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ON VECTOR BUNDLES ON 3-FOLDS WITH SECTIONAL GENUS 1

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ABSTRACT. Here we give a classification (in characteristic zero) of pairs (V, E) with V being a smooth, connected, complete 3-fold and E a rank-2 spanned ample vector bundle on V with sectional genus 1. The proof uses the partial classification of Fano 3-folds and Mori theory.

Let X be an integral complete manifold, $\dim(X) = n$, and E a rank-$(n-1)$ vector bundle on X. We define the sectional genus $g(E)$ using the following formula: $2g(E) - 2 := (K_X + c_1(E))c_{n-1}(E)$. It is easy to check (see Remark 4.1) that $g(E)$ is an integer. If E has a section with zero-locus C of codimension $n-1$, then $g(E) = p_a(C)$. In this paper we give a reasonable classification in characteristic zero (see Theorem 0) of the pairs (V, E) with V being a smooth, complete 3-fold, and E a rank-2 ample spanned vector bundle on E with $g(E) = 1$. Note that if $g(E) = 0$, then $K_V + c_1(E)$ is not nef; under this assumption (but a far weaker assumption on V, E) Wisniewski in [W] gave a classification, using Mori theory (see also 4.2 for a similar result). The case $g(E) = 1$ seems to be of a different order of difficulty. But we will use very much of Wisniewski's work [W], the partial classification of Fano 3-folds, and classifications (e.g., [Io]) based on Mori theory. In an interesting paper [F1] Fujita gave two definitions of sectional genus for higher rank vector bundles. The second one is the sectional genus of the tautological line bundle on $\mathbf{P}(E)$. The first one (and more interesting) is called the c_1-sectional genus of E; it is the sectional genus of the line bundle $\det(E)$, i.e., g is defined by: $2g - 2 := (K_X + (n-1)c_1(E))(c_1(E))^{n-1}$. For instance, if E is the direct sum of $n-1$ line bundles all isomorphic to a line bundle A, with our definition $2g(E) - 2 := (K_X + (n-1)A)A^{n-1}$ while the c_1-sectional genus g of E is defined by $2g - 2 := (n-1)^{(n-1)}(K_X + (n-1)^2 A)A^{n-1}$, which is much bigger if A is ample. For instance, the c_1-sectional genus of the bundle in (b) in the statement of Theorem 0 is 33.

The first three sections of this paper are devoted to the proof of the following result (over an algebraically closed base field with characteristic zero).

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Theorem 0. *Let V be a connected, tridimensional smooth complete variety V and E a rank-2 ample, spanned, vector bundle on V with $g(E) = 1$. Then either E is the direct sum of two isomorphic line bundles or (V, E) is isomorphic to one of the following pairs*:

(a) $V = \mathbf{P}^3$, $E = \boldsymbol{O}(3) \oplus \boldsymbol{O}(1)$;

(b) $V = \mathbf{P}^3$, $E = N(2)$ *with N a null-correlation bundle (i.e., N is stable $c_1(E) = 0$, $c_2(E) = 1$);*

(c) *V is a smooth quadric Q in $\mathbf{P}^4$ and* $E = \boldsymbol{O}(2) \oplus \boldsymbol{O}(1)$;

(d) *V is a smooth quadric Q in $\mathbf{P}^4$ and $E = F(1)$, with F restriction to Q of the universal quotient bundle on the Grassmannian $G(1, 3)$ (the 4-dimensional quadric)*;

(e) *V is a $\mathbf{P}^2$-bundle over a smooth elliptic curve C; the restriction of E to every fiber of the bundle is the direct sum of two line bundles of degree* 1.

The pairs (V, E) in Theorem 0 with E the direct sum of two isomorphic line bundles were completely classified [I, F].

The null-correlation bundles in (b) of Theorem 0 are completely described in [B] or [H], §8. For every $\mathbf{P}^2$-bundle V over an elliptic curve C, there are many bundles as in (e) of the statement of Theorem 0 which are ample and spanned (see the last lines of §1); however, we do not know exactly if a given E is ample and spanned. Only for this reason do we not have a complete classification of such pairs (V, E) (although we consider Theorem 0 a "reasonable classification," or if you prefer a "reasonable description," of the pairs (V, E)). During the proof of Theorem 0 we met a few interesting bundles (see in particular Case 10 at the end of §2) which although not ample or spanned should appear in more refined classifications or in more refined lists of "pathologies" and "extremal" bundles.

In the last section we give two remarks on the sectional genus of vector bundles, and prove the case $n = 2$ (i.e., on a surface) of a conjecture raised in [LPS].

Notations. We work over an algebraically closed field with characteristic zero. We will use often the equality sign "$=$" for an isomorphism, since this abuse here is harmless. V will be a smooth, complete, connected 3-fold. Let A be a closed subscheme of a scheme X; we denote with $\boldsymbol{I}_{A,B}$ (or $\boldsymbol{I}_A$ if there is no danger of misunderstanding) the ideal sheaf of A in B; we will always use the form $\boldsymbol{I}_A$ if $B = V$. Similarly we write often $\boldsymbol{O}$ instead of $\boldsymbol{O}_B$ or $\boldsymbol{O}_V$.

1. Reduction of proof of theorem

In this section we will reduce the proof of the theorem to the case in which V is a Fano 3-fold of index $r \geq 2$. For this reduction we will use [Io] and [W]; both references are based on Mori's theory.

Fix (V, E) as in Theorem 0 and set $L := \det(E)$. Let C be the 0-locus $(s)_0$ of a general section s of E. By Bertini's theorem, C is smooth. Using for instance Sommese's Lefschetz theory [S, L], we get that C is connected. Thus, by the assumption, C is an elliptic curve. By the adjunction formula we get $(K+L)C = 0$.

Remark 1.1. Since $L = \det(E)$ and E is ample spanned for all irreducible curves $T \subset V$, $LT \geq 2$. Hence (V, L) is its own reduction in the sense of [I, 0.11]. □

(a) First assume that $K + L$ is semi-ample. Then there is an integer $m > 0$ such that $m(K + L)$ is spanned by global sections, and hence induces a morphism $h: V \to \mathbf{P}$, $\mathbf{P}$ being some projective space. Since $C(K + L) = 0$, $h(C)$ is a point. Assume that $h(V)$ is not a point. Then there is a positive divisor $D > 0$ with $D \cap C = \varnothing$. Thus $c_{n-1}(E|D) = 0$, contradicting the ampleness of $E|D$. Thus $m(K + L)$ is trivial; hence, $-K$ is ample, and V is a Fano 3-fold. Since $\mathrm{Pic}(V)$ is torsion-free (see [Mu, p. 47]) we have $-K = L$. Let r be the index of V. First assume that V is of the first kind, i.e., $\mathrm{Pic}(V) = \mathbf{Z}$. By [W, 2.3], r is the length of V; the length is defined in [W] (the length of V is the minimal integer > 0 of the form $-K_V C$, C a curve on V); we need only to know that by 1.1 the length is at least 2. Thus, if V is a Fano 3-fold of the first kind, its index r is at least 2. Iskoviskih [I, 4.2], with a fundamental contribution of Shokurov (and a suggestion by Mori to fill in a gap, see [Mu, p. 57]) classified all Fano 3-folds with index $r \geq 2$ (see [Mu, p. 63], for a complete list). We will consider in §2 the pairs (V, E) with V a Fano 3-fold of index $r \geq 2$.

Now assume that V is a Fano 3-fold not of the first kind, i.e., with $b_2(V) \geq 2$. Such Fano 3-folds were classified by Mori and Mukai [MM]. Using their classification, we will classify in §3 the pairs (V, E) in Theorem 0 with V a Fano 3-fold with $b_2(V) \geq 2$.

(b) Now assume that $K + L$ is not semi-ample. By [Io, (1.7)], and the fact that (V, L) is its own reduction, we get that (V, L) belongs to one of the four classes, A, B, C, D considered in [Io]. Furthermore trivially the classes A and B cannot give any solution by 1.1.

We claim that the same is true for the class C. Again by 1.1 the only nontrivial case is when $-K = 2L$ (i.e., by definition (V, L) is a Del Pezzo 3-fold). Such pairs (V, L) are completely classified [I, F]: if $-K = 2L$, (V, L) has sectional genus 0. But since $K + L$ is not nef, by [W, Theorem 3.6], (V, E) is one of a few cases, and the only one with, possibly, sectional genus 0 is given by $\mathbf{P}^2$-bundles over a smooth curve B, with $E|F = 2\mathcal{O}_F(1)$ for every fiber F. This case will be considered below in subcase (iii).

Thus, we will assume that (V, L) is of class D of [Io, 0.6]. Thus, we have only three cases, (i), (ii), and (iii), to consider in the class D.

(i) Assume $(V, L) = (\mathbf{P}^3, \mathcal{O}(3))$. Since E is ample, the restriction of E to every line of $\mathbf{P}^3$ must be a direct sum of a line bundle of degree 1 and one of degree 2. Thus E is uniform. By [VdV], $E = \mathcal{O}(2) \oplus \mathcal{O}(1)$. Thus, E has sectional genus 0, contradiction.

(ii) Assume that V is a smooth quadric Q and $L = \mathcal{O}(2)$. Since E is ample, the restriction of E to any line in Q has splitting type (1,1), i.e., E is uniform. Again (V, E) has sectional genus 0, contradiction.

(iii) Assume that V is a $\mathbf{P}^2$-bundle $p: V \to X$ over a smooth curve C and that $L|A = \mathcal{O}_A(2)$ for all fibers $A = \mathbf{P}^2$ of p. Since $E|A$ is ample, we see again that it is uniform of type $(1, 1)$, hence by [VdV] isomorphic to $2\mathcal{O}(1)$. Let H be an effective divisor inducing $\mathcal{O}(1)$ on each fiber A of p; the existence of H follows for instance from the fact that $\dim(C) \leq \dim(A)$ and [Se]. By the theorem of changing basis $E(-H) = p^*(F)$ with F a rank-2 vector bundle on C, $F = p_*(E(-H))$. Thus, we see that (V, E) is of the type considered in (e) of the statement of Theorem 0. Taking a general section s of E, with $(s)_0$ a smooth curve D, we see that D intersects transversally and exactly at a point every fiber of p. Thus, D and C are isomorphic and in particular C has genus 1. Vice versa, fix any such V with C of genus 1, and fix H. Taking as F a sufficiently positive bundle on C, we see that $E := p^*(F)(H)$ is spanned; to check that E is ample it is sufficient to check that the restriction of E to every irreducible curve $T \subset V$ is ample [H1]; even less, it is sufficient to check that $E|T$ has not $\mathcal{O}_T$ as quotient [H1]. If $p(T)$ is a point, this is true for any F; if $p(T) = C$, it is sufficient (but not necessary) to assume that F is ample (use the fact that in characteristic zero the pull-back by a finite morphism of an ample bundle is ample [H1]). Fix any curve R which is a section of p. Using $p|R$ we may identity R and C, and hence think of $H|R$ as a line bundle, H', on C. Then $E|R = p^*(F(H'))$ and if E is ample and spanned, then $F(H')$ is ample and spanned on C.

But we do not know exactly all the pairs (V, E) as in (iii) with E ample and spanned. The classification of pairs (X, E) with X a $\mathbf{P}^{n-1}$-bundle over a smooth curve B, E ample and spanned of rank r on X, $E|A$ the direct sum of r line bundles of degree 1 for all fibers A of this fibration (π-uniform bundles in the sense of Ishimura [Is]) would be very interesting for many other classifications (even assuming further restrictions on X or E).

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In this section we will assume that V is a Fano 3-fold of index $r \geq 2$ and that $L := \det(E) = -K$. The possible V are classified (see [Mu, p. 63], and [I, 4.2]). For each possible V, we will study the possible bundles E. Hence, we will distinguish cases.

Case (1). Assume $r = 4$, i.e., $V = \mathbf{P}^3$. Hence, $L = \mathcal{O}(4)$. Take a general section s of E, s vanishing on a smooth elliptic curve C. Then s defines the following exact sequence:

$$0 \to \mathcal{O} \to E \to L \otimes \mathcal{I}_C \to 0 \tag{1}$$

with $L = \boldsymbol{O}(4)$. Vice versa, by [H, 1.1], given C we get a rank-2 vector bundle E which fits in the exact sequence (1). By (1) E is spanned if and only if $\boldsymbol{I}_C$ is generated by forms of degree 4. Set $a := \deg(C)$.

Assume the existence of a line D which is quadrisecant to C, i.e., such that $\text{length}(C \cap D) \geq 4$. Then $E|D$ has a section (the restriction of s) vanishing on a divisor of degree ≥ 4. Thus, $E|D$ cannot be the direct sum of two line bundles of degree > 0, since the sum of their degrees must be 4. Hence, $E|D$ is not ample. Thus, E is not ample. By the formula for the number of quadrisecant lines to C [LB, p. 116] and [LB1], we find $3 \leq a \leq 5$. If $a = 3$ or $a = 4$, C is a complete intersection. Using (1) we see easily that if $a = 3$ or 4, E is the direct sum of two line bundles, i.e., $E = \boldsymbol{O}(1) \oplus \boldsymbol{O}(3)$ or $E = 2\boldsymbol{O}(2)$. Vice versa, it is trivial to check that these bundles are ample and spanned.

Now assume $a = 5$. By (1) we get $h^0(E(-2)) = 0$, because no smooth elliptic curve of degree 5 is contained in a quadric surface (see [Ha, Example V.2.9], for the case of a quadric cone). Thus $E(-2)$ is a stable bundle with Chern classes $c_1 = 0$, $c_2 = 1$ [H, beginning of §2]. Thus, $E(-2)$ is one of the so-called null-correlation bundles (see [B] or [H], 8.4.1). Vice versa, for any null-correlation bundle N, by [H, 8.4.1], and the sequence corresponding to (1), $N(1)$ is spanned; hence, $N(2)$ is spanned and ample.

Case 2. Assume that $r = 3$; hence, V is a smooth quadric Q and $L = \boldsymbol{O}(3)$. Take again a general section of E and the associated exact sequence (1).

Assume the existence of a line $D \subset Q$ trisecant to C, i.e., with length $(C \cap D) \geq 3$. Then as in Case 1 we see that E is not ample. For degree reasons, any line in $\mathbf{P}^4$ which is trisecant to C is contained in Q. By the formula for the number of trisecant lines to a curve in $\mathbf{P}^4$ [LB1, p. 182], we see that if E is ample then $\deg(C)$ is 4 or 5. Conversely, if $\deg(C)$ is 4 or 5, $\boldsymbol{I}_C(3)$ is spanned by global sections and the bundle satisfying (1) constructed (for fixed C, L) by [H, 1.1 and 1.1.1] is spanned. If $\deg(C) = 4$, we see that $E = \boldsymbol{O}(1) \oplus \boldsymbol{O}(2)$ (either use [H, 1.1], or use that (1) and [H, beginning §2], implies that $E(-1)$ has a nowhere vanishing section). Now assume $t := \deg(C) = 5$. Every C by [H, 1.1 and 1.1.1] defines a bundle E which satisfies (1). Since $\boldsymbol{I}_C(3)$ is spanned, by (1) we get that E is spanned. To check that E is ample by [H1] it is sufficient to prove that for every irreducible curve $T \subset Q$, $E|T$ is an extension of two line bundles of degree > 0. Fix T; since E is spanned, we may find a section of E with zero-locus $C \neq T$; thus we may assume $C \neq T$. By (1) we have $h^0(E(-1)) = 0$, i.e.,. every nonzero section of E has zero-locus of codimension 2. Thus, we may find a section of E with zero-locus C' a curve intersecting T; thus, we may assume $C' \cap T \neq \varnothing$ (but now C' may be singular and even unreduced). By assumption, $\text{length}(C' \cap T)$ is finite, and > 0. Since $t \geq 2g(C) + 2$, $\boldsymbol{I}_C(2)$ has good properties, e.g., it is spanned; by (1) (with C smooth) we find $h^i(E(-i-1)) = 0$ for every $i > 0$. Thus, by (1) with C' instead of C we get $h^i(\boldsymbol{I}_{C'}(2-i)) = 0$ for every $i > 0$.

Remark. By Castelnuovo-Mumford's lemma, we get that $I_{C'}(2)$ is spanned and that the homogeneous ideal of C' in $\mathbf{P}^4$ is spanned by quadrics. Let $n: T' \to T$ be the normalization. Pulling-back to T' the section of E defining (1) for C', we get a section of $n^*(E)$ vanshing on a divisor with image $T \cap C$, hence degree > 0, and with degree $\leq 2\deg(T)$ by the remark. Thus, $n^*(E)$ is an extension of line bundles of degree > 0. By [H1] E is ample.

We could avoid this discussion (which should be useful to handle similar problems) exhibiting E. Indeed it is clear that (with the notations of (d) in Theorem 0) F is spanned; hence, $F(1)$ is spanned and ample. One checks that $g(F(1)) = 1$. Since $\deg(E|C) = \deg(F(1)|C) = 15$, by Riemann-Roch we get that both E and $F(1)$ have a section with the same C as zero-locus. Thus, $E = F(1)$ by the uniqueness part in [H, 1.1 and 1.1.1].

Case 3. In all the remaining cases we have $r = 2$, i.e., $L = 2H$ for some $H \in \operatorname{Pic}(V)$, H ample. Set $d := H^3$. We have $d \leq 7$. If $3 \leq d \leq 7$, H is very ample, and we consider V embedded by H. Now assume $d = 7$, i.e., V is the projection from one of its points of a Veronese embedding of $\mathbf{P}^3$ embedded by the quadrics. Thus, V is the blowing-up of $\mathbf{P}^3$ at one of its points. Let M be the exceptional divisor, M being isomorphic to $\mathbf{P}^2$. V is isomorphic to $\mathbf{P}(T\mathbf{P}^2(-1))$; let $p: V \to \mathbf{P}^2$ be the corresponding projection. By the adjunction formula the restriction of L to any fiber f of p is $\mathbf{O}(2)$; hence, as usual $E|f = 2\mathbf{O}_f(1)$ for every f. By changing basis, $E(-M) = p^*(A)$ with a rank-2 vector bundle on $\mathbf{P}^2$. By the adjunction formula, the restriction of L to every line of M has degree 2. By the ampleness of E and [VdV] we get in the usual way that $E(H)$ is a direct sum of two isomorphic line bundles, each of them of degree 2. Since M is a section of p, A has the same property and $E = p^*(2\mathbf{O}(2))(-H)$. For splitted bundles it is easy to check the ampleness and spannedness.

Case 4. If $d = 6$, there are exactly two possible V. Here assume $V = \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$. Since the restriction of L to every line in V has degree 2, E is uniform of splitting type $(1, 1)$. By [BN, Corollary, p. 221], E is the direct sum of two isomorphic line bundles, each of them the $\mathbf{O}(1)$ in the Segre embedding of V.

Case 5. Assume again $d = 6$, and that V is a hyperplane section of the Segre embedding of $\mathbf{P}^2 \times \mathbf{P}^2$. Consider the projection $p: V \to \mathbf{P}^2$. If some fiber of p is two-dimensional, $b_2(V) \geq 3$. We will see in the next section how to find a contradiction. Thus, we assume that each fiber of p is a $\mathbf{P}^1$. Taking a general hyperplane section of V, we see that p has a rational section. Thus, by [Se] p is $\mathbf{P}^1$-bundle. We make here an argument which works if V is any $\mathbf{P}^1$-bundle over $\mathbf{P}^2$. Set $V = \mathbf{P}(W)$, $\operatorname{rank}(W) = 2$ and let $p: V \to \mathbf{P}^2$ be the projection. By the adjunction formula and the ampleness of E, the restriction of E to every fiber of p has splitting type $(1, 1)$, i.e., by definition E is π-uniform in the sense of Ishimura [Is]. Fix any section M of p. By the π-uniformity of E and changing basis, $E = p^*(B)(H)$, where B is a rank-2 vector bundle on $\mathbf{P}^2$, $B = p_*(E(-H))$. Fix a line $D \subset \mathbf{P}^2$ and set $S = p^{-1}(D)$. S is an Hirzebruch surface F_e for some integer $e \geq 0$. Thus, $\mathrm{Pic}(S)$ has a base h, f with $h^2 = -e$, $hf = 1$, $f^2 = 0$. Since $E|S$ is π-uniform of type $(1, 1)$, by changing bases $E|S = (h + xf) \oplus (h + yf)$ for some integer $x, y, x \leq y$. Since E is ample, we have $x > e$ and $y > e$. By the adjunction formula, $L|S = (2h + (e+3)f)$. Hence $x + y = e + 3$, and we find either $e = 0$, $x = 1$, $y = 2$, or $e = 1$, $x = y = 2$. Since any small deformation of a surface F_k contains only surfaces F_{k+b} with b even, we see that the invariant e does not depend upon the choice of the line D in $\mathbf{P}^2$. Hence, also x and y do not depend upon the choice of D. This means that B is uniform. By the invariance of e, W is uniform, too. If $e = 1$, we find $B = 2\mathbf{O}(2-t)$, while if $e = 0$, either $B = T\mathbf{P}^2(-t)$ or $B = \mathbf{O}(2-t) \oplus \mathbf{O}(1-t)$ [VdV], with t defined by: $H|S = (h + tf)$.

If $e = 0$, W splits and $V = \mathbf{P}^1 \times \mathbf{P}^2$, while $-K$ is not isomorphic to $2H$ for any $H \in \mathrm{Pic}(V)$ (adjunction formula), contradiction.

Now assume $e = 1$; hence, B and hence E are a direct sum of two isomorphic line bundles (as in Case 3).

The referee suggested the following alternative argument. W is defined by the same exact sequence on $\mathbf{P}^2$ (the Euler's sequence)

$$0 \to \mathbf{O}(-1) \to 3\mathbf{O} \to W \to 0$$

which defines $T\mathbf{P}^2(-1)$; one checks that $B := \mathbf{P}(W) \subset \mathbf{P}^2 \times \mathbf{P}^2$ is smooth if and only if W is locally free; hence, $W = T\mathbf{P}^2(-1)$.

Case 6. Now assume $d = 3$; hence, V is a cubic hypersurface in $\mathbf{P}^4$. Take a general section, giving the exact sequence (1). Set $n = \deg(C)$. Set $\mathbf{P} := \mathbf{P}^4$ and note that $h^0(\mathbf{I}_C(2)) = h^0(\mathbf{I}_{C,\mathbf{P}}(2))$.

First assume $n = 3$ (plane cubic). By (1) we get $E = 2\mathbf{O}(1)$.

Now we prove that $h^0(E) \geq 5$. Assume by contradiction $h^0(E) = 4$ (the case $h^0(E) < 4$ being much easier). Since E is spanned, $H^0(E)$ defines a morphism $f: V \to G$, $G = G(1, 3)$ the Grassmannian of lines in $\mathbf{P}^3$, such that $E = f^*(U)$, U the universal quotient bundle on G. Since E is ample, f is finite; hence, $f(V)$ is divisor. Since $\mathrm{Pic}(G)$ is generated by the class of the hyperplane, for every plane P in G, $P \cap f(V)$ has positive dimension. The restriction of U to any plane P in one of the two families of planes in G has a trivial factor; hence, $U|(P \cap f(V))$ and E are not ample, contradiction.

Assume $n = 4$. By (1), as in Case 2, E is not ample if there is an irreducible curve T such that, for suitable C, $C \cap T \neq \varnothing$, $C \neq T$ and $\mathrm{length}(T \cap C) \geq 2\deg(T)$. Take a hyperplane A containing C and a general quadric q in A containing C. $V \cap q$ is the union of C and a curve O in Q of degree 2. Take as T an irreducible component of O.

Now assume $n = 5$; hence, by (1) $h^0(E) = 6$, $h^0(E(-1)) = 0$. Since for every line $M \subset V$, $E|M$ has splitting type $(1, 1)$, we find $b \in H^0(E)$ with

$b \neq 0$, $C' := (b)_0$ vanishing on a line M; apply the previous case to $C' \backslash M$.

Now assume $n = 6$. To obtain $h^0(E) \leq 4$, hence, a contradiction, it is sufficient to prove that $h^0(\mathbf{I}_{C,\mathbf{P}}(2)) = 3$, i.e., $h^1(\mathbf{I}_{C,\mathbf{P}}(2)) = 0$. We assume (check the smooth divisors on the quadric surfaces) that no smooth elliptic curve D of degree 5 in $\mathbf{P}^3$ is contained in a quadric surface. First, find a point $x \in C$ such that no trisecant line to C contains x; such a point exists, because C is not a complete intersection of V and a surface union of trisecant lines to C. Thus, the image of the projection of C from x into a tridimensional linear space R, $x \notin R$, is a smooth elliptic curve D, $\deg(D) = 5$. Choose homogeneous coordinates $x_0, \dots, x_4$ such that $x = (1; 0; 0; 0; 0)$. Consider the family of projective transformations g_t, $t \neq 0$, given by $g_t(x_0; x_1; \dots; x_4) = (tx_0; x_1; \dots; x_4)$. Since the Hilbert scheme is proper, the family of curves $\{g_t(C)\}$ has a limit X when t goes to 0. X contains D and for degree reasons, at least a line through P. Thus, we see that $h^0(\mathbf{I}_{X,\mathbf{P}}(2)) \leq 2$. The thesis follows by semicontinuity.

Now assume $n \geq 7$. Either modify the proof of the case $n = 6$, or check that such a curve C is neither on a minimal degree surface nor in the complete intersection of 3 quadrics (if $n = 7$ in the second case C would be linked to a line, hence, projectively normal).

Case 7. Now assume $d = 4$; hence, V is the complete intersection of 2 quadrics in $\mathbf{P}^5$. Take a general section of E and let C and the exact sequence (1) be the associated data. Set $n := \deg(C)$.

Since $\mathbf{I}_C(2)$ is spanned, we see that for every point $x \in C$, the space of quadrics containing C and singular at x has codimension 2 in the space of quadrics containing C. Hence, by the Bertini theorem, there is a smooth surface S containing C, S the intersection of 3 quadrics in $\mathbf{P}^5$. By the adjunction formula K_S is trivial and $C^2 = 0$. Let D be the class of $\mathbf{O}_S(1)$. Since $\mathbf{I}_{C,S}(2)$ is spanned, $(2D - C)$ is nef and effective; hence, $(2D - C)^2 \geq 0$, i.e., $n \leq 8$. Assume $n = 8$; hence, $(2D - C)^2 = 0$. Take a general $A \in |2D - C|$; $A > 0$ by the adjunction formula; we may assume that A has not C as component. We get $AC = 2\deg(A)$. Thus, there is an irreducible component T of A with $\text{length}(T \cap C) \geq 2\deg(T)$. As in Cases 1 and 2, this implies that E is not ample. If $n = 4$, by (1) we get $h^0(E(-1)) = 2$; we conclude (since $c_1(E(-1)) = 0$) that $E = 2\mathbf{O}(1)$.

Assume $n = 5$ or 6. The proof here could be applied also in Case 6 (and probably vice versa). Given V and C, by (1) and [H, 1.1.1], we get a bundle E. By (1) this bundle is spanned. By (1) and the proof of Case 2, this bundle is not ample if there is a line R with $\text{length}(R \cap C) \geq 2$, R not contained in C (if C is reducible). Since ampleness is an open condition, it is sufficient to prove that for general V and C, the corresponding bundle has a nonampleness line. Count dimensions. We have $h^0(E) = 16 - 2n$; hence, each bundle gives a family of dimension $15 - 2n$ of elliptic curves of degree n. Each of these curves has a 2-dimensional family of secant lines in $\mathbf{P}^5$. Furthermore, given C and a secant line R to C, by the Bertini theorem there is a smooth V containing $C \cup R$; it is this point that fails for $n = 4$.

Case 8. Now assume $d = 5$. By the description in [I, 4.2(iii)(a)], V is the blowing up of a quadric $W \subset \mathbf{P}^4$ along a smooth curve. Hence, V contains (many) smooth rational curves D with normal bundle of degree -1. By the adjunction formula, $LD = 1$; hence, there is no such E.

Case 9. Assume $d = 2$. Then [I, 4.2(v)], V is a double covering $h: V \to \mathbf{P}^3$ ramified on a smooth surface of degree 5. Fix a section of E, hence, C and (1). First assume $h^{-1}(h(C)) = C$. Then if $h(C)$ is not a line, there is a line $R \subset \mathbf{P}^3$ intersecting $h(C)$ at at least two points, hence, an irreducible component T of $h^{-1}(R)$ with length$(R \cap C) \geq \deg(L|C)$; as in Cases 1 and 2, E cannot be ample.

If $h(C)$ is a line, by (1) we see that $E = 2\boldsymbol{O}(H)$, with $H := h^*(\boldsymbol{O}(1))$.

Now assume that $h|C$ has degree 1. In [I, 4.2(v)], V was described also as a weighted hypersurface. From this description we see that $h^0(L) = 11 = 1 + h^0(\mathbf{P}^3, \boldsymbol{O}(2))$. By the adjunction formula, C is not the complete intersection of a section of L and a section of H. Thus, we see that $h(C)$ cannot be a plane curve. If $h(C)$ is not a plane curve (hence, it is an elliptic quartic), by (1) we have $h^0(E) = 4$. This possibility was excluded at the beginning of Case 6.

Case 10. Assume $d = 1$. The map $h := h_L$ induced by $H^0(L)$ is a degree 2 map over a cone $W \subset \mathbf{P}^6$ over the Veronese surface; h has a smooth ramification divisor D; D is the intersection with W of a hypersurface of degree 3 not containing the vertex b of W. One checks that $h^0(L) = 7$ and that $L \otimes \boldsymbol{I}_C$ is spanned only if $h(C)$ is a line of the cone W, hence, through the vertex b of W. Assume $h(C)$ is a line for every such C. Then every section of E vanishes on the point $h^{-1}(b)$. Thus, E is not spanned.

Remark. For a general such line D, $h^{-1}(D)$ is a smooth elliptic curve and the bundle E associated to C by (1), and [H, 1.1 and 1.1.1] is spanned except at the point $h^{-1}(b)$. As in Case 2 we check the ampleness of the restriction of E to all the curves of V.

3

Here we prove Theorem 0 for a Fano 3-fold V with $b_2(V) \geq 2$. However, most of the work was done in the previous section. We will use the classification announced in [MM].

We noted (Case 8 of §2) that, with the notations of [MM], V must be primitive. For primitive 3-folds the classification is very short [MM]: either $b_2(V) = 2$ and V is a conic bundle over $\mathbf{P}^2$ or $b_2(V) = 3$ and V is a conic bundle over $\mathbf{P}^1 \times \mathbf{P}^1$. In particular this gives the missing argument at the beginning of Case 5 of §2. By 1.1 each fibration with total space V and general fiber $\mathbf{P}^1$ must have all the fibers reduced and irreducible, i.e., isomorphic to $\mathbf{P}^1$, and the fibration is smooth. Thus, V is a Severi-Brauer variety over its base. To show that V is the projectivization of a rank-2 vector bundle A over the base, it is sufficient to prove that the fibration has a rational section. One can use general principles [DF] to prove the existence of a rational section, but we prefer to use the proof of Case 5 in §2. Let $h: V \to B$, $B = \mathbf{P}^2$ or $B = \mathbf{P}^1 \times \mathbf{P}^1$, be the fibration. Look at the proof of Case 5 in §2. Fix any line D in B; by [Se] $h|h^{-1}(D)$ has a section. If $h^{-1}(D) = F_1$ the existence of a divisor in V unisecant to h (hence, a rational section of h) follows from the deformation invariance of the exceptional curves. Assume $h^{-1}(D)$ is not isomorphic to F_1 for one line D in B (hence, for all lines A, $h^{-1}(A) = F_0$ by the proof of Case 5 of §2). Fix another line R in B with $\mathrm{card}(D \cap R) = 1$. Fix the image T of a section of $h|h^{-1}(R)$. We get a unisecant divisor taking the union for all lines u with $U \cap R \neq \emptyset$ of the unique line projected by h to U and intersecting T.

Anyway, after we know that $V = \mathbf{P}(A)$, we settle completely the case $b_2(V) = 2$ in Case 5 in §2. We use from now on the notations of Case 5 of §2 with D a line in $\mathbf{P}^1 \times \mathbf{P}^1$ instead of $\mathbf{P}^2$. If $e = 0$, W splits [Ba, Proposition 3] or [W, 3.6.1], and $V = \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$; this 3-fold was considered in Case 4 of §2. For the same reason if $e = 1$, A and hence E are a direct sum of two isomorphic line bundles.

4. Proof of elementary properties of sectional genus

In this section we prove a few elementary properties of the sectional genus: g is an integer and (if E is ample plus something more) $g \geq 0$. Then we give an affirmative answer (see 4.3) to the case $n = 2$ of [LPS, Conjecture 2.6] (restated from the first version of [LPS]).

We fix a smooth connected variety X, $\dim(X) = n > 1$, and a vector bundle E on X with $\mathrm{rank}(E) = n - 1$. Let K be the canonical divisor on X. The sectional genus $g(E)$ of E is defined by the formula: $2g(E) - 2 = (K + c_1, (E))c_{n-1}(E)$. By the adjunction formula, if E has a section with zero-locus C of codimension $n - 1$, then $g(E) = p_a(C)$. Set $L := \det(E)$.

Remark 4.1. With the notations just introduced, $g(E)$ is an integer.

Proof. Fix an ample line bundle M on X. Take a big even integer such that $E \otimes M^t$ has sections with a curve as zero-locus and note that

$$(K + c_1(E))c_{n-1}(E) \equiv (K + c_1(E \otimes M^t))c_{n-1}(E \otimes M^t) \mod(2)$$

(use [H, beginning of §]). Thus, $g(E)$ is an integer. □

In [W, 3.6], there is a classification of pairs (X, E) with E ample spanned and with $L + K$ not nef; in particular this covers the case $g(E) = 0$. As noted in §1, if E is spanned and ample, $g(E) \geq 0$. We do not know (although it seems very, very reasonable) if $g(E) \geq 0$ when E is ample but not spanned.

Remark 4.2. Assume E ample. Then $g(E) > 0$ unless we are in one of the following cases:

(a) X is a $\mathbf{P}^{n-1}$-bundle over a smooth curve B; E is the pull-back of a vector bundle over B, twisted by a relative ample line bundle which induces $\boldsymbol{O}(1)$ on each fiber; furthermore, $g(E) = 0$ and $p_a(B) = 0$.

(b) $X = \mathbf{P}^n$, E is the direct sum of $n-2$ copies of $\boldsymbol{O}(1)$ and either $\boldsymbol{O}(1)$ or $\boldsymbol{O}(2)$; here $g(E) = 0$.

(c) X is a quadric hypersurface and $E = (n-1)\boldsymbol{O}(1)$; here $g(E) = 0$.

(d) $\mathrm{Pic}(X) = \mathbf{Z}$, n is odd, $n \geq 5$; let H be the positive divisor of $\mathrm{Pic}(X)$; then $K = -aH$, $L = (a-1)H$, with $a := (n+1)/2$.

Proof. We modify the proofs in [W]. Assume $g \leq 0$, i.e., $(K+L)c_{n-1}(E) < 0$. First assume that $K+L$ is nef. By the Kawamata-Shokurov contraction theorem (see [K, Theorem 2.6]), $K+L$ is semi-ample and in particular there is an integer $m > 0$ such that $m(K+L)$ has a section. If $m(K+L)$ is trivial, we get $g(E) = 1$, contradiction. Hence, there is $D \in |m(K+L)|$. For every irreducible component T of D, we have $Tc_{n-1}(E) > 0$, because $E|T$ is ample. Thus, $g(E) > 1$, contradiction.

Hence, we may assume that $K+L$ is not nef. By Mori theory, there is an extremal ray R such that $(K+L)R < 0$. In [W, 3.1], there was defined an integer $\Delta(L, R) > 0$. In our case we have $\Delta(L, R) = 1$ or 2. If $\Delta(L, R) = 1$, the proof of [W, 3.6] works without assuming the spannedness of E, because in [W, 3.5], there is no such assumption on L. Assume $\Delta(L, R) = 2$. By the definitions in [W, §2 and §3], this means that there is a curve C with $(K+L)C = -2$, $KC = -(n+1)$ and an extremal ray with length $n+1$, hence [W, 2.4.1], $\mathrm{Pic}(X) = \mathbf{Z}$ and X is a Fano manifold. If $n = 3$, this implies $X = \mathbf{P}^n$ by [W, 2.6]. Assume $n > 3$. Take a positive generator H of $\mathrm{Pic}(X)$. Let $a > 0$, $b > 0$ such that $K = -aH$, $L = bH$. By assumption $(-a+b)HC = -2$; hence, either $HC = 1$ or $HC = 2$. If $HC = 1$, then $a = n+1$, $X = \mathbf{P}^n$ by the Kobayashi-Ochiai theorem [K-O], E is uniform, hence, the direct sum of line bundles [OSS]; hence, we are in case (c) (but then $\Delta(L, R) = 1$). Now assume $HC = 2$. Hence $2a = n+1$ and $b = a-1$, as we wanted. □

Proposition 4.3. *Let S be a smooth complete surface and E a rank-2 ample spanned vector bundle on S. Set $K := K_S$ and $L := \det(E)$. Assume $L^2 \geq 10$. Then either S is a $\mathbf{P}^1$-bundle and E restricted to every fiber has splitting type $(1,1)$ or $K+L$ is very ample.*

Proof. By 1.1 and [Re, Theorem 1], $K+L$ is very ample unless there is an irreducible curve D with $LD = 2$ and $D^2 = 0$. Fix such a curve D and let u be the genus of its normalization R.

If $u = 0$ and D is singular, by [W, Lemma 3.2], then $DL \geq 3$, contradiction.

Assume $u = 0$ and D smooth. Since $D^2 = 0$, the normal bundle of D in S is trivial. By deformation theory we get that D moves in an algebraic family.

By the classification of algebraic surfaces we get a fibration $f: S \to B$, B a smooth curve, with D as one of its fibers. Since E is ample, all the fibers of f must be irreducible and reduced. Thus, S is a $\mathbf{P}^1$-bundle and E is π-uniform as required.

Assume $u > 0$. Let F be the pull-back to R of $E|D$. Fix any point $P \in R$. If $h^0(F) \leq 2$, the spannedness of F implies its triviality, contradiction. Thus, we may assume $h^0(F) \geq 3$. Hence, there is a nonzero section s of F, s vanishing at P. We get an exact sequence on R:

$$0 \to U \to F \to V \to 0 \tag{2}$$

with U, V line bundles on R, $\deg(U) > 0$. Since F is ample, $\deg(V) > 0$. Thus, $\deg(U) = \deg(V) = 1$. Since $u > 0$, $h^0(F) \leq 2$, contradiction. □

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