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BOUNDS FOR PRIME SOLUTIONS OF SOME DIAGONAL EQUATIONS. II

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ABSTRACT. Let b_j and m be certain integers. In this paper we obtain a bound for prime solutions p_j of the diagonal equations of order k, $b_1 p_1^k + \cdots + b_s p_s^k = m$. The bound obtained is $C^{(\log B)^2} + C|m|^{1/k}$ where $B = \max_j\{e, |b_j|\}$ and C are positive constants depending at most on k.

1. Introduction. Throughout p denotes a prime number and $k \geqslant 2$ is an integer. Let $\theta \geqslant 0$ be the largest integer such that p^θ divides k. We write $p^\theta \| k$. Let

$$s_0 = \begin{cases} 3k-1 & \text{if there is a } p \text{ satisfying } p\,|\,k \text{ and } k = ((p-1)/2)p^\theta, \\ 2k & \text{otherwise.} \end{cases} \tag{1.1}$$

$$s_1 = \begin{cases} 2^k + 1 & \text{if } 2 \leqslant k \leqslant 11, \\ 2k^2(2\log k + \log\log k + 2.5) - 1 & \text{if } k \geqslant 12. \end{cases} \tag{1.2}$$

$$\nu = \begin{cases} \theta + 2 & \text{if } p = 2 \text{ and } 2\,|\,k, \\ \theta + 1 & \text{otherwise.} \end{cases} \tag{1.3}$$

$$K = \prod_{(p-1)|k} p^\nu. \tag{1.4}$$

In this paper we shall prove

THEOREM 1. *Let $b_1, \ldots, b_s$ be any nonzero integers which do not have the same sign. Let m be any integer satisfying*

$$\sum_{j=1}^{s} b_j \equiv m \pmod{K}. \tag{1.5}$$

If s is the least integer with $s \geqslant s_1$ and if no prime can divide more than $s - s_0$ b_j then there are constants $C_j(k)$ depending on k only such that the equation

$$\sum_{j=1}^{s} b_j p_j^k = m \tag{1.6}$$

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always has a solution in odd primes p_j *satisfying*

$$\max_{1 \leqslant j \leqslant s} p_j < C_1 |m|^{1/k} + C_2^{(\log B)^2} \tag{1.7}$$

where $B = \max\{|b_1|, \ldots, |b_s|, e\}$.

Investigations on bounds for *integral* solutions of diagonal equations similar to type (1.6) were made by Cassels [**3**], Birch and Davenport [**2**], Pitman and Ridout [**11**], Pitman [**12**]. On the other hand, results on bounds for *prime* solutions of (1.6) were obtained by Baker [**1**] and the author [**9**]. In all previous works on prime solutions, bounds obtained are of the form $C(k, \delta)^{(\max|b_j|)^{\delta}}$ for any $\delta > 0$. So (1.7) in Theorem 1 gives an essentially better bound than the previous one [**9**, (1.6)] and our Theorem 1 improves Theorem 1 in [**9**]. The new bound, $C^{(\log B)^2}$ is obtained by using [**5**, Theorem 6] a zero density estimate for L-functions which, as a consequence, replaces the Siegel-Walfisz theorem on prime distribution applied in both [**1**, Lemma 1] and [**9**, Lemma 6]. By this zero density estimate we can obtain a better error estimate as shown in our Lemma 2 which enables us to treat terms belonging to category (A) in §4 below. This change causes not only an improvement on the bound but also a greatly different emphasis in methods.

By (1.1) and (1.2) we see that the divisibility condition on b_j in Theorem 1 is better than (for $k \geqslant 4$) the condition, $(b_j, b_l) = 1$ for $j \neq l$, which is usually assumed in additive problems involving primes. By (1.4) and (1.5) our condition on m coincides with that in the Waring-Goldbach problem [**7**, p. 100 and p. 108] where the case $b_j = 1$ was considered.

2. Notation. Throughout we assume that N satisfies

$$\log N \geqslant N_0 (\log B)^2 \tag{2.1}$$

where $N_0 > 0$ is a large constant depending on k only.

$\chi \pmod q$ denotes a Dirichlet character and $\chi_0 \pmod q$ denotes the principal character. $\chi^* \pmod r$ is a primitive character, $\tilde{\chi} \pmod{\tilde{r}}$ is the exceptional primitive character and $\tilde{\beta}$ is the exceptional zero (see Lemma 1 below). Throughout the constants c_j and all implicit constants in the Vinogradov symbols $\ll$, the O-symbols are positive and depend at most on k. The constants A_j are positive absolute. $\phi(q)$ is the Euler function and for real α write $e(\alpha) = \exp(i2\pi\alpha)$. Let

$$P = P(N) = \exp\left(\sqrt{A_1 \log N}/10\right), \qquad Q = N^k P^{-1}, \tag{2.2}$$

where A_1 is given in Lemma 1. The constant $\sqrt{A_1}/10$ in (2.2) will be needed in the proof of Lemma 2. Let

$$W(a, \chi) = \sum_{n=1}^{q} \chi(n) e\left(\frac{an^k}{q}\right),$$

$$S(b\alpha) = \sum_{G < p \leqslant N} \log p \, e(b\alpha p^k), \qquad S(b\alpha, \chi) = \sum_{G < p \leqslant N} \chi(p) \log p \, e(b\alpha p^k),$$

where

$$G = N(6^k s|b|)^{-1/k}.$$

For $1 \leqslant a \leqslant q \leqslant P$, $(a,q)=1$ let $\mathscr{M}(q,a)$ be the major arc which is the set of real α satisfying $|\alpha - a/q| \leqslant \delta_q$ with

$$\delta_q = (qQ)^{-1}. \tag{2.3}$$

These major arcs are disjoint. Let $\mathscr{M}$ be the union of all major arcs and m denote minor arcs which is the complement of $\mathscr{M}$ with respect to the set of α satisfying $Q^{-1} \leqslant \alpha \leqslant 1 + Q^{-1}$.

For $\alpha \in \mathscr{M}(q,a)$ write $\alpha = a/q + \eta$. If $p > P$ then $(q,p)=1$, since $q \leqslant P$. It follows from the orthogonal relation of characters that

$$S(b\alpha) = \phi(q)^{-1}\sum_{\chi} W(ab,\bar{\chi})S(b\eta,\chi). \tag{2.4}$$

Note that if $p > P$ then

$$S(b\eta,\chi) = S(b\eta,\chi^*) \tag{2.5}$$

where χ^* (mod r) induces χ (mod q). Put

$$\begin{cases} I(b\eta) = \displaystyle\sum_{|b|G^k < n \leqslant |b|N^k} e(\pm\eta n)n^{-1+1/k}\left(k|b|^{1/k}\right)^{-1}, \\ \tilde{I}(b\eta) = -\displaystyle\sum_{|b|G^k < n \leqslant |b|N^k} e(\pm\eta n)n^{-1+\tilde{\beta}/k}\left(k|b|^{\tilde{\beta}/k}\right)^{-1} \end{cases} \tag{2.6}$$

where $\pm$ denotes the sign of b. $\tilde{I}(b\eta)$ is defined only if there is $\tilde{\beta}$. Let

$$\Delta(b\eta,\chi) = \begin{cases} S(b\eta,\chi_0) - I(b\eta) & \text{if } \chi = \chi_0, \\ S(b\eta,\tilde{\chi}\chi_0) - \tilde{I}(b\eta) & \text{if } \chi = \tilde{\chi}\chi_0, \\ S(b\eta,\chi) & \text{if } \chi \neq \chi_0 \text{ and } \chi \neq \tilde{\chi}\chi_0. \end{cases} \tag{2.7}$$

By (2.5) we have

$$\Delta(b\eta,\chi) = \Delta(b\eta,\chi^*). \tag{2.8}$$

3. Lemmas.

LEMMA 1. *Let $z = \sigma + it$. There is A_1 such that the Dirichlet L-function $L(z,\chi^*) \neq 0$ whenever $\sigma \geqslant 1 - A_1/\log(P(|t|+2))$ for all primitive characters χ^* (mod r) and $r \leqslant P$ with the possible exception of at most one primitive character, $\tilde{\chi}$ (mod $\tilde{r}$). If there is such an exceptional character then it is quadratic and the unique exceptional zero $\tilde{\beta}$ of $L(z,\tilde{\chi})$ is real and simple and satisfies*

$$A_2/\tilde{r}^{1/2}(\log\tilde{r})^2 \leqslant 1 - \tilde{\beta} \leqslant A_1/\log P. \tag{3.1}$$

PROOF. See [4, §14].

LEMMA 2. *For any real $\lambda \geqslant 1$ we have*

$$\sum_{r \leqslant P}\sum_{\chi^*}\left(\int_{-\delta_r}^{\delta_r} |\Delta(b\eta,\chi^*)|^{\lambda}\,d\eta\right)^{1/\lambda} \ll |b|N^{1-k/\lambda}P^{-2}$$

where the summation $\sum_{\chi^*}$ is taken over all χ^* (mod r).

PROOF. The proof is essentially the same as Theorem 7 [**5**]. In the proof we apply Theorem 6 [**5**] and put the T there to be P^7.

LEMMA 3. *Let* $q = q_1 \cdots q_t$ *with* $(q_j, q_l) = 1$ *for* $j \neq l$. *Let* $\chi \pmod{q}$ *be factorized into* $\prod_{j=1}^{t} \chi_j \pmod{q_j}$. *If* $(a, q) = 1$ *then there exist uniquely* $a_j \pmod{q_j}$ *with*

$$(a_j, q_j) = 1 \qquad (j = 1, \ldots, t),\ a = \sum_{j=1}^{t} \frac{a_j q}{q_j} \tag{3.2}$$

and

$$W(ab, \chi) = \prod_{j=1}^{t} W(a_j b, \chi_j).$$

PROOF. This is essentially Theorem 4.1 in [**8**, p. 159].

LEMMA 4. *Let* $h_1 = h/(h, q)$ *and* $q_1 = q/(h, q)$. *Let* $\chi^* \pmod{r}$ *induce* $\chi \pmod{q}$. *Then*

$$W(h, \chi) = \begin{cases} 0 & \text{if } r \nmid q_1, \\ \phi(q)\phi(q_1)^{-1} W(h_1, \chi_1) & \text{if } r \mid q_1 \text{ where } \chi_1 \pmod{q_1} \\ & \text{is induced by } \chi^* \pmod{r}. \end{cases}$$

REMARKS. Lemma 4 is parallel to the known result on the Ramanujan sum and its generalization [**6**, p. 450]. In fact, we can also prove that $W(h, \chi) = 0$ if $r \mid q_1$ and $(r, q_1/r) \nmid k$.

PROOF. Write $q_2 = q/q_1$ and $n = uq_1 + v$ with $u = 0, 1, \ldots, q_2 - 1$; $v = 1, 2, \ldots, q_1$. Then

$$\sum_{n=1}^{q} \chi(n) e\left(\frac{hn^k}{q}\right) = \sum_{\substack{v=1 \\ (v, q_1)=1}}^{q_1} e\left(\frac{h_1 v^k}{q_1}\right) T(v) \tag{3.3}$$

where $T(v) = \sum_{u=1}^{q_2} \chi(uq_1 + v)$.

Let $r \nmid q_1$. By the same argument as in showing $S(v) = 0$ in [**4**, p. 66] we can prove that $T(v) = 0$ and hence $W(h, \chi) = 0$.

Next consider $r \mid q_1$. Let $d = \prod_{p \mid q_2, p \nmid q_1} p$ and $\mathscr{J} = \{uq_1 + v\colon 1 \leqslant u \leqslant q_2\}$. If $(v, q_1) = 1$ then

$$\sum_{\substack{j \in \mathscr{J} \\ (j, q)=1}} 1 = \sum_{j \in \mathscr{J}} \sum_{n \mid (j, d)} \mu(n) = \sum_{n \mid d} \frac{\mu(n) q_2}{n} = q_2 \prod_{p \mid d} (1 - p^{-1}). \tag{3.4}$$

It follows from $\chi^*(uq_1 + v) = \chi^*(v)$ and (3.4) that if $(v, q_1) = 1$ then

$$T(v) = \chi^*(v) \sum_{\substack{u=1 \\ (uq_1 + v, q)=1}}^{q_2} 1 = \chi^*(v) \phi(q) \phi(q_1)^{-1}.$$

By (3.3) this proves Lemma 4.

LEMMA 5. (a) *If* $(a, p) = 1$ *and* p^l *is the modulus of* χ *then* $|W(a, \chi)| \leqslant 2kp^{l/2}$.

(b) *If* $(a, q) = 1$ *and* q *is the modulus of* χ *then for any* $\varepsilon > 0$ *there is a positive constant* $C(\varepsilon, k)$ *depending at most on* ε, k *such that*

$$|W(ab, \chi)| \leqslant C(k, \varepsilon)(q, b)^{1/2} q^{1/2+\varepsilon}.$$

PROOF. Part (a) follows from a similar argument as part 1 of the proof of Lemma 8.5 [7].

(b) Let χ^* (mod r) induce χ (mod q), $q' = q/(b, q)$, $b' = b/(b, q)$. Suppose that $r \mid q'$. Put $q' = \prod_{j=1}^{t} p_j^{l_j}$ and factorize χ' (mod q') into $\prod_{j=1}^{t} \chi_j$ (mod $p_j^{l_j}$), where χ' (mod q') is induced by χ^* (mod r). Then by Lemmas 4, 3, and Lemma 5(a)

$$\begin{aligned} |W(ab, \chi)| &= \phi(q)\phi(q')^{-1}|W(ab', \chi')| \leqslant (b, q) \prod_{j=1}^{t} |W(a_j b', \chi_j)| \\ &\leqslant (b, q)^{1/2}((b, q)q')^{1/2} \prod_{j=1}^{t} 2k. \end{aligned}$$

This proves Lemma 5(b).

4. Major arcs. I. Write

$$(4.1) \qquad \begin{cases} \mathscr{W}_j = \phi(q)^{-1} \sum_{\chi} W(ab_j, \bar{\chi}) \Delta(b_j \eta, \chi), \\ \mathscr{I}_j = \phi(q)^{-1} I(b_j \eta) W(ab_j, \chi_0), \\ \tilde{\mathscr{I}}_j = \phi(q)^{-1} \tilde{I}(b_j \eta) W(ab_j, \tilde{\chi}\chi_0), \end{cases}$$

where $\tilde{\mathscr{I}}_j$ is defined only when the exceptional character exists. By (2.4), (2.7) we have

$$(4.2) \qquad \begin{aligned} R_1(m) &= \sum_{q \leqslant P} \sum_{a}{}' \int_{-\delta_q}^{\delta_q} e\left(-m\left(\frac{a}{q} + \eta\right)\right) \prod_{j=1}^{s} S(b_j \alpha)\, d\eta \\ &= \sum_{q \leqslant P} \sum_{a}{}' e\left(\frac{-ma}{q}\right) \int_{-\delta_q}^{\delta_q} e(-m\eta) \prod_{j=1}^{s} (\mathscr{W}_j + \mathscr{I}_j + \tilde{\mathscr{I}}_j)\, d\eta \end{aligned}$$

where the sum $\sum_a'$ is taken over all a with $1 \leqslant a \leqslant q$ and $(a, q) = 1$.

There are two categories of terms in the last product of (4.2), namely, (A) terms having at least a factor $\mathscr{W}_j$; (B) terms having no factor $\mathscr{W}_j$. We shall treat category (A) in this section and category (B) in §6.

Let $\mathscr{I}_j'$ denote either $\mathscr{I}_j$ or $\tilde{\mathscr{I}}_j$. In category (A) for each fixed $h = 1, 2, \ldots, s$ we choose $\prod_{j=1}^{h} \mathscr{W}_j \prod_{j=h+1}^{s} \mathscr{I}_j'$ as the representative of those terms having exactly h factors $\mathscr{W}_j$. Put

$$(4.3) \qquad T_h(m) = \sum_{q \leqslant P} \sum_{a}{}' e\left(\frac{-ma}{q}\right) \int_{-\delta_q}^{\delta_q} \prod_{j=1}^{h} \mathscr{W}_j \prod_{j=h+1}^{s} \mathscr{I}_j' e(-m\eta)\, d\eta \qquad (h = 1, \ldots, s).$$

Let

$$(4.4) \qquad \chi_j' \ (\text{mod}\, q) = \chi_0 \ (\text{mod}\, q) \quad \text{or} \quad \tilde{\chi}\chi_0 \ (\text{mod}\, q).$$

$$(4.5) \qquad I'(b_j\eta) = I(b_j\eta) \quad \text{or} \quad \tilde{I}(b_j\eta).$$

Then by Schwarz's inequality and (4.1), (4.3) we have

(4.6)

$$|T_h(m)| \leqslant \sum_{p \leqslant P} \phi(q)^{-s} \sum_{\substack{\chi_j \\ j=1,\dots,h}} \left| \sum_a{}' e\left(\frac{-ma}{q}\right) \prod_{j=1}^{h} W(ab_j, \bar{\chi}_j) \prod_{j=h+1}^{s} W(ab_j, \chi_j') \right|$$
$$\times \prod_{j=1}^{h} \left(\int_{-\delta_q}^{\delta_q} |\Delta(b_j\eta, \chi_j)|^{n_j} d\eta \right)^{1/n_j} \prod_{j=h+1}^{s} \left(\int_{-\delta_q}^{\delta_q} |I'(b_j\eta)|^{n_j} d\eta \right)^{1/n_j},$$

where $\sum_{\chi_j, j=1,\dots,h}$ denotes h summations each of which is taken over all $\chi \pmod q$ and $n_j \geqslant 1$ are integers satisfying $\sum_{j=1}^{s} 1/n_j = 1$. Note that each $\chi_j \pmod q$ is induced by a unique $\chi_j^* \pmod{r_j}$ with $r_j | q$ and that each $\chi_j^* \pmod{r_j}$ and each q with $r_j | q$ induce a unique $\chi \pmod q$. Then by (2.8), (4.6) we have

(4.7)

$$|T_h(m)| \leqslant \sum_{\substack{r_j \leqslant P \\ j=1,\dots,h}} \sum_{\chi_j^*} \left\{ \sum_{\substack{q=1 \\ r_j | q,\, j=1,\dots,h}}^{\infty} \phi(q)^{-s} \left| \sum_a{}' e\left(\frac{-ma}{q}\right) \prod_{j=1}^{h} W(ab_j, \chi_0 \bar{\chi}_j^*) \right.\right.$$
$$\left.\left. \times \prod_{j=h+1}^{s} W(ab_j, \chi_j') \right| \right\}$$
$$\times \prod_{j=1}^{h} \left(\int_{-\delta_{r_j}}^{\delta_{r_j}} |\Delta(b_j\eta, \chi_j^*)|^{n_j} d\eta \right)^{1/n_j} \prod_{j=h+1}^{s} \left(\int_{-\delta_1}^{\delta_1} |I'(b_j\eta)|^{n_j} d\eta \right)^{1/n_j}.$$

By Lemma 5 with $\varepsilon = (10s)^{-1}$, the infinite sum inside the curly brackets of (4.7) is

$$\ll \sum_{q=1}^{\infty} \phi(q)^{-s+1} \prod_{j=1}^{s} |b_j|^{1/2} q^{1/2+1/10s} \ll B^{s/2} \tag{4.8}$$

since by (1.2) we have $s \geqslant 5$ for any $k \geqslant 2$. Also by (2.6) we have $I'(b_j\eta) \ll N$ and then by (2.3), (2.2)

$$\left(\int_{-\delta_1}^{\delta_1} |I'(b_j\eta)|^{n_j} d\eta \right)^{1/n_j} \ll N^{1-k/n_j} P^{1/n_j}. \tag{4.9}$$

It follows from (4.7), (4.8), (4.9) and Lemma 2 that

$$\begin{aligned} T_h(m) &\ll B^{s/2} \left(\prod_{j=1}^{h} |b_j| N^{1-k/n_j} P^{-2} \right) \left(\prod_{j=h+1}^{s} N^{1-k/n_j} P^{1/n_j} \right) \\ &\ll B^{3s/2} N^{s-k} P^{-1} = E_1, \quad \text{say,} \end{aligned} \tag{4.10}$$

since $\sum_{j=1}^{s} 1/n_j = 1$.

5. Singular series.

LEMMA 6. *For a given p let $p^\theta \| k$ and $p^\phi \| b$. Suppose that p^t and p^j are the moduli of χ_0 and χ_1 respectively and*

$$u = 2\phi + \theta + \begin{cases} 3 & \text{if } p = 2, \\ 1 & \text{if } p \geqslant 3. \end{cases}$$

If $1 \leqslant j \leqslant u - 2\phi - \theta$, $t \geqslant u + 1$ *and* $(a, p) = 1$ *then*

$$W(ab, \chi_0) = W(ab, \chi_1\chi_0) = 0.$$

PROOF. The proof is essentially the same as Lemma 1 [**9**].

LEMMA 7. *Let* $q = q_1q_2$, $(q_1, q_2) = 1$ *and factorize* $\chi_j \pmod q$ *into* $\prod_{l=1}^{2}\chi_{jl}$ $(\operatorname{mod} q_l)$ $(j = 1, 2, \ldots, s)$. *If*

$$B(m, q) = \phi(q)^{-s}\sum_a{}' e\left(\frac{-ma}{q}\right)\prod_{j=1}^{s} W(ab_j, \chi_j),$$

then

$$B(m, q) = B(m, q_1)B(m, q_2).$$

PROOF. Apply Lemma 3.

By Lemma 1 the exceptional character $\tilde{\chi} \pmod{\tilde{r}}$ is real and primitive. Then it is known [**8**, p. 159] that

$$\tilde{r} = 2^l p_2 \cdots p_t \tag{5.1}$$

where p_j are distinct odd primes and $l = 0$ or 2 or 3. If $\tilde{r} \mid q$ write

$$q = q_1q_2, \quad (q_1, q_2) = 1 \quad \text{and} \quad q_1 = 2^{l_1}p_2^{l_2} \cdots p_t^{l_t} \tag{5.2}$$

where $l_j \geqslant 1$ $(j = 2, \ldots, t)$; $l_1 \geqslant l$ if $l \neq 0$ and $l_1 = 0$ if $l = 0$. Put

$$B_h(m, q) = \phi(q)^{-s}\sum_a{}' e\left(\frac{-ma}{q}\right)\prod_{j=1}^{s} W\left(ab_j, \chi_j'\right) \qquad (h = 0, 1, \ldots, s), \tag{5.3}$$

where χ_j' is defined in (4.4) and there are exactly h $\chi_j' = \tilde{\chi}\chi_0 \pmod q$ in the last product of (5.3). Define singular series $(h = 0)$ and pseudosingular series $(h = 1, 2, \ldots, s)$ by

$$\mathscr{S}_0(m) = \sum_{q=1}^{\infty} B_0(m, q) \quad \text{and} \quad \mathscr{S}_h(m) = \sum_{\substack{q=1 \\ \tilde{r} \mid q}}^{\infty} B_h(m, q). \tag{5.4}$$

By Lemma 5(b) all series in (5.4) are absolutely convergent.

LEMMA 8. *Let* $\tilde{r}$ *and* q_1 *be defined as in* (5.1), (5.2). *If* $B_h(m, q_1) \neq 0$ *then* $q_1 = d_k\tilde{r}$ *or* $2d_k\tilde{r}$ *where* d_k *is a divisor of* k.

PROOF. For each p_j $(j = 1, \ldots, t)$ in (5.1) with $p_1 = 2$ let $p_j^{\theta_j} \| k$. Suppose that $l_1 \geqslant 4 + \theta_1$ or $l_j \geqslant 2 + \theta_j$ for some $j \geqslant 2$. For simplicity we only give the details for the case $j = 2$. Let

$$l_2 \geqslant \theta_2 + 2. \tag{5.5}$$

Since no prime can divide all b_j, we may assume that $p_2 \nmid b_1$. Factorizing the exceptional character $\tilde{\chi}$ and the character χ_1' in (5.3) we have

$$\tilde{\chi} \pmod{\tilde{r}} = \tilde{\chi}_1 \pmod{2^l}\prod_{j=2}^{t}\tilde{\chi}_j \pmod{p_j},$$

$$\chi_1' \pmod{q_1} = \prod_{j=1}^{t}\chi_{1j}' \left(\operatorname{mod} p_j^{l_j}\right),$$

where each χ'_{1j} is either $\chi_0 \pmod{p_j^{l_j}}$ or $\tilde{\chi}_j\chi_0 \pmod{p_j^{l_j}}$. By (3.2) for each a with $(a,q_1) = 1$ there are a_j ($j = 1,\ldots,t$) with $(a_j, p_j) = 1$ such that $W(ab_1, \chi'_1) = \prod_{j=1}^{t} W(a_j b_1, \chi'_{1j})$. Then by (5.5) and Lemma 6 with $\phi = 0$, for each a in Σ'_a of (5.3) we have $W(a_2 b_1, \chi'_{12}) = 0$. So by (5.3) if $B_h(m, q_1) \neq 0$ then $l \leqslant l_1 \leqslant l + 1 + \theta_1$ and $1 \leqslant l_j \leqslant 1 + \theta_j$ ($j = 2,3,\ldots,s$). This proves Lemma 8.

LEMMA 9. (a) $\mathscr{S}_0(m) \gg B^{s(1-s)}$ *and* (b) $\mathscr{S}_h(m) \ll \mathscr{S}_0(m)(\log N)^{-1/2}$ ($h = 1,2,\ldots,s$).

PROOF. Part (a) is Lemma 5 in [**9**].

We come now to prove part (b). For each q with $\tilde{r} \mid q$ define q_1 and q_2 as in (5.2). Since, by the hypothesis on b_j, no prime can divide more than $s - s_0$ b_j, we have

$$\prod_{j=1}^{s} (q_1, b_j) \leqslant q_1^{s-s_0}.$$

Then by (5.3) and Lemma 5 with $\varepsilon = (10s)^{-1}$ we have

$$B_h(m, q_1) \ll \phi(q_1)^{-s+1} q_1^{s/2+1/10} q_1^{(s-s_0)/2} \ll q_1^{6/5-s_0/2}.$$

Then by Lemma 8 and $s_0 \geqslant 2k \geqslant 4$ (see (1.1)) we have

$$B_h(m, q_1) \ll \tilde{r}^{-4/5}.$$

So by Lemma 8 again we have

$$\sum_{q_1=1}^{\infty} B_h(m, q_1) \ll \tilde{r}^{-4/5}. \tag{5.6}$$

On the other hand, by Lemma 5(a), the divisibility hypothesis on b_j and $|W(ab_j, \chi'_j)| \leqslant \phi(p^t)$, we see that the product in $B_0(m, p^t)$ in (5.3) satisfies

$$\left|\prod_{j=1}^{s} W\left(ab_j, \chi'_j\right)\right| \leqslant (2k)^{s_0} p^{ts_0/2} \phi(p^t)^{s-s_0}.$$

So by (5.3) and $s_0 \geqslant 4$ we have

$$\left|B_0(m, p^t)\right| \leqslant \phi(p^t)^{-s_0+1} (2k)^{s_0} p^{ts_0/2} \leqslant (4k)^s p^{t(1-s_0/2)} < c_1 p^{-t}. \tag{5.7}$$

For each p there exists some $b_j = b_1$, say, which is not divisible by p. By Lemma 6 for each a with $(a, p) = 1$ we have $W(ab_1, \chi_0) = 0$ if $t \geqslant \nu + 2$ where ν is defined in (1.3) and p^t is the modulus of χ_0. So by (5.3) we have $B_0(m, p^t) = 0$ if $t \geqslant \nu + 2$. Then by Lemma 7 and $(\tilde{r}, q_2) = 1$

$$\begin{aligned} \sum_{q_2=1}^{\infty} B_0(m, q_2) &= \prod_{p \nmid \tilde{r}} \left(1 + \sum_{t=1}^{\nu_1} B_0(m, p^t)\right) \\ &= \sum_{q=1}^{\infty} B_0(m, q) \Big/ \prod_{p \mid \tilde{r}} \left(1 + \sum_{t=1}^{\nu_1} B_0(m, p^t)\right) \end{aligned} \tag{5.8}$$

where $\nu_1 = \nu + 1$. Separate the last product $\prod_{p \mid \tilde{r}}$ into $\prod_{p \mid \tilde{r}, p \leqslant c_2}$ and $\prod_{p \mid \tilde{r}, p > c_2}$ where $c_2 = 4c_1$. Same as that in the proof of Lemma 5 in [**9**, see (4.16) and the product $\prod_1$ on p. 197] which depends essentially on (1.1)–(1.5) and the divisibility condition on b_j in Theorem 1, we have that the first product $\prod_{p|\tilde{r}, p \leqslant c_2}$ satisfies

$$\prod_{\substack{p|\tilde{r}\\ p\leqslant c_2}} \left(1 + \sum_{t=1}^{\nu_1} B_0(m, p^t)\right) \geqslant \prod_{\substack{p|\tilde{r}\\ p\leqslant c_2}} \phi(p^{\nu_1})^{-s} p^{\nu_1}$$

$$\geqslant \prod_{p \leqslant c_2} p^{\nu_1(1-s)} = c_3 > 0.$$

For the second product $\prod_{p|\tilde{r}, p > c_2}$, by (5.7) we have

$$\prod_{\substack{p|\tilde{r}\\ p> c_2}} \left(1 + \sum_{t=1}^{\nu_1} B_0(m, p^t)\right) > \prod_{c_2 < p \leqslant \tilde{r}} \left(1 - c_1 \sum_{t=1}^{\infty} p^{-t}\right) \tag{5.9}$$

$$> \prod_{c_2 < p \leqslant \tilde{r}} (1 - c_2/2p) \gg (\log \tilde{r})^{-c_2}.$$

The last inequality is a simple modification of Theorem 9.3 in [**8**, p. 92]. Now by (5.8), (5.9) we have

$$\sum_{q_2=1}^{\infty} B_0(m, q_2) \ll \mathscr{S}_0(m)(\log \tilde{r})^{c_2}. \tag{5.10}$$

Finally, by (5.2) we see that $\tilde{\chi}\chi_0 \pmod q$ can be factorized as the product of $\tilde{\chi}\chi_0$ $(\bmod\, q_1)$ and χ_0 $(\bmod\, q_2)$. Then by (5.4), Lemma 7, (5.6), (5.10) we have

$$\mathscr{S}_h(m) = \sum_{q_1=1}^{\infty} B_h(m, q_1) \sum_{q_2=1}^{\infty} B_0(m, q_2) \ll \mathscr{S}_0(m)(\log N)^{-1/2}$$

since by (3.1) we have

$$\tilde{r}^{4/5}(\log \tilde{r})^{-c_2} \gg (\log N)^{1/2}.$$

This proves Lemma 9.

6. Major arcs. II.

LEMMA 10. *We have*

$$\int_{(qQ)^{-1}}^{1/2} \left| \prod_{j=1}^{s} I'(b_j\eta) \right| d\eta \ll (qQ)^{s-1} N^{s(1-k)}$$

where $I'(b_j\eta)$ *is defined in* (4.5).

PROOF. If $0 < \eta \leqslant 1/2$ then for any $n \geqslant 1$ we have $\sum_{l=0}^{n} e(l\eta) \ll |\eta|^{-1}$. Let $\phi = 1/k$ or $\tilde{\beta}/k$. Then by Abel's partial summation formula and (2.6)

$$b^{\phi} I'(b\eta) \ll |\eta|^{-1}\left\{ |bN^k|^{\phi-1} + \int_{|b|G^k}^{|b|N^k} \left| \frac{d}{dy} y^{\phi-1} \right| dy \right\}$$

$$\ll |\eta|^{-1}(|b|G^k)^{\phi-1} \ll |\eta|^{-1} N^{1-k}.$$

So the lemma follows.

Let

$$J_h(m) = \int_{-1/2}^{1/2} \prod_{j=1}^{h} \tilde{I}(b_j\eta) \prod_{j=h+1}^{s} I(b_j\eta) e(-m\eta)\, d\eta \qquad (h = 0, 1, \ldots, s). \tag{6.1}$$

LEMMA 11. (a) $|J_h(m)| \leqslant J_0(m)$ $(h = 1, 2, \dots, s)$.
(b) *If*

$$|m| \leqslant (N/4)^k s^{-1} \tag{6.2}$$

then

$$J_0(m) \gg B^{-s/k} N^{s-k}.$$

PROOF. Part (a) follows from (6.1) and part (b) is essentially Lemma 8 [9].

We come now to treat those terms in category (B) defined in §4. In category (B) we choose $\prod_{j=1}^h \tilde{\mathscr{I}}_j \prod_{j=h+1}^s \mathscr{I}_j$ $(h = 0, 1, \dots, s)$ to represent those terms $\prod_{j=1}^s \mathscr{I}_j'$ having exactly h factors $\tilde{\mathscr{I}}_j$. Put

$$T_0(m) = \sum_{q \leqslant P} \sum_a{}' e\left(\frac{-ma}{q}\right) \int_{-\delta_q}^{\delta_q} \left(\prod_{j=1}^s \mathscr{I}_j \right) e(-m\eta)\, d\eta,$$

$$\tilde{T}_h(m) = \sum_{\substack{q \leqslant P \\ \tilde{r} | q}} \sum_a{}' e\left(\frac{-ma}{q}\right) \int_{-\delta_q}^{\delta_q} \left(\prod_{j=1}^h \tilde{\mathscr{I}}_j \prod_{j=h+1}^s \mathscr{I}_j \right) e(-m\eta)\, d\eta \tag{6.3}$$

$(h = 1, 2, \dots, s)$.

By (4.1), $s \geqslant 5$, Lemmas 10 and 5 with $\varepsilon = (10s)^{-1}$ we have

$$\begin{aligned} &\sum_{\substack{q \leqslant P \\ \tilde{r} | q}} \sum_a{}' e\left(\frac{-ma}{q}\right) \int_{\delta_q}^{1/2} \left(\prod_{j=1}^h \tilde{\mathscr{I}}_j \prod_{j=h+1}^s \mathscr{I}_j \right) e(-m\eta)\, d\eta \\ &\ll N^{s(1-k)} Q^{s-1} \sum_{q \leqslant P} \phi(q)^{-s+1} \left(\prod_{j=1}^s (q, b_j)^{1/2} q^{1/2+\varepsilon} \right) q^{s-1} \\ &\ll N^{s-k} B^{s/2} P^{-3/10} = E_2, \quad \text{say.} \end{aligned} \tag{6.4}$$

So, if we replace the integral $\int_{-\delta_q}^{\delta_q}$ in (6.3) by $\int_{-1/2}^{1/2}$ we have the error E_2 given in (6.4). Then by (6.1), (6.3), (4.1) we have

(6.5)

$$\tilde{T}_h(m) = J_h(m) \left\{ \sum_{\substack{q \leqslant P \\ \tilde{r} | q}} \phi(q)^{-s} \sum_a{}' e\left(\frac{-ma}{q}\right) \prod_{j=1}^h W(ab_j, \tilde{\chi}\chi_0) \prod_{j=h+1}^s W(ab_j, \chi_0) \right\} + E_2.$$

Similarly, by Lemma 5, if we replace the sum $\sum_{q \leqslant P, \tilde{r}|q}$ in (6.5) by $\sum_{q=1, \tilde{r}|q}^{\infty}$ we have an error $\ll B^{s/2} P^{-3/10}$. So by (5.4), (6.5) we have

$$\tilde{T}_h(m) = J_h(m)(\mathscr{S}_h(m) + E_3) + E_2 \qquad (h = 1, 2, \dots, s), \tag{6.6}$$

where $E_3 = O(E_2 N^{k-s})$. By the same argument we have

$$T_0(m) = J_0(m)(\mathscr{S}_0(m) + E_3) + E_2. \tag{6.7}$$

Note that each representative in either category (A) or (B) defined in §4 represents at most $O(1)$ terms. It follows from (4.2), (4.10), (6.7), (6.6) that

$$\begin{aligned} R_1(m) = J_0(m)(\mathscr{S}_0(m) + E_3) + O\left(\sum_{h=1}^s J_h(m) \{ \mathscr{S}_h(m) + E_3 \} \right) \\ + O(E_2 + E_1). \end{aligned} \tag{6.8}$$

By (4.10), (6.4), Lemmas 9(a) and 11(b) we see that for $j = 1, 2$

$$E_3/\mathscr{S}_0(m) \quad \text{and} \quad E_j/J_0(m)\mathscr{S}_0(m) \ll B^{2s^2}P^{-3/10}. \tag{6.9}$$

It follows from (6.8), (6.9), (2.1), Lemmas 11(a) and 9(b) that

$$R_1(m) > \tfrac{1}{2}J_0(m)\mathscr{S}_0(m). \tag{6.10}$$

7. Minor arcs.

LEMMA 12. *If* $\alpha \in \mathfrak{m}$ *then*

$$\sum_{p \leqslant N} e(\alpha b p^k) \ll N|b|P^{-\omega(k)}$$

where $\omega(k)^{-1} = 4^{(k+2)}(k+1)$.

PROOF. This is essentially Lemma 11 [**9**] (see also Lemma 5 [**1**]).

Let

$$R_2(m) = \int_{\mathfrak{m}} \prod_{j=1}^{s} S(b_j\alpha)e(-m\alpha)\,d\alpha.$$

Then by Lemma 12 and the same argument as Lemma 12 in [**9**] we have

$$R_2(m) \ll N^{s-k}B^sP^{-\omega(k)}(\log N)^{c_4}.$$

By (4.2), (6.10), Lemmas 9(a), 11(b) and (2.1)

$$\int_{Q^{-1}}^{1+Q^{-1}} \prod_{j=1}^{s} S(b_j\alpha)e(-m\alpha)\,d\alpha = R_1(m) + R_2(m) \tag{7.1}$$

$$\gg N^{s-k}B^{-s^2}\{1 - c_5B^{s(s+1)}P^{-\omega(k)}(\log N)^{c_4}\} > 0.$$

Choose the least N satisfying (2.1) and (6.2). So (7.1) implies the existence of a solution of $\sum_{j=1}^{s} b_jp_j^k = m$ in primes p_j and

$$\max_{1 \leqslant j \leqslant s} p_j \leqslant N \leqslant C_1(k)m^{1/k} + C_2(k)^{(\log B)^2}.$$

This completes the proof of Theorem 1.

REMARK. Combining the Circle Method with the Sieve Method, when $k = 1$ and $s = 3$, the author [**10**] is able to obtain a bound for solutions of (1.6) to be B^A where $A > 0$ is an absolute constant. However, for $k \geqslant 2$ it seems that these two methods do not combine well to replace the $(\log B)^2$ in (1.7) by $\log B$.

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