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# HOMOGENEOUS CONTINUA IN EUCLIDEAN $(n+1)$-SPACE WHICH CONTAIN AN $n$-CUBE ARE $n$-MANIFOLDS

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ABSTRACT. Let $X$ be a homogeneous continuum and let $E^n$ be Euclidean $n$-space. We prove that if $X$ is properly contained in a connected $(n+1)$-manifold, then $X$ contains no $n$-dimensional umbrella (i.e. a set homeomorphic to the set $\{(x_1,\dots,x_{n+1}) \in E^{n+1}: x_1^2+\dots+x_{n+1}^2 \le 1$ and $x_{n+1} \le 0$ and either $x_1 = \dots = x_n = 0$ or $x_{n+1} = 0\})$. Combining this fact with an earlier result of the author we conclude that if $X$ lies in $E^{n+1}$ and topologically contains $E^n$, then $X$ is an $n$-manifold.

The main purpose of this paper is to prove the following theorem.

**1. Theorem.** *Each homogeneous proper subcontinuum of a connected $(n+1)$-manifold contains no $n$-dimensional umbrella.*

The results of this paper are related to two classical results: the first one of S. Mazurkiewicz [M], and, the second one of R. H. Bing [B]. Namely, with the help of the result of [P], we give a full generalization of the result of [B] to all finite-dimensional cases (Theorem 7 below, and also, the statement formulated in the title). As it was emphasized in [P], the theorem of [B] may be obtained by combining two other theorems: $1^\circ$ each homogeneous locally connected nondegenerate plane continuum is a simple closed curve (this is the result of [M]), $2^\circ$ each homogeneous plane continuum that contains an arc is locally connected (this is the step really done in [B]), and thus $3^\circ$ each homogeneous plane continuum that contains an arc is a simple closed curve. (Bing's proof did not follow this scheme.) One can easily observe that Theorem 1 implies the result of [M] (for $n = 1$). Thus this paper generalizes step $1^\circ$. Step $2^\circ$ has already been extended in [P] to all finite-dimensional cases. Therefore we get Theorem 7 as a generalization of step $3^\circ$.

Finally, let us stress the fact that, similarly as in [P], the $\varepsilon$-push property (Theorem 4) plays a crucial role in the argument of the proof of Theorem 1. Probably, this is the real reason that the results of [P] and of this paper have not been earlier found.

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All spaces considered here will be either Euclidean $n$-spaces $E^n$ equipped with the usual Euclidean metric $d$, or (not necessarily compact) $n$-manifolds with a metric also denoted by $d$. The open ball of the space with center $c$ and radius $\varepsilon$ will be denoted by $B(c, \varepsilon)$. For two subsets $A$ and $B$ of the space we put $d(A, B) = \inf\{d(a, b): a \in A \text{ and } b \in B\}$. If $M$ is an $n$-manifold, the symbol $\partial M$ denotes its combinatorial boundary. An arc with end points $a$ and $b$ will often be denoted by $ab$. The symbol $I$ means the unit segment $[0, 1]$. Let a set $A$ be homeomorphic to the cube $I^n$ and let $ab$ be an arc. If $A \cap ab = \{a\}$ and $a \notin \partial A$, then the union $A \cup ab$ will be called an $n$-dimensional umbrella. A set $X$ is said to be homogeneous if for given $x, y \in X$ there is a homeomorphism $h: X \to X$ with $h(x) = y$. A mapping means a continuous function. A mapping (a homeomorphism) $f: X \to Y$ between subsets $X$ and $Y$ of the same space is called an $\varepsilon$-translation (an $\varepsilon$-homeomorphism) provided $d(x, f(x)) < \varepsilon$ for every $x \in X$. A point $x$ is said to be accessible from a set $V$ if there is an arc $xy$ with $xy \setminus \{x\} \subset V$. A set $C$ separates a set $V$ between two points $p, q \in V$, if $p$ and $q$ lie in distinct components of $V \setminus C$.

We start with two lemmas, which we need to prove Theorem 1.

**2. Lemma.** *If a point $c \in C \subset E^{n+1}$ has a neighborhood (in a set $C$) homeomorphic to $E^n$, then the number of components of $E^{n+1} \setminus C$ containing $c$ in their closures is either one or two. Moreover, $c$ is accessible from each of these components.*

*Proof.* Let a neighborhood $A$ of $c$ in $C$ be homeomorphic to $I^n$, and let a ball $B(c, \xi) \subset E^{n+1}$ be such that $C \cap B(c, \xi) = A \cap B(c, \xi)$ and $\partial A \cap B(c, \xi) = \varnothing$. Further, let $\{A_0, A_1, \ldots\}$ be the family (finite or infinite) of all components of $A \cap B(c, \xi)$ with $c \in A_0$. By Proposition 3 of [P] the set $A_0$ separates $B(c, \xi)$ into exactly two components $U_1^0$ and $U_2^0$. By the local connectedness of $A$ there is a ball $B(c, \tau)$ with $A \cap B(c, \tau) \subset A_0$. Since no pair of points of $U_i^0 \cap B(c, \tau)$ is separated by any $A_n$ in $B(c, \xi)$ for $i \in \{1, 2\}$ and $B(c, \xi)$ is homeomorphic to $E^{n+1}$, we see by Proposition 4 of [P] that $A$ also does not separate $B(c, \xi)$ between such points. This implies that $c \in \operatorname{cl} U_i$ for the component $U_i$ of $B(c, \xi) \setminus A$ containing $U_i^0 \cap B(c, \tau)$, for $i \in \{1, 2\}$, and $c$ does not lie in the closure of any other component of $B(c, \xi) \setminus A$. Thus the number of components of $E^{n+1} \setminus C$ containing $c$ in their closures is at most two, and, in fact, not less than one. Moreover, since $A_0$ is an ANR-set, the point $c$ is accessible from both $U_1^0$ and $U_2^0$ (see [Bo, p. 217]). Finally, the desired accessibility of $c$ follows by the previous argument.

**3. Lemma.** *Let a set $A \subset E^{n+1}$ be homeomorphic to $I^n$. Given a point $c \in E^{n+1}$ and a number $\varepsilon > 0$ such that $d(c, A) < \varepsilon < d(c, \partial A)$, let $A_1$ denote a component of $A \cap B(c, \varepsilon)$. For two given points $p$ and $q$ of distinct components of $B(c, \varepsilon) \setminus A_1$ let $pa$ and $aq$ be arcs in $B(c, \varepsilon)$ such that $(pa \cup aq) \cap A_1 = \{a\}$*

(*compare Proposition* 3 *of* [P] *and Lemma* 2). *Then for every* $\delta > 0$ *such that*

$$\delta < \frac{1}{2}\delta_0 = \frac{1}{2}\min\{d(pa \cup aq, E^{n+1}\backslash B(c,\varepsilon)), d(\{p,q\}, A_1)\}$$

*and for every* $\delta$*-homeomorphism* $h\colon A_1 \to h(A_1) \subset E^{n+1}$ *the component* $A_2$ *of* $h(A_1) \cap B(c,\varepsilon-\delta)$ *containing the point* $a' = h(a)$ *separates* $B(c,\varepsilon-\delta)$ *between* $p$ *and* $q$.

*Moreover, if* $\delta < \delta_0/4$, *there are arcs* $pa'$ *and* $a'q$ *in* $B(c,\varepsilon-\delta-\delta_0/4)$ *such that* $(pa' \cup a'q) \cap A_2 = \{a'\}$.

*Proof.* By Proposition 5 of [P] the set $h(A_1)$ separates the ball $B(c,\varepsilon-\delta)$ between $p$ and $q$. Therefore a component $A_2$ of $h(A_1) \cap B(c,\varepsilon-\delta)$ so does (see Proposition 4 of [P]). Again by Proposition 5 of [P], noting that $A_2$ is closed in $B(c,\varepsilon-\delta)$, we see that the set $h^{-1}(A_2) \subset A_1$ separates $B(c,\varepsilon-2\delta)$ between $p$ and $q$, and thus it intersects the set $pa \cup aq$. By the assumption the only point of this intersection is $a$, thus $a' = h(a) \in A_2$.

Let $\delta < \delta_0/4$. Put $\delta_1 = \varepsilon-\delta-\delta_0/4$. Suppose there is no arc $pa'$ in $B(c,\delta_1)$ with $pa' \cap A_2 = \{a'\}$. Let $A_3$ be the component of $B(c,\delta_1) \cap A_2$ containing $a'$. Thus there is an arc $Z \subset B(c,\delta_1)$ with end points $p$ and $a'$ such that $Z \cap A_3 = \{a'\}$ (see Proposition 3 of [P] and Lemma 2). Let a point $z \in Z\backslash\{a'\}$ be such that the arc $za' \subset Z$ intersects $A_2$ in the single point $a'$. Thus, by the above assumption, $A_2$ separates the ball $B(c,\delta_1)$ between $p$ and $z$. Therefore some component $A_4$ of $B(c,\delta_1) \cap A_2$ separates $B(c,\delta_1)$ between $p$ and $z$ (see Proposition 4 of [P]) and we have $A_4 \neq A_3$. Hence $A_4$ separates $B(c,\delta_1)$ between $p$ and $a'$. By the proved part of the conclusion of this lemma the set $A_3$ separates the ball $B(c,\delta_1)$ between $p$ and $q$ (for $h$ is also a $(\delta+\delta_0/4)$-homeomorphism). Therefore the set $A_4$ separates $B(c,\delta_1)$ between $p$ and $q$. Thus, by Proposition 5 of [P], the set $h^{-1}(A_4)$ separates $B(c,\delta_1-\delta)$ between $p$ and $q$. By the assumption on $\delta$ we have $pa \cup aq \subset B(c,\varepsilon-\delta_0) \subset B(c,\delta_1-\delta)$, therefore the set $h^{-1}(A_4) \subset A_1$ intersects the arc $pa \cup aq$ in a point distinct from $a$ (for $a' = h(a) \notin A_4$), a contradiction. The argument for the existence of an arc $a'q$ runs similarly.

Now recall the theorem (the so-called $\varepsilon$-push property) which is a corollary to the well-known Effros theorem.

**4. Theorem** (Lemma 4 of [H, p. 37]). *Let* $X$ *be a homogeneous metric continuum. Then for every* $\varepsilon > 0$ *there is* $\delta > 0$ (*the so-called Effros number for the number* $\varepsilon$) *such that for two given points* $x, y \in X$ *with* $d(x,y) < \delta$ *there is an* $\varepsilon$*-homeomorphism* $h\colon X \to X$ *sending* $x$ *to* $y$.

*Proof of Theorem* 1. Let $P$ be a homogeneous proper subcontinuum of a connected $(n+1)$-manifold $M$. Suppose, on the contrary, $P$ contains an $n$-dimensional umbrella. By the intrinsic invariance of open sets in the Euclidean spaces, since $\partial M$ is either an $n$-manifold or the empty set, $P$ cannot be contained in $\partial M$. Because $P$ is a boundary set in $M$, there is a point $a \in P\backslash\partial M$

accessible from $M\backslash P$. Let $d$ be a metric on $M$ such that the ball $B(a, 1) \subset M$ is isometric to the appropriate ball of Euclidean $(n+1)$-space. By the homogeneity of $P$ there is an $n$-dimensional umbrella $T \subset B(a, 1) \cap P$ such that $T = A \cup ab$, where the set $A$ is homeomorphic to $I^n$, $ab$ is an arc with ends $a$ and $b$, and $A \cap ab = \{a\} \subset A\backslash \partial A$. Let a number $\varepsilon > 0$ be such that $\partial A \cap B(a, \varepsilon) = \varnothing$. Without loss of generality we may assume that $ab \subset B(a, \varepsilon)$. Let $A_1$ be the component of $A \cap B(a, \varepsilon)$ containing $a$. Then $A_1$ separates the ball $B(a, \varepsilon)$ into exactly two components (see Proposition 3 of [P]): $V_b$ with $ab\backslash\{a\} \subset V_b$, and $V_c$ with some point $c \in V_c$. By the accessibility of $a$ there is an arc $ap \subset B(a, \varepsilon)$ with $ap \cap P = \{a\}$. By Lemma 2 there is an arc $ac \subset B(a, \varepsilon)$ with $ac\backslash\{a\} \subset V_c$.

(1) For any arc $ax$ with $ax\backslash\{a\} \subset V_c$ we have $P \cap (ax\backslash\{a\}) \neq \varnothing$.

In fact, let $cx \subset V_c$ be an arc (may be degenerate if $c = x$), and $\{z_m\}$ be a sequence of points of $ab\backslash\{a\}$ converging to $a$. By the $\varepsilon$-push property (Theorem 4) there are $\xi_m$-homeomorphisms $g_m \colon P \to P$ with $g_m(z_m) = a$ and $\lim \xi_m = 0$. By Proposition 5 of [P] the sets $g_m(A_1)$ separate the balls $B(a, \varepsilon - \xi_m)$ between $c$ and $b$, and, do not intersect $cx$ and some fixed ball $B(b, \xi) \subset V_b$ for sufficiently great $m$. Since $g_m(b) \in B(b, \xi)$, the sets $B(b, \xi) \cup g_m(bz_m) \cup ax \cup cx$ are connected for almost all $m$. Noting $g_m(bz_m) \cap g_m(A_1) = \varnothing$, where $bz_m \subset ab$, we see that the set $g_m(A_1) \subset P$ intersects $ax\backslash\{a\}$ for large $m$.

By (1) we see that there is a sequence $\{a_m\}$ converging to $a$ with $a_m \in P \cap ac\backslash\{a\}$, and also

(2) $$ap\backslash\{a\} \subset V_b.$$

Let $pb \subset V_b$ be an arc, and let $\tau$ be a number such that

$$0 < \tau < \tfrac{1}{2} \min\{d(ap \cup pb \cup ab \cup ac, M\backslash B(a, \varepsilon)), d(pb \cup \{c\}, A_1)\}.$$

Further, let $\psi > 0$ be an Effros number for the number $\tau/4$ (see Theorem 4), and let $\varphi > 0$ be an Effros number for the number $\psi$. Now, find $a_k$ with $d(a, a_k) < \varphi$, and let $f \colon P \to P$ be a $\psi$-homeomorphism such that $f(a) = a_k$. Let $A_2$ denote the component of $B(a, \varepsilon - \tau/4) \cap f(A_1)$ containing $f(a)$. By Lemma 3 this component separates the ball $B(a, \varepsilon - \tau/4)$ between $c$ and $b$, and also between $c$ and $p$ (for obviously we have $\psi \leq \tau/4$). Moreover, the same lemma guarantees the existence of an arc $ca_k \subset B(a, \varepsilon - \tau/4 - \tau/2) = B(a, \varepsilon - 3\tau/4)$ with $ca_k \cap A_2 = \{a_k\}$. Let $r$ be the first point of the arc $f(ab)$ (in the ordering from $f(a)$ to $f(b)$) intersecting $A_1$. Since $d(a, a_k) < \psi$, we may find a point $q \in a_k r \subset f(ab)$ with $a_k \neq q \neq r$ and $d(a, q) < \psi$.

Now consider the component $U$ of $B(a, \varepsilon - \tau/4)\backslash(A_1 \cup A_2)$ containing $q$. Observe that this component contains no point of the set $pb \cup \{c\}$. The point $a_k$ lies in the closures of $U$ and of the component $U_c$ of $B(a, \varepsilon - \tau/4)\backslash(A_2 \cup \operatorname{Bd} U)$ containing $c$, for there exist the arcs $ca_k$ and $a_k q \subset f(ab)$. Since $a_k$ has a neighborhood in $A_2 \cup \operatorname{Bd} U$ homeomorphic to $E^n$, there is, by Lemma 2, no other component of $B(a, \varepsilon - \tau/4) \backslash (A_2 \cup \operatorname{Bd} U)$ containing $a_k$ in its closure, in particular, $a_k$ does not lie in the closure of the component $U_b$ containing $b$. Let $B$ be a connected open neighborhood of $a_k$ in $A_2$ with a positive distance from $A_1 \cup \operatorname{Bd} U_b$. Then

(3) the set $(A_2 \cup \operatorname{Bd} U) \backslash B$ separates the ball $B(a, \varepsilon - \tau/4)$ between $c$ and $b$.

For every $x \in qb' \cap \operatorname{Bd} U \subset A_1 \backslash A_2$, where $b' = f(b)$ and $qb' \subset f(ab)$, find an open connected neighborhood $B_x$ of $x$ in $A_1 \backslash A_2$ such that

$$\operatorname{cl}\left(\bigcup\{B_x : x \in qb' \cap \operatorname{Bd} U\}\right) \cap \operatorname{cl} A_2 = \varnothing.$$

Put $B_1 = \bigcup\{B_x : x \in qb' \cap \operatorname{Bd} U\}$. Since $A_2 \subset (A_2 \cup \operatorname{Bd} U) \backslash B_1$, we get

(4) the set $(A_2 \cup \operatorname{Bd} U) \backslash B_1$ separates the ball $B(a, \varepsilon - \tau/4)$ between $c$ and $b$.

We also have

(5) the set $(A_2 \cup \operatorname{Bd} U) \backslash (B \cup B_1)$ does not separate the ball $B(a, \varepsilon - \tau/4)$ between $c$ and $b$.

For, the segment between $b$ and $f(b)$ does not intersect $A_1 \cup A_2$, and, the arc $ca_k \cup f(ab)$ does not intersect $(A_2 \cup \operatorname{Bd} U) \backslash (B \cup B_1)$.

Now, find a $(\tau/4)$-homeomorphism $h: P \to P$ such that $h(q) = a$. Then we obtain

(6) the set $h(A_2 \cup \operatorname{Bd} U) \backslash h(B \cup B_1)$ does not separate the ball $B(a, \varepsilon - 2\tau/4)$ between $c$ and $b$.

In fact, if not, then the set

$$h^{-1}(h(A_2 \cup \operatorname{Bd} U) \backslash h(B \cup B_1)) = (A_2 \cup \operatorname{Bd} U) \backslash (B \cup B_1)$$

would separate $B(a, \varepsilon - 3\tau/4)$ between $c$ and $b$ (see Proposition 5 of [P]), an impossibility, for the segment between $b$ and $f(b)$, as well as the arc $ca_k \cup f(ab)$, lie in $B(a, \varepsilon - 3\tau/4) \backslash ((A_2 \cup \operatorname{Bd} U) \backslash (B \cup B_1))$.

By Proposition 5 of [P] and by (3) and (4) we have

(7) each of sets $h(A_2 \cup \operatorname{Bd} U) \backslash h(B)$ and $h(A_2 \cup \operatorname{Bd} U) \backslash h(B_1)$ separates the ball $B(a, \varepsilon - 2\tau/4)$ between $c$ and $b$.

The following statement contradicts the previous one, so it completes the proof of Theorem 1.

(8) One of the sets $h(A_2 \cup \operatorname{Bd} U) \backslash h(B)$ and $h(A_2 \cup \operatorname{Bd} U) \backslash h(B_1)$ fails to separate the ball $B(a, \varepsilon - 2\tau/4)$ between $c$ and $b$.

Indeed, by (6) there is an arc $cb$ in $B(a, \varepsilon - 2\tau/4) \backslash (h(A_2 \cup \operatorname{Bd} U) \backslash h(B \cup B_1))$. By (7) this arc intersects the set $h(B \cup B_1)$. Since $d(B, B_1) > 0$, we have

$d(h(B), h(B_1)) > 0$. This implies that going from $c$ to $b$ along the arc $cb$, we may find a point $y \in cb$ such that either $y \in h(B)$ and $cy \cap h(B_1) = \emptyset$ or $y \in h(B_1)$ and $cy \cap h(B) = \emptyset$, where $cy \subset cb$. If $y \in h(B)$, then let $ya' \subset h(B)$ be an arc (where $a' = h(a_k) = hf(a)$), and put $J(y, a) = ya' \cup h(a_k q)$, where $a_k q \subset f(ab)$. If $y \in h(B_1)$, then $y \in h(B_x)$ for some $x \in qb' \cap \operatorname{Bd} U$, where $b' = f(b)$ and $qb' \subset f(ab)$. Let $yx' \subset h(B_x)$ be an arc, where $x' = h(x)$, and put $x'a = h(xq)$, where $xq \subset f(ab)$. Then put $J(y, a) = yx' \cup x'a$. Thus in the former case we get $J(y, a) \cap (h(A_2 \cup \operatorname{Bd} U) \backslash h(B)) = \emptyset$, and, in the latter case we have $J(y, a) \cap (h(A_2 \cup \operatorname{Bd} U) \backslash h(B_1)) = \emptyset$. But since both considered homeomorphisms are $(\tau/4)$-homeomorphisms, the set $h(A_2 \cup \operatorname{Bd} U)$ does not intersect the arc $pb$. By the assumption on $pa$ the set $h(A_2 \cup \operatorname{Bd} U) \subset P$ does not intersect the arc $pa$. Therefore the connected set $cy \cup J(y, a) \cup pa \cup pb \subset B(a, \varepsilon - 2\tau/4)$ does not intersect either $h(A_2 \cup \operatorname{Bd} U) \backslash h(B)$ or $h(A_2 \cup \operatorname{Bd} U) \backslash h(B_1)$. Thus we have (8).

The proof of Theorem 1 is complete.

A simple proof of the next fact is left to the reader.

**5. Fact.** *A homogeneous locally connected continuum that topologically contains the cube $I^n$ and contains no $n$-dimensional umbrella, is an $n$-manifold.*

Further, we get the following immediate consequence of Theorem 1.

**6. Corollary.** *A proper homogeneous locally connected subcontinuum of a connected $(n+1)$-manifold, that topologically contains the cube $I^n$, is an $n$-manifold.*

It was proved in [P] that each homogeneous subcontinuum of $E^{n+1}$, which topologically contains $I^n$ is locally connected. Thus we have the conclusion that forms the title of the paper.

**7. Theorem.** *Each homogeneous continuum that lies in the Euclidean space $E^{n+1}$ and topologically contains the cube $I^n$ is an $n$-manifold.*

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