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WELL-QUASI-ORDERING INFINITE GRAPHS WITH FORBIDDEN FINITE PLANAR MINOR

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ABSTRACT. We prove that given any sequence $G_1, G_2, \ldots$ of graphs, where G_1 is finite planar and all other G_i are possibly infinite, there are indices i, j such that $i < j$ and G_i is isomorphic to a minor of G_j. This generalizes results of Robertson and Seymour to infinite graphs. The restriction on G_1 cannot be omitted by our earlier result. The proof is complex and makes use of an excluded minor theorem of Robertson and Seymour, its extension to infinite graphs, Nash-Williams' theory of better-quasi-ordering, especially his infinite tree theorem, and its extension to something we call tree-structures over QO-categories, which includes infinitary version of a well-quasi-ordering theorem of Friedman.

1. INTRODUCTION

By a graph we shall mean in this paper a possibly infinite, undirected graph which may have loops and multiple edges. A graph is a *minor* of another if the first can be obtained from a subgraph of the second by edge contraction. A set Q, on which a quasi-ordering (i.e., reflexive and transitive relation) $\leq$ is defined, is said to be *well-quasi-ordered* (wqo) if for every infinite sequence $q_1, q_2, \ldots$ of elements of Q there are indices i, j such that $i < j$ and $q_i \leq q_j$.

The well-quasi-ordered sets have been studied for a while, but we mention the history and development very briefly, as this is well covered in [7]. The early years of wqo theory are closely tied with the following two conjectures of Vázsonyi.

(1.1) **Conjecture.** *All trees, finite or not, are wqo by the homeomorphic embedding (i.e., by the quasi-ordering $\leq$ such that $T \leq S$ if there is a homeomorphic embedding $V(T) \to V(S)$).*

The finite version of this conjecture was established by Kruskal [6], the proof was then simplified by Nash-Williams [12]. The general case was proved by Nash-Williams in [13] using his theory of better-quasi-ordering (bqo). This

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theorem and theory are of fundamental importance for this paper, but as both are discussed later on, we pass to the second Vázsonyi conjecture.

(1.2) **Conjecture.** *The set of finite graphs will all degrees* ≤ 3 *is wqo by the homeomorphic embedding.*

The restriction to graphs with degrees ≤ 3 in (1.2) is necessary, for otherwise it is easy to construct counterexamples. Namely, for $n \geq 3$ let G_n be the graph with n vertices $v_0, \ldots, v_n = v_0$ and with two edges joining v_i and v_{i+1} $(0 \leq i \leq n-1)$. Then for no $3 \leq i < j$ is there a homeomorphic embedding of G_i into G_j.

Since for graphs G, H with degrees ≤ 3, G is homeomorphically embeddable into H if and only if G is isomorphic to a minor of H, the following conjecture, named after K. Wagner, is a strengthening of (1.2).

(1.3) **Wagner's Conjecture.** *The set of all finite graphs is wqo by minors, i.e., by the quasi-ordering* $\leq$ *such that* $G \leq H$ *if* G *is isomorphic to a minor of* H.

There was only a little progress [11] on (1.3) in the past decades until its recent solution by Robertson and Seymour, which is being published in a series of lengthy papers under the collective title *Graph minors.*

There are some features of (1.3) and to a lesser extent of wqo theory itself which are worth mentioning.

(i) The truth of (1.3) implies the truth of a Kuratowski-type theorem for general classes of graphs. Recall that Kuratowski's theorem can be reformulated as "a finite graph is planar if and only if it has no K_5 or $K_{3,3}$ minor." Now let $\mathcal{G}$ be any class of graphs for which such a statement is in principle possible; that is, any minor of a member of $\mathcal{G}$ again belongs to $\mathcal{G}$. Then it follows easily from (1.3) that there is a *finite* list L of graphs such that a graph belongs to $\mathcal{G}$ if and only if it has no minor isomorphic to a member of L.

(ii) (1.3) also has interesting algorithmic aspects. For example, if $\mathcal{G}$ is as above then there is a polynomially bounded algorithm to test the membership of $\mathcal{G}$. As this is not a paper about algorithms we refer to [16] or [17] for more on this story.

(iii) Finally, (1.3) is very attractive from the metamathematical point of view, because its finite miniaturization yields a statement of finite combinatorics which is unprovable in $\Pi^1_1\text{-}CA_0$ or $\bigcup_n ID_n$, that is, relatively strong fragments of second-order arithmetics [2].

In this paper we are concerned with the analogue of (1.3) for infinite graphs. The only results known for infinite graphs so far were Nash-Williams' proof of (1.1) and the author's weaker version of (1.4) below (with $H_0 = K_4$) [26]. Related results are Laver's extension of (1.1) to a certain class of order-theoretical trees [9] and his order type theorem [8] (see also [21]). Since (1.3) is false in general as shown in [23], we are looking for some suitable class of infinite graphs for which (1.3) is true. We prove the following theorem.

(1.4) **Theorem.** *Let H_0 be a finite planar graph. Then the class of all (finite or not) graphs with no minor isomorphic to H_0 is wqo by minors.*

In fact, we prove that this class is better-quasi-ordered and even "well behaved," which is a technical strengthening of bqo defined in §4. Let us first reduce this theorem to another one. In [19], [20] Robertson and Seymour proved the following (for the definitions see next section).

(1.5) **Theorem.** *For every finite planar graph H_0 there exists an integer w such that every finite graph with no minor isomorphic to H_0 admits a linked tree-decomposition of width $< w$.*

A slightly stronger version of (1.5) (and with a better constant) is proved in [25]; this version is then used for extension of (1.5) to infinite graphs in [5]. To summarize:

(1.6) **Theorem.** *If G is such that its every finite subgraph admits a linked tree-decomposition of width $< w$, then G itself admits a linked tree-decomposition of width $< w$.*

Using (1.5) and (1.6) it is easily seen that in order to prove (1.4) it is sufficient to prove the following:

(1.7) **Theorem.** *For any integer w, the class of graphs which admit a linked tree-decomposition of width $< w$ is wqo by minors.*

This theorem is proved for finite graphs in [18], but our proof is completely independent of that paper. Our proof of (1.7) is self-contained; we obtain it as a corollary of a general theorem on QO-categories (Theorem (4.13)) which says that if a graph category $\mathscr{A}$ is well behaved, then the graph category $\mathscr{A}^w$ of all graphs which admit a "w-bounded" linked tree-decomposition "over $\mathscr{A}$" is well behaved. The main idea of the proof is to use the tree structure of the graphs involved and to imitate the Nash-Williams' proof of Conjecture (1.1). But there are several complications to this.

First we need a labeled version of Nash-Williams' result. The most natural labeling theorem was proved by Laver in [8], but we need a more general one. Each vertex t of our tree T is labeled by some graph from a "nice" class, but this graph is again labeled, this time by some possibly unbounded number of labels, each label corresponding to an edge of T incident with t.

The second complication is that we need not only the labeling of vertices as described above but also an infinitary version of a result proved by Friedman for finite trees [22], when the edges are numbered from a finite set of numbers. When a tree is homeomorphically embedded in another tree, each of its edges is represented by a path of the second tree. We need to insist that for each edge e of the first tree, its number should not exceed any number on the corresponding path P of the second tree and that the first and last edges of P should have numbers equal to the number of e. In fact we label vertices rather than edges, but that makes little difference. Since we have to build in these ingredients simultaneously, Theorem (8.2) includes them both. The pure extension of Friedman's result to infinite graphs is then derived in §9.

These two complications occur even in the finite case and are already treated in [18]. Hence the above project can be described by saying that we need to extend the method of Robertson and Seymour to infinite graphs. However, there are two more complications which are peculiar to infinite graphs.

The third one is that the methods of wqo theory alone are insufficient to obtain wqo results of infinite objects. We clarify this a bit more in §3.

Finally, the last complication is in fact caused by the second one. Namely, since we proceed by reversed induction on the tree-width, we need a stronger induction hypothesis than bqo. This is why we introduce the concept of "well behaved," which provides the suitable induction hypothesis. Some slight improvements of Nash-Williams' idea enable us to show that this property is preserved on each step (see also (10.1) for a discussion on the extensions of bqo).

§2 contains all definitions and notation; §3 introduces the bqo theory and presents the basic results. The exposition is self-contained and also includes an introduction to wqo as a motivation for bqo as well as an explanation of why the bqo theory is needed. In §4 we introduce the central notion of our study—the notion of QO-category—and the basic constructions (e.g., the QO-category of tree structures over a QO-category) and the notion of a well-behaved QO-category. We formulate the Main Theorem and derive (1.7) from it there. In §§5 and 6 we prove two auxiliary lemmas. Lemma (5.3) reduces the Main Theorem to proving that $(\leq w, \geq 0)$-structures (i.e., tree structures with bounded "amalgamation size") over a well-behaved QO-category are well behaved. Lemma (6.2) reduces the investigation of $(\leq w, \geq k)$-structures over $\mathscr{A}$ to the study of $(\leq k, \geq k)$-structures over $(\leq w, \geq k+1)$-structures over $\mathscr{A}$. This makes our reversed induction work and all we need is to show that $(\leq k, \geq k)$-structures over a well-behaved QO-category are well-behaved. This is done in §7, which is the cornerstone of the paper. Here the second complication does not occur and we may basically use the Nash-Williams' method. In §8 we prove the Main Theorem, in §9 the extension of Friedman's result to infinite graphs. Some conjectures are mentioned in §10.

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2. Definitions

(2.1) The letter Q will always stand for a set on which a quasi-ordering $\leq$ is defined. If V is any set then V^* denotes the set of all injective sequences of elements of V. If $p = (v_1, \dots, v_n) \in V^*$, then set$(p)$ denotes the set $\{v_1, \dots, v_n\} \subseteq V$. By $\exp V$ we denote the set of all subsets of V. As usual, ω denotes the first limit ordinal. If X, Y are sets, then $X \dot{\cup} Y$ denotes the disjoint union of X and Y. It will sometimes be convenient to introduce a new element $*$. Let us make the agreement that whenever this symbol occurs it will be assumed that it is distinct from all objects considered so far.

(2.2) *Multivalued mappings.* If X, Y are sets, then a mapping $f: X \to \exp Y - \{\varnothing\}$ is called a *multivalued mapping from* X *to* Y. It is called *injective* if $f(x) \cap f(y) = \varnothing$ for any distinct elements $x, y \in X$. The *identity*, denoted by id, is a multivalued mapping defined on any nonempty set X by $\mathrm{id}(x) = \{x\}$. We denote by $f(X)$ the set $\bigcup_{x\in X} f(x) \subseteq Y$. If f is an injective multivalued mapping from X to Y, then $f^{-1}: f(X) \to X$ is defined by $f^{-1}(y) = x$ iff $y \in f(x)$. If $x = (x_1, \dots, x_n) \in X^*$ and $y = (y_1, \dots, y_m) \in Y^*$ we write $x \xrightarrow{f} y$ to mean that $n = m$ and $y_i \in f(x_i)$ for $i = 1, \dots, n$. If f is a multivalued mapping such that $|f(x)| = 1$ for any $x \in X$, then f is called *single-valued*. We shall often identify single-valued mappings from X to Y with ordinary mappings $X \to Y$ in the obvious way. If f is a multivalued mapping from X to Y and g is a multivalued mapping from Y to Z, then the *composition* $f \circ g$ is a multivalued mapping h from X to Z defined by $h(x) = \bigcup_{y\in f(x)} g(y)$.

(2.3) *Graphs.* A *graph* G consists of a *vertex set* $V(G)$, an *edge set* $E(G)$, and a relation of *incidence* between these sets. Edges are either *loops* with one incident vertex or *links* with exactly two incident vertices. Incident vertices are also called the *end vertices* of an edge, and the endvertices of an edge are said to be *adjacent* in G. The *degree* of a vertex v is its number of incident edges, links counted once and loops counted twice. A graph G is a *subgraph* of a graph H if $V(G) \subseteq V(H)$, $E(G) \subseteq E(H)$, and incidences of G are incidences of H. The graph *union* and *intersection* are evident. A path in G is a subgraph of G with the usual property; no "repeated" vertices are allowed. A graph G is *connected* if any two vertices of G are connected by a path. If $A \subseteq V(G)$, then $G \restriction A$ denotes the graph spanned by A, that is, the graph whose vertex set is A, whose edge set consists of edges of G incident only with elements of A, and with the incidence relation inherited from G. If G and H are graphs then an injective multivalued mapping f from $V(G)$ to $V(H)$ is called an *expansion* of G into H provided

(2.3a) the graph $H \restriction f(v)$ is connected for every $v \in V(G)$, and

(2.3b) there exists an injective mapping $\varepsilon : E(G) \to E(H)$, called an *edge expansion*, such that if u, v are end vertices of an edge $e \in E(G)$, then $\varepsilon(e)$ has one end vertex in $f(u)$ and one in $f(v)$.

It follows from (2.3b) that an edge expansion maps loops onto loops and links onto links. Thus G is isomorphic to a minor of H iff there is an expansion f of G into H. The mapping f^{-1} is usually called a contraction (or a subcontraction) because of its physical meaning. If G and H are graphs then we say that G is *homeomorphically embeddable* into H if some subdivision of G is isomorphic to a subgraph of H.

(2.4) *Trees.* A *tree* T for our purposes is a possibly infinite oriented graph with a specified vertex, called the *root of* T and denoted by $\mathrm{root}(T)$, such that each edge is directed away from the root and for every vertex t of T there is a unique directed path from $\mathrm{root}(T)$ to t. (Thus a tree is not a graph in our terminology.) If T is a tree, then $V(T)$ and $E(T)$ denote the sets of vertices and edges of T, respectively. If $(t, t') \in E(T)$ we say that t' is the *successor* of t and that t is the *predecessor* of t'. Thus every $t \in V(T)$-$\{\mathrm{root}(T)\}$ has a unique predecessor. We say that t' follows t if t belongs to the directed path from $\mathrm{root}(T)$ to t'. A *subtree* of T is a tree S with $V(S) \subseteq V(T)$, $E(S) \subseteq E(T)$, and $\mathrm{root}(T) = \mathrm{root}(S) \in V(S)$. If $t \in V(T)$ then the *branch at* t, denoted by T_t, is the tree whose root is t and which consists of vertices of T that follow t and all edges of T incident with these vertices. Let us remark that T_t is not a subtree of T unless $t = \mathrm{root}(T)$, in which case $T_t = T$. If T is a tree and $t, t' \in V(T)$, then $[t, t']_T$ (or $[t, t']$ when no confusion is likely) denotes the set of vertices of the path between t and t' in the underlying undirected graph of T (which is a tree in the usual sense) so that $t, t' \in [t, t']_T \subseteq V(T)$. If T, S are trees, then a mapping $\varphi: V(T) \to V(S)$ is called *monotone* if $\varphi(t')$ follows $\varphi(t)$ whenever $(t, t') \in E(T)$. If $\varphi: V(T) \to V(S)$ is monotone, we define φ^*, $\varphi_*: V(T) \to V(S)$ as follows: We put $\varphi_*(\mathrm{root}(T)) = \mathrm{root}(S)$, and for $(t, t') \in E(T)$ let $s_1 = \varphi(t)$, $s_2, \ldots, s_{n-1}, s_n = \varphi(t')$ be the vertices of the directed path from $\varphi(t)$ to $\varphi(t')$ in the order in which they occur on this path; we put $\varphi_*(t') = s_2$, $\varphi^*(t) = s_{n-1}$. A mapping $\varphi: V(T) \to V(S)$ is called a *homeomorphic embedding* if it is monotone and for any $t, t_1, t_2 \in V(T)$ if t_1 and t_2 are distinct successors of t, then $\varphi_*(t_1) \neq \varphi_*(t_2)$. In other words, φ is a homeomorphic embedding if distinct successors of a vertex t follow distinct successors of $\varphi(t)$.

(2.5) *Tree-decompositions.* A *tree-decomposition* of a graph G is a pair (T, X), where T is a tree and $X = (X_t: t \in V(T))$ is a family of subsets of $V(G)$ with the following properties:

(2.5a) $\bigcup\{X_t: t \in V(T)\} = V(G)$,

(2.5b) for every edge e of G there exists $t \in V(T)$ such that e has both end vertices in X_t, and

(2.5c) for $t, t', t'' \in V(T)$, if $t' \in [t, t'']_T$ then $X_t \cap X_{t''} \subseteq X_{t'}$.

The *width* of a tree-decomposition is

$$\max\{|X_t| - 1: t \in V(T)\},$$

provided this max exists; otherwise it is undefined. The graph G has *tree-width* w if w is minimum such that G has a tree-decomposition of width w.

It can be shown, for example, that forests have tree-width ≤ 1, series-parallel graphs have tree-width ≤ 2, for $n \geq 1$ the complete graph K_n has tree-width $n - 1$, and for $n \geq 2$ the $n \times n$ grid (i.e., the adjacency graph of the $n \times n$ chessboard) has tree-width n. The tree-width of infinite graphs behaves nicely:

if w is the maximum of tree-widths of finite subgraphs of a graph G, then the tree-width of G is w (cf. [24, 27, 5]).

A tree-decomposition (T, X) of a graph G is called *linked* if for any oriented path $t_1, \ldots, t_n$ in T, oriented away from root(T), and any $k > 0$ such that $|X_{t_i} \cap X_{t_{i+1}}| \geq k$ $(i = 1, \ldots, n-1)$ there are k disjoint paths in G, each between $X_{t_1} \cap X_{t_2}$ and $X_{t_{n-1}} \cap X_{t_n}$. Linked tree-decompositions are important for our application. However, this property is not restrictive, since it is shown in [25] for finite graphs and extended to infinite ones in [5] that a graph G of tree-width w always has a linked tree-decomposition of width w.

(2.6) *The space* $[A]^\omega$. If A is any set, then $[A]^\omega$ denotes the set of all infinite subsets of A. We will consider $[A]^\omega$ only for $A \in [\omega]^\omega$. We need the so-called *classical topology* on $[A]^\omega$, that is, the topology $[A]^\omega$ gets as a subspace of 2^ω, which is given the product topology (the topology on 2 is discrete). The basic open sets for this topology are, for instance, the sets of the form

$$\{Z \in [A]^\omega : Z \cap \{1, \ldots, \max s\} = s\},$$

where s runs through all finite subsets of A. This topology is complete, separable, and metrizable. Of particular interest will be Borel subsets of $[A]^\omega$, that is, the sets belonging to the smallest σ-algebra generated by all open sets.

If $X \in [\omega]^\omega$ then $X-$ denotes the set $X - \{\min X\}$.

3. Bqo theory

In order to give some motivation for the better-quasi-ordering theory we start with an account of results of wqo theory. These are not needed in the sequel but are provided for the reader's convenience.

(3.1) **Definition.** For $q, q' \in Q$ we define $q < q'$ if $q \leq q'$ and $q' \not\leq q$, and $q \equiv q'$ if $q \leq q' \leq q$. The set Q is said to be *well-founded* if there is no infinite descending sequence $q_1 > q_2 > \cdots$ of elements of Q. We denote by $Q/\equiv$ the partially ordered set obtained from Q by identifying $\equiv$-equivalent elements. Any mapping $f: A \to Q$, where $A \in [\omega]^\omega$ is called a Q-sequence. A Q-sequence $f: A \to Q$ is called *good* if there are $i, j \in A$ such that $i < j$ and $f(i) \leq f(j)$, and is called *bad* otherwise. Thus Q is wqo if every Q-sequence is good.

We need to build up new quasi-ordered sets starting from old. If Q_1, Q_2 are quasi-ordered sets, then $Q_1 \dot\cup Q_2$ denotes the disjoint union of Q_1 and Q_2, whose quasi-ordering is the disjoint union of the quasi-orderings on Q_1 and Q_2. $Q_1 \times Q_2$ denotes the Cartesian product of Q_1 and Q_2 equipped with the product quasi-ordering. For α an ordinal, Q^α denotes the set of all α-sequences of elements of Q, $Q^{<\alpha} = \bigcup_{\beta<\alpha} Q^\beta$ and $Q^{0n} = \bigcup_{\alpha\in 0n} Q^\alpha$. Q^{0n} is quasi-ordered by the rule that $(a_\alpha)_{\alpha\in\lambda} \leq (b_\beta)_{\beta\in\mu}$ if there is a strictly increasing mapping $f: \lambda \to \mu$ such that $a_\alpha \leq b_{f(\alpha)}$ for all $\alpha \in \lambda$. Finally, $\exp Q$, the power set of Q, is quasi-ordered by $A \leq B$ if there is a 1-1 mapping $f: A \to B$

such that $a \leq f(a)$ for every $a \in A$. Construction of other wqo sets is more technical and is left to §4.

(3.2) **Theorem.** *The following conditions on a quasi-ordered set Q are equivalent.*

(i) *Q is wqo.*

(ii) *Q is well-founded and Q contains no infinite subset whose elements are pairwise $\leq$-incomparable.*

(iiii) *For every Q-sequence $f: A \to Q$ there exists $B \in [A]^{\omega}$ such that $f(i) \leq f(j)$ for every $i, j \in B$ such that $i < j$.*

(iv) *Every nonempty subset of Q contains at least one but finitely many minimal elements.*

(v) *Every linear extension of $\leq$ on $Q/\equiv$ is a well-ordering.*

Proof. Easy consequence of Ramsey's theorem. □

(3.3) **Proposition.**

(i) *Any well-ordered set is wqo.*

(ii) *If Q is the union of two subsets, each wqo in the induced quasi-ordering, then Q is wqo.*

(iii) *If Q_1 and Q_2 are wqo then $Q_1 \times Q_2$ is wqo.*

(iv) *For every Q-sequence $f: A \to Q$ there exists $B \in [A]^{\omega}$ such that either $f \restriction B$ is bad or $f(i) \leq f(j)$ for every $i, j \in B$ such that $i < j$.*

Proof. Easy exercise. □

(3.4) **Definition** (Laver [9]). A *partial ranking* on Q is a well-founded partial ordering $\leq'$ of the elements of Q such that $x \leq' y$ implies $x \leq y$. If $f: A \to Q$ and $g: B \to Q$ are Q-sequences, we define $g \leq' f$ if $B \subseteq A$ and $g(i) \leq' f(i)$ for every $i \in B$, and $g <' f$ if $B \subseteq A$ and $g(i) <' f(i)$ for every $i \in B$. Note that $g <' f$ is not equivalent to the conjunction $g \leq' f$ and $g \neq f$. A Q-sequence $f: A \to Q$ is called *minimal bad* if it is bad and there is no bad Q-sequence $g <' f$.

The following lemma, essentially due to Nash-Williams, is a powerful tool for proving the wqo property.

(3.5) **Lemma.** *Let $f: A \to Q$ be a bad Q-sequence. Then there exists a minimal bad Q-sequence $g \leq' f$.*

Proof. Let $A = \{i_1 < i_2 < \cdots\}$. Choose $g(i_1)$ such that it is a first term of a bad Q-sequence which is $\leq' f$ and there is no $q <' g(i_1)$ with this property. Then choose $g(i_2)$ such that $g(i_1)$, $g(i_2)$ (in that order) are the first two terms of a bad Q-sequence which is $\leq' f$ and there is no $q <' g(i_2)$ with this property. Continuing this process we get a bad $g: A \to Q$. We claim it is the desired one. For if there is a bad $h: B \to Q$, $h <' g$, we may define $k: C \to Q$ by $C = \{i \in A: i < \min B\} \cup B$ and

$$k(i) = \begin{array}{ll} g(i), & i < \min B,\ i \in A, \\ h(i), & i \in B. \end{array}$$

Now k is bad and $k(\min B) = h(\min B) <' g(\min B)$, which contradicts the choice of $g(\min B)$. $\square$

To illustrate the use of (3.5) we prove one of the basic theorems of wqo theory, namely Higman's finite sequence theorem. We formulate it in a form similar to corresponding theorems from bqo theory.

(3.6) **Theorem** (Higman [4]). *If $f: A \to Q^{<\omega}$ is a bad $Q^{<\omega}$-sequence, then there exist $B \in [A]^\omega$ and a bad Q-sequence $g: B \to Q$ such that for any $i \in B$, $g(i)$ is a term of $f(i)$. Hence if Q is wqo, then $Q^{<\omega}$ is wqo.*

Proof. For $s, t \in Q^{<\omega}$ define $s \leq' t$ to mean that s is a subsequence of t. Clearly $\leq'$ is a partial ranking and if $q \in Q$ is a term of $s \in Q^{<\omega}$ and $s \leq' t$, then q is a term of t. Hence by (3.5) we may safely assume that the $Q^{<\omega}$-sequence f is minimal bad. Let

$$f(i) = (q_1^i, \dots, q_{n_i}^i) \qquad (i \in A).$$

Since clearly $n_i \geq 1$ for any $i \in A$, we may define for $i \in A$

$$g(i) = q_1^i, \quad h(i) = (q_2^i, \dots, q_{n_i}^i).$$

By (3.3iv) let $B \in [A]^\omega$ be such that $g \restriction B$ is either bad or such that $g(i) \leq g(j)$ for any $i, j \in B$ such that $i < j$. In the first case g is as desired, so let us prove that the second one cannot hold. Suppose it does. Since $h \restriction B <' f$ and f is minimal bad, $h \restriction B$ cannot be bad. Hence there are $i, j \in B$ such that $i < j$ and $h(i) \leq h(j)$. But also $g(i) \leq g(j)$, which implies $f(i) \leq f(j)$, a contradiction to the badness of f. $\square$

The following example shows why methods of wqo theory do not suffice for the proofs of well-quasi-orderedness of infinitary objects. It turns out to be for the reason that the property of being wqo is too weak for inductive arguments about infinite objects to be carried out; one cannot pass from Q wqo to Q^ω wqo (or from Q wqo to exp Q wqo).

(3.7) **Example** (Rado [15]). There exists a wqo set Q such that Q^ω is not wqo, namely $Q = \{(i, j): i < j < \omega\}$ quasi-ordered by the rule $(i, j) \leq (k, l)$ if $i = k$ and $j \leq l$, or $j < k$. It is easy to verify that Q is as claimed.

However, the wqo spaces which "occur in nature" do not behave like the above example. The concept of better-quasi-ordering, invented by Nash-Williams, treats this situation. The property of being bqo is strong enough to be preserved by sufficiently many infinitary operations, and that makes a certain inductive argument work.

So let us begin with the bqo theory. The original Nash-Williams' definition is purely combinatorial, but we use a topological one, due to Simpson [21], which is easier to handle but perhaps misleading since bqo is a combinatorial property and not a topological one. We start with the fundamental theorem of Galvin and Prikry, which plays the rôle of Ramsey's theorem in bqo theory. It is sufficiently well known, so we omit its proof.

(3.8) **Theorem** (Galvin and Prikry [3]). *Let $A \in [\omega]^\omega$ and let $\mathscr{B}$ be a Borel set in $[A]^\omega$. Then there exists $X \in [A]^\omega$ such that either $[X]^\omega \subseteq \mathscr{B}$ or $[X]^\omega \cap \mathscr{B} = \emptyset$.*

(3.9) **Definition.** If $A \in [\omega]^\omega$ then a mapping $a: [A]^\omega \to Q$ is called an *array* if the range of a is countable and $a^{-1}(q)$ is a Borel set in $[A]^\omega$ for every $q \in Q$. An array is called *good* if there is $X \in [A]^\omega$ such that $a(X) \leq a(X-)$ (recall that $X- = X - \{\min X\}$) and is called *bad* otherwise. The set Q is called *better-quasi-ordered* (bqo) if every array $a: [A]^\omega \to Q$ is good.

(3.10) *Remark.* It can be shown that a is an array if and only if it is Borel measurable with respect to the discrete topology on Q and that Q is bqo if and only if every continuous array is good.

(3.11) **Proposition.**

(i) *Any well-ordered set is bqo.*

(ii) *If $a: [A]^\omega \to Q$ is a bad array and $Q = Q_1 \dot\cup Q_2$, then there exist $B \in [A]^\omega$ and $i \in \{1, 2\}$ such that $a \restriction [B]^\omega: [B]^\omega \to Q_i$ is a bad array. Hence if Q_1 and Q_2 are bqo, then Q is bqo. In particular, every finite set is bqo.*

(iii) *If $a: [A]^\omega \to Q_1 \times Q_2$ is a bad array, then there exist $B \in [A]^\omega$, $i \in \{1, 2\}$ and a bad array $b: [B]^\omega \to Q_i$ such that $b(Z)$ is the ith coordinate of $a(Z)$ for any $Z \in [B]^\omega$. Hence if Q_1 and Q_2 are bqo, then $Q_1 \times Q_2$ is bqo.*

(iv) *For any array $a: [A]^\omega \to Q$ there exists $B \in [A]^\omega$ such that either $a \restriction [B]^\omega$ is bad or $a(Z) \leq a(Z-)$ for every $Z \in [B]^\omega$.*

Proof. Easy consequence of the Galvin-Prikry theorem. □

(3.12) **Proposition.** *Every bqo set Q is wqo.*

Proof. Suppose that Q is not wqo and let $f: A \to Q$ be a bad Q-sequence. Define $a: [A]^\omega \to Q$ by $a(Z) = f(\min Z)$. Then a is a bad array. □

(3.13) **Definition.** Let Q be equipped with a partial ranking $\leq'$. If $a: [A]^\omega \to Q$ and $b: [B]^\omega \to Q$ are arrays, we write $b \leq' a$ if $B \subseteq A$ and $b(Z) \leq' a(Z)$ for all $Z \in [B]^\omega$, and $b <' a$ if $B \subseteq A$ and $b(Z) <' a(Z)$ for all $Z \in [B]^\omega$. An array $a: [A]^\omega \to Q$ is called *minimal bad* if there is no bad array $b <' a$.

The following theorem is essentially due to Nash-Williams [13] although it was first enunciated explicitly by Laver [9]. The proof we present here is due to [1] (see [21] for a different proof).

(3.14) **Theorem** (Minimal Bad Array Lemma). *Let $\leq'$ be a partial ranking on Q and let $a_0: [A_0]^\omega \to Q$ be a bad array. Then there is a minimal bad array $a \leq' a_0$.*

Proof. Suppose that there is no minimal bad array beneath a_0. We shall construct a sequence of bad arrays $(a_\alpha)_{\alpha \in \omega_1}$ (ω_1 is the first uncountable ordinal) with $a_\alpha: [A_\alpha]^\omega \to Q$ and for $\alpha < \beta$, $A_\beta \subseteq *A_\alpha \subseteq A_0$, and for all

$Z \in [A_\alpha \cap A_\beta]^\omega$, $a_\beta(Z) <' a_\alpha(Z)$. (Here $A \subseteq *B$ means $B - A$ finite). For successor steps $\alpha+1$, since a_α is not a minimal bad array, we can choose $a_{\alpha+1}$ with $A_{\alpha+1} \in [A_\alpha]^\omega$ as required. Now suppose $\delta < \omega_1$ is a limit ordinal and we have already got $(a_\alpha)_{\alpha<\delta}$. First note that for any $Z \in [\omega]^\omega$ $\{\alpha < \delta : Z \subseteq A_\alpha\}$ is finite, for otherwise if $Z \subseteq A_{\alpha_n}$, where $\alpha_n < \alpha_{n+1}$, then $a_{\alpha_0}(Z)' > a_{\alpha_1}(Z)' > \cdots$, contradicting the well-foundedness of $\leq'$. Let $B \in [A_0]^\omega$ be such that $B \subseteq *A_\alpha$ for every $\alpha < \delta$; such a B is easily constructed since δ is countable. We define $b: [B]^\omega \to Q$ by $b(Z) = a_\alpha(Z)$, where $\alpha = \max\{\beta : Z \subseteq A_\beta\}$. Then b is well-defined and we claim that it is a bad array. Clearly the range of b is countable and $b^{-1}(q) = \bigcup_{\alpha<\delta}(a_\alpha^{-1}(q) - \bigcup_{\alpha<\beta<\delta}[A_\beta]^\omega)$, hence b is an array. To show it is bad suppose that $b(Z) \leq b(Z-)$ for some $Z \in [B]^\omega$. Let $b(Z) = a_\alpha(Z)$ and $b(Z-) = a_\beta(Z-)$. Since $Z- \subseteq Z$ it must be $\alpha \leq \beta$ and hence $a_\beta(Z-) \leq' a_\alpha(Z-)$. But then $a_\alpha(Z) \leq a_\alpha(Z-)$, contradicting the badness of a_α.

Now apply the successor step to b and get $a_\delta : [A_\delta]^\omega \to Q$ bad with $a_\delta(Z) <' b(Z)$ for all $Z \in [A_\delta]^\omega$. Then $a_\delta(Z) <' a_\alpha(Z)$ for every $\alpha < \delta$ and every $Z \in [A_\delta \cap A_\alpha]^\omega$, thus completing the construction.

Next we construct a set $Z \in [\omega]^\omega$ and an increasing sequence α_n of countable ordinals such that $Z \in [A_{\alpha_n}]^\omega$ for every $n \in \omega$. Let $M_0 = \omega_1$ and assume that we have constructed distinct natural numbers $z_1, \ldots, z_n$, ordinals $\alpha_1 < \cdots < \alpha_n < \omega_1$, and uncountable sets $\omega_1 = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_n$ such that $z_n \in A_\alpha$ for any $\alpha \in M_n \cup \{\alpha_1, \ldots, \alpha_n\}$. Choose $\alpha_{n+1} \in M_n$ such that $\alpha_{n+1} > \alpha_n$, let $B = \bigcap_{i=1}^{n+1} A_{\alpha_i} - \{z_1, \ldots, z_n\}$, and let $i_\alpha \in B \cap A_\alpha$ for each $\alpha \in M_n$. Then there exists $i \in \omega$ and an uncountable set $M_{n+1} \subseteq M_n$ such that $i_\alpha = i$ for all $\alpha \in M_{n+1}$. Let $z_{n+1} = i$. Finally put $Z = \{z_1, \ldots, z_n, \ldots\}$.

Now if Z and α_n are as above, we have $a_{\alpha_1}(Z)' > a_{\alpha_2}(Z)' > \cdots$, contradicting the well-foundedness of $\leq'$. □

(3.15) **Lemma.** *If* $a = (a_\alpha)_{\alpha\in\lambda}$, $b = (b_\beta)_{\beta\in\mu} \in Q^{0n}$, *and* $a \not\leq b$, *then there exists* $\nu < \lambda$ *such that* $(a_\alpha)_{\alpha\in\nu} \leq b$ *and* $(a_\alpha)_{\alpha\in\nu+1}\square \not\leq b$.

Proof. Given $a \not\leq b$ define h by induction as follows. Let $h(\alpha)$ be the least $\beta \in \mu$ such that $a_\alpha \leq b_\beta$ and $\beta > h(\alpha')$ for all $\alpha' < \alpha$. Let ν be the least α such that $h(\alpha)$ is undefined. □

(3.16) **Theorem** (Nash-Williams [14]). *For a bad array* $a: [A]^\omega \to Q^{0n}$ *there exists a witnessing bad array, i.e., a bad array* $c: [C]^\omega \to Q$ *such that* $C \in [A]^\omega$ *and* $c(Z)$ *is a term of* $a(Z)$ *for any* $Z \in [C]^\omega$. *Hence if* Q *is bqo, then* Q^{0n} *is bqo.*

Proof [21]. For $s, t \in Q^{0n}$ define $s \leq' t$ to mean that s is an initial segment of t; that is, if $t = (t_\alpha)_{\alpha\in\lambda}$ then $s = (t_\alpha)_{\alpha\in\mu}$ for some $\mu \leq \lambda$. Clearly $\leq'$ is a partial ranking and if $q \in Q$ is a term of $s \in Q^{0n}$ and $s \leq' t$, then q is a term of t. Hence by the Minimal Bad Array Lemma (3.14) we may safely assume that the array a is minimal bad.

Let $a(Z) = (a_\alpha(Z))_{\alpha\in l(Z)}$; by (3.15) let $\nu(Z)$ be such that $(a_\alpha(Z))_{\alpha\in\nu(Z)} \leq a(Z-)$ and $(a_\alpha(Z))_{\alpha\in\nu(Z)+1} \nleq a(Z-)$. Letting $b(Z) = (a_\alpha(Z))_{\alpha\in\nu(Z)}$ we see that $b <' a$. By minimality of a, there is no bad array $\leq' b$. Hence, by (3.11iv) there exists $C \in [A]^\omega$ such that $b(Z) \leq b(Z-)$ for every $Z \in [C]^\omega$. Thus we have $(a_\alpha(Z))_{\alpha\in\nu(Z)} \leq (a_\alpha(Z-))_{\alpha\in\nu(Z-)}$ but $(a_\alpha(Z))_{\alpha\in\nu(Z)+1} \nleq (a_\alpha(Z-))_{\alpha\in\nu(Z-)+1}$ for every $Z \in [C]^\omega$. Thus $c\colon [C]^\omega \to Q$ defined by $c(Z) = a_{\nu(Z)}(Z)$ is the desired witnessing array. □

(3.17) **Theorem.** *For every bad array $a\colon [A]^\omega \to \exp Q$ there exists a witnessing bad array, i.e., a bad array $c\colon [C]^\omega \to Q$ such that $C \in [A]^\omega$ and $c(Z) \in a(Z)$ for any $Z \in [C]^\omega$. Hence, if Q is bqo then $\exp Q$ is bqo.*

Proof. Given a bad array $a\colon [A]^\omega \to \exp Q$ we may well-order each $a(Z) \subseteq Q$ to obtain a sequence $b(Z) \in Q^{0n}$. The array b thus obtained is clearly bad, hence the theorem follows from (3.16). □

4. QO-CATEGORIES

(4.1) **Definition.** A *QO-category* is a pair $\mathscr{A} = (\mathscr{O}, \mathscr{M})$ such that to each $\gamma \in \mathscr{O}$ are associated sets V_γ and P_γ, where $P_\gamma \subseteq V_\gamma^*$ contains the empty sequence, and $\mathscr{M} = \{\mathscr{A}(\gamma, \eta)\colon \gamma, \eta \in \mathscr{O}\}$, where $\mathscr{A}(\gamma, \eta)$ consists of some injective multivalued mappings from V_γ to V_η such that

(4.1a) $\mathrm{id} \in \mathscr{A}(\gamma, \gamma)$ for any $\gamma \in \mathscr{O}$, and

(4.1b) if $f_1 \in \mathscr{A}(\gamma_1, \gamma_2)$ and $f_2 \in \mathscr{A}(\gamma_2, \gamma_3)$, then the composition $f_1 \circ f_2 \in \mathscr{A}(\gamma_1, \gamma_3)$ for any $\gamma_1, \gamma_2, \gamma_3 \in \mathscr{O}$.

The elements of $\mathscr{O}$ are called *objects* and the elements of $\mathscr{A}(\gamma, \eta)$ are called *$\mathscr{A}$-morphisms* or simply *morphisms* when no confusion is likely. To simplify the notation we shall write $\gamma \in \mathscr{A}$ instead of $\gamma \in \mathscr{O}$.

(4.2) Each QO-category $\mathscr{A}$ turns into a quasi-ordered set by the rule that $\gamma \leq_{\mathscr{A}} \eta$ iff $\mathscr{A}(\gamma, \eta) \neq \emptyset$; if $f \in \mathscr{A}(\gamma, \eta)$ then we say that f is a *morphism corresponding to* $\gamma \leq_{\mathscr{A}} \eta$. Thus it makes sense to consider wqo and bqo properties of QO-categories.

(4.3) *Graph categories.* A QO-category $\mathscr{A}$ will be called a *graph category* if for each $\gamma \in \mathscr{A}$ there is a graph G_γ such that $V_\gamma = V(G_\gamma)$ and $f \in \mathscr{A}(\gamma, \eta)$ iff f is an expansion of G_γ into G_η. Let us remark that if $\mathscr{A}$ is a graph category, then $\mathscr{A}$ is wqo (bqo) if and only if $\{G_\gamma\colon \gamma \in \mathscr{A}\}$ is wqo (bqo) by minors, no matter what the sets P_γ are.

(4.4) *Q-labelings.* Let Q be a quasi-ordered set, $\mathscr{A}$ a QO-category, and let $\gamma \in \mathscr{A}$. A *Q-labeling* of γ is a triple $g = (Dg, \hat{g}, \check{g})$ such that Dg is an arbitrary set and

$$\hat{g}\colon Dg \to P_\gamma, \qquad \check{g}\colon Dg \to Q.$$

We define a new QO-category $\mathscr{A}[Q]$ of Q-labelings of elements of $\mathscr{A}$ as follows. Its objects are pairs (γ, g), where $\gamma \in \mathscr{A}$ and g is a Q-labeling of γ. We put $V_{(\gamma, g)} = V_\gamma$, $P_{(\gamma, g)} = P_\gamma$, and we define

$$f \in \mathscr{A}[Q]((\gamma, g), (\eta, h))$$

if $f \in \mathscr{A}(\gamma, \eta)$ and there is a 1-1 mapping $\iota: Dg \to Dh$ such that for any $x \in Dg$

$$\hat{g}(x) \xrightarrow{f} \hat{h}(\iota(x)), \quad \text{and} \tag{4.4a}$$

$$\check{g}(x) \le \check{h}(\iota(x)). \tag{4.4b}$$

We define $\mathrm{Im}(\gamma, g) := \mathrm{Im}\, \check{g} = \{\check{g}(x): x \in Dg\}$.

(4.5) *Multiple labelings.* The QO-category $\mathscr{A}[Q][Q']$ is isomorphic to $\mathscr{A}[Q\dot{\cup}Q']$ (in the usual category theory sense).

Proof. The isomorphism is given by

$$((\gamma, g), g') \to (\gamma, h),$$

where h is a $Q\dot{\cup}Q'$-labeling of γ defined by $h = (Dh, \hat{h}, \check{h})$, where $Dh = Dg\dot{\cup}Dg'$,

$$\hat{h}(x) = \begin{cases} \hat{g}(x), & x \in Dg, \\ \hat{g}'(x), & x \in Dg', \end{cases}$$

$$\check{h}(x) = \begin{cases} \check{g}(x), & x \in Dg, \\ \check{g}'(x), & x \in Dg'. \end{cases} \quad \square$$

(4.6) *Remark.* Definition (4.3) provides an important example of a QO-category; another example is obtained by replacing the expansion by a homeomorphic embedding. The reader should also check that the quasi-ordered sets introduced in (3.1) are in fact Q-labelings of some simpler QO-categories.

(4.7) *New graph categories.* Let $\mathscr{A}$ be a graph category; we wish to define a new graph category $\mathscr{A}^k$, where $k \ge 0$ is an integer. We define $\gamma = (T, (\gamma(t): t \in V(T))) \in \mathscr{A}^k$ if T is a tree and

(4.7a) $\gamma(t) \in \mathscr{A}$ for any $t \in V(T)$,

(4.7b) $(T, (V_{\gamma(t)}: t \in V(T)))$ is a linked tree-decomposition of the graph $G := \bigcup\{G_{\gamma(t)}: t \in V(T)\}$, and

(4.7c) for any $(t, t') \in E(T)$ and any $p \in V(G)^*$, $|V_{\gamma(t)} \cap V_{\gamma(t')}| \le k$ and if $V_{\gamma(t)} \cap V_{\gamma(t')} = \mathrm{set}(p)$, then $p \in P_{\gamma(t)} \cap P_{\gamma(t')}$.

We put $G_\gamma := G$, $V_\gamma := V(G)$, and

$$P_\gamma := \{p \in V_\gamma^*: \mathrm{set}(p) \subseteq V_{\gamma(t)} \text{ for some } t \in V(T) \text{ and } p \in P_{\gamma(t)} \text{for any such } t\}.$$

The $\mathscr{A}^k$-morphisms are defined by saying that $\mathscr{A}^k$ is a graph category.

(4.8) *Tree-structures.* Let $\mathscr{A}$ be a QO-category. A *tree-structure* over $\mathscr{A}$ is a pair (T, τ), where T is a tree and $\tau = (\tau(t): t \in V(T))$ is such that

$$\tau(t) = (\gamma(t), p_t, (p_{(t,t')}: (t, t') \in E(T))),$$

where $\gamma(t) \in \mathscr{A}$, $p_t, p_{(t,t')} \in P_{\gamma(t)}$, and $p_{(t,t')}$ and $p_{t'}$ have the same length. We denote by $N_{(T,\tau)}(t)$ the length of p_t.

Now we define a new QO-category $\mathscr{S}(\mathscr{A})$ whose objects are tree-structures over $\mathscr{A}$ and

$$V_{(T,\tau)} := \{(t, v): t \in V(T), v \in V_{\gamma(t)}\},$$
$$P_{(T,\tau)} := \{((t, v_1), \dots, (t, v_n)): t \in V(T), (v_1, \dots, v_n) \in P_{\gamma(t)}\}.$$

It is convenient to use the following notation: if $(T, \tau), (S, \sigma) \in \mathscr{S}(\mathscr{A})$ let

$$\tau(t) = (\gamma(t), p_t, (p_{(t,t')}: (t, t') \in E(T))) \qquad (t \in V(T)),$$
$$\sigma(s) = (\eta(s), q_s, (q_{(s,s')}: (s, s') \in E(S))) \qquad (s \in V(S)).$$

This notation will be used throughout the paper.

We define $\mathscr{S}(\mathscr{A})((T, \tau), (S, \sigma))$ as the set of all injective multivalued mappings λ from $V_{(T,\tau)}$ to $V_{(S,\sigma)}$ which are of the form $\lambda(t, v) = \{(\varphi(t), u): u \in f_t(v)\}$, where

(4.8a) $\varphi: V(T) \to V(S)$ is a homeomorphic embedding,

(4.8b) $f_t \in \mathscr{A}(\gamma(t), \eta(\varphi(t)))$ for any $t \in V(T)$,

(4.8c) $p_{(t,t')} \xrightarrow{f_t} q_{(\varphi(t), \varphi_*(t'))}$ for any $(t, t') \in E(T)$, $p_t \xrightarrow{f_t} q_{\varphi(t)}$ for any $t \in V(T)$,

(4.8d) $N_{(T,\tau)}(t) = N_{(S,\sigma)}(\varphi(t)) = N_{(S,\sigma)}(\varphi_*(t)) \le N_{(S,\sigma)}(s)$
for any $t \in V(T)$ and any $s \in [\varphi_*(t), \varphi(t)]_S$.

A tree-structure (T, τ) will be called a $(\ge i, \le k)$-structure if $i \le N_{(T,\tau)}(t) \le k$ for any $t \in V(T)$-$\{\mathrm{root}(T)\}$, the length of $p_{\mathrm{root}(T)}$ is at most k, and if $|V(T)| > 1$ then this length is i. The QO-category of $(\ge i, \le k)$-structures over $\mathscr{A}$ will be denoted by $\mathscr{S}_{\ge i}^{\le k}(\mathscr{A})$. The $(\ge k, \le k)$-structures will be called *k-structures*; the QO-category of k-structures over $\mathscr{A}$ will be denoted by $\mathscr{S}_k(\mathscr{A})$.

Let $(T, \tau) \in \mathscr{S}(\mathscr{A})$ and let R be either a subtree of T or a branch of T; for $r \in V(R)$ let

$$\rho(r) = (\gamma(r), p_r, (p_{(r,r')}: (r, r') \in E(R))).$$

Then the tree-structure $(R, \rho) \in \mathscr{S}(\mathscr{A})$ will be called the *restriction* of (T, τ) to R and will be denoted by (R, τ) for simplicity. Let us remark that $(R, \tau) \le (T, \tau)$, namely via the identity morphism.

(4.9) $\mathscr{S}(\mathscr{A})$, $\mathscr{S}_{\ge i}^{\le k}(\mathscr{A})$, and $\mathscr{S}_k(\mathscr{A})$ are QO-categories.

Proof. Since (4.1a) is clearly satisfied, we must verify (4.1b). So let $\lambda \in \mathscr{S}(\mathscr{A})((T, \tau), (S, \sigma))$, $\mu \in \mathscr{S}(\mathscr{A})((S, \sigma), (R, \rho))$, and we must show that

$\pi = \lambda \circ \mu \in \mathscr{S}(\mathscr{A})((T, \tau), (R, \rho))$. Let (T, τ), (S, σ) have the usual notation and let

$$\rho(r) = (\kappa(r), z_r, (z_{(r, r')}: (r, r') \in E(R))).$$

Let

$$\lambda(t, v) = \{(\varphi(t), u): u \in f_t(v)\} \quad \text{and}$$
$$\mu(s, w) = \{(\psi(s), u): u \in g_t(w)\}.$$

Then

$$\pi(t, v) = \{(\chi(t), u): u \in h_t(v)\},$$

where $\chi(t) = \psi(\varphi(t))$ and $h_t = f_t \circ g_t$. Hence conditions (4.8a) and (4.8b) are cleary fulfilled for π. Let us remark that $\chi_*(t) = \psi_*(\varphi_*(t))$. To prove (4.8c) we observe that

$$p_{(t, t')} \xrightarrow{f_t} q_{(\varphi(t), \varphi_*(t'))} \xrightarrow{g_{\varphi(t)}} z_{(\psi(\varphi(t)), \psi_*(\varphi_*(t')))} = z_{(\chi(t), \chi_*(t'))}$$

and similarly for p_t. Hence

$$p_{(t, t')} \xrightarrow{h_t} z_{(\chi(t), \chi_*(t'))}, \qquad p_t \xrightarrow{h_t} z_{\chi(t)},$$

and (4.8c) follows. To prove (4.8d) let us write

$$N_{(T, \tau)}(t) = N_{(S, \sigma)}(\varphi(t)) = N_{(R, \rho)}(\psi(\varphi(t))) = N_{(R, \rho)}(\chi(t)),$$
$$N_{(T, \tau)}(t) = N_{(S, \sigma)}(\varphi_*(t)) = N_{(R, \rho)}(\psi_*(\varphi_*(t))) = N_{(R, \rho)}(\chi_*(t)), \quad \text{and}$$
$$N_{(T, \tau)}(t) \le N_{(S, \sigma)}(s) \le N_{(R, \rho)}(r)$$

for $s \in [\varphi_*(t), \varphi(t)]_S$ and $r \in [\psi_*(s), \psi(s)]_R$. Hence

$$N_{(T, \tau)}(t) \le N_{(R, \rho)}(r)$$

for all $r \in \bigcup\{[\psi_*(s), \psi(s)]_R: s \in [\varphi_*(t), \varphi(t)]_S\} = [\chi_*(t), \chi(t)]_R$, which completes the proof. □

(4.10) *The isomorphism of* $\mathscr{S}^{\le k}_{\ge i}(\mathscr{A})[Q]$ *and* $\mathscr{S}^{\le k}_{\ge i}(\mathscr{A}[Q])$.

To each $((T, \tau), g) \in \mathscr{S}^{\le k}_{\ge i}(\mathscr{A})[Q]$ corresponds a unique $(T, \tau') \in \mathscr{S}^{\le k}_{\ge i}(\mathscr{A}[Q])$, where

$$\tau'(t) = ((\gamma(t), g_t), p_t, (p_{(t, t')}: (t, t') \in E(T)))$$

and $g_t = (Dg_t, \hat{g}_t, \check{g}_t)$ is defined by the rules

$$Dg_t = \{x \in Dg: \operatorname{set}(\hat{g}(x)) \subseteq \{(t, v): v \in V_{\gamma(t)}\}\},$$
$$\hat{g}_t = \hat{g} \restriction Dg_t,$$
$$\check{g}_t = \check{g} \restriction Dg_t.$$

The reader is invited to check that this defines an isomorphism between $\mathscr{S}^{\le k}_{\ge i}(\mathscr{A})[Q]$ and $\mathscr{S}^{\le k}_{\ge i}(\mathscr{A}[Q])$. Hence in the sequel we shall identify these QO-categories. In particular we define $\operatorname{Im}(T, \tau') := \operatorname{Im}(T, \tau)$.

(4.11) *Well-behavedness.* A QO-category $\mathscr{A}$ is said to be *well-behaved* if for any quasi-ordered set Q and any bad array $a\colon [A]^{\omega} \to \mathscr{A}[Q]$ such that $\operatorname{Im} a(Z)$ is bqo for any $Z \in [A]^{\omega}$ there exist $B \in [A]^{\omega}$ and a bad array $b\colon [B]^{\omega} \to Q$ such that $b(Z) \in \operatorname{Im} a(Z)$ for any $Z \in [B]^{\omega}$. The array b is called a *witnessing array for* a.

(4.12) **Lemma.** *If* $\mathscr{A}$ *is well-behaved, then it is bqo.*

Proof. Obvious. □

Now we are able to formulate the main result of this paper.

(4.13) **Main Theorem.** *If* $\mathscr{A}$ *is a well-behaved graph category and* $k \geq 0$ *an integer, then the graph category* $\mathscr{A}^k$ *is well-behaved.*

Let us observe how (1.7) can be proved using this theorem. Let $\mathscr{A}_k$ be the graph category whose objects are graphs with $\leq k$ vertices; for $G \in \mathscr{A}_k$ let $G_G = G$, $V_G = V(G)$, and $P_G = V(G)^*$.

(4.14) **Proposition.** $\mathscr{A}_k$ *is well-behaved for any* $k \geq 0$.

Proof. Let $a\colon [A]^{\omega} \to \mathscr{A}_k[Q]$ be a bad array, $a(Z) = (G(Z), g(Z))$. We may safely assume that $V(G(Z)) = V$, a fixed set (using (3.8)), and that $Dg(Z)$ is an ordinal for any $Z \in [A]^{\omega}$. Now each $a(Z)$ can be encoded by an element of

$$\mathrm{Card}^{V \cup \binom{V}{2}} \times (V^{<\omega} \times Q)^{Dg(Z)}.$$

(the first item encodes the number of edges with prescribed end vertices.) With this identification the existence of a witnessing array follows from (3.11ii), (3.11iii), and (3.16). □

(4.15) *Proof of* (1.7) (*assuming* (4.13)). Let $G_1, G_2, \ldots$ be as in (1.7) and let (T^n, X^n) be a linked tree-decomposition of G_n of width $< w$. Then

$$\gamma^n := (T^n, (G_n \restriction X_t^n : t \in V(T))) \in \mathscr{A}_w^w.$$

By (4.14), (4.13), (4.12), and (3.12) there are i, j such that $i < j$ and $\gamma^i \leq \gamma^j$. Hence G_i is isomorphic to a minor of G_j, as desired. □

5. First Encoding Lemma

In this section we prove the first auxiliary lemma, which enables us to replace the QO-category $\mathscr{A}^w[Q]$ by the QO-category $\mathscr{S}_{\geq 0}^{\leq w}(\mathscr{A})[Q]$.

(5.1) Let $(\gamma, g) = ((T, (\gamma(t)\colon t \in V(T))), g) \in \mathscr{A}^w[Q]$ be given, let $g = (Dg, \hat{g}, \check{g})$, and for any $x \in Dg$ let a vertex $t(x) \in V(T)$ be chosen in such a way that $\mathrm{set}(\hat{g}(x)) \in P_{\gamma(t(x))}$. We define the *encoding* $((T, \tau), \overline{g}) \in \mathscr{S}_{\geq 0}^{\leq w}(\mathscr{A})[Q]$ of (γ, g) by the rule

$$\tau(t) = (\gamma(t), p_t, (p_{(t,t')}\colon (t, t') \in E(T))),$$
$$\overline{g} = (Dg, \tilde{g}, \check{g}),$$

where (letting $\hat{g}(x) = (v_1, \dots, v_n)$)

$$\tilde{g}(x) = ((t(x), v_1), \dots, (t(x), v_n)),$$
$$p_{t'} = p_{(t,t')}, \quad \text{set}(p_{(t,t')}) = V_{\gamma(t)} \cap V_{\gamma(t')}, \quad \text{and}$$

(5.1a) if $t_1, \dots, t_n$ is a directed path in T, directed away from the root such that

$$|V_{\gamma(t_1)} \cap V_{\gamma(t_2)}| = |V_{\gamma(t_{n-1})} \cap V_{\gamma(t_n)}| = k$$

and

$$|V_{\gamma(t_i)} \cap V_{\gamma(t_{i+1})}| \geq k \qquad (i = 1, \dots, n-1),$$

then there are k disjoint paths $P_1, \dots, P_k$ in G_γ such that P_i joins the ith term of $p_{(t_1,t_2)}$ with the ith term of $p_{(t_{n-1},t_n)} = p_{t_n}$. Let us note that $\text{Im}(\gamma, g) = \text{Im}((T, \tau), \overline{g})$.

(5.2) **Lemma.** *Each* $(\gamma, g) \in \mathscr{A}^w[Q]$ *has an encoding.*

Proof. The encoding is almost fully determined; the only thing to specify is how to arrange the intersections $V_{\gamma(t)} \cap V_{\gamma(t')}$ $((t, t') \in E(T))$ into sequences to make condition (5.1a) fulfilled. This is done by induction. First we choose $p_{\text{root}(T)}$ to be the empty sequence. Now let $(t, t') \in E(T)$, let X be the set of edges of the path from $\text{root}(T)$ to t, and suppose that $p_{(s,s')}$ is arranged for every edge $(s, s') \in X$. Let $k = |V_{\gamma(t)} \cap V_{\gamma(t')}|$. If there is no edge $(s, s') \in X$ such that $|V_{\gamma(s)} \cap V_{\gamma(s')}| = k$ and $|V_{\gamma(r)} \cap V_{\gamma(r')}| \geq k$ for every edge (r, r') of the path from s to t', then arrange $p_{t'} = p_{(t,t')}$ arbitrarily. If there is one, then choose $(s, s') \in X$ with $|V_{\gamma(s)} \cap V_{\gamma(s')}| = k$ such that s' is as close to t as possible. By the linked property there are k disjoint paths each between $V_{\gamma(s)} \cap V_{\gamma(s')} = \text{set}(p_{(s,s')})$ and $V_{\gamma(t)} \cap V_{\gamma(t')}$. Now order $V_{\gamma(t)} \cap V_{\gamma(t')}$ in such a way that these paths would join the ith term of $p_{(s,s')}$ with the ith term of $p_{(t,t')}$. It is easily seen that this leads to an appropriate ordering of $V_{\gamma(t)} \cap V_{\gamma(t')}$. □

(5.3) **First Encoding Lemma.** *Let* $((T, \tau), \overline{g})$, $((S, \sigma), \overline{h}) \in \mathscr{S}_{\geq 0}^{\leq w}(\mathscr{A})[Q]$ *be encodings of* (γ, g), $(\eta, h) \in \mathscr{A}^w[Q]$ *respectively. If* $((T, \tau), \overline{g}) \leq ((S, \sigma), \overline{h})$ *as members of* $\mathscr{S}_{\geq 0}^{\leq w}(\mathscr{A})[Q]$, *then* $(\gamma, g) \leq (\eta, h)$ *as members of* $\mathscr{A}^w[Q]$.

Proof. Let

$$\gamma = (T, (\gamma(t): t \in V(T))), \qquad g = (Dg, \hat{g}, \check{g}),$$
$$\eta = (S, (\eta(s): s \in V(S))), \qquad h = (Dh, \hat{h}, \check{h}),$$
$$\tau(t) = (\gamma(t), p_t, (p_{(t,t')}: (t, t') \in E(T))), \qquad \overline{g} = (Dg, \tilde{g}, \check{g}),$$
$$\sigma(s) = (\eta(s), q_s, (q_{(s,s')}: (s, s') \in E(S))), \qquad \overline{h} = (Dh, \tilde{h}, \check{h}).$$

Let p_t^i, $p_{(t,t')}^i$, q_s^i, $q_{(s,s')}^i$ be the ith terms of p_t, $p_{(t,t')}$, q_s, $q_{(s,s')}$, respectively. We put $N(t) := N_{(T,\tau)}(t)$. Let $\lambda \in \mathscr{S}_{\geq 0}^{\leq w}(\mathscr{A})[Q](((T, \tau), \overline{g}), ((S, \sigma), \overline{h}))$ and let $\lambda(t, v) = \{(\varphi(t), u): u \in f_t(v)\}$. Then φ, f_t satisfy

(4.8a)–(4.8d); in particular f_t is an expansion of $G_{\gamma(t)}$ into $G_{\eta(\varphi(t))}$ and thus there is an edge expansion $\varepsilon_t : E(G_{\gamma(t)}) \to E(G_{\eta(\varphi(t))})$, and there is a 1-1 mapping $\imath : Dg \to Dh$ such that for every $x \in Dg$

(5.3a) $$\tilde{g}(x) \xrightarrow{\lambda} \tilde{h}(\imath(x)),$$

(5.3b) $$\check{g}(x) \leq \check{h}(\imath(x)).$$

By (4.8d) and (5.1a) there exist for every $(t, t') \in E(T)$ disjoint paths $Q^i_{t'}$ in G_η $(i = 1, \dots, N(t'))$ such that $Q^i_{t'}$ joins $q^i_{(\varphi(t), \varphi_*(t'))} = q^i_{\varphi_*(t')}$ with $q^i_{(\varphi^*(t), \varphi(t'))} = q^i_{\varphi(t')}$.

(5.4) *Claim.* $V(Q^i_{t'}) \subseteq \bigcup\{V_{\eta(s)} : \varphi_*(t') \in [\varphi(t), s]_S,\ \varphi^*(t) \in [\varphi(t'), s]_S\}$ for every $(t, t') \in E(T)$.

Proof. It is easily seen that a path joining a vertex from $V_{\eta(s)}$ to a vertex from $V_{\eta(s'')}$ must use any of the sets $V_{\eta(s')}$ $(s' \in [s, s'']_S)$. Now suppose that some $v \in V(Q^i_{t'})$ does not belong to the set on the right-hand side. But $v \in V_{\eta(s)}$ for some $s \in S$, and we may safely assume that $\varphi(t) \in [s, \varphi_*(t')]_S$. Then the two subpaths of $Q^i_{t'}$ obtained by cutting $Q^i_{t'}$ at v have only v in common and both must use a vertex from $V_{\eta(\varphi(t))} \cap V_{\eta(\varphi_*(t'))}$, which is impossible since $|V_{\eta(\varphi(t))} \cap V_{\mu(\varphi_*(t'))}| = k$ and any of the paths $Q^j_{t'}$ $(j \neq i)$ also use a vertex from $V_{\eta(\varphi(t))} \cap V_{\eta(\varphi_*(t'))}$. □

We define

$$f(v) = \bigcup\{f_t(v) : v \in V_{\gamma(t)}\} \cup \bigcup\{V(Q^i_t) : v = p^i_t\}$$

and $\varepsilon(e) = \varphi_t(e)$ for $e \in E(G)$ and $t \in V(T)$ such that e has both end vertices in $V_{\gamma(t)}$ and t is as close to root(T) as possible. Our aim is to show that $f \in \mathscr{A}^w[Q]((\gamma, g), (\eta, h))$. We proceed in a series of claims.

(5.5) *Claim.* Let $t, t' \in V(T)$ and $v \in V_{\gamma(t)}$. If $u \in f_t(v) \cap V_{\eta(\varphi(t'))}$, then $v \in V_{\gamma(t')}$ and $u \in f_{t'}(v)$.

Proof. First let $(t, t') \in E(T)$. Since $u \in V_{\eta(\varphi(t))} \cap V_{\eta(\varphi(t'))} \subseteq (V_{\eta(\varphi(t))} \cap V_{\eta(\varphi_*(t'))}) \cap (V_{\eta(\varphi^*(t))} \cap V_{\eta(\varphi(t'))})$ by (2.5c), there are $i, j \in \{1, \dots, N(t')\}$ such that $q^i_{(\varphi(t), \varphi_*(t'))} = u = q^j_{\varphi(t')}$. But since the paths $Q^i_{t'}$ are disjoint, it must be $i = j$. By (4.8c) $u \in f_t(p^i_{(t, t')})$ and $u \in f_{t'}(p^j_{t'})$. Since f_t is injective it follows that $v = p^i_{(t, t')} = p^j_{t'} \in V_{\gamma(t')}$ and hence $u \in f_{t'}(v)$.

A similar proof works for $(t', t) \in E(T)$. Now if $t, t' \in V(T)$ are arbitrary, we take vertices $r_0, r_1, \dots, r_n = t$, $r'_0 = r_0, r'_1, \dots, r'_{n'} = t'$ such that $r_1 \neq r'_1$, $(r_i, r_{i+1}) \in E(T)$ $(i = 0, \dots, n-1)$ and $(r'_i, r'_{i+1}) \in E(T)$ $(i = 0, \dots, n'-1)$. By (2.5c) $u \in V_{\eta(\varphi(r_i))} \cap V_{\eta(\varphi(r'_j))}$ for all $i = 0, 1, \dots, n$, $j = 0, \dots, n'$ and (5.5) follows from the special case proved above. □

(5.6) *Claim.* f is injective.

Proof. Let $u \in f(v) \cap f(v')$. We have to distinguish three cases.

1. Let $u \in f_t(v) \cap f_{t'}(v')$ for some $t, t' \in V(T)$. Then $u \in f_{t'}(v)$ by (5.5) and hence $v = v'$ by injectivity of $f_{t'}$.

2. Let $v' = p^i_{(t_1, t_2)} = p^i_{t_2}$ for $(t_1, t_2) \in E(T)$ and $u \in f_t(v) \cap V(Q^i_{t_2})$. First let $t_1 \in [t, t_2]_T$; then $u \in V_{\eta(\varphi(t_1))} \cap V_{\eta(\varphi_*(t_2))}$ by (5.4) and (2.5c). Hence $u \in f_{t_1}(v)$ by (5.5) and $u = q^i_{(\varphi(t_1), \varphi_*(t_2))}$. By (4.8c) $u = q^i_{(\varphi(t_1), \varphi_*(t_2))} \in f_{t_1}(p^i_{(t_1, t_2)})$ and hence $v = p^i_{(t_1, t_2)} = v'$ by the injectivity of f_{t_1}.

Second, let $t_2 \in [t_1, t]_T$; then $u \in V_{\eta(\varphi^*(t_1))} \cap V_{\eta(\varphi(t_2))}$ by (5.4) and (2.5c). Hence $u \in f_{t_2}(v)$ by (5.5) and $u = q^i_{\varphi(t_2)}$. By (4.8c) $u = q^j_{\varphi(t_2)} \in f_{t_2}(p^j_{t_2}))$ and hence $v = p^j_{t_2} = v'$ by the injectivity of f_{t_2}.

3. Let $v = p^i_{(t_1, t_2)}$, $v' = p^j_{(t'_1, t'_2)}$ for $(t_1, t_2) \in E(T)$, $(t'_1, t'_2) \in E(T)$ and let $u \in V(Q^i_{(t_1, t_2)}) \cap V(Q^j_{(t'_1, t'_2)})$. First assume that t_1, $t'_1 \in [t_2, t'_2]_T$. It follows from (5.4) and (2.5c) that $u \in (V_{\eta(\varphi(t_1))} \cap V_{\eta(\varphi_*(t_2))}) \cap (V_{\eta(\varphi(t'_1))} \cap V_{\eta(\varphi_*(t'_2))})$; hence there are $i \in \{1, \dots, N(t_2)\}$ and $j \in \{1, \dots, N(t'_2)\}$ such that $u = q^i_{(\varphi(t_1), \varphi_*(t_2))} = q^j_{(\varphi(t'_1), \varphi_*(t'_2))}$. Then $u \in f_{t_1}(p^i_{(t_1, t_2)}) \cap f_{t'_1}(p^j_{(t'_1, t'_2)}) = f_{t_1}(v) \cap f_{t'_1}(v')$ and hence $v = v'$ by part 1 above.

It remains to consider the case when $t_1, t'_2 \in [t'_1, t_2]_T$, but we hope that the reader is familiar enough with the techniques to supply the proof himself. □

(5.7) *Claim.* $G_\eta \restriction f(v)$ is connected for every $v \in V_\gamma$.

Proof. The proof is more or less obvious but tedious to write out in full, so we merely sketch it. It follows from the following facts:

1. Each $G_\eta \restriction f_t(v)$ is connected by (2.3a);

2. the (oriented) graph spanned by $\{t \in V(T) : v \in V_{\gamma(t)}\}$ in T is connected by (2.5c); and

3. if $v \in V_{\gamma(t)} \cap V_{\gamma(t')}$ for $(t, t') \in E(T)$, then $v = p^i_{(t, t')}$ for some i and $Q^i_{t'}$ joins $q^i_{(\varphi(t), \varphi_*(t'))} \in f_t(v)$ to $q^i_{(\varphi^*(t), \varphi(t'))} \in f_{t'}(v)$. □

(5.8) *Claim.* ε is an edge-expansion.

Proof. By (2.5b), ε is well-defined. From the definition of ε and the injectivity of ε_t we see that ε is injective; since each ε_t satisfies (2.3b) it follows that ε satisfies (2.3b). □

(5.9) *Claim.* $\hat{g}(x) \xrightarrow{f} \hat{h}(\iota(x))$ for any $x \in Dg$.

Proof. Immediate from (5.3a). □

From (5.6), (5.7), (5.8), and (5.3b) follows that $f \in \mathscr{A}^w[Q]((\gamma, g), (\eta, h))$, which completes the proof of Lemma (5.3). □

6. Second Encoding Lemma

The Second Encoding Lemma enables us to replace the QO-category $\mathcal{S}^{\leq w}_{\geq k}(\mathcal{A})$ by $\mathcal{S}_k(\mathcal{S}^{\leq w}_{\geq k+1}(\mathcal{A}))$, which together with the result of the next section makes our induction work.

(6.1) **Definition.** Put $\mathcal{B} = \mathcal{S}^{\leq w}_{\geq k+1}(\mathcal{A})$; we shall define a mapping

$$\Xi_k : \mathcal{S}^{\leq w}_{\geq k}(\mathcal{A}) \to \mathcal{S}_k(\mathcal{B}).$$

Let $(T, \tau) \in \mathcal{S}^{\leq w}_{\geq k}(\mathcal{A})$ and let

$$\tau(t) = (\gamma(t), p_t, (p_{(t,t')} : (t, t') \in E(T))).$$

Let us remove from T all edges (t, t') for which $N_{(T,\tau)}(t') = k$, and let $V(R)$ denote the set of components of the resulting forest. For $r, r' \in V(R)$ we put $(r, r') \in E(R)$ provided there are $t \in V(r)$, $t' \in V(r')$ such that $(t, t') \in E(T)$ and let $\operatorname{root}(R)$ be that component which contains $\operatorname{root}(T)$. Then R thus defined is a tree.

Every $r \in V(R)$ is again a tree in the obvious way; its root is its nearest vertex from $\operatorname{root}(T)$.

For $r \in V(R)$ we define (r, σ) by

$$\sigma(t) = (\gamma(t), p, (p_{(t,t')} : (t, t') \in E(r))),$$

where

$$\begin{aligned} p &= p_t && \text{if } |V(r)| = 1 \text{ or } t \neq \operatorname{root}(r), \\ &= p_{(t,t')} && \text{for some } t' \in V(r) \text{ such that } (t, t') \in E(r) \text{ otherwise.} \end{aligned}$$

Clearly $(r, \sigma) \in \mathcal{S}^{\leq w}_{\geq k+1}(\mathcal{A})$. We put

$$\rho(r) = ((r, \sigma), z_r, (z_{(r,r')} : (r, r') \in E(R))),$$

where

$$z_r = ((\operatorname{root}(r), v_1), \ldots, (\operatorname{root}(r), v_n)),$$
$$z_{(r,r')} = ((t, u_1), \ldots, (t, u_k)),$$

and $t \in V(r)$, $v_1, \ldots, v_n, u_1, \ldots, u_k$ are such that $p_{\operatorname{root}(r)} = (v_1, \ldots, v_n)$, $(t, t') \in E(T)$ for some $t' \in V(r')$ and $p_{(t,t')} = (u_1, \ldots, u_k)$. Finally we define $\Xi_k((T, \tau)) = (R, \rho)$.

(6.2) **Lemma.** *Let* (T^1, τ^1), $(T^2, \tau^2) \in \mathcal{S}^{\leq w}_{\geq k}(\mathcal{A})$; *and let* $(R^i, \rho^i) = \Xi_k((T^i, \tau^i))$ $(i = 1, 2)$. *If* $(R^1, \rho^1) \leq (R^2, \rho^2)$, *then* $(T^1, \tau^1) \leq (T^2, \tau^2)$.

Proof. Let

$$\tau^i(t) = (\gamma^i(t), p^i_t, (p^i_{(t,t')} : (t, t') \in E(T^i))) \qquad (t \in V(T^i)),$$
$$\rho^i(r) = ((r, \sigma^i), z^i_r, (z^i_{(r,r')} : (r, r') \in E(R^i))) \qquad (r \in V(R^i)),$$

where z_r^i and $z_{(r,r')}^i$ are constructed from p_t^i and $p_{(t,t')}^i$ as above. Let $\mu \in \mathscr{S}_k(\mathscr{B})((R^1, \rho^1), (R^2, \rho^2))$; then μ has the form $\mu(r, (t, v)) = \{(\psi(r), x): x \in \lambda_r(t, v)\}$ such that ψ and λ_r satisfy (4.8a)–(4.8c). Rewriting these conditions we get

(6.2a) $\psi: V(R^1) \to V(R^2)$ is a homeomorphic embedding,

(6.2b) $\lambda_r \in \mathscr{S}_{\geq k+1}^{\leq w}(\mathscr{A})((r, \sigma^1), (\psi(r), \sigma^2))$ for any $r \in V(R^1)$,

(6.2c) $z_{(r,r')}^1 \xrightarrow{\lambda_r} z_{(\psi(r),\psi_*(r'))}^2$ for any $(r, r') \in E(R^1)$, $z_r^1 \xrightarrow{\lambda_r} z_{\psi(r)}^2$ for any $r \in V(R^1)$.

Similarly, from (6.2b) it follows that λ_r has the form $\lambda_r(t, v) = \{(\varphi^r(t), u): u \in f_t^r(v)\}$, where φ^r and f_t^r satisfy (4.8a)–(4.8d). Again, rewriting these conditions we get

(6.2d) $\varphi^r: V(r) \to V(\psi(r))$ is a homeomorphic embedding,

(6.2e) $f_t^r \in \mathscr{A}(\gamma^1(t), \gamma^2(\varphi(t)))$ for any $t \in V(r)$,

(6.2f) $p_{(t,t')}^1 \xrightarrow{f_t^r} p_{(\varphi(t),\varphi_*(t'))}^2$ for any $(t, t') \in E(r)$,

$$p_t^1 \xrightarrow{f_t^r} p_{\varphi(t)}^2 \text{ for any } t \in V(r)\text{-}\{\text{root}(r)\}.$$

(6.2g) $N_{(r,\sigma^1)}(t) = N_{(\psi(r),\sigma^2)}(\varphi(t)) = N_{(\psi(r),\sigma^2)}(\varphi_*(t)) \leq N_{(\psi(r),\sigma^2)}(s)$ for any $r \in V(R^1)$, any $t \in V(r)\text{-}\{\text{root}(r)\}$, and any $s \in [\varphi_*(t), \varphi(t)]_{\psi(r)}$.

For $r \in V(R)$, $t \in V(r)$, and $v \in V_{\gamma(t)}$ we put $\varphi(t) = \varphi^r(t)$, $f_t(v) = f_t^r(v)$, and $\lambda(t, v) = \lambda_r(t, v) = \{(\varphi(t), u): u \in f_t(v)\}$. We shall prove that $\lambda \in \mathscr{S}_{\geq k}^{\leq w}(\mathscr{A})((T^1, \tau^1), (T^2, \tau^2))$, which will give the lemma. To this end we must show that φ and f_t satisfy (4.8a)–(4.8d), but first we prove that for any $(r, r') \in E(R^1)$

(6.2h) $p_t^1 \xrightarrow{f_t} p_{\varphi(t)}^2$, where $t = \text{root}(r)$,

(6.2i) $\varphi(\text{root}(r)) = \text{root}(\psi(r))$,

(6.2j) $p_{(t,t')}^1 \xrightarrow{f_t} p_{(\varphi(t),\varphi_*(t'))}^2$, where $(t, t') \in E(T^1)$ are such that $t \in V(r)$, $t' \in V(r')$,

(6.2k) $\varphi_*(\text{root}(r)) = \text{root}(\psi_*(r))$.

To prove (6.2h) and (6.2i) let $t = \text{root}(r)$, $s = \text{root}(\psi(r))$, $p_t^1 = (v_1, \dots, v_n)$, and $p_s^2 = (u_1, \dots, u_m)$; then $z_r^1 = ((t, v_1), \dots, (t, v_n))$ and $z_{\psi(r)}^2 = ((s, u_1), \dots, (s, u_m))$. By (6.2c) $n = m$, $s = \varphi(t)$, and $u_i \in f_t(v_i)$ $(i = 1, \dots, n)$, which proves (6.2h) and (6.2i).

To prove (6.2j) and (6.2k) let $(t, t') \in E(T^1)$ be such that $t \in V(r)$ and $t' \in V(r')$; then $t' = \text{root}(r')$. Let $s \in V(\psi(r))$, $s' \in V(\psi_*(r'))$ be such that $(s, s') \in E(T^2)$; then $s' = \text{root}(\psi_*(r'))$. Let $p_{(t,t')}^1 = (v_1, \dots, v_k)$, $p_{(s,s')}^2 = (u_1, \dots, u_k)$. Then $z_{(r,r')}^1 = ((t, v_1), \dots, (t, v_k))$, $z_{(\psi(r),\psi_*(r'))}^2 = ((s, u_1), \dots, (s, u_k))$. By (6.2c) $s = \varphi(t)$ and $u_i \in f_t(v_i)$ and by (6.2i) $\varphi(t') \in V(\psi(r'))$ which implies that $s' = \varphi_*(t')$. This proves (6.2j). Further,

$$\varphi_*(\text{root}(r')) = \varphi_*(t') = s' = \text{root}(\psi_*(r'))$$

and for $r = \mathrm{root}(R^1)$

$$\varphi_*(\mathrm{root}(r)) = \varphi_*(\mathrm{root}(T^1)) = \mathrm{root}(T^2) = \mathrm{root}(\psi_*(r)).$$

This proves (6.2k).

Now we are ready to show that φ^r and f_t^r satisfy (4.8a)–(4.8d). To prove (4.8a) let $t, t_1, t_2 \in V(T^1)$ be such that t_1 and t_2 are distinct successors of t. Let $t \in V(r)$, $t_1 \in V(r_1)$, and $t_2 \in V(r_2)$, where $r, r_1, r_2 \in V(R^1)$. We must show that $\varphi_*(t_1) \neq \varphi_*(t_2)$. If $r_1 = r_2$ then $r = r_1 = r_2$ and the assertion follows from (6.2d); if $r_1 \neq r_2$ then say $r \neq r_1$ and we have $\varphi_*(t_1) \in V(\psi_*(r_1))$ by (6.2k) and $\varphi_*(t_2) \in V(\psi_*(r_2)) \cup V(\psi(r))$ by (6.2k) or (6.2d) depending on whether or not $r \neq r_2$. In any case

$$V(\psi_*(r_1)) \cap (V(\psi_*(r_2)) \cup V(\psi(r))) = \varnothing$$

and (4.8a) follows.

Condition (4.8b) follows from (6.2e), condition (4.8c) from (6.2f), (6.2h), and (6.2j). Finally, to prove (4.8d) let $t \in V(T^1)$ and let $r \in V(R^1)$ be such that $t \in V(r)$. If $t \neq \mathrm{root}(r)$ then (4.8d) follows from (6.2g) and if $V(T^1) = \{t\}$ then it is obvious. So assume that $|V(T^1)| > 1$ and that $t = \mathrm{root}(r)$. We have by (6.2i) and (6.2k) for any $s \in V(T^2)$

$$\begin{aligned} N_{(T^1,\tau^1)}(t) = k &= N_{(T^2,\tau^2)}(\mathrm{root}(\psi(r)) = N_{(T^2,\tau^2)}(\varphi(t)), \\ k &= N_{(T^2,\tau^2)}(\mathrm{root}(\psi_*(r)) = N_{(T^2,\tau^2)}(\varphi_*(t)), \\ k &\leq N_{(T^2,\tau^2)}(s), \end{aligned}$$

which completes the proof of the lemma. □

7. Well-behavedness of k-structures

In this section we prove

(7.1) **Theorem.** *If $\mathscr{A}$ is a well-behaved QO-category, then the QO-category $\mathscr{S}_k(\mathscr{A})$ is well-behaved for any $k \geq 0$.*

The idea we use is essentially due to Nash-Williams but we follow Laver's paper [9], slightly improved in various places.

Let us fix $k \geq 0$ and a well-behaved QO-category $\mathscr{A}$. Let $\mathscr{B}$ be an arbitrary QO-category.

(7.2) **Definition.** Let T be a tree. The least ordinal γ (if such exists) such that there is a mapping $h: V(T) - \{\mathrm{root}(T)\} \to \gamma$ such that $h(t) > h(t')$ for any $(t, t') \in E(T)$, $t \neq \mathrm{root}(T)$ is called the *rank of T* and is denoted by $\mathrm{rank}(T)$. It is easy to see that T has a rank if and only if it contains no infinite path and that the rank of a branch of T distinct from T is smaller than the rank of T.

We introduce the following notation. Let $\mathscr{S}_k^0(\mathscr{B})$ denote the subcategory of $\mathscr{S}_k(\mathscr{B})$ consisting of all objects $(T, \tau) \in \mathscr{S}_k(\mathscr{B})$ such that $|V(T)| = 1$

and let $\mathscr{M}_k(\mathscr{B})$ denote the subcategory of $\mathscr{S}_k(\mathscr{B})$ consisting of all objects $(T, \tau) \in \mathscr{S}_k(\mathscr{B})$ such that T has a rank.

For $(T, \tau), (S, \sigma) \in \mathscr{S}_k(\mathscr{B})$ we define $(T, \tau) <' (S, \sigma)$ if $(T, \tau) = (S_s, \sigma) < (S, \sigma)$ for some $s \in V(S)\text{-}\{\operatorname{root}(S)\}$. Recall that S_s is the branch of S at s and (S_s, σ) is the restriction of (S, σ) to S_s. If $(S, \sigma) \in \mathscr{M}_k(\mathscr{B})$, then $\operatorname{rank}(T) < \operatorname{rank}(S)$, hence $<'$ is a partial ranking on $\mathscr{M}_k(\mathscr{B})$.

(7.3) **Lemma.** *$\mathscr{S}_k^0(\mathscr{A})$ is well-behaved.*

Proof. Let Q be a quasi-ordered set and let

$$a\colon [A]^\omega \to \mathscr{S}_k^0(\mathscr{A})[Q]$$

be a bad array such that $\operatorname{Im} a(Z) \subseteq Q$ is bqo for any $Z \in [A]^\omega$. By (4.10) we may safely assume that $a(Z) \in \mathscr{S}_k^0(\mathscr{A}[Q])$ $(Z \in [A]^\omega)$. For $Z \in [A]^\omega$ let

$$a(Z) = (T_Z, \tau_Z), \qquad V(T_Z) = \{t_Z\},$$
$$\tau_Z(t_Z) = ((\gamma_Z, g_Z), p_Z, \varnothing), \qquad p_Z \in P\gamma_Z,$$
$$g_Z = (Dg_Z, \hat{g}_Z, \check{g}_Z).$$

Let us define $h_Z = (Dh_Z, \hat{h}_Z, \check{h}_Z)$ by

$$Dh_Z = Dg_Z \cup \{*\},$$

$$\hat{h}_Z(x) = \hat{g}_Z(x) \quad \text{for } x \in Dg_Z, \qquad = p_Z \quad \text{for } x = *,$$

$$\check{h}_Z(x) = \check{g}_Z(x) \quad \text{for } x \in Dg_Z, \qquad = * \quad \text{for } x = *.$$

Finally, put $b(Z) = (\gamma_Z, h_Z) \in \mathscr{A}[Q \cup \{*\}]$. We claim that $b\colon [A]^\omega \to \mathscr{A}[Q \cup \{*\}]$ is a bad array. Indeed, let $b(Z) \le b(Z-)$ for some $Z \in [A]^\omega$; then there are an $f \in \mathscr{A}(\gamma_Z, \gamma_{Z-})$ and a 1-1 mapping $\iota\colon Dh_Z \to Dh_{Z-}$ such that (4.4a) and (4.4b) are satisfied. Clearly $\iota(*) = *$ and $\iota(x) \in Dh_{Z-} - \{x\}$ for $x \in Dh_Z - \{*\}$. If we define $\lambda\colon V_{(T_Z, \tau_Z)} \to V_{(T_{Z-}, \tau_{Z-})}$ by $\lambda(t_Z, v) = \{(t_{Z-}, u)\colon u \in f(v)\}$, then it is easily seen that

$$\lambda \in \mathscr{S}(\mathscr{A}[Q])((T_Z, \tau_Z), (T_{Z-}, \tau_{Z-})),$$

thus showing a contradiction $a(Z) \le a(Z-)$.

From (3.11ii) it follows that $\operatorname{Im} b(Z) \subseteq \operatorname{Im} a(Z) \cup \{*\}$ is bqo for any $Z \in [A]^\omega$. Hence by the well-behavedness of $\mathscr{A}$ there are $C \in [A]^\omega$ and a witnessing bad array $c\colon [C]^\omega \to Q \cup \{*\}$ such that $c(Z) \in \operatorname{Im} b(Z)$ $(Z \in [C]^\omega)$. By (3.11ii) we may assume that C is chosen in such a way that $c(Z) \in Q$ for any $Z \in [C]^\omega$. Now

$$c(Z) \in \operatorname{Im} b(Z) \cap Q \subseteq \operatorname{Im} a(Z),$$

hence c is the desired witnessing array. □

(7.4) **Definition.** Let (T, τ), $(S, \sigma) \in \mathscr{S}_k(\mathscr{B})$, let $\lambda \in \mathscr{S}_k(\mathscr{B})((T, \tau), (S, \sigma))$, let $\lambda(t, v) = \{(\varphi(t), u) : u \in f_t(v)\}$, and let R be a subtree of T. We put for $t \in V(R)$

$$B_t(T, R) = \{r \in V(T) - V(R) : (t, r) \in E(T)\},$$

$$C_t(S, R, \lambda) = \{s \in V(S) : (\varphi(t), s) \in E(S) \text{ and } s \text{ is followed by no } \varphi(t'),\ t' \in V(T)\}.$$

(7.5) **Lemma.** *Let (T, τ), $(S, \sigma) \in \mathscr{S}_k(\mathscr{B})$ with the usual notation (cf. (4.8)), let T^0 be a subtree of T, and suppose that there is a morphism*

$$\lambda^0 \in \mathscr{S}_k(\mathscr{B})((T^0, \tau), (S, \sigma)).$$

Let $\lambda^0(t, v) = \{(\varphi^0(t), u) : u \in f_t^0(v)\}$ and suppose that for any $t \in V(T^0)$ there is a 1-1 mapping $J_t : B_t(T, T^0) \to C_t(S, T^0, \lambda)$ such that for any $r \in B_t(T, T^0)$

(7.5a) $p_{(t,r)} \xrightarrow{f_t^0} q_{(\varphi^0(t), J_t(r))}$,

(7.5b) *there is a morphism $\lambda^r \in \mathscr{S}_k(\mathscr{B})((T_r, \tau), (S_{J_t(r)}, \sigma))$.*

Define λ by

$$\begin{aligned} \lambda(t, v) &= \lambda^0(t, v) \quad \text{for } t \in V(T^0),\ v \in V_{\gamma(t)}, \\ &= \lambda^r(t, v) \quad \text{for } t \in V(T_r),\ v \in V_{\gamma(t)},\ r \in B_{t_0}(T, T^0),\ t_0 \in V(T^0). \end{aligned}$$

Then $\lambda \in \mathscr{S}_k(\mathscr{B})((T, \tau), (S, \sigma))$ and hence $(T, \tau) \le (S, \sigma)$.

Proof. We have to verify that λ satisfies (4.8a)–(4.8d). Conditions (4.8a) and (4.8b) follow immediately; condition (4.8c) follows from the corresponding properties of λ^0 and λ^r and from (7.5a). Condition (4.8d) is vacuously true. □

(7.6) **Definition.** Let $(T, \tau) \in \mathscr{S}_k(\mathscr{B})$ and let T_0 be a subtree of T. Let (T, τ) have the usual meaning. We shall define a new k-structure $(T_0, \tau_{T_0}) \in \mathscr{S}_k(\mathscr{B}[\mathscr{S}_k(\mathscr{B})])$ as follows:

$$\tau_{T_0}(t) = ((\gamma(t), g_t), p_t, (p_{(t,t')} : (t, t') \in E(T_0))),$$

where $g_t = (Dg_t, \hat{g}_t, \check{g}_t)$, $Dg_t = B_t(T, T_0)$, and for $r \in Dg_t$, $\hat{g}_t(r) = p_{(t,r)}$ and $\check{g}_t(r) = (T_r, \tau)$.

Let us remark that if $(T, \tau) \in \mathscr{M}_k(\mathscr{B})$, then $(T_0, \tau_{T_0}) \in \mathscr{M}_k(\mathscr{B}[\mathscr{M}_k(\mathscr{B})])$.

(7.7) **Lemma.** *If $(T_0, \tau_{T_0}) \le (S_0, \sigma_{S_0})$ as elements of $\mathscr{S}_k(\mathscr{B}[\mathscr{S}_k(\mathscr{B})])$, then $(T, \tau) \le (S, \sigma)$ as elements of $\mathscr{S}_k(\mathscr{B})$.*

Proof. Let the notation be as in (7.6) and let

$$\sigma_{S_0}(s) = ((\eta(s), h_s), q_s, q_{(s,s')} : (s, s') \in E(S_0))),$$

where $h_s = (Dh_s, \hat{h}_s, \check{h}_s)$. Let $\lambda^0 \in \mathscr{S}_k(\mathscr{B}[\mathscr{S}_k(\mathscr{B})])((T_0, \tau_{T_0}), (S_0, \sigma_{S_0}))$ and let $\lambda^0(t, v) = \{(\varphi^0(t), u) : u \in f_t^0(v)\}$, where φ^0, f_t^0 satisfy (4.8a)–(4.8c). For

$t \in V(T_0)$ the mapping f_t^0 belongs to $\mathscr{B}[\mathscr{S}_k(\mathscr{B})]((\gamma(t), g_t), (\eta(\varphi^0(t)), h_{\varphi(t)}))$, hence $f_t^0 \in \mathscr{B}(\gamma(t), \eta(\varphi(t)))$ and there is a 1-1 mapping $\iota_t: Dg_t \to Dh_{\varphi^0(t)}$ such that for any $r \in Dg_t$

(7.7a) $p_{(t,r)} \xrightarrow{f_t^0} q_{(\varphi^0(t), \iota_t(r))}$, and
(7.7b) $(T_r, \tau) \le (S_{\iota_t(r)}, \sigma)$.

Let $\lambda^r \in \mathscr{S}_k(\mathscr{B})((T_r, \tau), (S_{\iota_t(r)}, \sigma))$; defining λ as in (7.5), we see that $(T, \tau) \le (S, \sigma)$. □

(7.8) **Lemma.** *$\mathscr{M}_k(\mathscr{A})$ is well-behaved.*

Proof. Suppose that the lemma is false for some quasi-ordered set Q and that $a: [A]^\omega \to \mathscr{M}_k(\mathscr{A})[Q]$ is a bad array such that $\operatorname{Im} a(Z)$ is bqo for every $Z \in [A]^\omega$, there is no witnessing bad array, and

$$\Gamma(a) := \sup\{\operatorname{rank}(T) + 1 : ((T, \tau), g) = a(Z) \text{ for some } Z \in [A]^\omega\}$$

is the least possible.

By the Minimal Bad Array Lemma (3.14) there are $B \in [A]^\omega$ and a minimal bad array $b: [B]^\omega \to \mathscr{M}_k(\mathscr{A})[Q]$, $b \le' a$. Then $\operatorname{Im} b(Z) \subseteq \operatorname{Im} a(Z)$ is bqo for any $Z \in [B]^\omega$, there is no witnessing array for b (since any witnessing array for b is a witnessing array for a), and $\Gamma(b) = \Gamma(a)$.

Now consider $b(Z)$ as an element of $\mathscr{M}_k(\mathscr{A}[Q])$. For $Z \in [B]^\omega$ let $b(Z) = (T, \tau)$, let T_0 be the subtree of T consisting of the root of T, and let $c(Z) = (T_0, \tau_{T_0}) \in \mathscr{S}_k^0(\mathscr{A}[Q][\mathscr{M}_k(\mathscr{A}[Q])])$. By (4.5) and (4.10) we can consider c as an array

$$c: [\mathscr{B}]^\omega \to \mathscr{S}_k^0(\mathscr{A})[Q \dot\cup \mathscr{M}_k(\mathscr{A}[Q])];$$

by (7.7) c is bad.

Our next aim is to show that $\operatorname{Im} c(Z)$ is bqo for any $Z \in [B]^\omega$. If for some $Z \in [B]^\omega$ $c(Z)$ was not bqo, there would be a bad array

$$i: [I]^\omega \to (Q \dot\cup \mathscr{M}_k(\mathscr{A}[Q])) \cap \operatorname{Im} c(Z).$$

By (3.11ii) we may safely assume that either

$$i: [I]^\omega \to Q \cap \operatorname{Im} c(Z) \subseteq \operatorname{Im} a(Z),$$

which is impossible since $\operatorname{Im} a(Z)$ is bqo, or else

$$i: [I]^\omega \to \mathscr{M}_k(\mathscr{A}[Q]) \cap \operatorname{Im} c(Z).$$

Now since $\Gamma(i) < \Gamma(b) = \Gamma(a)$, there is a witnessing bad array for i, thus showing that $\operatorname{Im} a(Z)$ is not bqo, which is a contradiction. Hence $\operatorname{Im} c(Z)$ is bqo for any $Z \in [B]^\omega$.

By (7.3) there are $D \in [B]^\omega$ and a witnessing bad array $d: [D]^\omega \to Q \dot\cup \mathscr{M}_k(\mathscr{A}[Q])$ for c. By (3.11ii) we may assume that either

$$d: [D]^\omega \to Q,$$

in which case d is a witnessing array for a, or else

$$d\colon [D]^{\omega} \to \mathscr{M}_k(\mathscr{A}[Q]),$$

in which case $d <' b$, contradicting the minimality of b. Since both cases lead to a contradiction, we are done. □

(7.9) **Definition.** A k-structure $(T, \tau) \in \mathscr{S}_k(\mathscr{B})$ is called *descentionally finite* if there is no infinite sequence $t_1, t_2, \ldots \in V(T)$ such that $t_{i+1} \in V(T_{t_i})$ and

$$(T_{t_1}, \tau) > (T_{t_2}, \tau) > \cdots .$$

The set of descentionally finite k-structures will be denoted by $\mathscr{F}_k(\mathscr{B})$. Let us remark that the inclusion $\mathscr{F}_k(\mathscr{B})[Q] \subseteq \mathscr{F}_k(\mathscr{B}[Q])$ is generally false. That is why Lemma (7.13) cannot be formulated by saying that $\mathscr{F}_k(\mathscr{A})$ is well-behaved. Note that $\leq'$ is a partial ranking on $\mathscr{F}_k(\mathscr{B})$.

(7.10) **Definition.** Let $(T, \tau) \in \mathscr{F}_k(\mathscr{B})$ and let $\{T^m\}_{m\in\omega}$ be a nondecreasing sequence of subtrees of T such that each T^m has a rank and $\bigcup_{m\in\omega} V(T^m) = V(T)$. We shall define k-structures $(T^m, \tau^m) \in \mathscr{M}_k(\mathscr{B}[\mathscr{F}_k(\mathscr{B}) \cup \{*\}])$. Let $\tau(t)$ be as usual; for $t \in V(T^m)$ we put $g_t^m = (Dg_t^m, \hat{g}_t^m, \check{g}_t^m)$, where

$$Dg_t^m = B_t(T, T^m)$$

and for $r \in Dg_t^m$

$$\begin{aligned} \hat{g}_t^m(r) &= p_{(t,r)}, \\ \check{g}_t^m(r) &= (T_r, \tau) \quad \text{if } (T_r, \tau) < (T, \tau), \\ &= * \qquad\quad\;\, \text{otherwise.} \end{aligned}$$

Now let

$$\tau^m(t) = ((\gamma(t), g_t^m), p_t, (p_{(t,t')}\colon (t, t') \in E(T^m))) \qquad (t \in V(T^m))$$

and put

$$\Phi(T, \tau) = \{(T^m, \tau^m)\colon m \in \omega\} \in \exp\mathscr{M}_k(\mathscr{B}[\mathscr{F}_k(\mathscr{B}) \cup \{*\}]).$$

It is worth noting that if $(T^m, \tau^m) \leq (S^k, \sigma^k)$, then $(T^m, \tau) \leq (S^k, \sigma)$.

(7.11) **Lemma.** *If* $\Phi(T, \tau) \leq \Phi(S, \sigma)$ *then* $(T, \tau) \leq (S, \sigma)$.

Proof. Let (T^m, τ^m) and (T, τ) be as in (7.10). Let

$$\begin{aligned} \sigma(s) &= (\eta(s), q_s, (q_{(s,s')}\colon (s, s') \in E(S))), \\ \Phi(S, \sigma) &= \{(S^m, \sigma^m)\colon m \in \omega\}, \\ \sigma^m(s) &= ((\eta(s), h_s^m), q_s, (q_{(s,s')}\colon (s, s') \in E(S^m))), \\ h_s^m &= (Dh_s^m, \hat{h}_s^m, \check{h}_s^m). \end{aligned}$$

Let R be a subtree of T. We say that a morphism $\lambda \in \mathscr{F}_k(\mathscr{B})((R, \tau), (S, \sigma))$ of the form $\lambda(t, v) = \{(\varphi(t), u)\colon u \in f_t(v)\}$ is *feasible* if for any $t \in V(R)$

there exists a 1-1 mapping $J_t: B_t(T, R) \to C_t(S, R, \lambda)$ such that for any $r \in B_t(T, R)$

(7.11a) $p_{(t,r)} \xrightarrow{f_t} q_{(\varphi(t), J_t(r))}$,
(7.11b) if $(T_r, \tau) < (T, \tau)$ then $(T_r, \tau) \le (S_{J_t(r)}, \sigma)$, and
(7.11c) if $(T_r, \tau) \equiv (T, \tau)$ then $(S_{J_t(r)}, \sigma) \equiv (S, \sigma)$.

We shall construct a morphism $\lambda \in \mathscr{F}_k(\mathscr{B})((T, \tau), (S, \sigma))$ by induction. The induction hypothesis at stage m is that there is a subtree R^m of T containing T^m and a feasible morphism $\lambda^m \in \mathscr{F}_k(\mathscr{B})((R^m, \tau), (S, \sigma))$.

For $m = 0$ there exists by assumption an n such that

$$(T^0, \tau^0) \le (S^n, \sigma^n);$$

let $\overline{\lambda}$ be the corresponding morphism and put $R^0 := T^0$, $\overline{\lambda}(t, v) = \{(\overline{\varphi}(t), u): u \in \overline{f}_t(v)\}$. In particular, $\overline{\lambda} \in \mathscr{F}_k(\mathscr{B})((R^0, \tau), (S, \sigma))$ and for every $t \in V(R^0)$ there exists a 1-1 mapping $\iota_t: Dg_t^0 \to Dh^n_{\overline{\varphi}(t)}$ such that for every $r \in Dg_t^0 = B_t(T, R^0)$

(7.11d) $p_{(t,r)} \xrightarrow{\overline{f}_t} q_{(\overline{\varphi}(t), \iota_t(r))}$,
(7.11e) if $(T_r, \tau) < (T, \tau)$, then $(T_r, \tau) \le (S_{\iota_t(r)}, \sigma)$, and
(7.11f) if $(T_r, \tau) \equiv (T, \tau)$ then $(S_{\iota_t(r)}, \sigma) \equiv (S, \sigma)$.

Thus letting $\lambda^0 := \overline{\lambda}$, $J_t^0(r) = \iota_t(r)$ we see that λ^0 is feasible.

Now assume that on the mth step we have constructed a subtree R^m of T containing T^m and a feasible morphism $\lambda^m \in \mathscr{F}_k(\mathscr{B})((R^m, \tau), (S, \sigma))$. Our aim is to extend λ^m to a feasible morphism

$$\lambda^{m+1} \in \mathscr{F}_k(\mathscr{B})((R^{m+1}, \tau), (S, \sigma)),$$

where R^{m+1} contains T^{m+1}.

Let $\lambda^m(t, v) = \{(\varphi^m(t), u): u \in f_t^m(v)\}$. Since λ^m is feasible, there is for any $t \in V(R^m)$ a 1-1 mapping $J_t^m: B_t(T, R^m) \to C_t(S, R^m, \lambda)$ satisfying (7.11a)–(7.11c). By assumption there exists $n \in \omega$ such that $(T^{m+1}, \tau^{m+1}) \le (S^n, \sigma^n)$; let $\overline{\lambda}(t, v) = \{(\overline{\varphi}(t), u): u \in \overline{f}_t(v)\}$ be the corresponding morphism. In particular $\overline{\lambda} \in \mathscr{F}_k(\mathscr{B})((T^{m+1}, \tau), (S^n, \sigma))$ and for every $t \in V(T^{m+1})$ there exists a 1-1 mapping $\iota_t: Dg_t^{m+1} \to Dh^n_{\overline{\varphi}(t)}$ such that for every $r \in Dg_t^{m+1} = B_t(T, T^{m+1})$ conditions (7.11d)–(7.11f) are satisfied.

Let $t_0 \in V(R^m)$ and $r_0 \in B_{t_0}(T, R^m)$. We distinguish two cases.

Case I: $(T_{r_0}, \tau) < (T, \tau)$. Then $(T_{r_0}, \tau) \le (S_{J^m_{t_0}(r_0)}, \sigma)$; let λ^{r_0} be the corresponding morphism. We put $R(r_0) := V(T_{r_0})$.

Case II: $(T_{r_0}, \tau) \equiv (T, \tau)$. Then $(S_{J^m_{t_0}(r_0)}, \sigma) \equiv (S, \sigma)$; let $\overline{\overline{\lambda}} = \overline{\overline{\lambda}}^{r_0}$ be the morphism corresponding to $(S, \sigma) \le (S_{j^m_{t_0}(r_0)}, \sigma)$ and let

$$\overline{\overline{\lambda}}(t, v) = \{(\overline{\overline{\varphi}}(t), u): u \in \overline{\overline{f}}_t(v)\}.$$

In particular, for every $t \in V(T^{m+1})$ and every $r \in Dg_t^{m+1} = B_t(T, T^{m+1})$

(7.11g) $$(S_{\iota_t(r)}, \sigma) \leq (S_{\overline{\overline{\varphi}}(\iota_t(r))}, \sigma).$$

We let $R(r_0) := V(T_{r_0}) \cap V(T^{m+1})$. The composition $\lambda^{r_0} := \overline{\lambda} \circ \overline{\overline{\lambda}}$ is a morphism showing that $(T^{m+1}, \tau) \leq (S_{J_{t_0}^m(r_0)}, \sigma)$. We claim that for every $t \in V(T^{m+1})$ and every $r \in Dg_t^{m+1} = B_t(T, T^{m+1})$

(7.11h) $p_{(t,r)} \overset{\overline{f}_t \circ \overline{\overline{f}}_{\varphi(t)}}{\longrightarrow} q_{(\overline{\overline{\varphi}}(\overline{\varphi}(t), \overline{\overline{\varphi}}_*(\iota_t(r)))}$,
(7.11i) if $(T_r, \tau) < (T, \tau)$ then $(T_r, \tau) \leq (S_{\overline{\overline{\varphi}}_*(\iota_t(r))}, \sigma)$, and
(7.11j) if $(T_r, \tau) \equiv (T, \tau)$ then $(S_{\overline{\overline{\varphi}}_*(\iota_t(r))}, \sigma) \equiv (S, \sigma)$.

Indeed, (7.11h) follows from (7.11d) and the fact that $\overline{\overline{\lambda}}$ satisfies (4.8c), (7.11i) follows from (7.11e) and (7.11g), and (7.11j) follows from (7.11f), (7.11g), and the inequalities

$$(S, \sigma) \leq (S_{\iota_t(r)}, \sigma) \leq (S_{\overline{\overline{\varphi}}(\iota_t(r))}, \sigma) \leq (S_{\overline{\overline{\varphi}}_*(\iota_t(r))}, \sigma) \leq (S, \sigma).$$

Now whichever case occurs, we let

$$R^{m+1} := T \restriction (V(R^m) \cup \bigcup\{R(r_0) \colon t_0 \in V(R^m),\ r_0 \in B_{t_0}(T, R^m)\}),$$

$$\begin{aligned} \lambda^{m+1}(t, v) &= \lambda^m(t, v) && \text{for } t \in V(R^m), \\ &= \lambda^{r_0}(t, v) && \text{for } t \in R(r_0),\ r_0 \in B_{t_0}(T, R^m),\ t_0 \in V(R^m), \\ J_t^{m+1}(r) &= J_t^m(r) && \text{for } t \in V(R^m),\ r \in B_t(T, R^{m+1}), \\ &= \overline{\overline{\varphi}}_*(\iota_t(r)) && \text{for } t \in V(R^{m+1}) - V(R^m),\ r \in B_t(T, R^{m+1}). \end{aligned}$$

It follows from Lemma (7.5) (applied to $T = R^{m+1}$, $T^0 = R^m$, $\lambda = \lambda^m$) and condition (7.11a) that $\lambda^{m+1} \in \mathscr{F}_k(\mathscr{B})((R^{m+1}, \tau), (S, \sigma))$. To show that λ^{m+1} is feasible we must verify that J_t^{m+1} satisfies (7.11a)–(7.11c). But that follows easily from (7.11h)–(7.11j) and the fact that J_t^m satisfies (7.11a)–(7.11c), completing the induction.

Now for $t \in V(T)$ and $v \in V_{\gamma(t)}$ let m be such that $t \in V(R^m)$; we put $\lambda(t, v) = \lambda^m(t, v)$. Clearly $\lambda \in \mathscr{F}_k(\mathscr{B})((T, \tau), (S, \sigma))$. □

(7.12) **Definition.** For $(T, \tau) \in \mathscr{F}_k(\mathscr{B})$ we define its height as follows:

$$\begin{aligned} \mathrm{ht}(T, \tau) &= 0 \quad \text{if } (T_t, \tau) \equiv (T, \tau) \text{ for every } t \in V(T) \\ &= \sup\{\mathrm{ht}(T_t, \tau) + 1 \colon (T_t, \tau) < (T, \tau), t \in V(T)\}. \end{aligned}$$

It follows from the definition of $\mathscr{F}_k(\mathscr{B})$ that $\mathrm{ht}(T, \tau)$ is well-defined for every $(T, \tau) \in \mathscr{F}_k(\mathscr{B})$.

(7.13) **Lemma.** *If $a \colon [A]^\omega \to \mathscr{F}_k(\mathscr{A}[Q])$ is a bad array such that* $\mathrm{Im}\, a(Z)$ *is bqo for any $Z \in [A]^\omega$, then there exists a witnessing bad array for a.*

Proof. Suppose that the lemma is false and let us take a bad array $a \colon [A]^\omega \to \mathscr{F}_k(\mathscr{A}[Q])$ such that $\mathrm{Im}\, a(Z)$ is bqo for every $Z \in [A]^\omega$, there is no witnessing bad array, and

$$\Lambda(a) := \sup\{\mathrm{ht}(a(Z)) + 1 : Z \in [A]^{\omega}\}$$

is the least possible. By the Minimal Bad Array Lemma (3.14) there are $B \in [A]^{\omega}$ and a minimal bad array $b: [B]^{\omega} \to \mathscr{M}_k(\mathscr{A}[Q])$, $b \leq' a$. Then $\mathrm{Im}\, b(Z) \subseteq \mathrm{Im}\, a(Z)$ is bqo for any $Z \in [B]^{\omega}$, there is no witnessing array for b, and $\Lambda(b) = \Lambda(a)$.

Let $c(Z) = \Phi(b(Z)) \in \exp \mathscr{M}_k(\mathscr{A}[Q][\mathscr{F}_k(\mathscr{A}[Q]) \cup \{*\}])$. By (7.11) c is a bad array, and by (3.17) there are $D \in [B]^{\omega}$ and a witnessing bad array

$$d: [D]^{\omega} \to \mathscr{M}_k(\mathscr{A}[Q][\mathscr{F}_k(\mathscr{A}[Q]) \cup \{*\}]).$$

Now we consider d as an array

$$d: [D]^{\omega} \to \mathscr{M}_k(\mathscr{A})[Q \dot\cup \mathscr{F}_k(\mathscr{A}[Q]) \dot\cup \{*\}].$$

Next we show that $\mathrm{Im}\, d(Z)$ is bqo for any $Z \in [D]^{\omega}$. If for some $Z \in [D]^{\omega}$ $d(Z)$ was not bqo, there would be a bad array

$$i: [I]^{\omega} \to (Q \dot\cup \mathscr{F}_k(\mathscr{A}[Q]) \dot\cup \{*\}) \cap \mathrm{Im}\, d(Z).$$

By (3.11ii) we may safely assume that either

$$i: [I]^{\omega} \to Q \cap \mathrm{Im}\, d(Z) \subseteq \mathrm{Im}\, a(Z),$$

which is impossible since $\mathrm{Im}\, a(Z)$ is bqo, or

$$i: [I]^{\omega} \to \{*\},$$

which is impossible since i is bad, or

$$i: [I]^{\omega} \to \mathscr{F}_k(\mathscr{A}[\mathrm{Im}\, a(Z)]) \cap \mathrm{Im}\, d(Z).$$

Now since $\Lambda(i) < \Lambda(b) = \Lambda(a)$ there is a witnessing bad array for i, thus giving a contradiction to $\mathrm{Im}\, a(Z)$ being bqo. Hence $\mathrm{Im}\, d(Z)$ is bqo for any $Z \in [D]^{\omega}$.

By (7.8) there are $E \in [D]^{\omega}$ and a witnessing bad array e for d,

$$e: [E]^{\omega} \to Q \dot\cup \mathscr{F}_k(\mathscr{A}[Q]) \dot\cup \{*\}.$$

By (3.11ii) we may assume that

$$e: [E]^{\omega} \to Q,$$

in which case e is a witnessing array for a, or

$$e: [E]^{\omega} \to \mathscr{F}_k(\mathscr{A}[Q]),$$

in which case $e <' a$, or

$$e: [E]^{\omega} \to \{*\}$$

which is impossible since e is bad. Since all possibilities lead to a contradiction, Lemma (7.13) is proved. □

(7.14) **Definition.** Let $(T, \tau) \in \mathscr{S}_k(\mathscr{B})$ and let $\{T^m\}_{m\in\omega}$ be as in (7.10). We shall define k-structures $(T^m, {}^m\tau) \in \mathscr{M}_k(\mathscr{B}[\mathscr{F}_k(\mathscr{B}) \cup \{*\}])$ as in (7.10). For $t \in V(T^m)$ we put

$$ {}^m g_t = (D^m g_t, {}^m \hat{g}_t, {}^m \check{g}_t), $$

where $D^m g_t = B_t(T, T^m)$, and for $r \in D^m g_t$

$$ {}^m \hat{g}_t(r) = p_{(t,r)}, $$

$$ \begin{aligned} {}^m \check{g}_t(r) &= (T_r, \tau) \quad \text{if } (T_r, \tau) \in \mathscr{F}_k(\mathscr{B}), \\ &= * \qquad\quad \text{otherwise.} \end{aligned} $$

Now let

$$ {}^m\tau(t) = ((\gamma(t), {}^m g_t), p_t, (p_{(t,t')}: (t, t') \in E(T^m))) \qquad (t \in V(T^m)) $$

and put

$$ \Psi(T, \tau) = \{(T^m, {}^m\tau): m \in \omega\} \in \exp \mathscr{M}_k(\mathscr{B}[\mathscr{F}_k(\mathscr{B}) \cup \{*\}]). $$

(7.15) **Lemma.** *Let (T, τ), $(S, \sigma) \in \mathscr{S}_k(\mathscr{B}) - \mathscr{F}_k(\mathscr{B})$ and let $\Psi(T, \tau) \leq \Psi(S_s, \sigma)$ for any $s \in V(S)$ such that $(S_s, \sigma) \notin \mathscr{F}_k(\mathscr{B})$. Then $(T, \tau) \leq (S, \sigma)$.*

Proof. Let (T, τ) and $(T^m, {}^m\tau)$ be as in (7.14). Let

$$ \begin{aligned} &\sigma(s) = (\eta(s), q_s, (q_{(s,s')}: (s, s') \in E(S))), \\ &\Psi(S, \sigma) = \{(S^m, {}^m\sigma): m \in \omega\}, \\ &{}^m\sigma(s) = ((\eta(s), {}^m h_s), q_s, (q_{(s,s')}: (s, s') \in E(S^m))), \\ &{}^m h_s = (D^m h_s, {}^m\hat{h}_s, {}^m\check{h}_s). \end{aligned} $$

For the purpose of this proof we change the definition of "feasible" as follows. Let R be a subtree of T. We say that a morphism $\lambda \in \mathscr{S}_k(\mathscr{B})((R, \tau), (S, \sigma))$ of the form $\lambda(t, v) = \{(\varphi(t), u): u \in f_t(v)\}$ is *feasible* if for any $t \in V(R)$ there exists a 1-1 mapping $J_t: B_t(T, R) \to C_t(S, R, \lambda)$ such that for any $r \in B_t(T, R)$

(7.15a) $p_{(t,r)} \xrightarrow{f_t} q_{(\varphi(t), J_t(r))}$,
(7.15b) if $(T_r, \tau) \in \mathscr{F}_k(\mathscr{B})$ then $(T_r, \tau) \leq (S_{J_t(r)}, \sigma)$, and
(7.15c) if $(T_r, \tau) \notin \mathscr{F}_k(\mathscr{B})$ then $(S_{J_t(r)}, \sigma) \notin \mathscr{F}_k(\mathscr{B})$.

We will construct a morphism $\lambda \in \mathscr{F}_k(\mathscr{B})((T, \tau), (S, \sigma))$ by induction as in the proof of (7.11). The induction hypothesis at stage m is that there is a subtree R^m of T containing T^m and a feasible morphism

$$ \lambda^m \in \mathscr{F}_k(\mathscr{B})((R^m, \tau), (S, \sigma)). $$

For $m = 0$ there exists by assumption an n such that

$$ (T^0, {}^0\tau) \leq (S^n, {}^n\sigma); $$

let λ^0 be the corresponding morphism and put $R^0 := T^0$, $\lambda^0(t, v) = \{(\overline{\varphi}(t), u) : u \in \overline{f}_t(v)\}$. In particular $\lambda^0 \in \mathscr{S}_k(\mathscr{B})((R^0, \tau), (S, \sigma))$ and for every $t \in V(R^0)$ there exists a 1-1 mapping $\iota_t : D^0 g_t \to D^n h_{\overline{\varphi}(t)}$ such that for every $r \in D^0 g_t = B_t(T, R^0)$

(7.15d) $p_{(t,r)} \xrightarrow{\overline{f}_t} q_{(\overline{\varphi}(t), \iota_t(r))}$,
(7.15e) if $(T_r, \tau) \in \mathscr{F}_k(\mathscr{B})$ then $(T_r, \tau) \leq (S_{\iota_t(r)}, \sigma)$, and
(7.15f) if $(T_r, \tau) \notin \mathscr{F}_k(\mathscr{B})$ then $(S_{\iota_t(r)}, \sigma) \notin \mathscr{F}_k(\mathscr{B})$.

Thus letting $J_t^0(r) = \iota_t(r)$ we see that λ^0 is feasible.

Now assume that on the mth step we have constructed a subtree R^m of T containing T^m and a feasible morphism $\lambda^m \in \mathscr{F}_k(\mathscr{B})((R^m, \tau), (S, \sigma))$. Our aim is to extend λ^m to a feasible morphism

$$\lambda^{m+1} \in \mathscr{F}_k(\mathscr{B})((R^{m+1}, \tau), (S, \sigma)),$$

where R^{m+1} contains T^{m+1}.

Let $\lambda^m(t, v) = \{(\varphi^m(t), u) : u \in f_t^m(v)\}$. Since λ^m is feasible, there exists for any $t \in V(R^m)$ a 1-1 mapping $J_t^m : B_t(T, R^m) \to C_t(S, R^m, \lambda^m)$ satisfying (7.15a)–(7.15c). Let $t_0 \in V(R^m)$ and $r_0 \in B_{t_0}(T, R^m)$. We distinguish two cases.

Case I: $(T_{r_0}, \tau) \in \mathscr{F}_k(\mathscr{B})$. Then $(T_{r_0}, \tau) \leq (S_{J_{t_0}^m(r_0)}, \sigma)$; let λ^{r_0} be the corresponding morphism. We let $R(r_0) := V(T_{r_0})$.

Case II: $(T_{r_0}, \tau) \notin \mathscr{F}_k(\mathscr{B})$. Then $(S_{J_{t_0}^m(r_0)}, \sigma) \notin \mathscr{F}_k(\mathscr{B})$.

By assumption there exists $n \in \omega$ such that

$$(T^{m+1}, {}^{m+1}\tau) \leq (\widetilde{S}^n, {}^n\tilde{\sigma}),$$

where $(\widetilde{S}, \tilde{\sigma}) = (S_{J_{t_0}^m(r_0)}, \sigma)$; let $\lambda^{r_0}(t, v) = \{(\overline{\varphi}(t), u) : u \in \overline{f}_t(v)\}$ be the corresponding morphism. In particular $\lambda^{r_0} \in \mathscr{F}_k(\mathscr{B})((T^{m+1}, \tau), (S_{J_{t_0}^m(r_0)}, \sigma))$ and for every $t \in V(T^{m+1})$ there exists a 1-1 mapping $\iota_t : D^{m+1} g_t \to D^n h_{\overline{\varphi}(t)}$ such that for every $r \in D^{m+1} g_t$ conditions (7.15d)–(7.15f) are satisfied. We put $R(r_0) := V(T_{r_0}) \cap V(T^{m+1})$.

Now whichever case occurs, we let

$$R^{m+1} := T \restriction (V(R^m) \cup \bigcup \{R(r_0) : t_0 \in V(R^m),\ r_0 \in B_{t_0}(T, R^m)\},$$

$$\begin{aligned} \lambda^{m+1}(t, v) &= \lambda^m(t, v) \quad \text{for } t \in V(R^m),\ v \in V_{\gamma(t)}, \\ &= \lambda^{r_0}(t, v) \quad \text{for } t \in R(r_0),\ v \in V_{\gamma(t)}, \\ J_t^{m+1}(r) &= J_t^m(r) \quad \text{for } t \in V(R^m),\ r \in B_t(T, R^{m+1}), \\ &= \iota_t(r) \quad \text{for } t \in V(R^{m+1}) - V(R^m),\ r \in B_t(T, R^{m+1}). \end{aligned}$$

It follows from Lemma (7.5) (applied to $T = R^{m+1}$, $T^0 = R^m$, $\lambda = \lambda^m$) and condition (7.15a) that $\lambda^{m+1} \in \mathscr{F}_k(\mathscr{B})((R^{m+1}, \tau), (S, \sigma))$. To show that

λ^{m+1} is feasible we must verify that J_t^{m+1} satisfies (7.15a)–(7.15c). But that follows from (7.15d)–(7.15f) and the fact that J_t^m satisfies (7.15a)–(7.15c). This completes the construction.

Now for $t \in V(T)$ and $v \in V_{\gamma(t)}$ let m be such that $t \in V(R^m)$; we put $\lambda(t, v) = \lambda^m(t, v)$. Clearly $\lambda \in \mathscr{F}_k(\mathscr{B})((T, \tau), (S, \sigma))$. □

(7.16) **Lemma.** *If* $(T, \tau) \in \mathscr{S}_k(\mathscr{A}[Q])$ *and* $\mathrm{Im}(T, \tau)$ *is bqo, then* $(T, \tau) \in \mathscr{F}_k(\mathscr{A}[Q])$.

Proof. Let $(T, \tau) \in \mathscr{S}_k(\mathscr{A}[Q]) - \mathscr{F}_k(\mathscr{A}[Q])$ and let $Q' = \mathrm{Im}(T, \tau)$ be bqo. We may assume that for $t \in V(T)$ the trees $(T_t)^m$ are chosen in such a way that $(T_t)^m = T_t \cap T^m$. Then for $t, r \in V(T)$ such that $r \in V(T_t)$ we have $\Psi(T_r, \tau) \le \Psi(T_t, \tau)$, namely via the identity embedding. Since

$$\{\Psi(T_t, \tau) : t \in V(T)\} \subseteq \exp \mathscr{M}_k(\mathscr{A}[Q'][\mathscr{F}_k(\mathscr{A}[Q']) \cup \{*\}]),$$

it follows from (4.10), (7.8), (7.13), and results of §3 that this set is bqo, in particular well-founded. Thus there exists $t \in V(T)$ such that $(T_t, \tau) \notin \mathscr{F}_k(\mathscr{A}[Q])$, but for any $r \in V(T_t) - \{t\}$ such that $(T_r, \tau) \notin \mathscr{F}_k(\mathscr{A}[Q])$ it holds $\Psi(T_r, \tau) \not< \Psi(T_t, \tau)$ (and hence $\Psi(T_r, \tau) \equiv \psi(T_t, \tau)$). Fix such a t. Since $(T_t, \tau) \notin \mathscr{F}_k(\mathscr{A}[Q])$, there exists $s \in V(T_t) - \{t\}$ such that $(T_s, \tau) < (T_t, \tau)$ and $(T_s, \tau) \notin \mathscr{F}_k(\mathscr{A}[Q])$. But (7.15) implies $(T_t, \tau) \le (T_s, \tau)$, which is a contradiction. □

Now we are ready to prove (7.1). Let

$$a: [A]^\omega \to \mathscr{S}_k(\mathscr{A})[Q]$$

be a bad array such that $\mathrm{Im}\, a(Z)$ is bqo for any $Z \in [A]^\omega$. If we consider each $a(Z)$ as an element of $\mathscr{S}_k(\mathscr{A}[Q])$, it follows by (7.16) that a is a bad array

$$a: [A]^\omega \to \mathscr{F}_k(\mathscr{A}[Q]).$$

Now (7.13) gives the desired witnessing array.

8. Proof of the Main Theorem

(8.1) **Lemma.** *If* $\mathscr{A}$ *is a well-behaved QO-category and* $0 \le i \le w$, *then the QO-category* $\mathscr{S}_{\ge i}^{\le w}(\mathscr{A})$ *is well-behaved.*

Proof. We proceed by induction on $w - i$. For $i = w$ it follows from (7.1), so let the lemma hold true for $i + 1$ and let

$$a: [A]^\omega \to \mathscr{S}_{\ge i}^{\le w}(\mathscr{A})[Q]$$

be a bad array such that $\mathrm{Im}\, a(Z)$ is bqo for any $Z \in [A]^\omega$. Since according to (4.10)

$$\mathscr{S}_{\ge i}^{\le w}(\mathscr{A})[Q] = \mathscr{S}_{\ge i}^{\le w}(\mathscr{A}[Q])$$

and

$$\mathscr{S}_i(\mathscr{S}_{\ge i}^{\le w}(\mathscr{A}[Q])) = \mathscr{S}_i(\mathscr{S}_{\ge i}^{\le w}(\mathscr{A}))[Q]$$

we may define an array

$$b\colon [A]^{\omega} \to \mathcal{S}_i(\mathcal{S}^{\leq w}_{\geq i+1}(\mathcal{A})[Q]$$

by the rule $b(Z) = \Xi_i(a(Z))$. By the Second Encoding Lemma (6.2) b is bad. Hence by the induction hypothesis and Theorem (7.1) there exist $C \in [A]^{\omega}$ and the desired witnessing array $c\colon [C]^{\omega} \to Q$. □

(8.2) **Theorem.** *If $\mathcal{A}$ is well-behaved, then $\mathcal{S}^{\leq w}_{\geq 0}(\mathcal{A})$ is well-behaved.*

(8.3) *Proof of* (4.13). Let $\mathcal{A}$ be a well-behaved graph category and Q a quasi-ordered set. Let

$$a\colon [A]^{\omega} \to \mathcal{A}^k[Q]$$

be a bad array such that $\operatorname{Im} a(Z)$ is bqo for every $Z \in [A]^{\omega}$. For $Z \in [A]^{\omega}$ let $b(Z)$ be an encoding of $a(Z)$. By the First Encoding Lemma (5.3)

$$B\colon [A]^{\omega} \to \mathcal{S}^{\leq w}_{\geq 0}(\mathcal{A})[Q]$$

is a bad array and hence by Theorem (8.2) there exist $C \in [A]^{\omega}$ and a witnessing array $c\colon [C]^{\omega} \to Q$, which is the desired witnessing array for a. This completes the proof of (4.13).

9. Friedman's gap condition

(9.1) **Definition.** Let n be a natural number and let $\mathcal{S}_n$ be the QO-category whose objects are pairs $\gamma = (T, l)$, where T is a tree and $l\colon V(T) \to \{1, \dots, n\}$, $V_\gamma = V(T)$, and P_γ consists of the empty sequence and all one-element sequences of elements of V_γ, and with morphisms defined by saying that $\varphi \in \mathcal{S}_n((T_1, l_1), (T_2, l_2))$ if

(9.1a) $\varphi\colon V(T_1) \to V(T_2)$ is a homeomorphic embedding, and

(9.1b) $l_1(t) = l_2(\varphi(t)) = l_2(\varphi_*(t)) \leq l_2(s)$ for every $s \in [\varphi_*(t), \varphi(t)]$.

Condition (9.1b) is called the *gap condition.* Let $\mathcal{S}^{\text{fin}}_n$ be the subcategory of all γ such that V_γ is finite. Friedman has shown that Q wqo implies $\mathcal{S}^{\text{fin}}_n[Q]$ wqo (cf. [22]). From our results it follows that $\mathcal{S}_n$ is well-behaved.

(9.2) **Theorem.** *The QO-category $\mathcal{S}_n$ is well-behaved.*

Proof. Let $\mathcal{A}$ be the QO-category whose only object is γ and $V_\gamma = \{1, \dots, n\}$, $P_\gamma = \{p_0, p_1, \dots, p_n\}$, where $p_i = (1, \dots, i)$. To each $((T, l), g) \in \mathcal{S}_n[Q]$ we assign a tree structure $((T, \tau), g') \in \mathcal{S}^{\leq n}_{\geq 0}(\mathcal{A})[Q \cup \{*\}]$ as follows:

$$\tau(t) = (\gamma, p, (p_{l(t')}\colon (t, t') \in E(T))), \quad g' = (Dg', \hat{g}', \check{g}'),$$

where

$$\begin{aligned} p &= p_{l(t)}, \quad t \neq \operatorname{root}(T), \\ &= \text{empty sequence}, \quad \text{otherwise}, \end{aligned}$$

$Dg' = Dg \cup \{*\}$ and for $x \in Dg'$

$$\begin{aligned} \hat{g}'(x) &= ((t, 1)) && \text{if } x \in Dg \text{ and } \hat{g}(x) = (t), \\ &= \text{empty sequence} && \text{if } x \in Dg \text{ and } \hat{g}(x) \text{ is the empty sequence}, \\ &= ((\text{root}(T), l(\text{root}(T)))) && \text{if } x = *, \end{aligned}$$

$$\begin{aligned} \check{g}'(x) &= \check{g}(x) && \text{if } x \in Dg, \\ &= * && \text{if } x = *. \end{aligned}$$

We claim that if $((T, \tau), g')$ and $((S, \sigma), h')$ are assigned to $((T, l), g)$ and $((S, k), h)$, respectively, then $((T, \tau), g') \le ((S, \sigma), h')$ implies $((T, l), g) \le ((S, k), h)$. Indeed, let λ be a morphism corresponding to the first relation. Then $\lambda(t, v) = \{(\varphi(t), v)\}$, where $\varphi\colon V(T) \to V(S)$ is a homeomorphic embedding, and there exists $\imath\colon Dg' \to Dh'$ such that (4.4a) and (4.4b) are satisfied. Clearly $\imath(*) = *$, hence $\varphi(\text{root}(T)) = \text{root}(S)$ and $l(\text{root}(T)) = k(\text{root}(S))$. For $t \ne \text{root}(T)$ we have

$$\begin{aligned} l(t) = N_{(T, \tau)}(t) &= N_{(S, \sigma)}(\varphi(t)) = k(\varphi(t)) \\ &= N_{(S, \sigma)}(\varphi_*(t)) = k(\varphi_*(t)) \\ &\le N_{(S, \sigma)}(s) = k(s) \end{aligned}$$

for any $s \in [\varphi_*(t), \varphi(t)]_S$ by (4.8c). Thus φ corresponds to the second relation and the claim is proved. The theorem now follows from (8.2). □

10. Conjectures

(10.1) For a QO-category $\mathscr{A}$ consider the following statements:

(i) Q bqo implies $\mathscr{A}[Q]$ bqo,
(ii) $\mathscr{A}$ is well-behaved,
(iii) every bad array $a\colon [A]^{\omega} \to \mathscr{A}[Q]$ admits a witnessing array.

Clearly (iii) ⇒ (ii) ⇒ (i). It seems to be an interesting technical question whether the converse implications hold or not.

(10.2) It is asked in [1] whether the QO-category of infinite trees (with homeomorphic embeddings as morphisms) satisfies (10.1iii). It follows from (7.1) that it is well-behaved, but the method used probably cannot give more.

Conjecture. *If a QO-category $\mathscr{A}$ satisfies* (10.1iii) *and $k \ge 0$, then the QO-category $\mathscr{S}_{\ge 0}^{\le k}(\mathscr{A})$ satisfies* (10.1iii).

(10.3) **Conjecture.** *The class of countable graphs is wqo by minors.*

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