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INVARIANT MEASURES AND EQUILIBRIUM STATES FOR PIECEWISE $C^{1+\alpha}$ ENDOMORPHISMS OF THE UNIT INTERVAL

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ABSTRACT. A differentiable function is said to be $C^{1+\alpha}$ if its derivative is a Hölder continuous function with exponent $\alpha > 0$. We show that three well-known results about invariant measures for piecewise monotonic and C^2 endomorphisms of the unit interval are in fact true for piecewise monotonic and $C^{1+\alpha}$ maps. We show the existence of unique, ergodic measures equivalent to Lebesgue measure for $C^{1+\alpha}$ Markov maps, extending a result of Bowen and Series for the C^2 case. We present a generalization of Adler's Folklore Theorem for maps which satisfy a restricted mixing condition, and we show that these $C^{1+\alpha}$ mixing endomorphisms possess unique equilibrium states, a result which was shown for the C^2 case by P. Walters.

0. INTRODUCTION

Let $I = [0,1]$ with the usual metric and denote by $(I, \mathscr{B}, \lambda)$ the measure space consisting of Lebesgue measure on the Borel subsets $\mathscr{B}$ of I. A mapping $g: I \to I$ is said to be *piecewise monotonic and* C^1 *with finite image* if

(1) There is a countable partition $\mathscr{J}$ of I into subintervals; $\mathscr{J} = \{J_n\}_{n\in N}$, $J_n = [a_n, b_n]$ with $\bigcup_{n\in N} J_n = I$ and $\lambda(J_n \cap J_m) = 0$ if $n \neq m$, so that:

(2) For every $n \in N$, g is monotonic and continuously differentiable on (a_n, b_n) and extends to a continuously differentiable function on an open interval $\tilde{J}_n \supseteq J_n$.

(3) $\Omega = \bigcup_{n\in N}\{\lim_{x\to a_n^+} g(x),\ \lim_{x\to b_n^-} g(x)\}$ is a finite set contained in $\bigcup_{n\in N}\{a_k, b_k\}$. (In particular, each interval gJ_n is λ-a.e. a union of atoms from $\mathscr{J}$.)

The mapping g is said to be *Markov* if it satisfies the above, plus

(4) $\bigcup_{n\geq 0} g^n J_k \supseteq J_l$ for all $k, l \in N$.

Let $\Omega \cup \{0,1\}$ be ordered by the usual order on I, that is, $\omega_0 = 0$ and ω_{i+1} lies immediately to the right of ω_i. Denote by $g\mathscr{J}$ the (finite) partition of I into intervals induced by the points $\Omega \cup \{0,1\}$. Thus atoms in gJ have the form $[\omega_i, \omega_{i+1}]$. Without loss of generality we may assume g maps each J_k onto (λ-a.e.) exactly one $[\omega_i, \omega_{i+1}]$. Let $I_0 = I - (\Omega \cup \{0,1\})$, a finite union of open intervals and let $I_{00} = I - \bigcup_{k\in N}\{a_k, b_k\}$ the union of the interiors of the

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J_n. Clearly $I_{00} \subseteq I_0 \subseteq I$. Write $M(I_0)$ for the collection of all Borel probability measures on I_0 and $M_g(I_0)$ to denote those measures in $M(I_0)$ for which g is measure-preserving. Members of $M_g(I_0)$ will also be called g-invariant. If $\mu \in M(I_0)$ and $\mathscr{B}_1, \mathscr{B}_2 < \mathscr{B}$ are sub-σ-algebras we write $I_\mu(\mathscr{B}_1|\mathscr{B}_2)$ for the conditional information function of $\mathscr{B}_1$ given $\mathscr{B}_2$ with respect to μ. If $\mathscr{B}_1$ is finite and $\mu \in M_g(I_0)$ we write $h_\mu(g, \mathscr{B}_1)$ for the entropy of g with respect to μ and the partition $\mathscr{B}_1$, and $h_\mu(g)$ for the entropy of g with respect to μ.

Given two countable partitions of I, say P and Q and $\varepsilon > 0$ we say that $P = \{P_i\}$, $Q = \{Q_j\}$ are ε-independent (for μ), denoted $P \perp^\varepsilon Q$, if

$$\sum_{Q_j} \mu(Q_j) \sum_{P_i} |\mu(P_i|Q_j) - \mu(P_i)| < \varepsilon.$$

We say P is *weak-Bernoulli* for g (with respect to μ) if, given $\varepsilon > 0$ there exists an $N = N(\varepsilon)$ so that for all $k, l > 0$ and $n > N$

$$\bigvee_0^k g^{-i} P \perp^\varepsilon \bigvee_{n+k}^{n+k+l} g^{-i} P.$$

Two basic facts we shall be appealing to (although indirectly) are:

(1) If P is a finite partition, denote by σ-P the smallest σ-algebra containing $\bigcup_{n \geq 0} \{g^{-n} P\}$. If one may find a finite partition P with σ-$P = \mathscr{B}$ and so that P is weak-Bernoulli for g with respect to μ then the natural extension of the system $(I, \mathscr{B}, \mu, g)$ is measurably isomorphic to a Bernoulli shift. See [F, O] and [Ro].

(2) If one may find a sequence of finite partitions P_n, with $P_n < P_{n+1}$ and so that the natural extension of each system $(I, \sigma\text{-}P_n, \mu, g)$ is isomorphic to a Bernoulli shift and $\sigma\text{-}P_n \uparrow \mathscr{B}$, then the natural extension of $(I, \mathscr{B}, \mu, g)$ is isomorphic to a (possibly generalized) Bernoulli shift, [O2].

The following definition appears in Walters [W]:

Suppose $\phi: I_{00} \to \mathbf{R}$ is continuous. We say that $\mu \in M_g(I_0)$ is an *equilibrium state* for ϕ if

$$\int I_\mu(\mathscr{B}|g^{-1}\mathscr{B}) + \phi \, d\mu \geq \int I_m(\mathscr{B}|g^{-1}\mathscr{B}) + \phi \, dm$$

for all $m \in M_g(I_0)$. This generalizes the usual definition of equilibrium state for finite entropy transformations to the (possibly) infinite entropy case. Since our mappings may have infinite entropy we use this more general definition.

If $gJ_k = (\omega_i, \omega_{i+1})$ and $x \in (\omega_i, \omega_{i+1})$ set $h_k(x) = (g|_{J_k})^{-1}(x)$ and extend this definition of h_k to all of I in such a way as to make h_k continuous and $h_k' = 0$ on the (at most) two intervals making up $I - [\omega_i, \omega_{i+1}]$.

It is our aim to prove the following results about the mapping g.

Theorem 0.1. *Let* $g: I \to I$ *be Markov and assume it satisfies*

(i) $|g'| \geq c > 0$ *and there exists an* $n_0 \geq 1$ *so that*

$$\inf_{n \in N} \left\{ \inf_{x \in (a_n, b_n)} |(g^{n_0})'(x)| \right\} = \lambda > 1.$$

(ii) *There exists an* M *and* $0 < \alpha \le 1$ *so that whenever* x *and* y *lie in the same atom of* $\mathcal{J}$ *one has*

$$|g'(x)/g'(y) - 1| \le M|g(x) - g(y)|^{\alpha}.$$

Then, there is a unique function p *on* I_0 *satisfying*

(a) $\nu = p d\lambda \in M_g(I_0)$ *and* g *is ergodic with respect to* ν.

(b) $\exists R < \infty$, $1/R \le p \le R$.

(c) $\int_I p\, d\lambda = 1$, (*viewing* p *as defined on* I).

(d) *There exists a* K *and* $0 < \alpha \le 1$ *so that whenever* x *and* y *lie in the same atom of* $g\mathcal{J}$, $|p(x) - p(y)| \le K|x - y|^{\alpha}$.

Remarks 0.2. (a) In hypothesis (i), the function $(g^{n_0})'(x)$ is not defined at the countably many points $g^{-i}(\bigcup_n \{a_n, b_n\})$, $i = 1, 2, \dots, n_0 - 1$, and by convention we shall assume the infimum is taken over the complement of this exceptional set.

(b) Keller [K] has already shown the existence of invariant measures for maps with the property that the functions $1/g'|_{J_n}$ are of universally bounded p-variation (which includes the piecewise $C^{1+\alpha}$ case) when there are only finitely many intervals of monotonicity but his techniques do not appear to work in our context.

(c) Many authors have shown the existence of invariant measures for piecewise monotonic and C^2 mappings of I. Perhaps the earliest the theorem of this type was given by Renyi [R]. Our present theorem most closely resembles the result of Bowen and Series [B, S] in which ergodic measures equivalent to Lebesgue measure are obtained for piecewise monotonic and C^2 Markov mappings $g: I \to I$ which satisfy the conditions:

(A1) there exists $N > 0$ such that

$$\inf_{x \in (0,1)} |(g^N)'(x)| \ge \lambda > 1$$

and

$$\text{(A2)} \qquad M = \sup_{x \in (0,1)} \left\{ \frac{|g''(x)|}{|g'(x)|^2} \right\} < \infty.$$

(A2) is known as Renyi's condition. Let us point out that (A2) implies our condition $|g'| \ge c > 0$. For $n \in N$ $\lambda(gJ_n) = |g'(x_n)|\lambda(J_n)$, some $x_n \in (a_n, b_n)$ and so

$$\inf_n |g'(x_n)| = \inf \frac{\lambda(gJ_n)}{\lambda(J_n)} > 0.$$

However, for $x, y \in (a_n, b_n)$, $x < y$ we have

$$\log \frac{g'(x)}{g'(y)} = \log|g'(x)| - \log|g(y)| = \int_x^y \frac{|g''(t)|}{|g'(t)|} dt \le M \int_x^y |g'(t)|\, dt \le M.$$

These two facts give $\inf_{x\in I}|g'(x)| > 0$. Moreover, with a little more care in the above calculation one sees that our Theorem 0.1 is indeed a generalization of the result in [B, S] for, if

$$\log\frac{g'(x)}{g'(y)} \le M|g(x)-g(y)|,$$

one obtains upon exponentiation,

$$\left|\frac{g'(x)}{g'(y)} - 1\right| \le M'|g(x)-g(y)|$$

and so in our condition (ii) one may choose $\alpha = 1$.

(c) Our condition (ii) is best viewed as a uniform piecewise Hölder condition on the derivatives of the inverse mappings h_k. If x, y lie in the same atom of $g\mathcal{J}$ one has

$$|h_k'(x)/h_k'(y) - 1| \le M|x-y|^\alpha.$$

It seems that one is led to this form, rather than the usual Hölder condition

$$|h_k'(x) - h_k'(y)| \le M|x-y|^\alpha$$

(note, however, if $|\mathcal{J}| < \infty$ these are equivalent) upon consideration of the classical example of continued fraction expansion; $g(x) = \frac{1}{x} - [\frac{1}{x}]$, where $[x]$ denotes the integer part of x. Indeed, one may check easily that for this example

$$\sup_{x,y\in J_n}\frac{|g'(x)-g'(y)|}{|g(x)-g(y)|^\alpha}$$

is unbounded in n for all $\alpha > 0$. Since this example is central to the theory as far back as Renyi we adopt our condition (ii).

Theorem 0.3. *Let* $g: I \to I$ *satisfy the hypothesis of Theorem* 0.1 *with the Markov condition* (4) *replaced by*

(iii) *There exists a* $D < \infty$ *so that for all* $d \ge D$, $k, l \in N$ *we have* $g^d J_k \cap J_l \ne \varnothing$.

Let p *and* ν *be the objects given by the conclusion of Theorem* 0.1. *Then the partition* $\mathcal{J}$ *is a weak-Bernoulli generator for the measure-preserving system* $(I, \mathcal{B}, \nu, g)$ *and the natural extension of this system is a Bernoulli shift with entropy*

$$\sum_{n\in N}\int_{J_n}\log|g'(x)|\,d\nu(x)$$

if this sum converges, (*a generalized Bernoulli shift if the above expression is divergent*).

Remarks 0.4. (a) Let $F \subseteq N$ be a finite subset, say $F = \{n_1, n_2, \dots, n_p\}$. Showing that the countable partition $\mathcal{J}$ is weak-Bernoulli implies the fact for the finite partition $\mathcal{J}_F = \{J_{n_1}, J_{n_2}, \dots, J_{n_p}, \bigcup_{n\in N-F} J_n\}$ and so the result will follow by taking a fixed sequence of finite subsets $F_n \subseteq F_{n+1} \subseteq N$ with $\bigcup_n F_n =$

N and applying the already-mentioned result about an increasing sequence of Bernoulli shifts.

(b) The sum in the expression for entropy is to be interpreted as

$$\sup_n \left\{ \sum_{k\in F_n} \int_{J_k} \log|g'(x)|\, d\nu(x) \right\}$$

where $\{F_n\}$ is an increasing sequence of finite subsets of N as in (a) above.

(c) This theorem can be viewed as a $C^{1+\alpha}$ version of the well-known Adler's Folklore Theorem which gave the same conclusions for piecewise monotonic and C^2 mappings satisfying conditions (A1) and (A2) already mentioned in Remarks 0.2 plus the condition

$$\text{(A3)} \qquad gJ_n = I \quad \text{for all } n \in N$$

instead of our mixing condition (iii). The argument in Remark 0.2(b) shows that our Theorem 0.3 is indeed a generalization of Adler's result.

(d) It is interesting that, in passing from Theorem 0.1 to Theorem 0.3, obtaining in the first instance, ergodic measures and in the second, isomorphism to a Bernoulli shift, we have replaced the Markov condition (4) which clearly is not carried to powers of g with the mixing condition (iii), which is. We shall make use of this fact in the proof of Theorem 0.3.

Before we state the last result we must introduce some notation. Denote by $\overline{I}$ the disjoint union of the collection of intervals $[\omega_i, \omega_{i+1}]$ and place on $\overline{I}$ a distance d so that

(1) $(\overline{I}, d)$ is a compact metric space,

(2) $d(x, y) = |x - y|$ if x and y lie in the same interval $[\omega_i, \omega_{i+1}]$ and $d(x, y) \geq 1$ if x and y do not lie in the same atom of $g\mathcal{J}$.

Here we naturally think of each $\omega_i \notin \{0, 1\}$ as two points ω_i^-, the rightmost endpoint of $[\omega_{i-1}, \omega_i]$ and ω_i^+ the leftmost endpoint of $[\omega_i, \omega_{i+1}]$. In particular, $d(\omega_i^+, \omega_i^-) \geq 1$. One also has a natural embedding $\phi\colon I_0 \to \overline{I}$ and $\phi\colon I_{00} \to \overline{I}$ where ϕ is an isometry when restricted to any one of the intervals (ω_i, ω_{i+1}). We shall, in fact, write I_0 (respectively I_{00}) when we really mean the more cumbersome $\phi(I_0)$ (respectively $\phi(I_{00})$). In the same spirit we write $\mathcal{J}$, $g\mathcal{J}$ and λ for the image under ϕ of these objects originally defined on I. What we have gained here is to have $g\colon I_{00} \to I_0 \subseteq \overline{I}$ with the inverse mappings h_k as previously defined on I now being C^1 functions on $\overline{I}$. We now present

Theorem 0.5. *Suppose g satisfies the hypothesis of Theorem 0.3 and suppose $\phi\colon I_{00} \to \mathbf{R}$ satisfies*

(iv) *There exists a $K < \infty$ so that*

$$\sum_{y\in\{g^{-1}x\}} \exp\{\phi(y)\} \leq K$$

for all $x \in I$.

(v) $|\phi(x)-\phi(y)| \le M_n|x-y|^\alpha$ *for all* $x, y \in (a_n, b_n)$ *where*

$$M = \sup_n \left\{ \sup_{x\in(a_n, b_n)} M_n|g'(x)|^{-\alpha} \right\} < \infty.$$

Then

(1) ϕ *has a unique equilibrium state* μ_ϕ.

(2) μ_ϕ *has no atoms and is positive on nonempty open subsets of* I.

(3) *The partition* $\mathcal{J}$ *is a weak-Bernoulli generator for* g *and so the natural extension of the system* $(I, \mathcal{B}, g, \mu_\phi)$ *is a (possibly generalized) Bernoulli shift.*

(4) *If* $\psi \in C(I_{00})$ *also satisfies* (iv) *and* (v) *above then* $\mu_\phi = \mu_\psi$ *if and only if there exists a* $c \in \mathbf{R}$ *and* $f \in C(\overline{I})$ *with* $\phi(x) = \psi(x) + f(gx) - f(x) + c$.

In particular, the measure $\nu = p d\lambda$ *from Theorem* 0.3 *is the unique equilibrium state for the function* $\phi_\nu = -\log|g'|$.

This can be considered as $C^{1+\alpha}$ analogue of a theorem of Walters [W]; in fact we can combine the major elements of the proof given there with the details of the proofs of Theorems 0.1 and 0.3 to obtain a proof of this result.

Returning to our set-up one can say even more about the h_k when g satisfies the hypothesis of Theorem 0.1. Let $\alpha > 0$ be the constant appearing the hypothesis of this theorem. If $f: \overline{I} \to \mathbf{R}$ we write

$${}_\alpha\|f\| = \sup_i \left\{ \sup_{\substack{x, y\in[\omega_i, \omega_{i+1}] \\ x\neq y}} \frac{|f(x)-f(y)|}{(d(x, y))^\alpha} \right\}$$

and denote by $H(\alpha, g\mathcal{J}) = \{f: \overline{I} \to \mathbf{R} |\, {}_\alpha\|f\| < \infty\}$. If $f \in H(\alpha, g\mathcal{J})$ with $|f| \ge c > 0$ then $f|_{I_0}$ is a piecewise Hölder continuous function with exponent α; more precisely one has

$$\left| \frac{f(x)}{f(y)} - 1 \right| \le \frac{{}_\alpha\|f\|}{c}|x-y|^\alpha = M|x-y|^\alpha$$

for all i and $x, y \in (\omega_i, \omega_{i+1})$. Suppose conversely, that $f: I_0 \to \mathbf{R}$ is piecewise Hölder continuous as above with constants M and α, and that the following limits exist for all i:

$$\lim_{x\uparrow\omega_i^-} f(x) \quad \lim_{x\downarrow\omega_i^+} f(x), \quad \lim_{x\downarrow 0} f(x), \quad \text{and} \quad \lim_{x\uparrow 1} f(x).$$

Then f has a unique extension to a continuous function on $\overline{I}$ (which we shall also call f) and this extension satisfies ${}_\alpha\|f\| \le M\|f\|_\infty$. In particular our functions h_k' may be viewed as elements of $H(\alpha, g\mathcal{J})$ with ${}_\alpha\|h_k'\| \le Mc^{-1}$ (α, M and c from hypothesis (ii) of Theorem 0.1). Another simple fact about $H(\alpha, g\mathcal{J})$ is

Lemma 0.6. *If* $f_n \in H(\alpha, g\mathcal{J})$ *and there exists* $M_0 < \infty$ *so* ${}_\alpha\|f_n\| \le M_0$ *and* $f_n \to f$ *uniformly* (*on* $\overline{I}$ *in* d) *then* ${}_\alpha\|f\| \le M_0$.

Proof. Let $\varepsilon > 0$ be given. If $x \neq y$ lie in the same atom $[\omega_i, \omega_{i+1}]$ then

$$\begin{aligned} |f(x) - f(y)| &\leq |f_n(x) - f(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \\ &\leq (2\varepsilon + M_0)(d(x,y))^\alpha \end{aligned}$$

for sufficiently large n. Hence, ${}_\alpha\|f\| \leq M_0 + 2\varepsilon$ where $\varepsilon > 0$ was arbitrary. □

Fix $n > 0$ and denote by $\mathscr{J}^n$ the partition $\bigvee_{p=0}^{n-1} g^{-p}\mathscr{J}$; atoms in $\mathscr{J}^n$ are naturally indexed by strings $k_0k_1\cdots k_{n-1} \in N^n$ and we shall write $J(k_0k_1\cdots k_{n-1})$ for the (possibly empty) interval $J_{k_0} \cap g^{-1}J_{k_1} \cap \cdots \cap g^{-n+1}J_{k_{n-1}}$. The reader may note that if $x \in J(k_0k_1\cdots k_{n-1})$ then the g-$\mathscr{J}$-name of x begins "$k_0k_1\cdots k_{n-1}$".

With respect to $\mathscr{J}^n$, g^n (the n-fold composition of g) is a piecewise monotonic C^1 map with finite image, in fact if $J(k_0k_1\cdots k_{n-1}) = (\alpha, \beta)$ then

$$\lim_{x\downarrow\alpha} g^n(x) \in \left\{ \lim_{y\downarrow a_{k_{n-1}}} g(y),\ \lim_{y\uparrow b_{k_{n-1}}} g(y) \right\} \subseteq \Omega$$

and similarly for $\lim_{x\uparrow\beta} g^n(x)$. Thus, (in an obvious extension of our notation) we have the partition $g^n\mathscr{J}^n$ contained in $g\mathscr{J}$. Define $h_{k_0k_1\cdots k_{n-1}}(x) = (g^n|_{J(k_0k_1\cdots k_{n-1})})^{-1}(x)$ whenever $x \in g^nJ(k_0k_1\cdots k_{n-1})$ and extend this definition to make $h_{k_0k_1\cdots k_{n-1}}$ continuous on I with $h'_{k_0k_1\cdots k_{n-1}} = 0$ on $I -$ closure$(g^nJ_{(k_0k_1\cdots k_{n-1})})$. Finally, extend these definitions to $\bar{I}$ so that each $h_{k_0k_1\cdots k_{n-1}}$ and $h'_{k_0k_1\cdots k_{n-1}}$ are continuous functions on $(\bar{I}, d)$. Incidentally, we shall see in Lemma 1.2 that under the hypothesis of Theorem 0.1 each $h'_{k_0k_1\cdots k_{n-1}}$ as defined above is in fact a member of $H(\alpha, g\mathscr{J})$.

If $f: \bar{I} \to \mathbf{R}^+$ is measurable we define the *Frobenius-Perron* operator applied to f by

$$F_g f(x) = \begin{cases} \sum_{k\in N} |h'_k(x)| f(h_k(x)), & \text{if this sum converges}, \\ 0, & \text{otherwise}. \end{cases}$$

As before, this sum is to be interpreted as the supremum over all finite subsets of our countable set N. We have the following elementary observations about the operator F_g:

(1) $F_g: L_1^+(\bar{I}, \lambda) \to L_1^+(\bar{I}, \lambda)$ with $\|F_g f\|_1 = \|f\|_1$.

(2) $p \in L_1^+(\bar{I}, \lambda)$ is the density of a g-invariant measure $\nu \ll \lambda$ if and only if $F_g p = p$.

(3) $F_g^m = F_{g^m}$ for all $m > 0$.

This last equality is most easily seen by induction, using the identity

$$h_{k_0k_1\cdots k_{n-1}}(x) = h_{k_0k_1\cdots k_{m-1}}(h_{k_mk_{m+1}\cdots k_{n-1}}(x))$$

whenever $0 \leq m \leq n-1$.

In the next section we shall present a proof of Theorem 0.1. Proofs of Theorems 0.3 and 0.5 will be given in §§2 and 3 respectively. Parts of this work

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1. Proof of Theorem 0.1

Throughout this section the hypotheses of Theorem 0.1 shall be assumed. We begin with the elementary

Lemma 1.1. *Let* $a_1, a_2, \ldots, a_p$ *and* $b_1, b_2, \ldots, b_p$ *be nonzero real numbers with* $a_i/b_i > 0$ *for* $i = 1, 2, \ldots, p$. *Assume there are numbers* $\lambda_i \geq 0$ *so*

$$\max\left\{\left|\frac{a_i}{b_i} - 1\right|, \left|\frac{b_i}{a_i} - 1\right|\right\} \leq \lambda_i, \qquad i = 1, 2, \ldots, p.$$

Then

$$\left|\prod_{i=1}^{p} \frac{a_i}{b_i} - 1\right| \leq \exp\left\{\sum_{i=1}^{p} \lambda_i\right\} - 1.$$

Proof. Estimate the size of $|\log \prod_{i=1}^{p} a_i/b_i|$. □

Our first use of this fact will be in

Lemma 1.2. (a) *There exists an* M_0 *so that for all* n *and* $k_0, k_1, \ldots, k_{n-1} \in N$ *and* x, y *elements of the same atom* $[\omega_i, \omega_{i+1}]$ *of* $g\mathcal{J}$ *one has*

$$\left|\frac{h'_{k_0 k_1 \cdots k_{n-1}}(x)}{h'_{k_0 k_1 \cdots k_{n-1}}(y)} - 1\right| \leq M_0 (d(x, y))^{\alpha}.$$

(*Here*, $\frac{0}{0} = 1$ *by convention.*)

(b) *There exists a* $C_1 > 0$ *so that for all* n *and* $k_0, k_1, \ldots, k_{n-1} \in N$, *if* x *and* y *lie in the same atom* $[\omega_i, \omega_{i+1}]$ *of* $g\mathcal{J}$

$$\frac{1}{C_1} \leq \frac{h'_{k_0, k_1 \cdots k_{n-1}}(x)}{h'_{k_0 k_1 \cdots k_{n-1}}(y)} \leq C_1.$$

(c) *There exists a* $C_2 > 0$ *so that for all* n *and* $k_0, k_1, \ldots, k_{n-1} \in N$ *and* x *such that* $h'_{k_0 k_1 \cdots k_{n-1}}(x) \neq 0$, $x \in [\omega_i, \omega_{i+1}]$

$$\frac{1}{C_2}\lambda(h_{k_0 k_1 \cdots k_{n-1}}[\omega_i, \omega_{i+1}]) \leq |h'_{k_0 k_1 \cdots k_{n-1}}(x)| \leq C_2 \lambda(h_{k_0 k_1 \cdots k_{n-1}}[\omega_i, \omega_{i+1}]).$$

Remark 1.3. It follows from (c) in the above lemma that if $f\colon \bar{I} \to \mathbf{R}$ is bounded then

$$F_g^n f(x) = F_{g^n} f(x) \sum_{k_0 k_1 \cdots k_{n-1}} |h'_{k_0 k_1 \cdots k_{n-1}}(x)| f(h_{k_0 k_1 \cdots k_{n-1}}(x)),$$

the sum on the right-hand side being uniformly convergent. If f is continuous on $\overline{I}$, so is $F_g^n f$ for all n, moreover, if $f_l \to f$ uniformly on $\overline{I}$, $F_g^n f_l \to F_g^n f$ and the convergence is uniform on $\overline{I}$.

Proof of Lemma 1.2. Let c, n_0 and $\lambda > 1$ be from hypothesis (i) of the theorem and assume $c < 1$. Fix $n > 0$ and write $n = dn_0 + d_0$ where $d_0 < n_0$. Write

$$h'_{k_0k_1\cdots k_{n-1}}(\cdot) = h'_{k_0k_1\cdots k_{n_0-1}}(h_{k_{n_0}\cdots k_{n-1}}(\cdot))h'_{k_{n_0}\cdots k_{2n_0-1}}(h_{k_{2n_0}\cdots k_{n-1}}(\cdot))\cdots h'_{k_{dn_0}\cdots k_{n-1}}(\cdot),$$

and observe that for $0 \le p \le d$,

$$|h_{k_{pn_0}\cdots k_{n-1}}(x) - h_{k_{pn_0}\cdots k_{n-1}}(y)| \le \left(\frac{1}{\lambda}\right)^{d-p}\left(\frac{1}{c}\right)^{d_0} d(x,y) \le M'\left(\frac{1}{\lambda}\right)^{d-p} d(x,y).$$

Evidently when x and y lie in the same atom of $g\mathcal{J}$, $h'_{k_0k_1\cdots k_{n-1}}(x)$ and $h'_{k_0k_1\cdots k_{n-1}}(y)$ are either both zero or both nonzero. Assume the latter and apply the above chain rule expansion to the expression $|h'_{k_0k_1\cdots k_{n-1}}(x)/h'_{k_0k_1\cdots k_{n-1}}(y) - 1|$ from which one obtains, upon applying Lemma 1.1, an upper bound of

$$\exp\left\{M\sum_{p=1}^{d}\left(M'\left(\frac{1}{\lambda}\right)^{d-p} d(x,y)\right)^{\alpha} + M\sum_{p=1}^{d_0}\left(\left(\frac{1}{c}\right)^{d_0-p} d(x,y)\right)^{\alpha}\right\} - 1$$
$$\le \exp\{M''(d(x,y))^{\alpha}\} - 1,$$

since the first of these sums inside the braces is geometric, and in the second sum $d_0 < n_0$. This last quantity is clearly independent of n and $k_0k_1\cdots k_{n-1}$. If x and y are in the same atom $[\omega_i, \omega_{i+1}]$ then $(d(x,y))^{\alpha} \le 1$ and so we may bound this exponential by $M_0(d(x,y))^{\alpha}$ with M_0 independent of x and y.

(b) follows immediately from (a).

Take absolute values in the inequality from statement (b), multiply through by $|h'_{k_0k_1\cdots k_{n-1}}(y)|$, provided this is nonzero, and finally integrate with respect to y obtaining the inequalities in (c). □

Lemma 1.4. *Suppose* $A \subseteq \overline{I}$ *is such that*

$$\min_i\{\lambda(A|[\omega_i, \omega_{i+1}])\} = m > 0.$$

Then there exists an $\varepsilon_0 > 0$ *so that for all* $d > 0$ *we have* $\lambda(g^{-d}A) \ge \varepsilon_0 m > 0$.

Proof. Set $A_j = A \cap [\omega_j, \omega_{j+1}]$ and use (c) of Lemma 1.2 to get

$$\begin{aligned}\lambda(g^{-d}A_j) &= \sum_{k_0k_1\cdots k_{d-1}} \int_{A_j} |h'_{k_0k_1\cdots k_{d-1}}(x)|\, d\lambda(x) \\ &\geq \sum_{k_0k_1\cdots k_{d-1}} \frac{1}{C_2}\lambda(h_{k_0k_1\cdots k_{d-1}}[\omega_j, \omega_{j+1}])\lambda(A_j) \\ &\geq \sum_{k_0k_1\cdots k_{d-1}} \frac{1}{C_2}\lambda(h_{k_0k_1\cdots k_{d-1}}[\omega_j, \omega_{j+1}])m\lambda([\omega_j, \omega_{j+1}]) \\ &\geq \sum_{k_0k_1\cdots k_{d-1}} \varepsilon_0 m\lambda(h_{k_0k_1\cdots k_{d-1}}[\omega_j, \omega_{j+1}]).\end{aligned}$$

Summing this over j and then $k_0, k_1, \ldots, k_{d-1}$ one obtains

$$\lambda(g^{-d}A) \geq \sum_{k_0k_1\cdots k_{d-1}} \varepsilon_0 m\lambda(J(k_0k_1\cdots k_{d-1})) = \varepsilon_0 m. \quad \square$$

Lemma 1.5. (a) *The sequence* $\{F_g^n 1\}$ *is norm bounded in* $H(\alpha, g\mathcal{J})$.

(b) *Setting* $p_n = (1/n)\sum_{i=0}^{n-1} F_g^i 1$ *there is a subsequence* p_{n_k} *converging uniformly on* $\overline{I}$ *to* $p \in H(\alpha, g\mathcal{J})$.

The function p *also satisfies*:

(c) *There exists* $R < \infty$ *so that* $1/R < p < R$.

(d) $\int_{\overline{I}} p\, d\lambda = 1$.

(e) $F_g p = p$ *and so* $p|_{I_0}$ *is the density of a* g*-invariant measure*; *i.e.*, $\nu = p d\lambda \in M_g(I_0)$.

Proof. Fix $n \geq 0$ and x, y in the same atom $[\omega_i, \omega_{i+1}] \in g\mathcal{J}$. Then

$$\begin{aligned}|F_g^n 1(x) - F_g^n 1(y)| &= \left| \sum_{k_0k_1\cdots k_{n-1}} |h'_{k_0k_1\cdots k_{n-1}}(x)| - \sum_{k_0k_1\cdots k_{n-1}} |h'_{k_0k_1\cdots k_{n-1}}(y)| \right| \\ &\leq \sum_S \big|\, |h'_{k_0k_1\cdots k_{n-1}}(x)| - |h'_{k_0k_1\cdots k_{n-1}}(y)| \,\big|\end{aligned}$$

where S is the collection of $k_0k_1\cdots k_{n-1}$ where $h'_{k_0k_1\cdots k_{n-1}} \neq 0$. Apply to this last expression first (a) and then (c) of Lemma 1.2 to obtain a further upper bound

$$\begin{aligned}&\leq \sum_S M_0 |h'_{k_0k_1\cdots k_{n-1}}(y)|(d(x, y))^\alpha \\ &\leq \sum_S M_0 C_2 (d(x, y))^\alpha \lambda(h_{k_0k_1\cdots k_{n-1}}[\omega_i, \omega_{i+1}]) \\ &\leq M_0 C_2 (d(x, y))^\alpha.\end{aligned}$$

Hence ${}_\alpha\|F_g^n 1\| \leq M_0 C_2$.

We now show there exists a $B > 0$ and $N < \infty$ so that for all $n \geq N$ and $x \in \bar{I}$ one has $p_n(x) \geq B$. It will be sufficient to restrict our attention to x in one particular $[\omega_i, \omega_{i+1}]$. Observe first that by (c) of Lemma 1.2 one has

$$\begin{aligned} F_g^n 1(x) &= \sum_{k_0 k_1 \cdots k_{n-1}} |h'_{k_0 k_1 \cdots k_{n-1}}(x)| \\ &\geq \frac{1}{C_2} \sum_{k_0 k_1 \cdots k_{n-1}} \lambda(h'_{k_0 k_1 \cdots k_{n-1}}[\omega_i, \omega_{i+1}]) \\ &= \frac{1}{C_2} \lambda(g^{-n}[\omega_i, \omega_{i+1}]), \end{aligned}$$

and hence, for all k, $D > 0$

$$(*) \qquad \begin{aligned} \frac{1}{D} \sum_{n=k}^{k+D-1} F_g^n 1(x) &\geq \frac{1}{C_2 D} \lambda\left(\bigcup_{n=k}^{k+D-1} g^{-n}[\omega_i, \omega_{i+1}] \right) \\ &= \frac{1}{C_2 D} \lambda\left(g^{-k} \left(\bigcup_{n=0}^{D-1} g^{-n}[\omega_i, \omega_{i+1}] \right) \right). \end{aligned}$$

For each j, by the Markov hypothesis (4) one obtains D_j so

$$\lambda(g^{D_j}[\omega_j, \omega_{j+1}] \cap [\omega_i, \omega_{i+1}]) > 0$$

and hence

$$\lambda(g^{-D_j}[\omega_i, \omega_{i+1}] \cap [\omega_j, \omega_{j+1}]) > 0.$$

Take $D = \max_j \{D_j + 1\}$, set $A = \bigcup_{n=0}^{D-1} g^{-n}[\omega_i, \omega_{i+1}]$ and let

$$m = \min_j \lambda(A | [\omega_j, \omega_{j+1}]) > 0.$$

Applying Lemma 1.4 to the expression $(*)$ one obtains

$$\frac{1}{D} \sum_{n=k}^{k+D-1} F_g^n 1(x) \geq \frac{1}{C_2 D} \varepsilon_0 m = \varepsilon_1 > 0,$$

where ε_1 is independent of $x \in [\omega_i, \omega_{i+1}]$. If $N = l_1 D + l_2$ where $0 \leq l_2 < D$, then

$$\begin{aligned} \frac{1}{N} \sum_{n=0}^{N-1} F_g^n 1(x) = \frac{D}{N} \Bigg(\frac{1}{D} \sum_{n=1}^{D-1} F_g^n 1(x) + \frac{1}{D} \sum_{n=D}^{2D-1} F_g^n 1(x) \\ + \cdots + \frac{1}{D} \sum_{n=(l_1-1)D}^{l_1 D-1} F_g^n 1(x) \Bigg) + \frac{l_2}{N} \frac{1}{l_2} \sum_{l_1 D}^{N-1} F_g^n 1(x) \\ > \frac{\varepsilon_1}{2} \quad \text{for all sufficiently large } N. \end{aligned}$$

Further, if $x \in [\omega_i, \omega_{i+1}]$

$$F_g^n 1(x) \leq C_2 \sum_{k_0 k_1 \cdots k_{n-1}} \lambda(h_{k_0 k_1 \cdots k_{n-1}}[\omega_i, \omega_{i+1}]) \leq C_2$$

and so one concludes there exists $R < \infty$ and $N < \infty$ so that if $n > N$, $x \in \bar{I}$ we have $1/R \leq p_n(x) \leq R$. Evidently ${}_\alpha\|p_n\| \leq M_0C_2$ and so $\{p_n\}$ is a uniformly bounded, equicontinuous family on $\bar{I}$. By the Arzela-Ascoli Theorem there is a subsequence $\{p_{n_k}\}$ and a continuous function $p: \bar{I} \to \mathbf{R}$ with $p_{n_k} \to p$ uniformly. By the above, $1/R \leq p \leq R$, by Lemma 0.6 ${}_\alpha\|p\| \leq M_0C_2$ and clearly $\int_0^1 p\,d\lambda = 1$. Moreover,

$$F_g p = \lim_{k\to\infty} F_g p_{n_k} = \lim_{k\to\infty} \frac{1}{n_k} \sum_{i=0}^{n_k-1} F_g^{i+1} 1$$

$$= \lim_{k\to\infty} \left\{ \frac{1}{n_k} \sum_{i=0}^{n_k-1} F_g^i 1 + \frac{1}{n_k}\{F_g^{n_k} 1 - 1\} \right\} = p,$$

this last equality following from the uniform bound on the terms $F_g^{n_k}1$. This completes the proof of Lemma 1.5. □

The use of the Arzela-Ascoli Theorem in this context has already appeared in the literature on twice differentiable mappings. See, for example, [H, Theorem 8] for finite-to-one C^2 mappings and [T, Lemma 3] for the countable-to-one case.

It remains to show the ergodic and uniqueness properties. It will be enough to show that λ has no nontrivial invariant subsets, from which ergodicity of $p\,d\lambda$ and uniqueness of the function p will follow. This will be the content of the following two lemmas. The arguments here are simple modifications of those in Lemma 1.1 and Theorem 1.2 of [B,S].

Lemma 1.6. *If $\lambda(E) > 0$ and E is g-invariant then $\lambda(E \cap [\omega_i, \omega_{i+1}]) > 0$ for all i.*

Proof. Fix i and obtain, using the Markov hypothesis (4) on g, an n so that $\lambda(g^n[\omega_i, \omega_{i+1}] \cap E) > 0$. Now $[\omega_i, \omega_{i+1}]$ is a countable union of intervals from $\mathcal{J}^n$ and we may find a $J(k_0k_1\cdots k_{n-1}) \subseteq [\omega_i, \omega_{i+1}]$ with

$$\lambda(g^n J(k_0k_1\cdots k_{n-1}) \cap E) > 0.$$

On $g^n J(k_0k_1\cdots k_{n-1})$, $h_{k_0k_1\cdots k_{n-1}}$ is C^1 with derivative bounded away from zero and hence

$$0 < \lambda(h_{k_0k_1\cdots k_{n-1}}(g^n J(k_0k_1\cdots k_{n-1}) \cap E))$$
$$= \lambda(J(k_0k_1\cdots k_{n-1}) \cap g^{-n}E)$$
$$= \lambda(J(k_0k_1\cdots k_{n-1}) \cap E).$$

Evidently then, $\lambda([\omega_i, \omega_{i+1}] \cap E) > 0$. □

Lemma 1.7. *If E is g-invariant then $\lambda(E) \in \{0, 1\}$.*

Proof. Fix n and write each $J(k_0k_1\cdots k_{n-1}) = \bigcup_i J_i(k_0k_1\cdots k_{n-1})$, a union of finitely many (possibly empty) intervals, each

$$J_i(k_0k_1\cdots k_{n-1}) = h_{k_0k_1\cdots k_{n-1}}[\omega_i, \omega_{i+1}].$$

If $\lambda(E \cap [\omega_i, \omega_{i+1}]) > 0$ and $\lambda(J_i(k_0 k_1 \cdots k_{n-1})) > 0$ we may write

$$\frac{\lambda(g^{-n}E \cap J_i(k_0 k_1 \cdots k_{n-1}))}{\lambda(E \cap [\omega_i, \omega_{i+1}])} \Big/ \frac{\lambda(J_i(k_0 k_1 \cdots k_{n-1}))}{\lambda([\omega_i, \omega_{i+1}])}$$
$$= \frac{\lambda(h_{k_0 k_1 \cdots k_{n-1}}(E \cap [\omega_i, \omega_{i+1}]))}{\lambda(E \cap [\omega_i, \omega_{i+1}])} \Big/ \frac{\lambda(h_{k_0 k_1 \cdots k_{n-1}}[\omega_i, \omega_{i+1}])}{\lambda([\omega_i, \omega_{i+1}])}$$

and since $h_{k_0 k_1 \cdots k_{n-1}}\colon (\omega_i, \omega_{i+1}) \to J_i(k_0 k_1 \cdots k_{n-1})$ is a C^1-diffeomorphism we may find $x_1, x_2 \in [\omega_i, \omega_{i+1}]$ and express this latter quantity as

$$h'_{k_0 k_1 \cdots k_{n-1}}(x_1)/h'_{k_0 k_1 \cdots k_{n-1}}(x_2).$$

Using (b) of Lemma 1.2 and reorganizing the first big quotient above obtain the bounds

$$C_1^{-1} \le \frac{\lambda(g^{-n}E | J_i(k_0 k_1 \cdots k_{n-1}))}{\lambda(E | [\omega_i, \omega_{i+1}])} \le C_1$$

and hence

$$\lambda(g^{-n}E | J_i(k_0 k_1 \cdots k_{n-1})) \ge C_1^{-1} \lambda(E | [\omega_i, \omega_{i+1}]).$$

Now suppose $0 < \lambda(E) < 1$ and E is g-invariant. Set

$$\varepsilon_0 = C_1^{-1} \min_i \lambda(E | [\omega_i, \omega_{i+1}]) > 0,$$

and conclude

$$\lambda(E | J_i(k_0 k_1 \cdots k_{n-1})) \ge \varepsilon_0$$

for all i and so

$$\lambda(E | J(k_0 k_1 \cdots k_{n-1})) \ge \varepsilon_0,$$

with ε_0 independent of n.

The expansive condition (i) of the theorem implies that the partition $\mathcal{J}$ is a generator (since the maximal length of an interval in $\mathcal{J}^n$ goes to zero as $n \to \infty$) and so, for sufficiently large n, one obtains an atom $J(k_0 k_1 \cdots k_{n-1})$ with

$$\lambda(E | J(k_0 k_1 \cdots k_{n-1})) < \varepsilon_0,$$

a contradiction in view of our calculation above. Hence $\lambda(E) \in \{0, 1\}$. □

This completes the proof of Theorem 0.1.

2. Generalized baker's transformations and a proof of Theorem 0.3

In Corollary 4.1 of [B1] we have already proved a version of Theorem 0.3 for a more restrictive class of endomorphisms. To be precise, assume $g\colon I \to I$ satisfies the hypotheses of Theorem 0.3 but with the expansive condition (i) replaced by

(i′) $\inf_{x \in I_0} g'(x) = \lambda > 1$ (in particular g is increasing on each interval J_n), and with the added condition

(vi) $F_g 1 = 1$ (that is, g is Lebesgue-measure-preserving).
Then by the above-mentioned corollary the conclusions of Theorem 0.3 are verified for g and $\mathscr{J}$ where, of course $p \equiv 1$ and $\nu = \lambda$. Moreover, in Corollary 4.1 we have identified the natural extension of (g, λ) as the *generalized baker's transformation* $T_{\underline{f}}$ (see [B1] for a definition of this) associated with the family of functions on I: $\underline{f} = \{h'_n\}_{n \in N}$. An examination of the arguments given there reveal that the assumption that g be increasing on each of the intervals J_n is not necessary; one may construct a variant of the generalized baker's transformation so as to be the natural extension of g provided that only

(i″) $\inf_{x \in I_0} |g'(x)| = \lambda > 1$

and provided that g is Lebesgue-measure-preserving. The details of this construction may be found in [B2]. The results of Theorem 0.3 remain true for this not-necessarily increasing, Lebesgue-measure-preserving endomorphism as in the increasing case. We now proceed to show that under the hypotheses of Theorem 0.3 one may alway assume that (up to measurable isomorphism) the mapping g satisfies (i″) and (vi) above.

Let n_0 be from hypothesis (i) and recall that g^{n_0} is piecewise monotonic and C^1 with finite image with respect to the interval partition $\mathscr{J}^{n_0}$. One shows by induction that g^{n_0} satisfies hypothesis (ii) of Theorem 0.1 as follows:

Write first

$$\frac{(g^{n_0})'(x)}{(g^{n_0})'(y)} = \frac{g'(g^{n_0-1}(x))(g^{n_0-1})'(x)}{g'(g^{n_0-1}(y))(g^{n_0-1})'(y)}.$$

If $x, y \in J(k_0 k_1 \cdots k_{n_0-1})$ and if

$$|(g^{n_0-1})'(x)/(g^{n_0-1})'(y) - 1| \leq M_{n_0-1} |g^{n_0-1}(x) - g^{n_0-1}(y)|^\alpha$$

where $M_{n_0-1} \geq M$, then using Lemma 1.1 and the fact that $|g(x) - g(y)| \geq c|x-y|$ one obtains

$$\left| \frac{(g^{n_0})'(x)}{(g^{n_0})'(y)} - 1 \right| \leq \exp\left\{ \left(\frac{1}{c} M_{n_0-1} + M \right) |g^{n_0}(x) - g^{n_0}(y)|^\alpha \right\} - 1.$$

This gives an $M \leq M_{n_0} < \infty$ so that

$$\left| \frac{(g^{n_0})'(x)}{(g^{n_0})'(y)} - 1 \right| \leq M_{n_0} |g^{n_0}(x) - g^{n_0}(y)|^\alpha.$$

Let $J(k_0 k_1 \cdots k_{n_0-1})$ be fixed. Clearly $J(k_0 k_1 \cdots k_{n_0-1}) \subseteq J_{k_0}$ and we may find n such that $J_{k_0} \subseteq [\omega_n, \omega_{n+1}]$. For any $A = J(k'_0 k'_1 \cdots k'_{n_0-1})$ we have $g^{n_0} A$ a union of intervals from $\mathscr{J}$ so by hypothesis (iii) of Theorem 0.3 for g one has $\lambda(g^{n_0+d} A \cap J_{k_0}) > 0$ for all $d \geq D$. But $g^{n_0+d} A$ covers $[\omega_n, \omega_{n+1}]$ and so $\lambda(g^{n_0+d} A \cap J(k_0 k_1 \cdots k_{n_0-1})) > 0$ for all $d \geq D$. In particular

$$\lambda((g^{n_0})^d A \cap J(k_0 k_1 \cdots k_{n_0-1})) > 0$$

for all $d \geq D$ (assume, without loss of generality $D > n_0/(n_0 - 1)$) and we obtain finally that g^{n_0} satisfies the hypothesis of Theorems 0.1 and 0.3 when g does. Let p_{n_0} and ν_{n_0} be the objects given by Theorem 0.1 applied to g^{n_0} and $\mathscr{J}^{n_0}$. Set

$$p = \frac{1}{n_0} \sum_{n=0}^{n_0-1} F_g^i p_{n_0}.$$

Then $F_g p = p$ and so $F_{g^{n_0}} p = F_g^{n_0} p = p$. Since we may assume (by taking powers $(g^{n_0})^k$) that $n_0 > D$ one may conclude that there exists an $\overline{R} < \infty$ with $p > 1/\overline{R}$, from which one obtains, by ergodicity of g^{n_0} with respect to ν_{n_0} that $p = p_{n_0}$. (Actually, we conclude this λ-a.e. but since both are continuous functions on I_0 one may conclude the pointwise equality.)

Thus $\nu = p d\lambda$ is the unique ergodic measure for g given by Theorem 0.1. A standard interpolation argument gives the weak-Bernoulli condition for $\mathscr{J}$ with respect to g and ν, knowing it for $\mathscr{J}^{n_0}$ with respect to g^{n_0} and ν_{n_0} $(= \nu)$. To obtain the entropy formula one calculates as follows:

$$n_0 h(g) = h(g^{n_0}) = \sum_{k_0 k_1 \cdots k_{n_0-1}} \int_{J(k_0 k_1 \cdots k_{n_0-1})} \log |(g^{n_0})'(x)| p(x)\, dx.$$

Each term in the above sum may be written as a further sum of n_0 terms, each of the form

$$\int_{J(k_0 k_1 \cdots k_{n_0-1})} \log |g'(g^{n_0-i-1}(x))| p(x)\, dx$$

for $0 \leq i \leq n_0 - 1$. A change of variable gives this term as

$$\int_{g^{n_0-i-1} J(k_0 k_1 \cdots k_{n_0-1})} \log |g'(u)| p(h_{k_0 k_1 \cdots k_{n_0-i-2}}(u)) |h'_{k_0 k_1 \cdots k_{n_0-i-2}}(u)|\, du.$$

Exchanging the order of summation, and bringing in the sum over k_{n_0-i-1}, $k_{n_0-i}, \ldots, k_{n_0-1}$ gives

$$\int_{g^{n_0-i-1} J(k_0 k_1 \cdots k_{n_0-i-2})} \log |g'(u)| p(h_{k_0 k_1 \cdots k_{n_0-i-2}}(u)) |h'_{k_0 k_1 \cdots k_{n_0-i-2}}(u)|\, du$$
$$= \int_0^1 \log |g'(u)| p(h_{k_0 k_1 \cdots k_{n_0-i-2}}(u)) |h'_{k_0 k_1 \cdots k_{n_0-i-2}}(u)|\, du,$$

by the definition of $h'_{k_0 k_1 \cdots k_{n_0-i-2}}$. Now bringing in the sum over $k_0, k_1, \ldots,$ k_{n_0-i-2} yields

$$\int_0^1 \log |g'(u)| F_g^{n_0-i-1} p(u)\, du = \int_0^1 \log |g'(u)| p(u)\, du.$$

Finally, summing over i, $0 \leq i \leq n_0 - 1$ and dividing by n_0 gives the desired entropy formula when the integral is written as a sum of integrals over the individual atoms of $\mathscr{J}$.

We see that in proving Theorem 0.3 it is enough to assume the expansiveness condition (i$''$) instead of the weaker condition (i) of Theorem 0.1.

Let p and $\nu = pd\lambda$ be given by Theorem 0.1 applied to g and $\mathcal{J}$. if $x \in I$ set $\phi(x) = \int_0^x pd\lambda$. Then $\phi: I \to I$ is continuous, strictly increasing and when $x \in I - \bigcup_n \{a_n, b_n\}$, $\phi'(x) = p(x)$. As before we may unambiguously think of ϕ and ϕ' as functions defined on $\overline{I}$, $\phi: \overline{I} \to I$ being continuous with $\phi' \in H(\alpha, g\mathcal{J})$. We also have $\phi: (I, \beta, pd\lambda) \to (I, \beta, \lambda)$ a measurable isomorphism which carries g to $\hat{g} = \phi \circ g \circ \phi^{-1}: I \to I$, carries $\mathcal{J}$ to $\hat{\mathcal{J}} = \phi\mathcal{J}$, a partition of I into countably many intervals $\hat{J}_n = \phi J_n = [\phi(a_n), \phi(b_n)]$ and carries $g\mathcal{J}$ to $\hat{g}\hat{\mathcal{J}} = \phi(g\mathcal{J})$, a finite partition of I into intervals $[\phi(\omega_i), \phi(\omega_{i+1})]$, each such interval the (λ-a.e.) union of atoms from $\hat{\mathcal{J}}$. We also have the corresponding inverse functions $\hat{h}_{k_0k_1\cdots k_{n-1}} = \phi \circ h_{k_0k_1\cdots k_{n-1}} \circ \phi^{-1}$ where we observe that $\hat{h}_{k_0k_1\cdots k_{n-1}}(x) = (\hat{g}^n|_{\hat{J}(k_0k_1\cdots k_{n-1})})^{-1}(x)$ whenever $x \in \hat{g}^n(\hat{J}(k_0k_1\cdots k_{n-1}))$ and $\hat{h}_{k_0k_1\cdots k_{n-1}}$ is defined so as to be continuous on I with $\hat{h}'_{k_0k_1\cdots k_{n-1}} = 0$ outside the interval $\hat{g}^n(\hat{J}(k_0k_1\cdots k_{n-1}))$. Evidently $\hat{g}: I \to I$ is a piecewise monotonic and C^1 mapping which has finite image with respect to the partition $\hat{\mathcal{J}}$ and $\hat{g}$ preserves Lebesgue measure. We collect a few more facts about $\hat{g}$.

Lemma 2.1. (1) *If* $d \geq D$ (D *from* (iii) *of Theorem* 0.3) *then for all* $k, l \in N$ *we have*

$$\lambda(\hat{g}^d \hat{J}_k \cap \hat{J}_l) > 0.$$

(2) *There exists a* $\lambda > 1$ *so*

$$\inf_{k_0k_1\cdots k_{D-1}} \left\{ \inf_{\substack{x \in \text{interior of} \\ \hat{J}(k_0k_1\cdots k_{D-1})}} |(\hat{g}^D)'|(x) \right\} \geq \lambda.$$

(3) *There exists an* M *and* $0 < \alpha \leq 1$ *so that whenever* x *and* y *lie in the same atom of* $\hat{\mathcal{J}}$ *one has*

$$|\hat{g}'(x)/\hat{g}'(y) - 1| \leq M|\hat{g}(x) - \hat{g}(y)|^\alpha.$$

Proof. (1) is immediate from the corresponding fact for g and $\mathcal{J}$. To see (2) we will show there exists a $\sigma < 1$ so that for all $k_0k_1\cdots k_{D-1}$, $|\hat{h}'_{k_0k_1\cdots k_{D-1}}(x)| \leq \sigma$ for all $x \in \bigcup_i (\phi(\omega_i), \phi(\omega_{i+1})) = \hat{I}_0$. But this follows from two facts

(i) $\sum_{k_0k_1\cdots k_{D-1}} |\hat{h}'_{k_0k_1\cdots k_{D-1}}(x)| = 1$ for all $x \in \hat{I}_0$.

(ii) There exists an $\varepsilon_0 > 0$ so that for all $x \in \hat{I}_0$ there are (at least) two strings $k_0k_1\cdots k_{D-1} \neq k_0'k_1'\cdots k_{D-1}'$ such that

$$\min\{|\hat{h}'_{k_0k_1\cdots k_{D-1}}(x)|, |\hat{h}'_{k_0',k_1'\cdots k_{D-1}'}(x)|\} \geq \varepsilon_0.$$

Now (i) follows trivially for λ-a.e. $x \in \hat{I}_0$ since $\hat{g}^D$ is Lebesgue-measure-preserving. To obtain the equality for all $x \in \hat{I}_0$ we compute

$$\sum_{k_0k_1\cdots k_{D-1}} |\hat{h}'_{k_0k_1\cdots k_{D-1}}(x)| = \sum_{k_0k_1\cdots k_{D-1}} \frac{p(h_{k_0k_1\cdots k_{D-1}} \circ \phi^{-1}(x))|h'_{k_0k_1\cdots k_{D-1}}| \circ \phi^{-1}(x)}{p(\phi^{-1}(x))}$$
$$= \frac{F_g^n p(\phi^{-1}(x))}{p(\phi^{-1}(x))} = 1 \quad \text{for all } \phi^{-1}(x) \in \overline{I}.$$

For (ii) it will be enough to find $\varepsilon_0(i) > 0$ that works for all $x \in (\phi(\omega_i), \phi(\omega_{i+1}))$, for we may then take $\varepsilon_0 = \min_i \varepsilon_0(i)$.

(iii) of Theorem 0.3 implies there exist at least two strings $k_0k_1\cdots k_{D-1} \neq k'_0k'_1\cdots k'_{D-1}$ so that (ω_i, ω_{i+1}) is contained in both $g^D(J(k_0k_1\cdots k_{D-1}))$ and $g^D(J(k'_0k'_1\cdots k'_{D-1}))$. Using first (c) of Lemma 1.2 to get an inequality for $h'_{k_0k_1\cdots k_{D-1}}$ and $h'_{k'_0k'_1\cdots k'_{D-1}}$ and then passing to the corresponding statements via the conjugacy conclude

$$\min\left\{\inf_{x\in(\phi(\omega_i),\phi(\omega_{i+1}))} |\hat{h}'_{k_0k_1\cdots k_{D-1}}(x)|, \inf_{x\in(\phi(\omega_i),\phi(\omega_{i+1}))} |\hat{h}'_{k'_0k'_1\cdots k'_{D-1}}(x)|\right\} > 0$$

and for $\varepsilon_0(i)$ we may take the quantity on the left side of this last inequality.

Finally, to obtain (3) we estimate, when x and y lie in the interior of the same atom of $\hat{\mathscr{J}}$ the quantity $|\hat{g}'(x)/\hat{g}'(y) - 1|$. By the chain rule this is

$$\left|\frac{p(g \circ \phi^{-1}(x))g'(\phi^{-1}(x))p(\phi^{-1}(y))}{p(g \circ \phi^{-1}(y))g'(\phi^{-1}(y))p(\phi^{-1}(x))} - 1\right|.$$

Using first the fact that $p \in H(\alpha, g\mathscr{J})$, with $p \geq 1/R$ and then the straightforward inequality

$$\frac{1}{R}|x-y| \leq |\phi(x) - \phi(y)| \leq R|x-y|$$

one obtains

$$\left|\frac{p(g \circ \phi^{-1}(x))}{p(g \circ \phi^{-1}(y))} - 1\right| \leq {}_\alpha\|p\| R^{\alpha+1} |\hat{g}(x) - \hat{g}(y)|^\alpha.$$

By a similar argument, but using (ii) of Theorem 0.1 get

$$\left|\frac{g'(\phi^{-1}(x))}{g'(\phi^{-1}(y))} - 1\right| \leq MR^\alpha |\hat{g}(x) - \hat{g}(y)|^\alpha.$$

Also,

$$\left|\frac{p(\phi^{-1}(y))}{p(\phi^{-1}(x))} - 1\right| \leq {}_\alpha\|p\| R^{\alpha+1} |\hat{g}(x) - \hat{g}(y)|^\alpha$$

where we have used the fact that $|g'| > 1$. Apply Lemma 1.1 and obtain

$$\left|\frac{\hat{g}'(x)}{\hat{g}'(y)} - 1\right| \leq \exp\left\{\left(M + 2_\alpha\|p\|R\right) R^\alpha |\hat{g}(x) - \hat{g}(y)|^\alpha\right\} - 1$$

whenever x and y lie in the interior of the same atom of $\hat{\mathscr{J}}$. This gives (3). □

We have shown that $\hat{g}$ is a Lebesgue-measure-preserving map on $\overline{I}$ which satisfies the hypotheses of Theorem 0.3 and as we have seen before, by passing to the power $\hat{g}^D$ one may assume that $\hat{g}$ satisfies the condition (i$''$) instead of the weaker expansive condition (i). Conclude that $\hat{\mathscr{J}}$ is a weak-Bernoulli generator for $\hat{g}$ with respect to λ and the natural extension of $(\hat{g}, \lambda)$ is a Bernoulli shift with entropy

$$\sum_n \int_{\hat{J}_n} \log |\hat{g}'| \, d\lambda \, .$$

if this sum converges (a generalized Bernoulli shift if this sum diverges). These conclusions are carried to the partition $\mathscr{J}$ and the mapping g with respect to the measure ν by the isomorphism ϕ. The entropy formula becomes

$$\begin{aligned} h(g) = h(\hat{g}) &= \sum_n \int_{\hat{J}_n} \log |\hat{g}'(u)| \, du \\ &= \sum_n \int_{\hat{J}_n} \log \left| \frac{p(g \circ \phi^{-1}(u)) g'(\phi^{-1}(u))}{p(\phi^{-1}(u))} \right| du \\ &= \sum_n \int_{J_n} \log \left| \frac{p(g(x)) g'(x)}{p(x)} \right| p(x) \, dx \, , \end{aligned}$$

by a change of variable. Since we have

$$\sum_n \int_{J_n} \log \left| \frac{p(g(x))}{p(x)} \right| p(x) \, dx = 0$$

one obtains the desired entropy formula:

$$h(g) = \sum_n \int_{J_n} \log |g'(x)| \, d\nu(x) \, ,$$

and this completes the proof of Theorem 0.3.

3. Equilibrium states and a proof of Theorem 0.5

References in this section, unless stated otherwise shall be to the article of P. Walters [W]. Following the ideas presented there we first change slightly the metric d we have been using on $\overline{I}$ to an equivalent metric d^* so as to make g expansive on I_{00}, that is, for each $x, y \in I_{00}$ we desire

$$d^*(g(x), g(y)) \geq d^*(x, y).$$

If x and x' lie in the interior of the same atom $[\omega_i, \omega_{i+1}]$ of $g\mathscr{J}$, and $l \geq 1$, then to each $y \in \{g^{-1}(x)\}$ there is one and only on y' so both $g^l y' = x'$ and y and y' lie in the same atom of $\mathscr{J}^l$. With this notation we set

$$d^*(x, x') = \sup_{n \geq 0} \left\{ \sup_{y \in \{g^{-n}(x)\}} d(y, y') \right\}$$

if x and x' lie in the interior of the same atom of $g\mathcal{J}$ and $d^*(x,x') = d(x,x')$ when x and x' lie in different atoms of $g\mathcal{J}$. If $x, x' \in (\omega_i, \omega_{i+1})$ then since $|g'| \geq c > 0$ and $|(g^{n_0})'| \geq \lambda > 1$ one has

$$|x - x'| \leq d^*(x,x') \leq \max(1, c^{-n_0})|x - x'| = L|x - x'|,$$

and so d^* extends to a metric on $\overline{I}$ which is clearly equivalent to d. Evidently g is expansive with respect to d^*. Theorem 0.5 will now follow from Theorem 16 of [W, p. 140] upon replacement of $(\overline{X}, d)$ with our $(\overline{I}, d^*)$, X with our I_0, X_0 with our I_{00}, and T with our continuous and surjective mapping $g: I_{00} \to I_0$.

To apply this result we must first verify that our map g satisfies Properties I and II (pp. 123 and 125 respectively). For Property I take $\varepsilon_0 = \frac{1}{2}$, for then if $x \in I_0$, $g^{-1}(B_{2\varepsilon_0}(x) \cap I_0)$ is clearly a disjoint union of countably many open intervals (at most one in each J_n) denoted $A_n(x)$, with $g|A_n(x)$ an expanding homeomorphism.

To see Property II observe that for any $x \in I_0$, $\{g^{-d}x\}$ intersects each and every J_n when $d \geq D$ by hypothesis (iii) of Theorem 0.3. Given $\varepsilon > 0$ choose N so that $L(1/\lambda)^N > \varepsilon/2$, ($L$ from above) and let $M = Nn_0 + D$. Clearly then $\{g^{-M}(x)\}$ intersects every atom of $\mathcal{J}^{Nn_0}$.

Since the maximal length of an interval in $\mathcal{J}^{Nn_0}$ is $(1/\lambda)^N$ one has $\{g^{-M}(x)\}$ ε-dense in I_0 (in the d^* metric). Incidentally, the above observation on $\mathcal{J}^{Nn_0}$ implies, as we have seen before that $\mathcal{J}$ is a generator for g.

As for the conditions on $\phi \in C(I_0)$, our condition (iv) is exactly condition (i) on p. 123 and we now show that our (v) implies (iii) on p. 125.

Let $x, x' \in (\omega_i, \omega_{i+1})$. Computing

$$\begin{aligned}
&\sup_{y \in \{g^{-n}(x)\}} \sum_{i=0}^{n-1} \{\phi(g^i(y)) - \phi(g^i(y'))\} \\
&\quad = \sup_{k_0 k_1 \cdots k_{n-1}} \sum_{i=0}^{n-1} \{\phi(h_{k_0 k_1 \cdots k_i}(x)) - \phi(h_{k_0 k_1 \cdots k_i}(x'))\} \\
&\quad \leq \sup_{k_0 k_1 \cdots k_{n-1}} \sum_{i=0}^{n-1} M_{k_0} |h_{k_0 k_1 \cdots k_i}(x) - h_{k_0 k_1 \cdots k_i}(x')|^\alpha \\
&\quad = \sup_{k_0 k_1 \cdots k_{n-1}} \sum_{i=0}^{n-1} M_{k_0} |h'_{k_0 k_1 \cdots k_i}(\bar{x})|^\alpha |x - x'|^\alpha
\end{aligned}$$

where $\bar{x} = \bar{x}(k_0, k_1 \cdots k_i, x, x')$. Now using the chain rule and our hypothesis (v) one bounds this last quantity by

$$M|x - x'|^\alpha \left\{ \sup_{k_0 k_1 \cdots k_{n-1}} \left\{ \sum_{i=0}^{n-1} |h'_{k_1 k_2 \cdots k_i}(\bar{x})|^\alpha + 1 \right\} \right\}.$$

Writing $i = dn_0 + l$ with $0 \leq l \leq n_0 - 1$, $d \geq 0$ one observes

$$|h_{k_1 k_2 \cdots k_i}(\bar{x})|^\alpha \leq \lambda^{-d\alpha} c^{-l\alpha} \leq \lambda^{-d\alpha} c^{-n_0\alpha},$$

(recall, we assume $c \leq 1$).

Using this in our sum one obtains a further upper bound of

$$M|x - x'|^\alpha c^{-n_0\alpha} n_0 \sum_{d=0}^{[(n-1)/n_0]} \lambda^{-\alpha d} \leq P|x - x'|^\alpha.$$

Thus setting

$$C_\phi(x, x') = \sup_n \left\{ \sup_{y \in \{g^{-n}(x)\}} \sum_{i=0}^{n-1} [\phi(g^i(y)) - \phi(g^i(y'))] \right\}$$

one obtains $C_\phi(x, x') \leq P|x - x'|^\alpha \leq (d^*(x, x'))^\alpha \leq PL$, and $C_\phi(x, x') \to 0$ as $d^*(x, x') \to 0$.

The conclusions of our Theorem 0.5 then follow from Theorem 16 upon checking that the partition $\mathcal{J}$ satisfies the properties listed in Theorem 13 (p. 136). To this end, properties (a)–(c) are trivially satisfied. We have (d) since the maximal length of an interval in $\mathcal{J}^n$ is $(1/\lambda)^{[n/n_0]} L$. In (e), if $g(a_n, b_n) = (\omega_i, \omega_{i+1})$ take $x_n = (\omega_i + \omega_{i+1})/2$. (Here we have used again our assumption from the Introduction that g maps each J_n onto one and only one (ω_i, ω_{i+1}).)

Finally, if $\phi_0 = -\log|g'|$ we have

$$\sum_{y \in \{g^{-1}(x)\}} \exp\{\phi_0(y)\} = \sum_{y \in \{g^{-1}(x)\}} \frac{1}{|g'(y)|} = \sum_{k \in N} |h_k'(x)| \leq C_2$$

by Lemma 1.2.

We also have

$$|\phi_0(x) - \phi_0(y)| = \left| \log \frac{g'(x)}{g'(y)} \right| \leq M \left\{ \sup_{\bar{x} \in (a_n b_n)} |g'(\bar{x})|^\alpha \right\} |x - y|^\alpha$$

whenever $x, y \in (a_n, b_n)$. Thus, conditions (iv) and (v) are satisfied for ϕ_0. Conclude that $\mu_{\phi_0} = \nu$ since both are ergodic for g and they cannot be disjointly supported. This completes the proof of Theorem 0.5.

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