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THE SPECTRAL MEASURE AND HILBERT TRANSFORM OF A MEASURE-PRESERVING TRANSFORMATION

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ABSTRACT. V. F. Gaposhkin gave a condition on the spectral measure of a normal contraction on L^2 sufficient to imply that the operator satisfies the pointwise ergodic theorem. We prove that unitary operators which come from measure-preserving transformations satisfy a stronger version of this condition. This follows from the fact that the rotated ergodic Hilbert transform is a continuous function of its parameter. The maximal inequality on which the proof depends follows from an analytic inequality related to the Carleson-Hunt Theorem on the a.e. convergence of Fourier series.

There is a large body of work on the question of when a given operator on an L^p space satisfies the pointwise ergodic theorem, that is, when the Cesàro means of powers of the operator applied to an element of L^p converge a.e. (See the book by Krengel [1985] and the article by Duncan [1977] for surveys.) Gaposhkin [1981] gave a necessary and sufficient condition involving the spectral measure of the operator for the case when $p = 2$ and the operator in question is a normal contraction. In this paper we develop a connection (Proposition 1) between this condition and the ergodic Hilbert transform. This connection allows us to prove directly that operators induced by measure-preserving transformations satisfy a strengthened version of Gaposhkin's condition (Theorem 1). The fundamental result is a form of continuity of the rotated ergodic Hilbert transform (Theorem 2), the proof of which depends on a new kind of maximal inequality involving a supremum over a parameter (Lemma 1). We reduce the proof of this inequality to an analytic maximal inequality (Lemma 2), which is proved from the Carleson-Hunt estimate on maxima of partial sums of Fourier series.

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Let $(X, \mathscr{B}, \mu)$ denote a measure space, and let T be a normal contraction on $L^2(X, \mathscr{B}, \mu)$. Let E_T denote the spectral measure for T, supported on the closed unit disc in the complex plane, and for each $n = 1, 2, \ldots$ let

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$V_n = \{z \in \mathbf{C}: 0 < |1 - z| < 2^{-n}\}$. In 1981 V. F. Gaposhkin proved that for a given $f \in L^2$ the pointwise ergodic theorem holds for T and f, i.e.,

$$\text{(1)} \qquad \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} T^k f(x) \text{ exists a.e.}$$

if and only if

$$\text{(2)} \qquad \lim_{n\to\infty} [E_T(V_n)f](x) = 0 \quad \text{a.e.}$$

Moreover, he showed that the sequence 2^{-n} used in the definition of V_n may be replaced by any sequence q_n converging monotonically to 0 and satisfying $Q \geq q_n/q_{n+1} \geq q > 1$ for every n.

Gaposhkin observed that since the pointwise ergodic theorem is known to hold for all $f \in L^2$, if T is an invertible measure-preserving (m.p.t.) on X, it would be interesting to verify (2) directly for operators induced by such transformations. (If the measure-preserving transformation is not invertible, then the operator that is induces is not normal.) Our main result shows that for m.p.t.'s a strengthened version of (2) actually holds:

Theorem 1. *Let T be an invertible m.p.t. on a measure space $(X, \mathscr{B}, \mu)$ with associated spectral representation*

$$\text{(3)} \qquad T = \int_{-\pi}^{\pi} e^{i\lambda}\, dE(\lambda)$$

as an operator on L^2. If $\{\varepsilon_k\}$ is any nonnegative sequence which tends to 0 as $k \to \infty$, then

$$\text{(4)} \qquad \lim_{k\to\infty} [E(-\varepsilon_k, 0)f](x) = 0 \quad \textit{a.e. for all } f \in L^2.$$

We prove this theorem by formulating a condition equivalent to (4) and then verifying this new condition. The condition we will formulate depends upon the ergodic Hilbert transform (e.H.t.). Recall that for $f \in L^2$ the e.H.t. of f (induced by T) is the a.e. limit

$$\text{(5)} \qquad \lim_{n\to\infty} \frac{1}{\pi} {\sum_{k=-n}^{n}}' \frac{f(T^k x)}{k} = \text{P.V.} \frac{1}{\pi} {\sum_{k=-\infty}^{\infty}}' \frac{f(T^k x)}{k} = H_T f(x),$$

where $'$ denotes omission of the term for which $k = 0$. Usually T is understood and we just write Hf. The map which sends f to Hf is a bounded operator on L^2 (see Cotlar [1955]). If T has the representation (3), then H may be represented via the spectral integral

$$\text{(6)} \qquad H = i \int_{-\pi}^{\pi} \eta(\lambda)\, dE(\lambda),$$

where $\eta(\lambda)$ is the odd function on $[-\pi, \pi]$ whose value for $\lambda \in (0, \pi]$ is $(\pi - \lambda)/\pi$ and $\eta(0) = 0$. (See Campbell [1986].)

By considering a simple product transformation it is easy to see that for fixed ε in $[-\pi, \pi]$, the rotated e.H.t., $H_\varepsilon f$, defined by

$$H_\varepsilon f(x) = \text{P.V.}\ \frac{1}{\pi} \sum_{k=-\infty}^{\infty}{}' \frac{e^{ik\varepsilon} f(T^k x)}{k} \tag{7}$$

exists a.e. for each $f \in L^2$, and the map sending f to $H_\varepsilon f$ is a bounded operator on L^2. We claim that (4) above is equivalent to a form of continuity of the rotated Hilbert transform at $\varepsilon = 0$:

Proposition 1. *For $f \in L^2$, T, and E as above,*

$$\lim_{k\to\infty} [E(-\varepsilon_k, 0)f](x) = 0 \quad a.e.$$

if and only if

$$\lim_{k\to\infty} H_{\varepsilon_k} f(x) = Hf(x) + i[E\{0\}f](x) \quad a.e. \tag{8}$$

Proof. We apply the functional calculus. If $\eta(\lambda)$ is the "representing function" for H as in (6), then $\eta(\lambda + \varepsilon_k)$ is the representing function for H_{ε_k}, so that $H_{\varepsilon_k} - H$ has the representation

$$H_{\varepsilon_k} - H = i \int_{-\pi}^{\pi} [\eta(\lambda + \varepsilon_k) - \eta(\lambda)]\, dE(\lambda). \tag{9}$$

The difference $\eta(\lambda + \varepsilon_k) - \eta(\lambda)$ may be written as

$$(-\varepsilon_k/\pi)1 + \chi_{\{-\varepsilon_k\}}(\lambda) + 2\chi_{(-\varepsilon_k, 0)}(\lambda) + \chi_{\{0\}}(\lambda), \tag{10}$$

where 1 is the function which is identically 1 on $[-\pi, \pi]$ and χ_A is the characteristic function of set A. Since $\int_{-\pi}^{\pi} 1\, dE(\lambda) = I$, the identity operator on L^2, we may conclude that (8) holds if and only if

$$\lim_{k\to\infty} \left(\frac{-\varepsilon_k}{\pi}\right) f(x) + E\{-\varepsilon_k\} f(x) + 2E(-\varepsilon_k, 0) f(x) = 0 \quad \text{a.e.} \tag{11}$$

Clearly, $\lim_{k\to\infty} -\varepsilon_k f(x)/\pi = 0$ a.e. for each $f \in L^2$. Also,

$$\lim_{k\to\infty} E(\{-\varepsilon_k\}) f(x) = 0 \quad \text{a.e.},$$

for the following reason. If $e^{-i\varepsilon_k}$ is an eigenvalue for T, then $E\{-\varepsilon_k\}$ is the projection onto the corresponding eigenspace; otherwise it is the 0 projection. No matter what order the eigenvalues are given, as $k \to \infty$ these projections applied to f give the tail blocks of the eigenfunction expansion for the projection of f into the Kronecker factor. Since these blocks are L^2 summable, the tail blocks tend to 0 a.e. Thus (8) holds if and only if the third piece of the sum, $2E((-\varepsilon_k, 0))f(x)$, tends to 0 a.e., as k tends to ∞.

To prove Theorem 1 it is sufficient to prove that (8) holds:

Theorem 2. *If T is an invertible m.p.t. on a measure space $(X, \mathscr{B}, \mu)$, and $f \in L^2(X, \mathscr{B}, \mu)$, then for any nonnegative sequence $\{\varepsilon_k\}$ tending to 0 as $k \to \infty$ we have*

$$\lim_{k\to\infty} H_{\varepsilon_k} f(x) = Hf(x) + iE\{0\}f(x) \quad a.e.$$

Proof. As usual, we prove a.e. convergence for a dense subset of L^2, and then establish an appropriate maximal inequality (Lemma 1). The functional calculus may be used to provide the dense subset. If f is in the range of $E\{0\}$, so that $f(x) = f(Tx) = E\{0\}f(x)$ a.e. and $Hf(x) = 0$ a.e., then $H_{\varepsilon_k} f(x) = i\eta(\varepsilon_k)f(x)$ a.e., and $\lim_{k\to\infty} i\eta(\varepsilon_k)f(x) = if(x)$ a.e. Suppose now that $\delta > 0$ is fixed and f is in the range of $E([-\pi, -\delta) \cup (\delta, \pi])$. Then $E\{0\}f(x) = 0$ a.e., and if $0 < \varepsilon_k < \delta$ we have (see (11)) $(H_{\varepsilon_k} - H)f(x) = i\varepsilon_k f(x)/\pi$ a.e., which clearly tends to 0 a.e. as $k \to \infty$. The union over all positive δ of such functions, along with those in the range of $E\{0\}$, is dense in L^2.

To complete the proof of Theorem 2 we prove the following maximal inequality which involves a double supremum:

Lemma 1. *For each $f \in L^2$, $\varepsilon > 0$ and $n \in \mathbf{N}$, define*

$$H_{n,\varepsilon} f(x) = \frac{1}{\pi} \sum_{k=-n}^{n}{}' \frac{e^{ik\varepsilon} f(T^k x)}{k}, \tag{12}$$

and

$$H^* f(x) = \sup_{n,\varepsilon} |H_{n,\varepsilon} f(x)|. \tag{13}$$

Then there exists a constant $C > 0$ such that

$$\mu\{x : H^* f(x) > \lambda\} \le \frac{C}{\lambda^2}\|f\|_2^2 \quad \text{for all } \lambda > 0 \text{ and } f \in L^2. \tag{14}$$

To prove Lemma 1 we need first to prove Lemma 2, an interesting maximal inequality from harmonic analysis. Then we use Lemma 2 to prove Lemma 3, a sequence version of Lemma 1, which finally transfers to the ergodic setting.

Lemma 2. *For $h \in L^2[-\pi, \pi]$ (with respect to Lebesgue measure) define a sequence of nonnegative numbers by*

$$I^* h(j) = \sup_{\varepsilon>0} \left| \int_{-\varepsilon}^{\varepsilon} h(t) e^{ijt}\, dt \right|, \qquad j \in \mathbf{Z}. \tag{15}$$

Then there is a constant $C > 0$ such that

$$\|I^* h(j)\|_{l^2(\mathbf{Z})} \le C \|h(t)\|_{L^2[-\pi,\pi]} \quad \text{for all } h \in L^2[-\pi, \pi]. \tag{16}$$

Remark. The Carleson-Hunt estimate for the maxima of partial sums of Fourier series of L^2 functions says that if

$$S^* \hat{f}(x) = \sup_{n>0} \left| \sum_{k=-n}^{n} \hat{f}(k) e^{ikx} \right|,$$

then

$$\|S^* \widehat{f}\|_{L^2[-\pi,\pi]} \leq C\|\widehat{f}\|_{l^2(\mathbf{Z})}.$$

Thus Lemma 2 may be regarded as the Fourier transform of the Carleson-Hunt Theorem. Kenig and Tomas [1980] derived analogues of the Carleson-Hunt result for other dual pairings; the following argument uses their transference of the Carleson-Hunt Theorem to the $(\mathbf{R}, \mathbf{R})$ pairing to prove the version stated in Lemma 2.

Proof of Lemma 2. Fix h and for each complex number s and $\varepsilon > 0$ define

$$G_\varepsilon(s) = \left| \int_{-\infty}^{\infty} \chi(t)_{(-\varepsilon,\varepsilon)} h(t) e^{ist}\, dt \right|, \tag{17}$$

and

$$G(s) = \sup_{\varepsilon>0} G_\varepsilon(s). \tag{18}$$

Easy calculations show that

(i) The family $\{G_\varepsilon : 0 \leq \varepsilon \leq \pi\}$ is equicontinuous on the complex plane.

(ii) For each ε, $G_\varepsilon(s)$ is a subharmonic function of s.

These statements together are sufficient to imply that G is subharmonic in the plane (see Hörmander [1973, p. 16]). We want to bound the sum of the squares of the values of G at the integers by a constant times the L^2 norm of h. If D_j is the disc of radius $1/2$ centered at j, then by the mean-value property of subharmonic functions and Hölder's inequality we have

$$G(j)^2 \leq C \left(\iint_{D_j} G(x+iy)\, dx\, dy \right)^2 \leq C \iint_{D_j} G(x+iy)^2\, dx\, dy. \tag{19}$$

Note that C is independent of j and h. The disjoint union of the D_j's is contained in the strip $-1/2 \leq y \leq 1/2$, and we can estimate the L^2 norm of G over this strip as follows. For each $y \in [-1/2, 1/2]$, let $G^y(x) = G(x+iy)$, so that

$$G^y(x) = \sup_{\varepsilon>0} \left| \int_{-\varepsilon}^{\varepsilon} h(t) e^{-yt} e^{ixt}\, dt \right|. \tag{20}$$

For each such y, $F(t) = e^{-yt} h(t)$ is in $L^2(\mathbf{R})$, and hence we may apply the result of Kenig-Tomas [1980] to transfer the Carleson-Hunt maximal estimate to this setting:

$$\|G^y(x)\|^2_{L^2(\mathbf{R})} \leq C\|e^{-yt} h(t)\|^2_{L^2[-\pi,\pi]}. \tag{21}$$

Since the y's are being chosen from a compact set, there is a constant C such that $\|e^{-yt} h(t)\|^2 \leq C\|h(t)\|^2$. Applying Fubini's Theorem, we have

$$\int_{y=1/2}^{1/2} \int_{x=-\infty}^{\infty} G(x+iy)^2\, dx\, dy \leq \int_{y=-1/2}^{1/2} C\|h\|^2_{L^2[-\pi,\pi]}\, dy \leq C\|h\|^2_{L^2[-\pi,\pi]}. \tag{22}$$

Remark. It follows from Lemma 2 that if $h(t) = \sum_{k=-\infty}^{\infty} a_k e^{ikt}$ and

$$R^* h(j) = \sup_{\varepsilon>0} \left| \sum_{k=-\infty}^{\infty}{}' \frac{e^{i(k+j)\varepsilon} a_{k+j}}{k} \right|,$$

then $\|R^*(h)\|_{l^2(\mathbf{Z})} \leq C\|h\|_{L^2[-\pi,\pi]}$ (because R^* differs from I^* by an operator which is clearly strong (2, 2)). This is not a robust enough result to survive transference to the ergodic context. We need instead to provide an estimate for

$$a_*^*(j) = \sup_{n\geq 1} \sup_{\varepsilon>0} \left| \sum_{k=-n}^{n}{}' \frac{e^{i(k+j)\varepsilon} a_{k+j}}{k} \right|;$$

this is the content of Lemma 3, which follows from Lemma 2 by means of the covering lemma and disjointification techniques also used in Petersen [1983]. We conjecture that even strong (2, 2) holds: $\|a_*^*\|_{l^2(\mathbf{Z})} \leq C\|a\|_{l^2(\mathbf{Z})}$.

Lemma 3. *There is a constant* $C > 0$ *such that if* $\{a_k\} \in l^2(\mathbf{Z})$ *and* $\lambda > 0$ *then*

$$\operatorname{card}\{j : a_*^*(j) > \lambda\} \leq \frac{C}{\lambda^2} \sum_{k=-\infty}^{\infty} |a_k|^2. \tag{23}$$

Proof. Let A be a bounded subset of

$$\left\{ j : \sup_{n\geq 1} \sup_{\varepsilon>0} \left| \sum_{k=-n}^{n}{}' \frac{e^{i(k+j)\varepsilon} a_{k+j}}{k} \right| > \lambda \right\},$$

so that $A \subset [-N, N]$, say. For each $j \in A$ there is a block in $\mathbf{Z}$ of the form $[j - n_j, j + n_j]$ and an $\varepsilon_j > 0$ such that

$$\left| \sum_{k\in I_j}{}' \frac{e^{ik\varepsilon_j} a_k}{k - j} \right| > \lambda. \tag{24}$$

Since

$$\sum_{k\in I_j}{}' = \sum_{k=-\infty}^{\infty}{}' - \sum_{k\notin I_j},$$

we may apply the triangle inequality to see that

$$\begin{aligned} A \subset & \left\{ j : \sup_{\varepsilon>0} \left| \sum_{k=-\infty}^{\infty}{}' \frac{e^{ik\varepsilon} a_k}{k - j} \right| > \frac{\lambda}{2} \right\} \\ & \cup \left\{ j \in [-N, N] : \left| \sum_{k\notin I_j} \frac{e^{ik\varepsilon_j} a_k}{k - j} \right| > \frac{\lambda}{2} \right\} = A_1 \cup A_2. \end{aligned} \tag{25}$$

If $j \in A_1$, then by Lemma 2 (since strong (2, 2) implies weak (2, 2)), j falls into a single (independent of j) set of cardinality no more than $(4C/\lambda^2)\|a\|_2^2$. To count the j's in A_2, we first make each of the numerators real and nonnegative by taking the positive and negative parts of the real and imaginary parts. Thus we define $r_j^+(k) = [\mathrm{Re}(e^{ik\varepsilon_j}a_k)]^+$, and similarly define r_j^-, i_j^+, and i_j^-. We will count

$$\left\{ j \in A_2 \colon \sum_{k \notin I_j} \frac{r_j^+(k)}{k-j} < -\frac{\lambda}{8} \right\} = r^+(A_2). \tag{26}$$

A similar argument will count how often the same sum is larger than $\lambda/8$, and also how often the same sum with r_j^+ replaced by r_j^-, i_j^+, and i_j^- is greater than $\lambda/8$ and less than $-\lambda/8$, giving the same estimate each time.

Replace the family $\{I_j\}$ by a disjoint subfamily $\{\widetilde{I}_j\}$ which still covers at least $1/3$ of A_2. Index the centers of the new intervals by $\widetilde{J}$. If for $t \in \mathbf{R}$ we let

$$h_j(t) = \sum_{k \notin \widetilde{I}_j} \frac{r_j^+(k)}{k-t}, \tag{27}$$

then $h_j'(t) > 0$. Since $h_j(j) < -\lambda/8$, so is $h_j(n) < -\lambda/8$ for $n \in [j-n_j, j]$. Hence we find

$$\begin{aligned}
\operatorname{card} r^+(A_2) &\le 6 \sum_{j \in \widetilde{J}} (n_j + 1) \le 6 \operatorname{card} \bigcup_{j \in \widetilde{J}} \left\{ n \colon \sum_{k \notin \widetilde{I}_j} \frac{r_j^+(k)}{k-n} < -\frac{\lambda}{8} \right\} \\
&\le 6 \operatorname{card} \left[\bigcup_{j \in \widetilde{J}} \left(\left\{ n \colon \sum_{k=-\infty}^{\infty}{}' \frac{r_j^+(k)}{k-n} < -\frac{\lambda}{16} \right\} \cup \left\{ n \colon \sum_{k \in \widetilde{I}_j} \frac{r_j^+(k)}{k-n} > \frac{\lambda}{16} \right\} \right) \right] \\
&\le 6 \operatorname{card} \left[\bigcup_{j \in \widetilde{J}} \left\{ n \colon \sum_{k=-\infty}^{\infty}{}' \frac{r_j^+(k)}{k-n} < -\frac{\lambda}{16} \right\} \right] + 6 \sum_{j \in \widetilde{J}} \left(\frac{(16)^2 C}{\lambda^2} \sum_{k \in \widetilde{I}_j} |a_k|^2 \right),
\end{aligned}$$

by Lemma 2. Continuing, because the $\widetilde{I}_j$ are disjoint all of this is

$$\begin{aligned}
&\le 6 \operatorname{card} \left\{ n \colon \sup_{\varepsilon > 0} \left| \sum_{k=-\infty}^{\infty}{}' \frac{e^{ik\varepsilon} a_k}{k-n} \right| > \frac{\lambda}{16} \right\} + \frac{6(16)^2 C}{\lambda^2} \sum_{k=-\infty}^{\infty} |a_k|^2 \\
&\le \frac{6(16)^2 C}{\lambda^2} \sum_{k=-\infty}^{\infty} |a_k|^2 + \frac{6(16)^2 C}{\lambda^2} \sum_{k=-\infty}^{\infty} |a_k|^2,
\end{aligned}$$

where we have again applied Lemma 2, as in the remark above. The estimate on $r^+(A_2)$ is now complete with constant equal to $(12)(16)^2 C$, where C is the constant from Lemma 2. We may similarly estimate $r^-(A_2)$, $i^+(A_2)$, and $i^-(A_2)$, where the definitions of these sets are obvious. Combining these estimates on A_2 with the previously obtained estimate on A_1, we obtain the estimate in the conclusion of Lemma 3, where the constant for the estimate in line (22) may be taken as $(2)(4)(12)(16)^2 C$.

Proof of Lemma 1. For each fixed $N \geq 1$ define

$$A_N = \left\{ x: \sup_{\varepsilon > 0} \sup_{1 \leq n \leq N} \left| \frac{1}{\pi} \sum_{k=-n}^{n}{}' \frac{e^{ik\varepsilon} f(T^k x)}{k} \right| > \lambda \right\}, \tag{28}$$

and for this N and $K \in \mathbf{N}$ define

$$\overline{A}_{N,K} = \{(x, j): -K \leq j \leq K \text{ and } T^j x \in A_N\}. \tag{29}$$

An argument analogous to that given in Petersen [1983] to prove Lemma 1 of that paper shows that

$$\mu(A_N) = \frac{1}{2K+1} \mu x \operatorname{card}(\overline{A}_{N,K}) \leq \left(\frac{C}{\lambda^2}\right) \frac{[2(K+N)+1]}{2K+1} \|f\|_2^2,$$

where card denotes counting measure on $\mathbf{Z}$. Lemma 1 follows by letting K tend to infinity.

To complete the proof of Theorem 2, fix any nonnegative sequence $\{\varepsilon_k\}$ tending to 0. We have shown that the a.e. convergence claimed in the conclusion of Theorem 2 holds for a dense set of functions in L^2. The maximal inequality provided by Lemma 1 implies, by Banach's Principle, that the set of functions in L^2 for which this same a.e. convergence holds must be closed. Hence it must be all of L^2.

Remark. Probably a Wiener-Wintner type theorem holds in this situation; that is, for each $f \in L^2$ there probably exists a single set of measure 0 outside of which $H_\varepsilon f(x)$ exists for all ε. Then these arguments would extend to show that in fact

$$\lim_{\varepsilon \to 0^+} H_\varepsilon f(x) = Hf(x) + iE\{0\} f(x) \quad \text{a.e. for each } f \in L^2.$$

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