TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 316, Number 2, December 1989

ISOMETRIC DILATIONS FOR INFINITE SEQUENCES OF NONCOMMUTING OPERATORS

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ABSTRACT. This paper develops a dilation theory for $\{T_n\}_{n=1}^{\infty}$ an infinite sequence of noncommuting operators on a Hilbert space, when the matrix $[T_1, T_2, \dots]$ is a contraction. A Wold decomposition for an infinite sequence of isometries with orthogonal final spaces and a minimal isometric dilation for $\{T_n\}_{n=1}^{\infty}$ are obtained. Some theorems on the geometric structure of the space of the minimal isometric dilation and some consequences are given. This results are used to extend the Sz.-Nagy-Foiaş lifting theorem to this noncommutative setting.

This paper is a continuation of [5] and develops a dilation theory for an infinite sequence $\{T_n\}_{n=1}^{\infty}$ of noncommuting operators on a Hilbert space $\mathscr{H}$ when $\sum_{n=1}^{\infty} T_n T_n^* \leq I_{\mathscr{H}}$ ($I_{\mathscr{H}}$ is the identity on $\mathscr{H}$).

Many of the results and techniques in dilation theory for one operator [8] and also for two operators [3, 4] are extended to this setting.

First we extend Wold decomposition [8, 4] to the case of an infinite sequence $\{V_n\}_{n=1}^{\infty}$ of isometries with orthogonal final spaces.

In §2 we obtain a minimal isometric dilation for $\{T_n\}_{n=1}^{\infty}$ by extending the Schaffer construction in [6, 4]. Using these results we give some theorems on the geometric structure of the space of the minimal isometric dilation. Finally, we give some sufficient conditions on a sequence $\{T_n\}_{n=1}^{\infty}$ to be simultaneously quasi-similar to a sequence $\{R_n\}_{n=1}^{\infty}$ of isometries on a Hilbert space $\mathscr{K}$ with $\sum_{n=1}^{\infty} R_n R_n^* = I_{\mathscr{K}}$.

In §3 we use the above-mentioned theorems to obtain the Sz.-Nagy-Foiaş lifting theorem [7, 8, 1, 4] in our setting.

In a subsequent paper we will use the results of this paper for studying the "characteristic function" associated to a sequence $\{T_n\}_{n=1}^{\infty}$ with $\sum_{n=1}^{\infty} T_n T_n^* \leq I_{\mathscr{H}}$.

1

Throughout this paper Λ stands for the set $\{1, 2, \dots, k\}$ $(k \in \mathbf{N})$ or the set $\mathbf{N} = \{1, 2, \dots\}$.

Received by the editors September 1, 1987.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 47A20; Secondary 47A45.
Key words and phrases. Isometric dilation, Wold decomposition, lifting theorem.

For every $n \in \mathbf{N}$ let $F(n, \Lambda)$ be the set of all functions from the set $\{1, 2, \dots, n\}$ to Λ and

$$\mathscr{F} = \bigcup_{n=0}^{\infty} F(n, \Lambda), \quad \text{where } F(0, \Lambda) = \{0\}.$$

Let $\mathscr{H}$ be a Hilbert space and $\mathscr{V} = \{V_\lambda\}_{\lambda \in \Lambda}$ be a sequence of isometries on $\mathscr{H}$. For any $f \in F(n, \Lambda)$ we denote by V_f the product $V_{f(1)} V_{f(2)} \cdots V_{f(n)}$ and $V_0 = I_{\mathscr{H}}$.

A subspace $\mathscr{L} \subset \mathscr{H}$ will be called *wandering* for the sequence $\mathscr{V}$ if for any distinct functions f, $g \in \mathscr{F}$ we have

$$V_f \mathscr{L} \perp V_g \mathscr{L} \quad (\perp \text{ means orthogonal}).$$

In this case we can form the orthogonal sum

$$M_{\mathscr{F}}(\mathscr{L}) = \bigoplus_{f \in \mathscr{F}} V_f \mathscr{L}.$$

A sequence $\mathscr{V} = \{V_\lambda\}_{\lambda \in \Lambda}$ of isometries on $\mathscr{H}$ is called a Λ-orthogonal shift if there exists in $\mathscr{H}$ a subspace $\mathscr{L}$, which is wandering for $\mathscr{V}$ and such that $\mathscr{H} = M_{\mathscr{F}}(\mathscr{L})$.

This subspace $\mathscr{L}$ is uniquely determined by $\mathscr{V}$: indeed we have $\mathscr{L} = \mathscr{H} \ominus (\bigoplus_{\lambda \in \Lambda} V_\lambda \mathscr{H})$. The dimension of $\mathscr{L}$ is called the multiplicity of the Λ-orthogonal shift. One can show, by an argument similar to the classical unilateral shift, that a Λ-orthogonal shift is determined up to unitary equivalence by its multiplicity. It is easy to see that for $\Lambda = \{1\}$ we find again the classical unilateral shift.

Let us make some simple remarks whose proofs will be omitted.

Remark 1.1. If $\mathscr{V} = \{V_\lambda\}_{\lambda \in \Lambda}$ is a Λ-orthogonal shift on $\mathscr{H}$, with the wandering subspace $\mathscr{L}$, then for any $n \in \mathbf{N}$, $\lambda \in \Lambda$ and $f \in F(n, \Lambda)$ we have

(a)

$$V_\lambda^* V_f = \begin{cases} V_{f(2)} V_{f(3)} \cdots V_{f(n)} & \text{if } f(1) = \lambda, \\ 0 & \text{if } f(1) \neq \lambda, \end{cases}$$

and $V_\lambda^* \ell = 0 \quad (\ell \in \mathscr{L})$.

(b) $\sum_{\lambda \in \Lambda} V_\lambda V_\lambda^* + P_{\mathscr{L}} = I_{\mathscr{H}}$, where $P_{\mathscr{L}}$ stands for the orthogonal projection from $\mathscr{H}$ into $\mathscr{L}$.

Remark 1.2. If $\mathscr{V} = \{V_\lambda\}_{\lambda \in \Lambda}$ is a Λ-orthogonal shift on $\mathscr{H}$ then

(a) $\lim_{n \to \infty} \sum_{f \in F(n, \Lambda)} \|V_f^* h\|^2 = 0$, for any $h \in \mathscr{H}$.

(b) $V_\lambda^{*k} \to 0$ (strongly) as $k \to \infty$, for any $\lambda \in \Lambda$.

(c) There exists no nonzero reducing subspace $\mathscr{H}_0 \subset \mathscr{H}$ for each V_λ $(\lambda \in \Lambda)$ such that $(I_{\mathscr{H}} - \sum_{\lambda \in \Lambda} V_\lambda V_\lambda^*)|_{\mathscr{H}_0} = 0$.

Let us consider a model Λ-orthogonal shift.

Form the Hilbert space

$$l^2(\mathscr{F},\mathscr{H}) = \left\{ (h_f)_{f\in\mathscr{F}};\ \sum_{f\in\mathscr{F}} \|h_f\|^2 < \infty, h_f \in \mathscr{H} \right\}.$$

We embed $\mathscr{H}$ in $l^2(\mathscr{F},\mathscr{H})$ as a subspace, by identifying the element $h \in \mathscr{H}$ with the element $(h_f)_{f\in\mathscr{F}}$, where $h_0 = h$ and $h_f = 0$ for any $f \in \mathscr{F}$, $f \neq 0$.

For each $\lambda \in \Lambda$ we define the operator S_λ on $l^2(\mathscr{F},\mathscr{H})$ by $S_\lambda((h_f)_{f\in\mathscr{F}}) = (h'_g)_{g\in\mathscr{F}}$, where $h'_0 = 0$ and for $g \in F(n,\Lambda)$ $(n \geq 1)$

$$h'_g = \begin{cases} h_0 & \text{if } g \in F(1,\Lambda) \text{ and } g(1) = \lambda, \\ h_f & \text{if } g \in F(n,\Lambda)\ (n \geq 2),\ f \in F(n-1,\Lambda) \text{ and } g(1) = \lambda, \\ & g(2) = f(1),\ g(3) = f(2), \ldots, g(n) = f(n-1), \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that $\{S_\lambda\}_{\lambda\in\Lambda}$ is the Λ-orthogonal shift, acting on $l^2(\mathscr{F},\mathscr{H})$, with the wandering subspace $\mathscr{H}$.

This model plays an important role in this paper. The following theorem is our version of Wold decomposition for a sequence of isometries.

Theorem 1.3. *Let $\mathscr{V} = \{V_\lambda\}_{\lambda\in\Lambda}$ be a sequence of isometries on a Hilbert space $\mathscr{K}$ such that $\sum_{\lambda\in\Lambda} V_\lambda V_\lambda^* \leq I_{\mathscr{K}}$.*

Then $\mathscr{K}$ decomposes into an orthogonal sum $\mathscr{K} = \mathscr{K}_0 \oplus \mathscr{K}_1$ such that $\mathscr{K}_0$ and $\mathscr{K}_1$ reduce each operator V_λ $(\lambda \in \Lambda)$ and we have $(I_{\mathscr{K}} - \sum_{\lambda\in\Lambda} V_\lambda V_\lambda^*)|_{\mathscr{K}_1} = 0$ and $\{V_\lambda|_{\mathscr{K}_0}\}_{\lambda\in\Lambda}$ is a Λ-orthogonal shift acting on $\mathscr{K}_0$.

This decomposition is uniquely determined, indeed we have

$$\mathscr{K}_1 = \bigcap_{n=0}^{\infty} \left(\bigoplus_{f\in F(n,\Lambda)} V_f \mathscr{K} \right);$$

$\mathscr{K}_0 = M_{\mathscr{F}}(\mathscr{L})$, *where* $\mathscr{L} = \mathscr{K} \ominus (\bigoplus_{\lambda\in\Lambda} V_\lambda \mathscr{K})$.

Proof. It is easy to see that the subspace $\mathscr{L} = \mathscr{K} \ominus (\bigoplus_{\lambda\in\Lambda} V_\lambda \mathscr{K})$ is wandering for $\mathscr{V}$.

Now let $\mathscr{K}_0 = M_{\mathscr{F}}(\mathscr{L})$ and $\mathscr{K}_1' = \mathscr{K} \ominus \mathscr{K}_0$. Observe that $k \in \mathscr{K}_1'$ if and only if $k \perp \bigoplus_{f\in\mathscr{F}_n} V_f \mathscr{L}$ for every $n \in \mathbf{N}$, where $\mathscr{F}_n$ stands for $\bigcup_{k=0}^{n} F(k,\Lambda)$.

We have

$$\mathscr{L} \oplus \left(\bigoplus_{f\in F(1,\Lambda)} V_f \mathscr{L} \right) \oplus \cdots \oplus \left(\bigoplus_{g\in F(n,\Lambda)} V_g \mathscr{L} \right) = \left[\mathscr{K} \ominus \left(\bigoplus_{f\in F(1,\Lambda)} V_f \mathscr{K} \right) \right]$$
$$\oplus \left[\left(\bigoplus_{f\in F(1,\Lambda)} V_f \mathscr{K} \right) \ominus \left(\bigoplus_{f\in F(2,\Lambda)} V_f \mathscr{K} \right) \right]$$
$$\oplus \cdots \oplus \left[\left(\bigoplus_{f\in F(n,\Lambda)} V_f \mathscr{K} \right) \ominus \left(\bigoplus_{f\in F(n+1,\Lambda)} V_f \mathscr{K} \right) \right]$$
$$= \mathscr{K} \ominus \left(\bigoplus_{f\in F(n+1,\Lambda)} V_f \mathscr{K} \right).$$

Thus $k \in \mathscr{K}_1'$ if and only if $k \in \bigoplus_{f\in F(n+1,\Lambda)} V_f \mathscr{K}$ for every $n \in \mathbf{N}$. Since

$$\bigoplus_{f\in F(n,\Lambda)} V_f \mathscr{K} \supset \bigoplus_{f\in F(n+1,\Lambda)} V_f \mathscr{K} \qquad (n \in \mathbf{N})$$

it follows that

$$\mathscr{K}_1' = \bigcap_{n=0}^{\infty} \left(\bigoplus_{f\in F(n,\Lambda)} V_f \mathscr{K} \right) = \mathscr{K}_1.$$

Let us notice that

$$V_\lambda \mathscr{K}_1 \subset \bigcap_{n=0}^{\infty} \left(\bigoplus_{f\in F(n,\Lambda)} V_\lambda V_f \mathscr{K} \right) \subset \bigcap_{n=0}^{\infty} \left(\bigoplus_{g\in F(n+1,\Lambda)} V_g \mathscr{K} \right) = \mathscr{K}_1,$$

$$V_\lambda^* \mathscr{K}_1 \subset \bigcap_{n=1}^{\infty} \left(V_\lambda^* \left(\bigoplus_{\substack{g\in F(n,\Lambda)\\ g(1)=\lambda}} V_g \mathscr{K} \right) \right) = \bigcap_{n=1}^{\infty} \left(\bigoplus_{f\in F(n-1,\Lambda)} V_f \mathscr{K} \right) = \mathscr{K}_1.$$

Therefore $\mathscr{K}_1$ reduces each V_λ $(\lambda \in \Lambda)$. Hence $\mathscr{K}_0$ also reduces each V_λ $(\lambda \in \Lambda)$.

Since $\mathscr{K}_1 \subset \bigoplus_{\lambda\in\Lambda} V_\lambda \mathscr{K}$ it follows that $(I_{\mathscr{K}} - \sum_{\lambda\in\Lambda} V_\lambda V_\lambda^*)|_{\mathscr{K}_1} = 0$. The fact that $\{V_\lambda|_{\mathscr{K}_0}\}_{\lambda\in\Lambda}$ is a Λ-orthogonal shift is obvious. The uniqueness of the decomposition follows by an argument similar to the classical Wold decomposition [8, Chapter I, Theorem 1.1]. The proof is completed.

Remark 1.4. The subspaces $\mathscr{K}_0$, $\mathscr{K}_1$ from Wold decomposition can be described as follows:

$$\mathscr{K}_0 = \left\{ k \in \mathscr{K} : \lim_{n\to\infty} \sum_{f\in F(n,\Lambda)} \|V_f^* k\|^2 = 0 \right\},$$

$$\mathscr{K}_1 = \left\{ k \in \mathscr{K} : \sum_{f\in F(n,\Lambda)} \|V_f^* k\|^2 = \|k\|^2 \text{ for every } n \in \mathbf{N} \right\}.$$

We call the sequence $\mathscr{V} = \{V_\lambda\}_{\lambda\in\Lambda}$ in Theorem 1.3 *pure* if $\mathscr{K}_1 = 0$, that is, if $\mathscr{V}$ is a Λ-orthogonal shift on $\mathscr{K}$.

2

Let $\mathscr{T} = \{T_\lambda\}_{\lambda\in\Lambda}$ a sequence of contractions on a Hilbert space $\mathscr{H}$ such that $\sum_{\lambda\in\Lambda} T_\lambda T_\lambda^* \leq I_{\mathscr{H}}$.

We say that a sequence $\mathscr{V} = \{V_\lambda\}_{\lambda\in\Lambda}$ of isometries on a Hilbert space $\mathscr{K} \supset \mathscr{H}$ is a minimal isometric dilation of $\mathscr{T}$ if the following conditions hold:

(a) $\sum_{\lambda\in\Lambda} V_\lambda V_\lambda^* \leq I_{\mathscr{K}}$.
(b) $\mathscr{H}$ is invariant for each V_λ^* $(\lambda\in\Lambda)$ and $V_\lambda^*|_{\mathscr{H}} = T_\lambda^*$ $(\lambda\in\Lambda)$.
(c) $\mathscr{K} = \bigvee_{f\in\mathscr{F}} V_f\mathscr{H}$.

Let D_* on $\mathscr{H}$ and D on $\bigoplus_{\lambda\in\Lambda}\mathscr{H}_\lambda$ ($\mathscr{H}_\lambda$ is a copy of $\mathscr{H}$) be the positive operators uniquely defined by $D_* = (I_{\mathscr{H}} - \sum_{\lambda\in\Lambda} T_\lambda T_\lambda^*)^{1/2}$ and $D = D_T$, where T stands for the matrix $[T_1, T_2, \ldots]$ and $D_T = (I - T^*T)^{1/2}$.

Let us denote $\mathscr{D}_* = \overline{D_*\mathscr{H}}$ and $\mathscr{D} = \overline{D\left(\bigoplus_{\lambda\in\Lambda}\mathscr{H}_\lambda\right)}$.

Theorem 2.1. *For every sequence $\mathscr{T} = \{T_\lambda\}_{\lambda\in\Lambda}$ of noncommuting operators on a Hilbert space $\mathscr{H}$ such that $\sum_{\lambda\in\Lambda} T_\lambda T_\lambda^* \leq I_{\mathscr{H}}$, there exists a minimal isometric dilation $\mathscr{V} = \{V_\lambda\}_{\lambda\in\Lambda}$ on a Hilbert space $\mathscr{K}\supset\mathscr{H}$, which is uniquely determined up to an isomorphism.*

Proof. Let us consider the Hilbert space $\mathscr{K} = \mathscr{H}\oplus l^2(\mathscr{F},\mathscr{D})$. We embed $\mathscr{H}$ and $\mathscr{D}$ into $\mathscr{K}$ in a natural way. For each $\lambda\in\Lambda$ we define the isometry $V_\lambda\colon \mathscr{K}\to\mathscr{K}$ by the relation

$$V_\lambda(h\oplus(d_f)_{f\in\mathscr{F}}) = T_\lambda h \oplus (D(\underbrace{0,\ldots,0}_{\lambda-1\text{ times}},h,0,\ldots) + S_\lambda(d_f)_{f\in\mathscr{F}}) \tag{2.1}$$

where $\{S_\lambda\}_{\lambda\in\Lambda}$ is Λ-orthogonal shift on $l^2(\mathscr{F},\mathscr{D})$ (see §1).

Obviously, for any λ, $\mu\in\Lambda$, $\lambda\neq\mu$ we have $\operatorname{range} S_\lambda \perp \operatorname{range} S_\mu$ and

$$(T_\mu^* T_\lambda h, h') = -(D^2(\underbrace{0,\ldots,0}_{\lambda-1\text{ times}},h,0,\ldots),(\underbrace{0,\ldots,0}_{\mu-1\text{ times}},h',0,\ldots)).$$

Hence, taking into account (2.1), it follows that

$$\operatorname{range} V_\lambda \perp \operatorname{range} V_\mu \qquad (\lambda,\mu\in\Lambda,\lambda\neq\mu)$$

therefore $\sum_{\lambda\in\Lambda} V_\lambda V_\lambda^* \leq I_{\mathscr{K}}$.

It is easy to show that $\mathscr{H}$ is invariant for each V_λ^* $(\lambda\in\Lambda)$ and $V_\lambda^*|_{\mathscr{H}} = T_\lambda^*$ $(\lambda\in\Lambda)$.

Finally, we verify that $\mathscr{V} = \{V_\lambda\}_{\lambda\in\Lambda}$ is the minimal isometric dilation of $\mathscr{T}$.

Let $\mathscr{H}_1 = \mathscr{H}\vee(\bigvee_{f\in F(1,\Lambda)} V_f\mathscr{H})$ and

$$\mathscr{H}_n = \mathscr{H}_{n-1}\vee\left(\bigvee_{f\in F(1,\Lambda)} V_f\mathscr{H}_{n-1}\right) \quad \text{if } n\geq 2.$$

It is easy to see that $\mathscr{H}_1 = \mathscr{H} \oplus \mathscr{D}$ and

$$\mathscr{H}_n = \mathscr{H} \oplus \mathscr{D} \oplus \left(\bigoplus_{f \in F(1,\Lambda)} S_f \mathscr{D} \right) \oplus \cdots \oplus \left(\bigoplus_{f \in F(n-1,\Lambda)} S_f \mathscr{D} \right) \quad \text{if } n \geq 2.$$

Clearly $\mathscr{H}_n \subset \mathscr{H}_{n+1}$ and we have

$$\bigvee_1^\infty \mathscr{H}_n = \mathscr{H} \oplus M_{\mathscr{F}}(\mathscr{D}) = \mathscr{H} \oplus l^2(\mathscr{F}, \mathscr{D}) = \mathscr{K}.$$

Therefore $\mathscr{K} = \bigvee_{f \in \mathscr{F}} V_f \mathscr{H}$.

Following Theorem 4.1 in [8, Chapter I] it is easy to show that the minimal isometric dilation $\mathscr{V}$ of $\mathscr{T}$ is unique up to a unitary operator. To be more precise, let $\mathscr{V}' = \{V_\lambda'\}_{\lambda \in \Lambda}$ be another minimal isometric dilation of $\mathscr{T}$, on a Hilbert space $\mathscr{K}' \supset \mathscr{H}$. Then there exists a unitary operator $U: \mathscr{K} \to \mathscr{K}'$ such that $V_\lambda' U = U V_\lambda$ $(\lambda \in \Lambda)$ and $Uh = h$ for every $h \in \mathscr{H}$.

This completes the proof.

Remark 2.2. For each $\lambda \in \Lambda$, $V_\lambda^{*n} \to 0$ (strongly) as $n \to \infty$ if and only if $T_\lambda^{*n} \to 0$ (strongly) as $n \to \infty$.

From this remark and Theorem 2.1 one can easily deduce Proposition 1.1 in [5].

The following is a generalization of [2] or Theorem 1.2 in [8, Chapter II].

Propostion 2.3. *Let $\mathscr{V} = \{V_\lambda\}_{\lambda \in \Lambda}$ be the minimal isometric dilation of $\mathscr{T} = \{T_\lambda\}_{\lambda \in \Lambda}$. Then $\mathscr{V}$ is pure if and only if*

$$\lim_{n \to \infty} \sum_{f \in F(n,\Lambda)} \|T_f^* h\|^2 = 0 \tag{2.2}$$

for any $h \in \mathscr{H}$.

Proof. Assume that $\mathscr{V}$ is pure. Then, by Theorem 1.3 it follows that $\mathscr{V}$ is a Λ-orthogonal shift on the space $\mathscr{K} \supset \mathscr{H}$ of the minimal isometric dilation of $\mathscr{T}$.

Taking into account Remark 1.2 and the fact that for each $f \in \mathscr{F}$, $V_f^*|_{\mathscr{H}} = T_f^*$, we have

$$\lim_{n \to \infty} \sum_{f \in F(n,\Lambda)} \|T_f^* h\|^2 = \lim_{n \to \infty} \sum_{f \in F(n,\Lambda)} \|V_f^* h\|^2 = 0 \quad \text{for any } h \in \mathscr{H}.$$

Conversely, assume that (2.2) holds. We claim that

$$\lim_{n \to \infty} \sum_{f \in F(n,\Lambda)} \|V_f^* k\|^2 = 0 \quad \text{for any } k \in \mathscr{K} = \bigvee_{f \in \mathscr{F}} V_f \mathscr{H}. \tag{2.3}$$

By (2.2) we have

$$\lim_{n \to \infty} \sum_{f \in F(n,\Lambda)} \|V_f^* h\|^2 = 0 \qquad (h \in \mathscr{H}).$$

For each $k \in \bigvee_{f\in\mathscr{F};f\neq 0} V_f\mathscr{H}$ and any $\varepsilon > 0$, there exists

$$k_\varepsilon = \sum_{g\in\mathscr{F};g\neq 0}' V_g h_g \qquad (h_g \in \mathscr{H})$$

such that $\|k - k_\varepsilon\| < \varepsilon$. (Here $\sum'$ stands for a finite sum.)

Since the isometries V_λ $(\lambda \in \Lambda)$ have orthogonal final spaces, it follows that

$$\lim_{n\to\infty} \sum_{f\in F(n,\Lambda)} \|V_f^* k\|^2 = \lim_{n\to\infty} \sum_{f\in F(n,\Lambda)} \|V_f^*(k - k_\varepsilon)\|^2 \leq \|k - k_\varepsilon\|^2 < \varepsilon^2,$$

for any $\varepsilon > 0$. Thus, (2.3) holds and by Remark 1.4 we have that $\mathscr{V}$ is pure. This completes the proof.

Corollary 2.4. *If $\sum_{\lambda\in\Lambda} T_\lambda T_\lambda^* \leq rI_{\mathscr{H}}$, $r < 1$, then the minimal isometric dilation of $\mathscr{T} = \{T_\lambda\}_{\lambda\in\Lambda}$ is pure.*

Now let us establish when the minimal isometric dilation $\mathscr{V} = \{V_\lambda\}_{\lambda\in\Lambda}$ cannot contain a Λ-orthogonal shift. The notations being the same as above we have

Proposition 2.5. *$\sum_{\lambda\in\Lambda} V_\lambda V_\lambda^* = I_{\mathscr{K}}$ if and only if $\sum_{\lambda\in\Lambda} T_\lambda T_\lambda^* = I_{\mathscr{H}}$.*

Proof. $(\Rightarrow)$ Since $V_\lambda^*|_{\mathscr{H}} = T_\lambda^*$ $(\lambda \in \Lambda)$ it follows that $\sum_{\lambda\in\Lambda} T_\lambda T_\lambda^* h = h$ $(h \in \mathscr{H})$.

$(\Leftarrow)$ If $\sum_{\lambda\in\Lambda} T_\lambda T_\lambda^* = I_{\mathscr{H}}$ then $\sum_{f\in F(n,\Lambda)} \|T_f^* h\|^2 = \|h\|^2$ for any $n \in \mathbf{N}$ and $h \in \mathscr{H}$. Taking into account Theorem 1.3 let us assume that there exists $k \in \mathscr{K} \ominus (\bigoplus_{\lambda\in\Lambda} V_\lambda \mathscr{K})$, $k \neq 0$. Using Remark 1.4 it follows that

$$\lim_{n\to\infty} \sum_{f\in F(n,\Lambda)} \|V_f^* k\|^2 = 0. \tag{2.4}$$

On the other hand, since

$$\mathscr{K} = \mathscr{H} \vee \left(\bigvee_{\substack{f\in\mathscr{F}\\ f\neq 0}} V_f \mathscr{H} \right) \quad \text{and} \quad \bigvee_{\substack{f\in\mathscr{F}\\ f\neq 0}} V_f\mathscr{H} \subset \bigoplus_{\lambda\in\Lambda} V_\lambda \mathscr{K}$$

it follows that $k \in \mathscr{H}$ and by (2.4) that $\lim_{n\to\infty} \sum_{f\in F(n,\Lambda)} \|T_f^* k\|^2 = 0$, contradiction. Thus we have $\sum_{\lambda\in\Lambda} V_\lambda V_\lambda^* = I_{\mathscr{K}}$ and the proof is complete.

Dropping out the minimality condition in the definition of the isometric dilation of a sequence $\mathscr{T} = \{T_\lambda\}_{\lambda\in\Lambda}$, we can prove the following.

Proposition 2.6. *For any sequence $\mathscr{T} = \{T_\lambda\}_{\lambda\in\Lambda}$ of operators on a Hilbert space $\mathscr{H}$ such that $\sum_{\lambda\in\Lambda} T_\lambda T_\lambda^* \leq I_{\mathscr{H}}$ there exists an isometric dilation $\mathscr{V} = \{V_\lambda\}_{\lambda\in\Lambda}$ on a Hilbert space $\mathscr{K} \supset \mathscr{H}$ such that $\sum_{\lambda\in\Lambda} V_\lambda V_\lambda^* = I_{\mathscr{K}}$.*

Proof. Taking into account Theorems 2.1 and 1.3, we show, without loss of generality, that the Λ-orthogonal shift $\mathscr{S} = \{S_\lambda\}_{\lambda\in\Lambda}$ on $\mathscr{K}_0 = l^2(\mathscr{F}, \mathscr{E})$ ($\mathscr{E}$ is a Hilbert space) can be extended to a sequence $\mathscr{V} = \{V_\lambda\}_{\lambda\in\Lambda}$ of isometries on a Hilbert space $\mathscr{K}_0 \supset \mathscr{H}_0$ such that

$$\sum_{\lambda\in\Lambda} V_\lambda V_\lambda^* = I_{\mathscr{K}_0} \quad \text{and} \quad V_\lambda|_{\mathscr{H}_0} = S_\lambda \qquad (\lambda\in\Lambda). \tag{2.5}$$

Consider the Hilbert space

$$\mathscr{K} = [l^2(\mathscr{F}, \mathscr{E}) \ominus \mathscr{E}] \oplus l^2(\mathscr{F}, \mathscr{E}).$$

We embed $l^2(\mathscr{F}, \mathscr{E})$ into $\mathscr{K}$ by identifying the element $\{e_f\}_{f\in\mathscr{F}} \in l^2(\mathscr{F}, \mathscr{E})$ with the element $0 \oplus \{e_f\}_{f\in\mathscr{F}} \in \mathscr{K}$.

Let us define the isometries V_λ $(\lambda\in\Lambda)$ on $\mathscr{K}$. For $\lambda \geq 2$ we set $V_\lambda = S_\lambda|_{l^2(\mathscr{F},\mathscr{E})\ominus\mathscr{E}} \oplus S_\lambda$.

Consider the countable set

$$\mathscr{F}' = \{f \in \mathscr{F}\backslash F(1, \Lambda): f(1) = 1\} \cup F(1, \Lambda) \cup \{0\}$$

and a one-to-one map $\gamma: \mathscr{F}\backslash\{0\} \to \mathscr{F}'$.

For $\{e_f^*\}_{f\in\mathscr{F}\backslash\{0\}} \oplus \{e_f\}_{f\in\mathscr{F}} \in \mathscr{K}$ the isometry V_1 is defined as follows

$$V_1(0 \oplus \{e_f\}_{f\in\mathscr{F}}) = 0 \oplus S_1(\{e_f\}_{f\in\mathscr{F}}),$$

$$V_1(\{e_f^*\}_{f\in\mathscr{F}\backslash\{0\}} \oplus 0) = \{e_g'^*\}_{g\in\mathscr{F}\backslash\{0\}} \oplus \{e_g'\}_{g\in\mathscr{F}},$$

where

$$e_g'^* = \begin{cases} e_f^* & \text{if } g = \gamma(f), \\ 0 & \text{otherwise} \end{cases}$$

and

$$e_0' = e_f^* \quad \text{if } \gamma(f) = 0, \qquad e_g' = 0 \quad \text{if } g \in \mathscr{F}\backslash\{0\}.$$

Now it is easy to see that the relations (2.5) hold.

Following the classification of contractions from [8] we give, in what follows, a classification of the sequences of contractions.

Let $\mathscr{T} = \{T_\lambda\}_{\lambda\in\Lambda}$ on a Hilbert space $\mathscr{H}$ such that $\sum_{\lambda\in\Lambda} T_\lambda T_\lambda^* \leq I_{\mathscr{H}}$.

Consider the following subspace of $\mathscr{H}$:

$$\mathscr{H}_0 = \left\{ h \in \mathscr{H}: \lim_{n\to\infty} \sum_{f\in F(n,\Lambda)} \|T_f^* h\|^2 = 0 \right\}, \tag{2.6}$$

$$\mathscr{H}_1 = \left\{ h \in \mathscr{H}: \sum_{f\in F(n,\Lambda)} \|T_f^* h\|^2 = \|h\|^2 \text{ for any } n \in \mathbf{N} \right\}. \tag{2.7}$$

Remark 2.7. The subspaces $\mathscr{H}_0$ and $\mathscr{H}_1$ are orthogonal and invariant for each operator T_λ^* $(\lambda\in\Lambda)$.

Proof. Taking into account Theorem 2.1, 1.3 and Remark 1.4 the proof is immediately.

Thus, the Hilbert space $\mathscr{H}$ decomposes into an orthogonal sum $\mathscr{H} = \mathscr{H}_0 \oplus \mathscr{H}_1 \oplus \mathscr{H}_2$.

For each $k \in \{0, 1, 2\}$ we shall denote by $C^{(k)}$ (respectively $C_{(k)}$) the set of all sequences $\mathcal{T} = \{T_\lambda\}_{\lambda\in\Lambda}$ on $\mathcal{H}$ for which we have $\mathcal{H}_k = \{0\}$ (respectively $\mathcal{H} = \mathcal{H}_k$).

Let us mention that $\mathcal{H}_1$ is the largest subspace in $\mathcal{H}$ on which the matrix

$$\begin{bmatrix} T_1^* \\ T_2^* \\ \vdots \end{bmatrix}$$

acts isometrically.

Consequently, a sequence $\mathcal{T} \in C^{(1)}$ will be also called completely noncoisometric (c.n.c).

In the particular case when $\mathcal{T} = \{T\}$ ($\|T\| \leq 1$) we have that $\mathcal{T} \in C^{(1)}$ if and only if T^* is completely nonisometric, that is, if there is no nonzero invariant subspace for T^* on which T^* is an isometry.

We continue this section with the study of the geometric structure of the space of the minimal isometric dilation.

For this, let $\mathcal{T} = \{T_\lambda\}_{\lambda\in\Lambda}$ be a sequence of operators on a Hilbert space $\mathcal{H}$ such that $\sum_{\lambda\in\Lambda} T_\lambda T_\lambda^* \leq I_{\mathcal{H}}$ and $\mathcal{V} = \{V_\lambda\}_{\lambda\in\Lambda}$ be the minimal isometric dilation of $\mathcal{T}$ on the Hilbert space $\mathcal{K} = \mathcal{H} \oplus l^2(\mathcal{F}, \mathcal{D})$ (see Theorem 2.1).

Considering the subspaces of $\mathcal{K}$

$$\mathcal{L} = \bigvee_{\lambda\in\Lambda} (V_\lambda - T_\lambda)\mathcal{H} \quad \text{and} \quad \mathcal{L}_* = \overline{\left(I_{\mathcal{K}} - \sum_{\lambda\in\Lambda} V_\lambda T_\lambda^*\right)\mathcal{H}}$$

we can generalize some of the results from [8, Chapter II, §§1,2] concerning the geometric structure of the space of the minimal isometric dilation.

Theorem 2.8. (i) *The subspaces $\mathcal{L}$ and $\mathcal{L}_*$ are wandering subspaces for $\mathcal{V}$ and*

$$\dim \mathcal{L} = \dim \mathcal{D}; \quad \dim \mathcal{L}_* = \dim \mathcal{D}_*.$$

(ii) *The space $\mathcal{K}$ can be decomposed as follows:*

$$\mathcal{K} = \mathcal{R} \oplus M_{\mathcal{F}}(\mathcal{L}_*) = \mathcal{H} \oplus M_{\mathcal{F}}(\mathcal{L}),$$

and the subspace $\mathcal{R}$ reduces each operator V_λ $(\lambda \in \Lambda)$.

(iii) $\mathcal{L} \cap \mathcal{L}_* = 0$.

(iv) *The subspace $\mathcal{R}$ reduces to $\{0\}$ if and only if $\mathcal{T} \in C_{(0)}$.*

Proof. The Wold decomposition (see Theorem 1.3) for the minimal isometric dilation $\mathcal{V}$ on the space $\mathcal{K} = \mathcal{H} \oplus l^2(\mathcal{F}, \mathcal{D})$ gives $\mathcal{K} = \mathcal{R} \oplus M_{\mathcal{F}}(\mathcal{L}_*')$, where $\mathcal{R} = \bigcap_{n=0}^\infty [\bigoplus_{f\in F(n,\Lambda)} V_f \mathcal{K}]$ reduces each operator V_λ $(\lambda \in \Lambda)$ and $\mathcal{L}_*' = \mathcal{K} \ominus (\bigoplus_{\lambda\in\Lambda} V_\lambda \mathcal{K})$ is a wandering subspace for $\mathcal{V}$.

It is easy to see that $\mathcal{L}_*' = \mathcal{L}_*$ and that the operator $\Phi_* : \mathcal{L}_* \to \mathcal{D}_*$ defined by

$$\Phi_* \left(I_{\mathcal{K}} - \sum_{\lambda\in\Lambda} V_\lambda T_\lambda^*\right) h = D_* h \qquad (h \in \mathcal{H})$$

is unitary. Hence it follows that $\dim \mathscr{L}_* = \dim \mathscr{D}_*$. Equation (2.1) yields

$$\sum_{\lambda\in\Lambda}(V_\lambda - T_\lambda)h_\lambda = 0 \oplus D((h_\lambda)_{\lambda\in\Lambda}) \quad \text{for } (h_\lambda)_{\lambda\in\Lambda} \in \bigoplus_{\lambda\in\Lambda} \mathscr{H}_\lambda$$

($\mathscr{H}_\lambda$ is a copy of $\mathscr{H}$).

By this relation we deduce that there exists a unitary operator $\Phi: \mathscr{L} \to \mathscr{D}$ defined by equation

$$\Phi\left(\sum_{\lambda\in\Lambda}(V_\lambda - T_\lambda)h_\lambda\right) = D((h_\lambda)_{\lambda\in\Lambda})$$

and hence that $\dim \mathscr{L} = \dim \mathscr{D}$.

The fact that $\mathscr{L}$ is a wandering subspace for $\mathscr{V}$ and that $\mathscr{H} \perp M_{\mathscr{G}}(\mathscr{L})$ follows from the form of the isometries V_λ $(\lambda \in \Lambda)$ defined by (2.1).

Taking into account the minimality of $\mathscr{K}$ it follows that $\mathscr{K} = \mathscr{H} \oplus M_{\mathscr{G}}(\mathscr{L})$.

Let us now show that $\mathscr{L} \cap \mathscr{L}_* = 0$. First we need to prove that

$$\mathscr{L}_* \oplus \left(\bigoplus_{\lambda\in\Lambda} V_\lambda \mathscr{H}\right) = \mathscr{H} \oplus \mathscr{L}. \tag{2.8}$$

This follows from the fact that, for an element $u \in \mathscr{K}$, the possibility of a representation of the form

$$u = \left(I_{\mathscr{H}} - \sum_{\lambda\in\Lambda} V_\lambda T_\lambda^*\right) h_0 + \sum_{\lambda\in\Lambda} V_\lambda h_\lambda, \qquad h_0 \in \mathscr{H},\ (h_\lambda)_{\lambda\in\Lambda} \in \bigoplus_{\lambda\in\Lambda} \mathscr{H}_\lambda,$$

is equivalent to the possibility of a representation of the form

$$u = h^{(0)} + \sum_{\lambda\in\Lambda}(V_\lambda - T_\lambda)h^{(\lambda)}, \qquad h^{(0)} \in \mathscr{H},\ (h^{(\lambda)})_{\lambda\in\Lambda} \in \bigoplus \mathscr{H}_\lambda.$$

Indeed, we have only to set

$$h_0 = h^{(0)} - \sum_{\lambda\in\Lambda} T_\lambda h^{(\lambda)}, \quad h_\lambda = T_\lambda^* h^{(0)} + h^{(\lambda)}$$

and, conversely,

$$h^{(0)} = \sum_{\lambda\in\Lambda} T_\lambda h_\lambda + \left(I_{\mathscr{H}} - \sum_{\lambda\in\Lambda} T_\lambda T_\lambda^*\right) h_0, \quad h^{(\lambda)} = h_\lambda - T_\lambda^* h_0.$$

Thus (2.8) holds. On the other hand, since

$$\mathscr{L} \subset \left(\bigoplus_{\lambda\in\Lambda} V_\lambda \mathscr{H}\right) \vee \mathscr{H} \quad \text{and} \quad \bigoplus_{\lambda\in\Lambda} V_\lambda \mathscr{H} \subset \mathscr{L} \oplus \mathscr{H}$$

we have that $\mathscr{H} \vee (\bigoplus_{\lambda\in\Lambda} V_\lambda \mathscr{H}) = \mathscr{H} \oplus \mathscr{L}$. This relation and (2.8) show that $\mathscr{L} \cap \mathscr{L}_* = \{0\}$.

The statement (iv) is contained in Proposition 2.3. The proof is complete.

Propostion 2.9. *For every sequence* $\mathscr{T} = \{T_\lambda\}_{\lambda\in\Lambda}$ *of operators on* $\mathscr{H}$ *and for its minimal isometric dilation* $\mathscr{V} = \{V_\lambda\}_{\lambda\in\Lambda}$ *on* $\mathscr{K}$ *, we have*

$$M_{\mathscr{F}}(\mathscr{L}) \vee M_{\mathscr{F}}(\mathscr{L}_*) = \mathscr{K} \ominus \mathscr{H}_1 \tag{2.9}$$

where $\mathscr{H}_1$ *is given by* (2.7) *.*

In particular, if $\mathscr{T}$ *is c.n.c., then*

$$M_{\mathscr{F}}(\mathscr{L}) \vee M_{\mathscr{F}}(\mathscr{L}_*) = \mathscr{K}. \tag{2.10}$$

Proof. Taking into account Theorem 2.8 and that $\mathscr{H}_1 \subset \mathscr{R}$ it follows that $\mathscr{H}_1 \perp M_{\mathscr{F}}(\mathscr{L}) \vee M_{\mathscr{F}}(\mathscr{L}_*)$.

Now let $k \in \mathscr{K}$ be such that $k \perp M_{\mathscr{F}}(\mathscr{L})$ and $k \perp M_{\mathscr{F}}(\mathscr{L}_*)$.

From the same theorem it follows that $k \in \mathscr{H}$ and $k \perp V_f\mathscr{L}_*$ for every $f \in \mathscr{F}$. Hence we have

$$0 = \left(k, V_f\left(I_{\mathscr{H}} - \sum_{\lambda\in\Lambda} V_\lambda T_\lambda^*\right)h\right) = (T_f^*k, h) - \sum_{\lambda\in\Lambda}(T_\lambda^* T_f^* k, T_\lambda^* h)$$

for every $h \in \mathscr{H}$.

Choosing $h = T_f^*k$ $(f \in \mathscr{F})$ we obtain

$$\|T_f k\|^2 = \sum_{\lambda\in\Lambda} \|T_\lambda^* T_f^* k\|^2$$

for any $f \in \mathscr{F}$.

Hence we deduce

$$\sum_{g\in F(n,\Lambda)} \|T_g^* k\|^2 = \|k\|^2$$

for any $n \in \mathbf{N}$. We conclude that $k \in \mathscr{H}_1$. Conversely, for every $k \in \mathscr{H}_1$ it is easy to see that $k \perp M_{\mathscr{F}}(\mathscr{L}) \vee M_{\mathscr{F}}(\mathscr{L}_*)$. The relation (2.10) follows because for $\mathscr{T}$ c.n.c. we have $\mathscr{H}_1 = \{0\}$.

The last aim of this section is to generalize some of the results from [8, Chapter II, §3]. Throughout $\mathscr{V} = \{V_\lambda\}_{\lambda\in\Lambda}$ is the minimal isometric dilation of $\mathscr{T} = \{T_\lambda\}_{\lambda\in\Lambda}$. The space of the minimal isometric dilation is

$$\mathscr{K} = \mathscr{R} \oplus M_{\mathscr{F}}(\mathscr{L}_*) = \mathscr{H} \oplus l^2(\mathscr{F}, \mathscr{D}). \tag{2.11}$$

Proposition 2.10. *For every* $h \in \mathscr{H}$ *we have*

$$P_{\mathscr{R}} h = \lim_{n\to\infty} \sum_{f\in F(n,\Lambda)} V_f T_f^* h \tag{2.12}$$

and consequently

$$\|P_{\mathscr{R}} h\|^2 = \lim_{n\to\infty} \sum_{f\in F(n,\Lambda)} \|T_f^* h\|^2 \tag{2.13}$$

where $P_{\mathscr{R}}$ *denotes the orthogonal projection of* $\mathscr{K}$ *into* $\mathscr{H}$ *.*

Proof. An easy computation shows that

$$\left\| \sum_{f\in F(n+1,\Lambda)} V_f T_f^* h - \sum_{f\in F(n,\Lambda)} V_f T_f^* h \right\|^2 = \sum_{f\in F(n+1,\Lambda)} \|T_f^* h\|^2 - \sum_{f\in F(n,\Lambda)} \|T_f^* h\|^2 \leq 0$$

for every $n \in \mathbf{N}$. This implies the convergence of $\{\sum_{f\in F(n,\Lambda)} V_f T_f^* h\}_{n=1}^{\infty}$ the sequence in $\mathscr{K}$. Setting

$$k = \lim_{n\to\infty} \sum_{f\in F(n,\Lambda)} V_f T_f^* h,$$

let us show that $k = P_{\mathscr{R}} h$, i.e. $k \perp M_{\mathscr{F}}(\mathscr{L}_*)$ and $h - k \in M_{\mathscr{F}}(\mathscr{L}_*)$.

Since for every $g \in \mathscr{F}$ there exists $n_0 \in N$ such that

$$\sum_{f\in F(n,\Lambda)} V_f T_f^* h \perp V_g \mathscr{L}_*$$

for any $n \geq n_0$, it follows that $k \perp M_{\mathscr{F}}(\mathscr{L}_*)$.

On the other hand we have

$$h - \sum_{f\in F(n,\Lambda)} V_f T_f^* h = \left(I_{\mathscr{H}} - \sum_{\lambda\in\Lambda} V_\lambda T_\lambda^*\right) h + \sum_{f\in F(1,\Lambda)} V_f \left(I_{\mathscr{H}} - \sum_{\lambda\in\Lambda} V_\lambda T_\lambda^*\right) T_f^* h$$
$$+ \cdots + \sum_{f\in F(n-1,\Lambda)} V_g \left(I_{\mathscr{H}} - \sum_{\lambda\in\Lambda} V_\lambda T_\lambda^*\right) T_g^* h \in M_{\mathscr{F}}(\mathscr{L}_*)$$

and therefore

$$h - k = \lim_{n\to\infty} \left(h - \sum_{f\in F(n,\Lambda)} V_f T_f^* h\right) \in M_{\mathscr{F}}(\mathscr{L}_*).$$

This ends the proof.

Proposition 2.11. *Let $\mathscr{T} = \{T_\lambda\}_{\lambda\in\Lambda}$ be a sequence of operators on $\mathscr{H}$ such that the matrix $[T_1, T_2, \ldots]$ is an injection. Then $\overline{P_{\mathscr{R}}\mathscr{H}} = \mathscr{R}$.*

Proof. Let us suppose that there exists $k \in \mathscr{R}$, $k \neq 0$ such that $k \perp P_{\mathscr{R}}\mathscr{H}$, or equivalently, such that $k \perp M_{\mathscr{F}}(\mathscr{L}_*)$ and $k \perp \mathscr{H}$.

By Theorem 2.8 we have $\mathscr{K} = \mathscr{H} \oplus M_{\mathscr{F}}(\mathscr{L})$. It follows that $k \in M_{\mathscr{F}}(\mathscr{L})$ and hence $k = \sum_{f\in\mathscr{F}} V_f l_f$ where $l_f \in \mathscr{L}$ $(f \in \mathscr{F})$ and $\sum_{f\in\mathscr{F}} \|l_f\|^2 < \infty$. Since $k \neq 0$ there exists $f_0 \in \mathscr{F}$, such that $V_{f_0} l_{f_0} \neq 0$ and

$$V_{f_0}^* k = l_{f_0} + \sum_{\substack{g\in\mathscr{F}\\ g\neq 0}} V_g l'_g \qquad (l'_g \in \mathscr{L}).$$

One can easily show that for every $g \in \mathscr{F}$, $g \neq 0$, $V_g \mathscr{L} \perp \mathscr{L}_*$. Since $V_{f_0}^* k \perp \mathscr{L}_*$ it follows that $l_{f_0} \perp \mathscr{L}_*$. By the relation (2.8) we deduce that $l_{f_0} \in \bigoplus_{\lambda\in\Lambda} V_\lambda \mathscr{H}$.

Therefore, there exists a nonzero $\bigoplus_{\lambda\in\Lambda} h_\lambda \in \bigoplus_{\lambda\in\Lambda} \mathscr{H}_\lambda$ such that $l_{f_0} = \sum_{\lambda\in\Lambda} V_\lambda h_\lambda$. Since $\mathscr{L} \perp \mathscr{H}$, it follows that $\sum_{\lambda\in\Lambda} T_\lambda h_\lambda = 0$ which is a contradiction with the hypothesis.

Thus $\overline{P_{\mathscr{R}}\mathscr{H}} = \mathscr{R}$ and the proof is complete.

For each $\lambda \in \Lambda$ let us denote by R_λ the operator $V_\lambda|_{\mathscr{R}}$. Taking into account the Wold decomposition (Theorem 1.3) we have $\sum_{\lambda\in\Lambda} R_\lambda R_\lambda^* = I_{\mathscr{R}}$.

The following theorem is a generalization of Proposition 3.5 in [8, Chapter II].

Proposition 2.12. *Let $\mathscr{T} = \{T_\lambda\}_{\lambda\in\Lambda}$ a sequence of operators on $\mathscr{H}$ such that $\mathscr{T} \in C^{(0)}$ and the matrix $[T_1, T_2, \ldots]$ is an injective contraction.*

Then $\mathscr{T}$ is quasi-similar to $\{R_\lambda\}_{\lambda\in\Lambda}$, i.e., there exists a quasi-affinity Y from $\mathscr{R}$ to $\mathscr{H}$ such that $T_\lambda Y = YR_\lambda$ for every $\lambda \in \Lambda$.

Proof. According to Proposition 2.10 we have

$$\begin{aligned} V_\lambda^* P_{\mathscr{R}} h &= \lim_{n\to\infty} \sum_{f\in F(n,\Lambda)} V_\lambda^* V_f T_f^* h \\ &= \lim_{n\to\infty} \sum_{g\in F(n-1,\Lambda)} V_g T_g^* T_\lambda^* h = P_{\mathscr{R}} T_\lambda^* h \end{aligned}$$

for all $h \in \mathscr{H}$ and each $\lambda \in \Lambda$.

Setting $X = P_{\mathscr{R}}|_{\mathscr{H}}$ it follows that $R_\lambda^* X = XT_\lambda^*$ for every $\lambda \in \Lambda$. Let us show that X is a quasi-affinity.

Since $\mathscr{T} \in C^{(0)}$ we have that

$$\lim_{n\to\infty} \sum_{f\in F(n,\Lambda)} \|T_f^* h\|^2 = 0 \quad \text{for every nonzero } h \in \mathscr{H}.$$

By Proposition 2.10 we deduce that $P_{\mathscr{R}} h \neq 0$ for every nonzero $h \in \mathscr{H}$, i.e., X is an injection.

On the other hand, Proposition 2.11 shows that $\overline{X\mathscr{H}} = \mathscr{R}$.

If we take $Y = X^*$, this finishes the proof.

3

In this section we extend the Sz.-Nagy-Foiaş lifting theorem [7, 8, 1, 4] to our setting.

Let $\mathscr{T} = \{T_\lambda\}_{\lambda\in\Lambda}$ be a sequence of operators on $\mathscr{H}$ with $\sum_{\lambda\in\Lambda} T_\lambda T_\lambda^* \leq I_{\mathscr{H}}$ and $\mathscr{V} = \{V_\lambda\}_{\lambda\in\Lambda}$ be the minimal isometric dilation of the Hilbert space $\mathscr{K} = \mathscr{H} \oplus l^2(\mathscr{F}, \mathscr{D})$ (see Theorem 2.1).

Consider the following subspaces of $\mathscr{K}$

$$\mathscr{H}_1 = \mathscr{H} \vee \left(\bigvee_{f\in F(1,\Lambda)} V_f \mathscr{H} \right)$$

and

$$\mathscr{H}_n = \mathscr{H}_{n-1} \vee \left(\bigvee_{f\in F(1,\Lambda)} V_f \mathscr{H}_{n-1} \right) \quad \text{for } n \geq 2.$$

Note that $\mathscr{H}_n \subset \mathscr{H}_{n+1}$ and that all the space $\mathscr{H}_n$ $(n \geq 1)$ are invariant for each operator V_λ^* $(\lambda \in \Lambda)$.

As in [7, 8, 1, 4] the n-stepped dilation of $\mathscr{T}$ is the sequence $\mathscr{T}_n = \{(T_\lambda)_n\}_{\lambda\in\Lambda}$ of operators defined by $(T_\lambda)_n^* = V_\lambda^*|_{\mathscr{H}_n}$ $(n \geq 1, \lambda \in \Lambda)$.

One can easily show that $\mathscr{V}$ is the minimal isometric dilation on $\mathscr{T}_n$ and that $\mathscr{T}_{n+1}$ is the one-step dilation of $\mathscr{T}_n$.

Let us observe that $\mathscr{H}_1 = \mathscr{H} \oplus \mathscr{D}$ and

$$\mathscr{H}_n = \mathscr{H} \oplus \mathscr{D} \oplus \left(\bigoplus_{f\in F(1,\Lambda)} S_f \mathscr{D} \right) \oplus \cdots \oplus \left(\bigoplus_{f\in F(n-1,\Lambda)} S_f \mathscr{D} \right) \qquad (n \geq 2)$$

where $\mathscr{S} = \{S_\lambda\}_{\lambda\in\Lambda}$ is the Λ-orthogonal shift acting on $l^2(\mathscr{F}, \mathscr{D})$.

Now Lemma 2 and Theorem 3 in [4] can be easily extended to our setting. Thus, we omit the proofs in what follows.

Lemma 3.1. *Let P_n be the orthogonal projection from $\mathscr{K}$ into $\mathscr{H}_n$. Then $\bigvee_{n\geq 1} \mathscr{H}_n = \mathscr{K}$ and for each $\lambda \in \Lambda$ we have*

$$(T_\lambda)_n^* P_n \to V_\lambda^* \quad (\textit{strongly}) \textit{ as } n \to \infty.$$

Let $\mathscr{T}' = \{T_\lambda'\}_{\lambda\in\Lambda}$ be another sequence of operators on a Hilbert space $\mathscr{H}'$ with $\sum_{\lambda\in\Lambda} T_\lambda' T_\lambda'^* \leq I_{\mathscr{H}'}$ and $\mathscr{V}' = \{V_\lambda'\}_{\lambda\in\Lambda}$ be the minimal isometric dilation of $\mathscr{T}'$ acting on the Hilbert space $\mathscr{K}' = \mathscr{H}' \oplus l^2(\mathscr{F}, \mathscr{D}')$.

Theorem 3.2. *Let $A\colon \mathscr{H} \to \mathscr{H}'$ be a contraction such that for each $\lambda \in \Lambda$ $T_\lambda' A = A T_\lambda$. Then there exists a contraction $B\colon \mathscr{K} \to \mathscr{K}'$ such that for each $\lambda \in \Lambda$ $V_\lambda' B = B V_\lambda$ and $B^*|_{\mathscr{H}'} = A^*$.*

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