha$.

For $F \in \mathcal{S}_t\binom{\gamma}{\beta}$ and $G \in \mathcal{S}_t\binom{\beta}{\alpha}$ the composite $F \cdot G \in \mathcal{S}_t\binom{\gamma}{\alpha}$ is defined via the usual composition of mappings (however, in reversed order), viz., $(F \cdot G)(i) = G(F(i))$.

REMARK. $\mathcal{S}_t$ is the *category of parameter words over alphabet t*. Using a different notation, these have been introduced and studied by Graham and Rothschild **[GR71]**, generalizing an earlier result of Hales and Jewett **[HJ63]**. The present notation goes essentially back to Leeb **[Le73]**. The original motivation for studying parameter words lies in the fact that $\mathcal{S}_t\binom{\beta}{0}$ is isomorphic to the set of β-sequences $(b_i)_{i<\beta}$ with entries in $t = \{0, \ldots, t-1\}$. This is the β-dimensional cube over alphabet t. Having in mind this isomorphism, i.e., $(0, \ldots, t-1, b_0, b_1, \ldots) \Leftrightarrow (b_0, b_1, \ldots)$, parameter words $F \in \mathcal{S}_t\binom{\beta}{\alpha}$ describe α-dimensional subcubes, viz., $\{F \cdot G \mid G \in \mathcal{S}_t\binom{\alpha}{0}\}$.

The Graham-Rothschild partition theorem says that for every mapping Δ: $\mathscr{S}_t\binom{n}{k} \to r$, where $n \geqslant n(t, r, k, m)$ is sufficiently large, there exists an $F \in \mathscr{S}_t\binom{n}{m}$ such that $\Delta(F \cdot G) = \Delta(F \cdot \hat{G})$ for all $G, \hat{G} \in \mathscr{S}_t\binom{m}{k}$.

For a short proof and further explanations see [**DV82**].

DEFINITION. Let t be a positive and k a nonnegative integer. By $\mathscr{S}_t^*\binom{\omega}{k}$ we denote the set of all surjective mappings f: $\omega \to (t + k) \cup \{*\}$ satisfying

(1) $f(i) = i$ for all $i < t$,

(2) $\min f^{-1}(i) < \min f^{-1}(j)$ for all $i < j < t + k$,

(3) if $f(i) = *$ for some $i < \omega$, then also $f(i + 1) = *$.

For $F \in \mathscr{S}_t\binom{\omega}{\omega}$ and $f \in \mathscr{S}_t^*\binom{\omega}{k}$, the composite $F \cdot f \in \mathscr{S}_t^*\binom{\omega}{k}$ is defined by

$$(F \cdot f)(j) = \begin{cases} * & \text{if } (F \cdot f)(i) = * \text{ for some } i < j, \\ f(F(j)) & \text{otherwise.} \end{cases}$$

The infinite *-version of Graham-Rothschild's partition theorem is

THEOREM A [**Voixx**]. *Let* Δ: $\mathscr{S}_t^*\binom{\omega}{k} \to r$ *be a mapping. Then there exists an* $F \in \mathscr{S}_t\binom{\omega}{\omega}$ *such that* $\Delta(F \cdot g) = \Delta(F \cdot \hat{g})$ *for all* $g, \hat{g} \in \mathscr{S}_t^*\binom{\omega}{k}$.

REMARK. The finite version of this result is due to [**Voi80**]. In [**DV82**] it is shown how the finite version with $k = 0$ can be used in order to give short proofs for the Graham-Rothschild partition theorem for parameter words as well as for the Graham-Leeb-Rothschild partition theorem for finite affine (resp., projective) spaces.

Independently, the case $k = 0$ of Theorem A has already been proven in [**CS84**], where it serves as a kind of 'key-lemma'.

3. Baire sets in $\mathscr{S}_t\binom{\omega}{k}$ are Ramsey. Using the axiom of choice, one easily defines mappings Δ: $\mathscr{S}_t\binom{\omega}{k} \to 2$ such that for every $F \in \mathscr{S}_t\binom{\omega}{\omega}$ there exist $G, \hat{G} \in \mathscr{S}_t\binom{\omega}{k}$ with $\Delta(F \cdot G) \neq \Delta(F \cdot \hat{G})$ (cf., [**CS84**]).

However, this is no longer true for mappings which are, in some sense, constructive.

We endow $\mathscr{S}_t\binom{\omega}{k}$ with the Tychonoff product topology. Define a metric d: $\mathscr{S}_t\binom{\omega}{k} \times \mathscr{S}_t\binom{\omega}{k} \to \mathbf{R}$ by $d(G, \hat{G}) = 1/(i + 1)$ iff $i = \min\{j | G(j) \neq \hat{G}(j)\}$. The topology induced by the metric d is the same as the one the Tychonoff product topology on $(t + k)^\omega$ yields for $\mathscr{S}_t\binom{\omega}{k}$.

Note that $\mathscr{S}_t\binom{\omega}{k}$ is an open subset of $(t + k)^\omega$. So the *Baire category theorem*, saying that a countable intersection of dense open sets again is dense, is valid in $\mathscr{S}_t\binom{\omega}{k}$.

Carlson and Simpson [**CS84**] showed that for every Borel-measurable mapping Δ: $\mathscr{S}_t\binom{\omega}{k} \to r$ there exists an $F \in \mathscr{S}_t\binom{\omega}{\omega}$ with $\Delta(F \cdot G) = \Delta(F \cdot \hat{G})$ for all $G, \hat{G} \in \mathscr{S}_t\binom{\omega}{k}$. As a matter of fact, it has been observed by Pierre Matet that the Carlson-Simpson proof remains valid for $\mathscr{C}$-measurable mappings, whenever $\mathscr{C}$ is a σ-algebra which is closed under continuous preimages and such that each member of $\mathscr{C}$ has the property of Baire (cf., [**CS84**, Remark 2.6]).

For $k = 0$ the Carlson-Simpson argument is valid for all Baire mappings Δ: $\mathscr{S}_t\binom{\omega}{0} \to r$, but for $k > 0$ apparently it does not work in that generality.

Here we show that all Baire sets are Ramsey:

THEOREM B. *Let* $\Delta: \mathcal{S}_t\binom{\omega}{k} \to r$ *be a Baire mapping, i.e, for every* $i < r$ *the preimage* $\Delta^{-1}(i)$ *has the property of Baire. Then there exists an* $F \in \mathcal{S}_t\binom{\omega}{\omega}$ *such that* $\Delta(F \cdot G) = \Delta(F \cdot \hat{G})$ *for all* $G, \hat{G} \in \mathcal{S}_t\binom{\omega}{k}$.

As mentioned before, the case $k = 0$ is already due to Carlson and Simpson. We do not prove this here. Also, as we do not proceed by induction on k, the case $k = 0$ is not needed in order to establish the remaining cases.

So, fix positive integers t and k.

The remainder of this section is devoted to proving Theorem B.

Notation. For $f \in \mathcal{S}_t^*\binom{\omega}{k}$ let

$$\mathcal{T}(f) = \left\{ F \in \mathcal{S}_t\binom{\omega}{k} \middle| F(i) = f(i) \text{ for all } i < \min f^{-1}(*) \right\}$$

be the *Tychonoff-cone* generated by f.

The set of all $\mathcal{T}(f)$, $f \in \mathcal{S}_t^*\binom{\omega}{k}$, forms a basis for the topology on $\mathcal{S}_t\binom{\omega}{k}$.

Notation. For nonnegative integers m we denote by $(t + m)^*$ the set of all finite sequences with entries in $(t + m)$. Formally, $(t + m)^*$ consists of all mappings h: $\omega \to (t + m) \cup \{*\}$ such that $f(i) = *$ for some $i < \omega$ and if $f(i) = *$, then also $f(i + 1) = *$.

Notation. For $f \in \mathcal{S}_t^*\binom{\omega}{m}$ and $h \in (t + m)^*$ the *concatenation* $f \otimes h \in \mathcal{S}_t^*\binom{\omega}{m}$ is defined by

$$(f \otimes h)(i) = \begin{cases} f(i) & \text{if } i < \min f^{-1}(*), \\ h(i - \min f^{-1}(*)) & \text{if } \min f^{-1}(*) \leqslant i. \end{cases}$$

Notation. For $g \in \mathcal{S}_t^*\binom{\omega}{m}$, the parameter word $g^+ \in \mathcal{S}_t^*\binom{\omega}{m+1}$ is defined by juxtaposition of a new parameter, viz.,

$$g^+(i) = \begin{cases} g(i) & \text{if } i < \min g^{-1}(*), \\ t + m & \text{if } i = \min g^{-1}(*), \\ * & \text{otherwise.} \end{cases}$$

The following lemma is obvious, but it will be used throughout.

LEMMA 1. *Let* $f \in \mathcal{S}_t^*\binom{\omega}{m}$, $g \in \mathcal{S}_t\binom{m}{k}$ *and let* $h \in (t + k)^*$. *Define* $\tilde{h} \in (t + m)^*$ *by* $\tilde{h}(i) = \min g^{-1}(h(i))$. *Then* $(f \otimes \tilde{h}) \cdot g = (f \cdot g) \otimes h$. □

The crucial lemma for proving Theorem B is Lemma 4. The proof is based on a Baire category argument. As a technical device, Lemmas 2 and 3 will be needed. For $f \in \mathcal{S}_t^*\binom{\omega}{m}$ and $g \in \mathcal{S}_t\binom{m}{k}$ the composition $f \cdot g \in \mathcal{S}_t^*\binom{\omega}{k}$ is defined in the obvious way, viz., $(f \cdot g)(j) = *$ if $f(j) = *$ and $(f \cdot g)(j) = g(f(j))$ otherwise.

LEMMA 2. *Let* r *be a positive integer and let* $B_i \subseteq \mathcal{S}_k\binom{\omega}{k}$, $i < r$, *be open subsets such that* $\bigcup_{i<r} B_i$ *is dense. Let* $f \in \mathcal{S}_t^*\binom{\omega}{k+m}$. *Then there exists* $\tilde{h} \in (t + k + m + 1)^*$ *such that* $f^+ \otimes \tilde{h}$ *has the following property: for every* $g \in \mathcal{S}_t\binom{k+m}{k-1}$ *there exists* $i < r$ *such that* $\mathcal{T}((f^+ \otimes \tilde{h}) \cdot g^+) \subseteq B_i$.

PROOF. Let $(g_i)_{i<s}$ be an enumeration of $\mathscr{S}_t\binom{k+m}{k-1}$. By induction, let

$$\tilde{h}_j \in (t+k+m+1)^*$$

be such that for every $i<j$ there exists an $i'<r$ such that $\mathscr{T}((f^+\otimes \tilde{h}_j)\cdot g_i^+)\subseteq B_{i'}$. For constructing $\tilde{h}_{j+1}$, consider $\mathscr{T}((f^+\otimes \tilde{h}_j)\cdot g_j^+)$. As $\bigcup_{i<r}B_i$ is dense, there exists $j'<r$ such that $B_{j'}\cap \mathscr{T}((f^+\otimes \tilde{h}_j)\cdot g_j^+)\neq \varnothing$. So the intersection contains some basic open set, i.e., there exists an $h\in (t+k)^*$ such that $\mathscr{T}(((f^+\otimes \tilde{h}_j)\cdot g_j^+)\otimes h)\subseteq B_{j'}$. By Lemma 1, there exists $\tilde{h}'\in (t+k+m+1)^*$ such that $(f^+\otimes \tilde{h}_j\otimes \tilde{h}')\cdot g_j^+ = ((f^+\otimes \tilde{h}_j)\cdot g_j^+)\otimes h$. Then $\tilde{h}_{j+1}=\tilde{h}_j\otimes \tilde{h}'$ again satisfies the inductive assumption.

Finally, $\tilde{h}_{s+1}$ satisfies the assertion of the lemma. □

LEMMA 3. *Let $D\subseteq \mathscr{S}_t\binom{\omega}{k}$ be dense open and let $f\in \mathscr{S}_t^*\binom{\omega}{k+m+1}$. Then there exists $\tilde{h}\in (t+k+m+1)^*$ such that $f\otimes \tilde{h}$ has the property*

$$\mathscr{T}((f\otimes \tilde{h})\cdot g)\subseteq D \quad \textit{for every } g\in \mathscr{S}_t\binom{k+m+1}{k}.$$

PROOF. Let $(g_i)_{i<s}$ be an enumeration of $\mathscr{S}_t\binom{k+m+1}{k}$. By induction, let

$$\tilde{h}_j\in (t+k+m+1)^*$$

be such that $\mathscr{T}((f\otimes h_j)\cdot g_i)\subseteq D$ for every $i<j$. For constructing $\tilde{h}_{j+1}$, consider $\mathscr{T}((f\otimes \tilde{h}_j)\cdot g_j)$. As D is dense open, there exists $h\in (t+k)^*$ such that

$$\mathscr{T}\Big(\big((f\otimes \tilde{h}_j)\cdot g_j\big)\otimes h\Big)\subseteq D.$$

By Lemma 1, there exists $\tilde{h}'\in (t+k+m+1)^*$ such that $(f\otimes \tilde{h}_j\otimes \tilde{h}')\cdot g_j = ((f\otimes \tilde{h}_j)\cdot g_j)\otimes h$. Hence, $\tilde{h}_{j+1}=\tilde{h}_j\otimes \tilde{h}'$ again satisfies the inductive assumption.

Finally, $\tilde{h}_{s+1}$ satisfies the assertion of the lemma. □

LEMMA 4. *Let $M\subset \mathscr{S}_t\binom{\omega}{k}$ be meager and let $B_i\subseteq \mathscr{S}_t\binom{\omega}{k}$, $i<r$, be open such that $\bigcup_{i<r}B_i$ is dense. Then there exists an $F\in \mathscr{S}_t\binom{\omega}{\omega}$ such that*

(1) *for every $g\in \mathscr{S}_t^*\binom{\omega}{k-1}$ there exists an $i<r$ such that $F\cdot G\in B_i$ for all $G\in \mathscr{T}(g^+)$,*

(2) *$F\cdot G\notin M$ for all $G\in \mathscr{S}_t\binom{\omega}{k}$.*

PROOF. As M is meager, there exist dense open subsets $D_n\subseteq \mathscr{S}_t\binom{\omega}{k}$, $n<\omega$, such that $M\subseteq \mathscr{S}_t\binom{\omega}{k}\setminus \bigcap_{n<\omega}D_n$. For convenience, put $D_n^*=\bigcap_{l\leqslant n}D_n$.

To start the construction of F, pick any $f\in \mathscr{S}_t^*\binom{\omega}{k}$. Let $i<r$ be such that $\mathscr{T}(f)\cap D_0^*\cap B_i\neq\varnothing$. Such an i exists, as $\bigcup_{i<r}B_i$ as well as D_0^* are dense. Then let $f_0\in \mathscr{S}_t^*\binom{\omega}{k}$ be such that $\mathscr{T}(f_0)\subseteq \mathscr{T}(f)\cap D_0^*\cap B_i$.

Note that actually $f_0=f\otimes h$ for some $h\in (t+k)^*$. By induction, let $f_m\in \mathscr{S}_t^*\binom{\omega}{k+m}$ be such that

(3) for every $g\in \mathscr{S}_t\binom{k+m-1}{k-1}$ there exists an $i<r$ such that $\mathscr{T}(f_m\cdot g^+)\subseteq B_i$,

(4) $\mathscr{T}(f_m\cdot g)\subseteq D_m^*$ for every $g\in \mathscr{S}_t\binom{k+m}{k}$,

(5) $f_l(i)=f_m(i)$ for every $i<\min f_m^{-1}(t+k+l)$ and every $l<m$.

By Lemma 2, there exists $\tilde{h} \in (t + k + m + 1)^*$ such that $f_m^+ \otimes \tilde{h}$ satisfies (3) for $m + 1$. By Lemma 3, there exists $\tilde{\tilde{h}} \in (t + k + m + 1)^*$ such that $f_{m+1} = f_m^+ \otimes \tilde{h} \otimes \tilde{\tilde{h}}$ also satisfies (4) for $m + 1$. By construction, f_{m+1} also satisfies (5) for $m + 1$. Finally, let $F = \lim f_m$, i.e., $F(i) = f_i(i)$. By (5), F is defined consistently.

We verify properties (1) and (2).

ad(1). Let $g \in \mathscr{S}_t^*\binom{\omega}{k-1}$, say, $t + k + m - 1 = \min g^{-1}(*)$. So, g can be viewed as an element of $\mathscr{S}_t\binom{k+m-1}{k-1}$. By (3), there exists an $i < r$ such that $\mathscr{T}(f_m \cdot g^+) \subseteq B_i$. According to (5) and the definition of F it follows that $\{F \cdot G | G \in \mathscr{T}(g^+)\} \subseteq \mathscr{T}(f_m \cdot g^+) \subseteq B_i$.

ad(2). Let $G \in \mathscr{S}_t\binom{\omega}{k}$. We show that $F \cdot G \in \bigcap_{n<\omega} D_n^*$. So, let $m < \omega$. Say, without restriction, that $\min G^{-1}(t + k - 1) < t + k + m$. Thus, $g \in G \urcorner t + k + m$ is an element of $\mathscr{S}_t\binom{k+m}{k}$. By (4), (5) and the definition of F it then follows that $F \cdot G \in \mathscr{T}(f_m \cdot g) \subseteq D_m^*$. □

Proof of Theorem B. Let $\Delta: \mathscr{S}_t\binom{\omega}{k} \to r$ be a Baire mapping, i.e., for every $i < r$ the preimage $\Delta^{-1}(i)$ has the property of Baire, viz., each $\Delta^{-1}(i)$ is open modulo some meager set. So there exist open sets $B_i \subseteq \mathscr{S}_t\binom{\omega}{k}$, $i < r$, such that the symmetric differences

$$M_i = \left(\Delta^{-1}(i) \setminus B_i\right) \cup \left(B_i \setminus \Delta^{-1}(i)\right)$$

are meager.

Put $M = \bigcup_{i<r} M_i$. Then $\mathscr{S}_t\binom{\omega}{k} \setminus M \subseteq \bigcup_{i<r} B_i$ and thus, by the Baire category theorem, $\bigcup_{i<r} B_i$ is dense. Apply Lemma 4. Let $F \in \mathscr{S}_t\binom{\omega}{\omega}$ be such that (1) and (2) are satisfied. Note that for every $g \in \mathscr{S}_t^*\binom{\omega}{k-1}$ and all $G, \hat{G} \in \mathscr{T}(g^+)$ it follows that $\Delta(F \cdot G) = \Delta(F \cdot \hat{G})$.

Define a mapping $\Delta^*: \mathscr{S}_t^*\binom{\omega}{k-1} \to r$ by $\Delta^*(g) = \Delta(F \cdot G)$ for any $G \in \mathscr{T}(g^+)$. Apply Theorem A. Let $F^* \in \mathscr{S}_t\binom{\omega}{\omega}$ be such that $\Delta^*(F^* \cdot g) = \Delta^*(F^* \cdot \hat{g})$ for all g, $\hat{g} \in \mathscr{S}_t^*\binom{\omega}{k-1}$. But then, by choice of F and the definition of Δ^* it follows that $\Delta(F \cdot F^* \cdot G) = \Delta(F \cdot F^* \cdot \hat{G})$ for all G, $\hat{G} \in \mathscr{S}_t\binom{\omega}{k}$, i.e, $F \cdot F^* \in \mathscr{S}_t\binom{\omega}{\omega}$ has the desired properties. □

4. Baire sets of partial $\mathscr{G}$-partitions are Ramsey. An $F \in \mathscr{S}_1\binom{\beta}{\alpha}$ gives rise to a partial partition of $\{i | 1 \leqslant i < 1 + \beta\}$ into α mutually disjoint and nonempty blocks, viz., $F^{-1}(j)$ for $1 \leqslant j < 1 + \alpha$. Conversely, every partial partition is described by a (uniquely determined) parameter word over alphabet 1.

Dowling [**Dow73**] introduced a class of geometric lattices which is closely related to the original concept of parameter words, resp., to partial partitions. These *Dowling lattices* are based on finite groups.

Definition. Let $\mathscr{G}$ be a finite group and let $e \in \mathscr{G}$ denote the unit element of $\mathscr{G}$. Furthermore, let $\mathscr{A}$ be a symbol not occurring in $\mathscr{G}$ and let $\alpha \leqslant \beta \leqslant \omega$ be ordinals. By $\mathscr{S}_{\mathscr{G}}\binom{\beta}{\alpha}$ we denote the set of all mappings $F: \beta \to \{\mathscr{A}\} \cup (\alpha \times \mathscr{G})$ satisfying

(1) for every $j < \alpha$ there exists an $i < \beta$ with $F(i) = (j, e)$ and $F(i') \notin \{j\} \times \mathscr{G}$ for all $i' < i$,

(2) $\min F^{-1}(i, e) < \min F^{-1}(j, e)$ for all $i < j < \alpha$.

$\mathscr{S}_{\mathscr{G}}$ is the *category of partial $\mathscr{G}$-partitions*. Mappings $F \in \mathscr{S}_{\mathscr{G}}\binom{\beta}{\alpha}$ are partial $\mathscr{G}$-partitions of β into α blocks.

DEFINITION. For partial $\mathcal{G}$-partitions $F \in \mathcal{S}_{\mathcal{G}}\binom{\gamma}{\beta}$ and $G \in \mathcal{S}_{\mathcal{G}}\binom{\beta}{\alpha}$ the composition $F \cdot G \in \mathcal{S}_{\mathcal{G}}\binom{\gamma}{\alpha}$ is defined by

$$(F \cdot G)(i) = \begin{cases} \mathscr{A} & \text{if } F(i) = \mathscr{A}, \\ \mathscr{A} & \text{if } F(i) = (j, b) \text{ and } G(j) = \mathscr{A}, \\ (k, b \cdot c) & \text{if } F(i) = (j, b) \text{ and } G(j) = (k, c), \end{cases}$$

where $b \cdot c$ refers to multiplication in $\mathcal{G}$.

Partial $\mathcal{G}$-partitions arise from 'ordinary' partial partitions $F \in \mathcal{S}_1\binom{\beta}{\alpha}$ by labeling the parameters $1, \ldots, \alpha$ with elements from the group $\mathcal{G}$. Composition of labels means multiplication. The constant $\mathscr{A}$ acts as a kind of annihilator. For the trivial group $\mathcal{G} = \{e\}$, the categories $\mathcal{S}_1$ and $\mathcal{S}_{\{e\}}$ are isomorphic. Let $\mathcal{S}_{\mathcal{G}}(\beta) = \bigcup_{\alpha \leqslant \beta} \mathcal{S}_{\mathcal{G}}\binom{\beta}{\alpha}$ be the set of all partial $\mathcal{G}$-partitions of β. For $G \in \mathcal{S}_{\mathcal{G}}\binom{\beta}{\alpha}$ and $G^* \in \mathcal{S}_{\mathcal{G}}\binom{\beta}{\alpha^*}$ let $G^* \geqslant G$ iff $G^* = G \cdot H$ for some $H \in \mathcal{S}_{\mathcal{G}}\binom{\alpha}{\alpha^*}$. Dowling **[Dow73]** shows that for each finite group $\mathcal{G}$ and for each nonnegative integer n the set $(\mathcal{S}_{\mathcal{G}}(n), \leqslant)$ of partial $\mathcal{G}$-partitions of n is a geometric lattice. Also, different groups yield nonisomorphic lattices.

The Graham-Rothschild partition theorem **[GR71]** implies that for every mapping $\Delta\colon \mathcal{S}_{\mathcal{G}}\binom{n}{k} \to r$, where $n \geqslant n(\mathcal{G}, k, r, m)$ is sufficiently large, there exists an $F \in \mathcal{S}_{\mathcal{G}}\binom{n}{m}$ such that $\Delta(F \cdot G) = \Delta(F \cdot \hat{G})$ for all $G, \hat{G} \in \mathcal{S}_{\mathcal{G}}\binom{m}{k}$.

As before, we define a metric on $\mathcal{S}_{\mathcal{G}}\binom{\omega}{k}$ by $d(G, \hat{G}) = 1/(i+1)$ iff $i = \min\{j \mid G(j) \neq \hat{G}(j)\}$. The topology induced by the metric d is the same as the one the Tychonoff topology on $(\{\mathscr{A}\} \cup k \times \mathcal{G})^\omega$ yields for $\mathcal{S}_{\mathcal{G}}\binom{\omega}{k}$. $\mathcal{S}_{\mathcal{G}}\binom{\omega}{k}$ is an open subset of $(\{\mathscr{A}\} \cup k \times \mathcal{G})^\omega$. We show that, with respect to this topology, every Baire set in $\mathcal{S}_{\mathcal{G}}\binom{\omega}{k}$ is Ramsey.

THEOREM C. *Let $\Delta\colon \mathcal{S}_{\mathcal{G}}\binom{\omega}{k} \to r$ be a Baire mapping. Then there exists an $F \in \mathcal{S}_{\mathcal{G}}\binom{\omega}{\omega}$ such that $\Delta(F \cdot G) = \Delta(F \cdot \hat{G})$ for all $G, \hat{G} \in \mathcal{S}_{\mathcal{G}}\binom{\omega}{k}$.*

Theorem C can be proved following the pattern of the proof of Theorem B. Thus, we first introduce *partial $\mathcal{G}$-partitions of variable length*, viz., the category $\mathcal{S}_{\mathcal{G}}^*$. This will be done in §5.

Now, in order to prove Theorem C we proceed step by step as in the proof of Theorem B, substituting the categories $\mathcal{S}_t$, resp. $\mathcal{S}_t^*$, by the categories $\mathcal{S}_{\mathcal{G}}$, resp. $\mathcal{S}_{\mathcal{G}}^*$. We omit the details.

5. An infinite *-version for $\mathcal{S}_{\mathcal{G}}$. By $\mathcal{S}_{\mathcal{G}}^*\binom{\omega}{k}$ we denote the set of all mappings $f\colon \omega \to \{\mathscr{A}\} \cup k \times \mathcal{G} \cup \{*\}$ satisfying:

(1) for every $j < k$ there exists $i < \omega$ with $f(i) = (j, e)$ and $f(i') \notin \{j\} \times \mathcal{G}$ for all $i' < i$,

(2) $\min f^{-1}(i, e) < \min f^{-1}(j, e)$ for all $i < j < k$,

(3) there exists a $j < \omega$ such that $f(i) \neq *$ for all $i < j$ and $f(i) = *$ for all $j \leqslant i$.

DEFINITION. For $F \in \mathcal{S}_{\mathcal{G}}\binom{\omega}{\omega}$ and $g \in \mathcal{S}^*_{\mathcal{G}}\binom{\omega}{k}$ the composite $F \cdot g \in \mathcal{S}^*_{\mathcal{G}}\binom{\omega}{k}$ is defined by

$$(F \cdot g)(j) = \begin{cases} * & \text{if } (F \cdot g)(i) = * \text{ for some } i < j \\ & \quad \text{or } F(j) = (k, e) \text{ and } g(k) = *, \\ \mathcal{A} & \text{if } (F \cdot g)(i) \neq * \text{ for every } i < j \\ & \quad \text{and either } F(j) = \mathcal{A} \text{ or } F(j) = (k, b) \\ & \quad \text{and } g(k) = \mathcal{A}, \\ (l, b \cdot c) & \text{if } (F \cdot g)(i) \neq * \text{ for every } i < j \\ & \quad \text{and } F(j) = (k, b), g(k) = (l, c). \end{cases}$$

For $f \in \mathcal{S}^*_{\mathcal{G}}\binom{\omega}{m}$ and $g \in \mathcal{S}_{\mathcal{G}}\binom{m}{k}$ the composite $f \cdot g \in \mathcal{S}^*_{\mathcal{G}}\binom{\omega}{k}$ is defined analogously, where $(f \cdot g)(j) = *$ if $f(j) = *$.

This section is devoted to the proof of the following theorem.

THEOREM D. *Let $\Delta: \mathcal{S}^*_{\mathcal{G}}\binom{\omega}{k} \to r$ be a mapping. Then there exists an $F \in \mathcal{S}_{\mathcal{G}}\binom{\omega}{\omega}$ such that $\Delta(F \cdot g) = \Delta(F \cdot \hat{g})$ for all $g, \hat{g} \in \mathcal{S}^*_{\mathcal{G}}\binom{\omega}{k}$.*

Notation. For nonnegative integers k let $(\{\mathcal{A}\} \cup k \times \mathcal{G})^*$ denote the set of finite sequences with entries from $\{\mathcal{A}\} \cup k \times \mathcal{G}$. Formally $(\{\mathcal{A}\} \cup k \times \mathcal{G})^*$ is the set of all mappings $h: \omega \to \{\mathcal{A}\} \cup k \times \mathcal{G} \cup \{*\}$ such that $g(i) = *$ for some $i < \omega$ and $g(i) = *$ implies $g(i+1) = *$ for every $i < \omega$. Thus, $\mathcal{S}^*_{\mathcal{G}}\binom{\omega}{k}$ is a subset of $(\{\mathcal{A}\} \cup k \times \mathcal{G})^*$.

Let $f \in \mathcal{S}^*_{\mathcal{G}}\binom{\omega}{k}$ and $g \in (\{\mathcal{A}\} \cup k \times \mathcal{G})^*$. Then the concatenation $f \otimes g \in \mathcal{S}^*_{\mathcal{G}}\binom{\omega}{k}$ is defined by

$$(f \otimes g)(i) = \begin{cases} f(i) & \text{if } i < \min f^{-1}(*), \\ g(i - \min f^{-1}(*)) & \text{if } \min f^{-1}(*) \leqslant i. \end{cases}$$

REMARK. For $f \in \mathcal{S}^*_{\mathcal{G}}\binom{\omega}{m}$, $g \in \mathcal{S}_{\mathcal{G}}\binom{m}{k}$ and $h \in (\{\mathcal{A}\} \cup k \times \mathcal{G})^*$ there exists $\tilde{h}(\{\mathcal{A}\} \cup m \times \mathcal{G})^*$ such that $(f \otimes \tilde{h}) \cdot g = (f \cdot g) \otimes h$. Define, for example, $\tilde{h}(i) = \mathcal{A}$ if $h(i) = \mathcal{A}$ and $\tilde{h}(i) = (\min g^{-1}(j, e), b)$ if $h(i) = (j, b)$. This is the analogue of Lemma 1 for $\mathcal{S}_{\mathcal{G}}$.

Notation. For $g \in \mathcal{S}^*_{\mathcal{G}}\binom{\omega}{k}$ let $g^+ \in \mathcal{S}^*_{\mathcal{G}}\binom{\omega}{k+1}$ be defined by

$$g^+(i) = \begin{cases} g(i) & \text{if } i < \min g^{-1}(*), \\ (k, e) & \text{if } i = \min g^{-1}(*), \\ * & \text{otherwise.} \end{cases}$$

Analogously for $h \in \mathcal{S}_{\mathcal{G}}\binom{l}{k}$, where l is a nonnegative integer, let $h^+ \in \mathcal{S}_{\mathcal{G}}\binom{l+1}{k+1}$ be given by $h^+(i) = h(i)$ for every $i < l$ and $h^+(l) = (k, e)$.

The main tool in proving Theorem D is Lemma 6. As a technical device we need the following

LEMMA 5. *Let $\Delta: \mathcal{S}^*_{\mathcal{G}}\binom{\omega}{k+1} \to r$ be a mapping and let $l \geqslant k$ be a nonnegative integer. Then there exists an $F \in \mathcal{S}_{\mathcal{G}}\binom{\omega}{\omega}$ with $F(i) = (i, e)$ for every $i < l+1$ such that for every $g \in \mathcal{S}_{\mathcal{G}}\binom{l}{k}$ it follows that*

$$\Delta(F \cdot (g^+ \otimes h)) = \Delta(F \cdot (g^+ \otimes \hat{h})) \quad \textit{for all } h, \hat{h} \in (\{\mathcal{A}\} \cup (k+1) \times \mathcal{G})^*.$$

PROOF. Let $(g_i)_{i<s}$ be an enumeration of $\mathcal{S}_{\mathcal{G}}\binom{l}{k}$. By induction, let $F_j \in \mathcal{S}_{\mathcal{G}}\binom{\omega}{\omega}$ with $F(i) = (i, e)$ for every $i < l + 1$ be such that for every $i < j$ it follows that

$$\Delta\left(F_j \cdot (g_i^+ \otimes h)\right) = \Delta\left(F_j \cdot (g_i^+ \otimes \hat{h})\right) \quad \text{for all } h, \hat{h} \in (\{\mathcal{A}\} \cup (k+1) \times \mathcal{G})^*.$$

Let $\sigma\colon\ 1 + (k+1)\cdot|\mathcal{G}| \to \{\mathcal{A}\} \cup (k+1) \times \mathcal{G}$ be any bijection. For $h \in \mathcal{S}^*_{1+(k+1)\cdot|\mathcal{G}|}\binom{\omega}{0}$ define $h^\sigma \in (\{\mathcal{A}\} \cup (k+1) \times \mathcal{G})^*$ by $h^\sigma(i) = \sigma(h(t+i))$. Consider the mapping $\Delta^*\colon \mathcal{S}^*_{1+(k+1)\cdot|\mathcal{G}|}\binom{\omega}{0} \to r$ which is defined by

$$\Delta^*(h) = \Delta\left(F_j \cdot \left(g_j^+ \otimes h^\sigma\right)\right) \quad \text{for all } h \in \mathcal{S}^*_{1+(k+1)\cdot|\mathcal{G}|}\binom{\omega}{0}.$$

According to Theorem A for $k = 0$ there exists $F \in \mathcal{S}_{1+(k+1)\cdot|\mathcal{G}|}\binom{\omega}{\omega}$ with $\Delta^*(F \cdot h) = \Delta^*(F \cdot \hat{h})$ for all $h, \hat{h} \in \mathcal{S}^*_{1+(k+1)\cdot|\mathcal{G}|}\binom{\omega}{0}$. Consider $\hat{F} \in \mathcal{S}_{\mathcal{G}}\binom{\omega}{\omega}$, which is defined as follows:

$$\hat{F}(i) = \begin{cases} (i, e) & \text{if } i < l+1, \\ \mathcal{A} & \text{if } i \geqslant l+1,\ F(t+i-l-1) < 1 + (k+1)\cdot|\mathcal{G}| \\ & \quad \text{and } \sigma\left(F(t+i-l-1)\right) = \mathcal{A}, \\ \left(\min\left(g_j^+\right)^{-1}(m, e), b\right) & \text{if } i \geqslant l+1,\ F(t+i-l-1) < 1 + (k+1)\cdot|\mathcal{G}| \\ & \quad \text{and } \sigma\left(F(t+i-l-1)\right) = (m, b), \\ \left(F(t+i-l-1) + l - (k+1)\cdot|\mathcal{G}|, e\right) & \text{otherwise.} \end{cases}$$

Then $\hat{F} \cdot (g_j^+ \otimes h^\sigma) = g_j^+ \otimes (F \cdot h)^\sigma$ for all $h \in \mathcal{S}^*_{i+(k+1)\cdot|\mathcal{G}|}\binom{\omega}{0}$. Hence, $F_{j+1} = F_j \cdot \hat{F}$ satisfies the inductive assumption for $j + 1$ and, finally, F_{s+1} satisfies the assertion of the lemma. □

LEMMA 6. *Let $\Delta\colon \mathcal{S}^*_{\mathcal{G}}\binom{\omega}{k+1} \to r$ be a mapping. Then there exists an $F \in \mathcal{S}_{\mathcal{G}}\binom{\omega}{\omega}$ such that $\Delta(F \cdot g) = \Delta(F \cdot \hat{g})$ for all $g, \hat{g} \in \mathcal{S}^*_{\mathcal{G}}\binom{\omega}{k+1}$ satisfying* $\min g^{-1}(k, e) = \min \hat{g}^{-1}(k, e)$ *and $g(i) = \hat{g}(i)$ for all $i < \min g^{-1}(k, e)$.*

PROOF. By induction, let $F_l \in \mathcal{S}_{\mathcal{G}}\binom{\omega}{\omega}$ be such that

(1) $\Delta(F_l \cdot g) = \Delta(F_l \cdot \hat{g})$ for all $g, \hat{g} \in \mathcal{S}^*_{\mathcal{G}}\binom{\omega}{k+1}$ satisfying $\min g^{-1}(k, e) = \min \hat{g}^{-1}(k, e) < l$ and $g(i) = \hat{g}(i)$ for all $i < \min g^{-1}(k, e)$.

(2) $F_l(i) = F_{l'}(i)$ for every $l' < l$ and every $i < \min F^{-1}(l', e)$.

By Lemma 5, there exists an $\hat{F} \in \mathcal{S}_{\mathcal{G}}\binom{\omega}{\omega}$ with $\hat{F}(i) = (i, e)$ for every $i < l + 1$ and such that $F_{l+1} = F_l \cdot \hat{F}$ satisfies again (1) and (2), but now for $l + 1$. Finally, $F = \lim F_l$, i.e., $F(l) = F_l(l)$, satisfies the assertion of the lemma. □

PROOF OF THEOREM D. By induction on k. For $k = 0$ we are done by the pigeon-hole principle. Thus, assume the validity of the theorem for some $k \geqslant 0$ and let $\Delta\colon \mathcal{S}^*_{\mathcal{G}}\binom{\omega}{k+1} \to r$ be a mapping. By Lemma 6 we can assume that $\Delta(g) = \Delta(\hat{g})$ for all $g, \hat{g} \in \mathcal{S}^*_{\mathcal{G}}\binom{\omega}{k+1}$ satisfying $\min g^{-1}(k, e) = \min \hat{g}^{-1}(k, e)$ and $g(i) = \hat{g}(i)$ for all $i < \min g^{-1}(k, e)$.

Let $\Delta^*\colon\ \mathcal{S}^*_{\mathcal{G}}\binom{\omega}{k} \to r$ be given by $\Delta^*(g) = \Delta(g^+)$. According to the inductive hypothesis there exists $F \in \mathcal{S}_{\mathcal{G}}\binom{\omega}{\omega}$ such that $\Delta^*(F \cdot g) = \Delta^*(F \cdot \hat{g})$ for all $g, \hat{g} \in \mathcal{S}^*_{\mathcal{G}}\binom{\omega}{k}$. But then $\Delta(F \cdot g^+) = \Delta(F \cdot \hat{g}^+)$ for all $g, \hat{g} \in \mathcal{S}^*_{\mathcal{G}}\binom{\omega}{k}$. Thus, F fulfills the assertion of Theorem D. □

6. Ascending parameter words.

DEFINITION. For ordinals $\alpha \leqslant \beta \leqslant \omega$ we define

$$\mathscr{S}_t^<\binom{\beta}{\alpha} = \left\{F \in \mathscr{S}_t\binom{\beta}{\alpha}\middle| F^{-1}(j) \text{ is finite and } \max F^{-1}(t+i) < \min F^{-1}(t+j) \text{ for all } i < j < \alpha\right\}.$$

As one easily observes, $\mathscr{S}_t^<$ is closed under composition. We call $\mathscr{S}_t^<$ the *category of ascending parameter words over alphabet t*.

Using a different notation, the categories $\mathscr{S}_t^<$ have been studied by Milliken [**Mil75**] and Carlson [**Carxx**]; see also [**Pri82**].

Note that $\mathscr{S}_0^<\binom{\omega}{k} = \varnothing$ by definition. However, $\mathscr{S}_0^<\binom{\omega}{\omega}$ describes $[\omega]^\omega$, the infinite subsets of ω.

With respect to $t = 1$, the first interesting case appears for 1-parameter words. Hindman's theorem [**Hin74**] follows from saying that for every mapping Δ: $\mathscr{S}_1^<\binom{\omega}{1} \to r$ there exists an $F \in \mathscr{S}_1^<\binom{\omega}{\omega}$ such that $\Delta(F \cdot G) = \Delta(F \cdot \hat{G})$ for all G, $\hat{G} \in \mathscr{S}_1^<\binom{\omega}{1}$. This has been generalized by Milliken [**Mil75**], viz., for every mapping Δ: $\mathscr{S}_1^<\binom{\omega}{k} \to r$ there exists an $F \in \mathscr{S}_1^<\binom{\omega}{\omega}$ such that $\Delta(F \cdot G) = \Delta(F \cdot \hat{G})$ for all G, $\hat{G} \in \mathscr{S}_1^<\binom{\omega}{k}$. Such a result does not hold for $t > 1$. Again, this can be seen using the axiom of choice.

As a subset of $\mathscr{S}_t\binom{\omega}{k}$, $\mathscr{S}_t^<\binom{\omega}{k}$ is a metric space. (Note, with respect to the usual metric, $\mathscr{S}_1\binom{\omega}{k}$ becomes discrete.)

It is the purpose of this section to show that Baire sets of $\mathscr{S}_t^<\binom{\omega}{k}$ are Ramsey:

THEOREM E. *For every Baire mapping Δ: $\mathscr{S}_t^<\binom{\omega}{k} \to r$ there exists an $F \in \mathscr{S}_t^<\binom{\omega}{\omega}$ such that $\Delta(F \cdot G) = \Delta(F \cdot \hat{G})$ for all G, $\hat{G} \in \mathscr{S}_t^<\binom{\omega}{k}$.*

To prove this, we use essentially the same method as in the preceding sections. The case $k = 0$ follows from the Baire category construction of Carlson and Simpson [**CS84**]. So we are left with the cases $k > 0$.

DEFINITION.

$$\mathscr{S}_t^{<*}\binom{\omega}{k} = \left\{f \in \mathscr{S}_t^*\binom{\omega}{k}\middle| \max f^{-1}(t+i) < \min f^{-1}(t+j) \text{ for all } i < j < k\right\}.$$

For $F \in \mathscr{S}_t^<\binom{\omega}{\omega}$ and $f \in \mathscr{S}_t^{<*}\binom{\omega}{k}$ the composite $F \cdot f \in \mathscr{S}_t^{<*}\binom{\omega}{k}$ is defined as before, i.e.,

$$(F \cdot f)(j) = \begin{cases} * & \text{if } (F \cdot f)(i) = * \text{ for some } i < j, \\ f(F(j)) & \text{otherwise.} \end{cases}$$

The required result about $\mathscr{S}_t^{<*}\binom{\omega}{k}$ is

THEOREM F. *For every mapping Δ: $\mathscr{S}_t^{<*}\binom{\omega}{k} \to r$ there exists an $F \in \mathscr{S}_t^<\binom{\omega}{\omega}$ such that $\Delta(F \cdot g) = \Delta(F \cdot \hat{g})$ for all g, $\hat{g} \in \mathscr{S}_t^{<*}\binom{\omega}{k}$.*

This can be proved most easily from the partition theorem of Carlson [**Carxx**] (see also [**Pri82**]) for $\mathscr{S}_t^<\binom{\omega}{\omega}$. As before, $\mathscr{S}_t^<\binom{\omega}{\omega}$ is a metric space with the usual metric, i.e, $d(F, \hat{F}) = 1/(i+1)$ iff $i = \min\{j < \omega | F(j) \neq \hat{F}(j)\}$. Carlson's theorem implies that for every continuous mapping Δ: $\mathscr{S}_t^<\binom{\omega}{\omega} \to r$ there exists an $F \in \mathscr{S}_t^<\binom{\omega}{\omega}$

such that $\Delta(F \cdot G) = \Delta(F \cdot \hat{G})$ for all $G, \hat{G} \in \mathcal{S}_t^<\binom{\omega}{\omega}$. Note that the case $t = 0$ is a result of Nash-Williams **[N-W65]**, $t = 1$ is due to Milliken **[Mil75]**. As a matter of fact, Carlson's results (as well a Milliken's for $t = 1$) is much more general.

PROOF OF THEOREM F. Given $G \in \mathcal{S}_t^<\binom{\omega}{\omega}$, define $G^* \in \mathcal{S}_t^{<*}\binom{\omega}{k}$ by $G^*(i) = G(i)$ if $i < \min G^{-1}(t + k)$ and $G^*(i) = *$ otherwise. Given Δ: $\mathcal{S}_t^{<*}\binom{\omega}{k} \to r$, define Δ^*: $\mathcal{S}_t^<\binom{\omega}{\omega} \to r$ by $\Delta^*(G) = \Delta(G^*)$ for every $G \in \mathcal{S}_t^<\binom{\omega}{\omega}$. Then Δ^* is continuous. So, by Carlson's result, there exists an $F \in \mathcal{S}_t^<\binom{\omega}{\omega}$ such that $\Delta^*(F \cdot G) = \Delta^*(F \cdot \hat{G})$ for all $G, \hat{G} \in \mathcal{S}_t^<\binom{\omega}{\omega}$. As $(F \cdot G)^* = F \cdot G^*$, it follows that $\Delta(F \cdot g) = \Delta(F \cdot \hat{g})$ for all g, $\hat{g} \in \mathcal{S}_t^{<*}\binom{\omega}{k}$. □

Theorem E then is established in a way similar to the proof of Theorem B. However, we have to be a bit careful to assure that the parameters in the desired F are really *ascending*, i.e., $\max F^{-1}(t + i) < \min F^{-1}(t + i + 1)$. For the reader's convenience, we briefly recall the needed lemmas, which are slight modifications of Lemmas 1–4, resp.

Notation. For $f \in \mathcal{S}_t^{<*}\binom{\omega}{k}$ the *Tychonoff cone* generated by f is defined by

$$\mathcal{T}^<(f) = \left\{F \in \mathcal{S}_t^<\binom{\omega}{k} \middle| F(i) = f(i) \text{ if } i < \min f^{-1}(*) \text{ and } F(i) < t \text{ otherwise}\right\}.$$

The set of all Tychonoff cones $\mathcal{T}^<(f)$, $f \in \mathcal{S}_t^{<*}\binom{\omega}{k}$, forms a basis for the topology on $\mathcal{S}_t^<\binom{\omega}{k}$.

Note that every subcone of $\mathcal{T}^<(f)$ can be written as $\mathcal{T}^<(f \otimes h)$ for some $h \in t^*$.

LEMMA $1^<$. *Let $f \in \mathcal{S}_t^{<*}\binom{\omega}{m}$, let $g \in \mathcal{S}_t^<\binom{m}{k}$ and let $h \in t^*$. Then $(f \otimes h) \cdot g^+ = (f \cdot g^+) \otimes h$.*

PROOF. Obvious. □

LEMMA $2^<$. *Let r be a positive integer and let $B_i \subseteq \mathcal{S}_t^<\binom{\omega}{k}$, $i < r$, be open subsets such that $\bigcup_{i<r} B_i$ is dense. Let $f \in \mathcal{S}_t^{<*}\binom{\omega}{k+m}$. Then there exists $\tilde{h} \in t^*$ such that $f^+ \otimes \tilde{h}$ has the following property: for every $g \in \mathcal{S}_t^<\binom{k+m+1}{k}$ with $g(t + k + m) = t + k - 1$ there exists an $i < r$ such that $\mathcal{T}^<((f^+ \otimes \tilde{h}) \cdot g) \subseteq B_i$.*

PROOF. Cf. proof of Lemma 2. □

LEMMA $3^<$. *Let $D \subseteq \mathcal{S}_t^<\binom{\omega}{k}$ be dense open and let $f \in \mathcal{S}_t^{<*}\binom{\omega}{k+m+1}$. Then there exists $\tilde{h} \in t^*$ such that $f \otimes \tilde{h}$ has the following property*:

$$\mathcal{T}^<((f \otimes \tilde{h}) \cdot g) \subseteq D \quad \textit{for every } g \in \mathcal{S}_t^<\binom{k+m+1}{k}.$$

PROOF. Cf. proof of Lemma 3. □

LEMMA $4^<$. *Let $M \subseteq \mathcal{S}_t^<\binom{\omega}{k}$ be meager and let $B_i \subseteq \mathcal{S}_t^<\binom{\omega}{k}$, $i < r$, be open such that $\bigcup_{i<r} B_i$ is dense. Then there exists an $F \in \mathcal{S}_t^<\binom{\omega}{\omega}$ such that*

(1) *$F \cdot G \notin M$ for all $G \in \mathcal{S}_t^<\binom{\omega}{k}$,*

(2) *for every $g \in \mathcal{S}_t^{<*}\binom{\omega}{k}$ there exists an $i < r$ such that $F \cdot G \in B_i$ for all $G \in \mathcal{T}^<(g)$.*

PROOF. Cf. proof of Lemma 4; note that it suffices to assure (2) for all $g \in \mathscr{S}_t^{<}*\binom{\omega}{k}$ with $g(\min g^{-1}(*) - 1) = t + k - 1$. □

PROOF OF THEOREM E. Cf. proof of Theorem B; however, define $\Delta^*\colon \mathscr{S}_t^{<}*\binom{\omega}{k} \to r$ by $\Delta^*(g) = \Delta(F \cdot G)$ for any $G \in \mathscr{T}^{<}(g)$. Then Theorem F is applied. □

7. Concluding remarks. (1) Lemma 4 (for $\mathscr{S}_t$) and the corresponding results for $\mathscr{S}_{\mathscr{G}}$ and $\mathscr{S}_t^{<}$ imply that meager sets in these categories are *Ramsey null.* More precisely:

THEOREM G. *Let $M \subseteq \mathscr{S}_t\binom{\omega}{k}$ (resp. $M \subseteq \mathscr{S}_{\mathscr{G}}\binom{\omega}{k}$, resp. $M \subseteq \mathscr{S}_t^{<}\binom{\omega}{k}$) be meager sets. Then there exists an $F \in \mathscr{S}_t\binom{\omega}{\omega}$ (resp. $F \in \mathscr{S}_{\mathscr{G}}\binom{\omega}{\omega}$, resp. $F \in \mathscr{S}_t^{<}\binom{\omega}{\omega}$) such that $F \cdot G \notin M$ for all $G \in \mathscr{S}_t\binom{\omega}{k}$ (resp. $G \in \mathscr{S}_{\mathscr{G}}\binom{\omega}{k}$, resp. $G \in \mathscr{S}_t^{<}\binom{\omega}{k}$).* □

Let X, Y be metric spaces and assume that Y is separable. A result of Kuratowski says that every Baire mapping $f\colon X \to Y$ is continuous apart from a meager set, i.e., there exists a meager set $M \subseteq X$, such that $f \urcorner X \setminus M$ is continuous. In fact, the converse is also true (cf. [**Kur66**]).

Hence, for every Baire mapping $\Delta\colon S_t\binom{\omega}{k} \to Y$, where Y is a separable metric space, there exists an $F \in \mathscr{S}_t\binom{\omega}{\omega}$ such that $\Delta \urcorner \{F \cdot G | G \in \mathscr{S}_t\binom{\omega}{k}\}$ is continuous. From a result of Emeryk, Frankiewicz and Kulpa [**EFK79**] follows that in these three cases the separability of Y can be dismissed, i.e., it suffices to require Y to be a metric space. More precisely:

THEOREM H. *Let Y be a metric space and let $\Delta\colon \mathscr{S}_t\binom{\omega}{k} \to Y$ (resp. $\Delta\colon \mathscr{S}_{\mathscr{G}}\binom{\omega}{k} \to Y$, resp. $\Delta\colon \mathscr{S}_t^{<}\binom{\omega}{k} \to Y$) be a Baire mapping. Then there exists an $F \in \mathscr{S}_t\binom{\omega}{\omega}$ (resp. $F \in \mathscr{S}_{\mathscr{G}}\binom{\omega}{\omega}$, resp. $F \in \mathscr{S}_t^{<}\binom{\omega}{\omega}$) such that $\Delta \urcorner \{F \cdot G | G \in \mathscr{S}_t\binom{\omega}{k}\}$ (resp. $\Delta \urcorner \{F \cdot G | G \in \mathscr{S}_{\mathscr{G}}\binom{\omega}{k}\}$, resp. $\Delta \urcorner \{F \cdot G | G \in \mathscr{S}_t^{<}\binom{\omega}{k}\}$) is continuous.* □

This result can be applied for establishing a canonization theorem for Baire mappings $\Delta\colon \mathscr{S}_t\binom{\omega}{k} \to Y$, where Y is a metric space (cf., [**PSVxx**]). So far, almost nothing is known about canonical forms of continuous mappings $\Delta\colon \mathscr{S}_{\mathscr{G}}\binom{\omega}{k} \to Y$, resp. $\Delta\colon \mathscr{S}_t^{<}\binom{\omega}{k} \to Y$, in a metric space Y. The only exception is Taylor's result [**Tay76**], which describes canonical forms of mappings $\Delta\colon \mathscr{S}_1^{<}\binom{\omega}{1} \to \omega$. However, no topology is involved here, as $\mathscr{S}_1^{<}\binom{\omega}{\omega}$ is countably discrete.

(2) Let us call a set $A \subseteq \mathscr{S}_t\binom{\omega}{k}$ *completely Ramsey* iff for every $F \in \mathscr{S}_t\binom{\omega}{\omega}$ there exists $G \in \mathscr{S}_t\binom{\omega}{\omega}$ such that either

$$F \cdot G \cdot H \in A \text{ for every } H \in \mathscr{S}_t\binom{\omega}{k} \quad \text{or} \quad F \cdot G \cdot H \notin A \text{ for every } H \in \mathscr{S}_t\binom{\omega}{k}.$$

A set $A \subseteq \mathscr{S}_t\binom{\omega}{k}$ has the *property of Baire in the restricted sense* iff for every $B \subseteq \mathscr{S}_t\binom{\omega}{k}$ the intersection $A \cap B$ has the property of Baire with respect to B. Obviously, Theorem B implies that every Baire set in the restricted sense is completely Ramsey. However, we do not know whether there exists a set A which is completely Ramsey but lacks the property of Baire in the restricted sense. Possibly, using the axiom of choice, such a set exists.

References

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