

HANKEL OPERATORS ON THE BERGMAN SPACE OF BOUNDED SYMMETRIC DOMAINS

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ABSTRACT. Let Ω be a bounded symmetric domain in \mathbb{C}^n with normalized volume measure dV . Let P be the orthogonal projection from $L^2(\Omega, dV)$ onto the Bergman space $L_a^2(\Omega)$ of holomorphic functions in $L^2(\Omega, dV)$. Let \bar{P} be the orthogonal projection from $L^2(\Omega, dV)$ onto the closed subspace of antiholomorphic functions in $L^2(\Omega, dV)$. The "little" Hankel operator h_f with symbol f is the operator from $L_a^2(\Omega)$ into $L^2(\Omega, dV)$ defined by $h_f g = \bar{P}(fg)$. We characterize the boundedness, compactness, and membership in the Schatten classes of the Hankel operators h_f in terms of a certain integral transform of the symbol f . These characterizations are further studied in the special cases of the open unit ball and the poly-disc in \mathbb{C}^n .

1. INTRODUCTION

Let Ω be a bounded symmetric domain in \mathbb{C}^n with (normalized) volume measure dV . For any $1 \leq p < +\infty$, the Bergman space $L_a^p(\Omega)$ consists of holomorphic functions in $L^p(\Omega, dV)$. It is easy to see that $L_a^p(\Omega)$ is a closed subspace of $L^p(\Omega, dV)$. In particular, there exists an orthogonal projection

$$P : L^2(\Omega, dV) \rightarrow L_a^2(\Omega).$$

Let $K(z, w)$ be the Bergman kernel of Ω , then

$$Pf(z) = \int_{\Omega} K(z, w)f(w) dV(w).$$

The above integral formula extends P to $L^1(\Omega, dV)$. Let \bar{P} be the orthogonal projection from $L^2(\Omega, dV)$ onto $L_a^2(\Omega) = \{\bar{f} : f \in L_a^2(\Omega)\}$, then

$$\begin{aligned} \bar{P}f(z) &= \int_{\Omega} \overline{K(z, w)}f(w) dV(w) \\ &= \int_{\Omega} K(w, z)f(w) dV(w). \end{aligned}$$

This formula also extends \bar{P} to $L^1(\Omega, dV)$.

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Given $\phi \in L^2(\Omega, dV)$, define two operators H_ϕ and h_ϕ with domain $L^2_a(\Omega)$ as follows:

$$H_\phi f = (I - P)(\phi f), \quad h_\phi f = \bar{P}(\phi f),$$

where I is the identity operator. H_ϕ is called the Hankel operator with symbol ϕ and h_ϕ is called the reduced (or little) Hankel operator with symbol ϕ . The word "reduced" (or little) is justified by the inequality

$$\bar{P} - P_0 \leq I - P,$$

where P_0 is the orthogonal projection (rank 1) from $L^2(\Omega, dV)$ onto the constants, that is

$$P_0 f(z) = \int_\Omega f(z) dV(z).$$

The main problems that we are interested in here are the following:

Problem 1. When are h_ϕ and H_ϕ bounded operators from $L^2_a(\Omega)$ into $L^2(\Omega, dV)$?

Problem 2. When are h_ϕ and H_ϕ compact?

Problem 3. When are h_ϕ and H_ϕ in the Schatten-Von Neumann ideals S_p ?

An immediate consequence of the inequality $\bar{P} - P_0 \leq I - P$ is that H_ϕ is bounded implies h_ϕ is bounded; H_ϕ is compact implies h_ϕ is compact; and H_ϕ is in S_p implies h_ϕ is in S_p .

By now it is fairly well-known [2, 3] that the properties of H_ϕ and $H_{\bar{\phi}}$ are governed by the Berezin transform. More specifically, for any f in $L^1(\Omega, dV)$, the Berezin transform of f is the function Bf defined on Ω by

$$Bf(z) = \int_\Omega f(w) |k_z(w)|^2 dV(w),$$

where

$$k_z(w) = \frac{K(w, z)}{\sqrt{K(z, z)}}$$

is a unit vector in $L^2_a(\Omega)$ for each $z \in \Omega$. For any $\phi \in L^2(\Omega, dV)$, let

$$MO(\phi)(z) = [B(|\phi|^2)(z) - |B\phi(z)|^2]^{1/2}, \quad z \in \Omega.$$

$MO(\phi)$ describes a type of mean oscillation of ϕ in Ω [2, 3]. It was shown in [3] that

- (1) $H_\phi, H_{\bar{\phi}}$ are bounded $\Leftrightarrow MO(\phi) \in B\mathbb{C}$;
- (2) $H_\phi, H_{\bar{\phi}}$ are compact $\Leftrightarrow MO(\phi) \in \mathbb{C}_0$.

Here $B\mathbb{C}$ is the space of bounded continuous functions on Ω and \mathbb{C}_0 is the space of continuous functions on Ω vanishing on the boundary.

In view of the above results, we see that the Berezin transform carries a lot of information on the Hankel operators H_ϕ . The main purpose of this article is to demonstrate that there exists a similar integral transform on Ω which carries a lot of information on the reduced Hankel operators. We give a unified treatment on the size estimates of h_ϕ using the integral transform V defined below. It should be pointed out that [6] gives a fairly complete theory on reduced Hankel operators. However, the theory in [6] does not cover our results here because Hankel operators (forms) in [6] are defined on a different space (or at least for the main results there), namely, the space of square integrable holomorphic functions with respect to the measure $K(z, z)^{-1} dV(z)$.

The integral transform V is defined as follows. Given f in $L^1(\Omega, dV)$, Vf is the function on Ω defined by

$$Vf(z) = \lambda_\Omega \int_\Omega f(w) \overline{k_z^2(w)} dV(w), \quad z \in \Omega,$$

where

$$\lambda_\Omega^{-1} = \int_\Omega K(z, z)^{-1} dV(z).$$

Note that in the definition of the Berezin transform we used $|k_z(w)|^2$, while in the definition of V we used $\overline{k_z^2(w)}$.

Theorem. *Let Ω be any bounded symmetric domain in \mathbb{C}^n . If $\phi \in L^2(\Omega, dV)$ and $p \geq 1$, then*

- (1) $h_{\overline{\phi}}$ is bounded $\Leftrightarrow V\phi \in BC$;
- (2) $h_{\overline{\phi}}$ is compact $\Leftrightarrow V\phi \in C_0$;
- (3) $h_{\overline{\phi}}$ is in S_p $\Leftrightarrow V\phi \in L^p(d\lambda)$, where $d\lambda(z) = K(z, z) dV(z)$.

Note that for $\phi \in L^2(\Omega, dV)$, we always have $h_{\overline{\phi}} = h_{\overline{P\phi}}$, where P is the Bergman projection. Thus in considering reduced Hankel operators, we can content ourselves with antiholomorphic symbols.

When ϕ is holomorphic in Ω , $V\phi$ is “computable” in the most interesting cases. If $\Omega = B_n$ and ϕ is holomorphic in B_n , then we will show that $V\phi$ behaves like the functions

$$(1 - |z|^2)^{n+1} \frac{\partial^{n+1}}{\partial z^\alpha}(z) \quad (|\alpha| = n + 1),$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ is an ordered n -tuple of nonnegative integers, $|\alpha| = \alpha_1 + \dots + \alpha_n$, and

$$\frac{\partial^{n+1} \phi}{\partial z^\alpha}(z) = \frac{\partial^{n+1} \phi}{\partial z_1^{\alpha_1} \dots \partial z_n^{\alpha_n}}(z)$$

for $|\alpha| = n + 1$. Therefore, we obtain

Corollary 1 [6]. *If ϕ is holomorphic in B_n and $p \geq 1$, then*

- (1) $h_{\overline{\phi}}$ is bounded $\Leftrightarrow (1 - |z|^2)^{n+1} \frac{\partial^{n+1} \phi}{\partial z^\alpha}(z) \in BC$ for all $|\alpha| = n + 1$;

- (2) $h_{\bar{\phi}}$ is compact $\Leftrightarrow (1 - |z|^2)^{n+1} \frac{\partial^{n+1} \phi}{\partial z^{\alpha^2}}(z) \in \mathbb{C}_0$ for all $|\alpha| = n + 1$;
- (3) $h_{\bar{\phi}}$ is in S_p $\Leftrightarrow (1 - |z|^2)^{n+1} \frac{\partial^{n+1} \phi}{\partial z^{\alpha^2}}(z) \in L^p(d\lambda)$ for all $|\alpha| = n + 1$.

There is another case when we can “compute” $V\phi$ for ϕ holomorphic, namely, when $\Omega = \mathbb{D}^n$, the poly-disc in \mathbb{C}^n . For simplicity of notation, we will confine ourselves to the bidisc \mathbb{D}^2 . In this case, we show that $V\phi$ is determined by the following five functions:

$$\begin{aligned}
 T_1\phi(z_1) &= (1 - |z_1|^2)^2 \frac{\partial^2 \phi}{\partial z_1^2}(z_1, 0), & T_2\phi(z_2) &= (1 - |z_2|^2)^2 \frac{\partial^2 \phi}{\partial z_2^2}(0, z_2), \\
 T_3\phi(z_1) &= (1 - |z_1|^2)^2 \frac{\partial^3 \phi}{\partial z_1^2 \partial z_2}(z_1, 0), \\
 T_4\phi(z_2) &= (1 - |z_2|^2)^2 \frac{\partial^3 \phi}{\partial z_1 \partial z_2^2}(0, z_2), \\
 T\phi(z_1, z_2) &= (1 - |z_1|^2)^2 (1 - |z_2|^2)^2 \frac{\partial^4 \phi}{\partial z_1^2 \partial z_2^2}(z_1, z_2).
 \end{aligned}$$

It is possible to use lower order derivatives here (as well as in the case of B_n) if we only consider S_p for $p > 1$ [10, 12].

Corollary 2. *If $p \geq 1$ and ϕ is holomorphic in \mathbb{D}^2 , then we have*

- (1) $h_{\bar{\phi}}$ is bounded $\Leftrightarrow T_i\phi \in BC(\mathbb{D}) \quad (1 \leq i \leq 4), \quad T\phi \in BC(\mathbb{D}^2)$;
- (2) $h_{\bar{\phi}}$ is compact $\Leftrightarrow T_i\phi \in \mathbb{C}_0(\mathbb{D}) \quad (1 \leq i \leq 4), \quad T\phi \in \mathbb{C}_0(\mathbb{D}^2)$;
- (3) $h_{\bar{\phi}}$ is in $S_p \Leftrightarrow T_i\phi \in L^p(\mathbb{D}, d\lambda) \quad (1 \leq i \leq 4), \quad T\phi \in L^p(\mathbb{D}^2, d\lambda)$.

2. BASIC PROPERTIES OF THE TRANSFORM V

In this section we collect the basic properties of the integral transform V defined by

$$Vf(z) = \lambda_{\Omega} \int_{\Omega} f(w) \overline{k_z^2(w)} dV(w).$$

Lemma 1. *The operator Q defined by*

$$Qf(z) = \lambda_{\Omega} \int_{\Omega} \frac{K(z, w)^2}{K(w, w)} f(w) dV(w)$$

is a bounded projection from $L^1(\Omega, dV)$ onto $L_a^1(\Omega)$.

Proof. See [9] or use Fubini’s theorem.

Proposition 2. *If $f \in L^2(\Omega, dV)$, then*

- (1) $PVf = Pf$;
- (2) $VPf = Vf$;
- (3) $V^2f = Vf$.

Proof. All the equalities follow from Fubini's theorem and Lemma 1 as well as the reproducing property of the Bergman projection P . We check (1) for example.

$$\begin{aligned}
 PVf(z) &= \int_{\Omega} K(z, w)Vf(w) dV(w) \\
 &= \int_{\Omega} K(z, w) dV(w)\lambda_{\Omega} \int_{\Omega} \frac{K(w, u)^2}{K(w, w)} f(u) dV(u) \\
 &= \lambda_{\Omega} \int_{\Omega} f(u) dV(u) \int_{\Omega} \frac{K(w, u)^2}{K(w, w)} K(z, w) dV(w) \\
 &= \int_{\Omega} f(u) dV(u)\lambda_{\Omega} \int_{\Omega} \frac{K(u, w)^2}{K(w, w)} K(w, z) dV(w) \\
 (\text{Lemma 1}) &= \int_{\Omega} f(u)\overline{K(u, z)} dV(u) \\
 &= \int_{\Omega} K(z, u)f(u) dV(u) = Pf(z).
 \end{aligned}$$

The other two equalities can be checked similarly.

Proposition 3. V is a bounded operator on $L^p(\Omega, d\lambda)$ for all $1 \leq p \leq +\infty$.

Proof. Since each k_z is a unit vector in $L^2_a(\Omega)$, we have

$$\begin{aligned}
 |Vf(z)| &\leq \lambda_{\Omega} \int_{\Omega} |f(w)||k_z(w)|^2 dV(w) \\
 &\leq \lambda_{\Omega} \|f\|_{\infty} \int_{\Omega} |k_z(w)|^2 dV(w) \\
 &= \lambda_{\Omega} \|f\|_{\infty}
 \end{aligned}$$

for all z in Ω . Thus $\|Vf\|_{\infty} \leq \lambda_{\Omega} \|f\|_{\infty}$ and V is bounded on $L^{\infty}(d\lambda)$.

If $p \geq 1$, then

$$|Vf(z)|^p \leq \lambda_{\Omega}^p \int_{\Omega} |f(w)|^p |k_z(w)|^2 dV(w)$$

since $|k_z(w)|^2 dV(w)$ is a probability measure for each $z \in \Omega$. Therefore,

$$\begin{aligned}
 \int_{\Omega} |Vf(z)|^p d\lambda(z) &\leq \lambda_{\Omega}^p \int_{\Omega} K(z, z) dV(z) \int_{\Omega} |f(w)|^p \frac{|K(z, w)|^2}{K(z, z)} dV(w) \\
 &= \lambda_{\Omega}^p \int_{\Omega} |f(w)|^p dV(w) \int_{\Omega} |K(z, w)|^2 dV(z) \\
 &= \lambda_{\Omega}^p \int_{\Omega} |f(w)|^p K(w, w) dV(w) \\
 &= \lambda_{\Omega}^p \int_{\Omega} |f(w)|^p d\lambda(w).
 \end{aligned}$$

Proposition 4. V is an orthogonal projection on the Hilbert space $L^2(\Omega, d\lambda)$.

Proof. Since V is a bounded operator on $L^2(d\lambda)$, it suffices to check that $V^2 = V$ and $V^* = V$ on $L^2(d\lambda)$, where V^* is the adjoint of V on $L^2(d\lambda)$.

$V^2 = V$ follows from Proposition 2. To prove $V^* = V$ on $L^2(d\lambda)$, it suffices to prove $\langle Vf, g \rangle_\lambda = \langle f, Vg \rangle_\lambda$ for all f, g in $L^2(d\lambda)$, where $\langle \cdot, \cdot \rangle_\lambda$ is the (complex) integral pairing with respect to the measure $d\lambda$. Once again, Fubini's theorem gives what we want:

$$\begin{aligned} \langle Vf, g \rangle_\lambda &= \int_\Omega \overline{g(z)} d\lambda(z) \lambda_\Omega \int_\Omega \frac{K(z, w)^2}{K(z, z)} f(w) dV(w) \\ &= \lambda_\Omega \int_\Omega f(w) dV(w) \int_\Omega K(z, w)^2 \overline{g(z)} dV(z) \\ &= \int_\Omega f(w) d\lambda(w) \lambda_\Omega \int_\Omega \frac{K(w, z)^2}{K(w, w)} g(z) dV(z) \\ &= \int_\Omega f(w) \overline{Vg(w)} d\lambda(w) = \langle f, Vg \rangle_\lambda. \end{aligned}$$

Remark 1. The boundedness of V on $L^p(d\lambda)$ ($1 \leq p \leq +\infty$) implies that

$$\langle Vf, g \rangle_\lambda = \langle f, Vg \rangle_\lambda$$

for all $f \in L^p(d\lambda)$, $g \in L^q(d\lambda)$ with $\frac{1}{p} + \frac{1}{q} = 1$, $1 \leq p \leq \infty$.

Remark 2. Under the usual integral pairing $\langle \cdot, \cdot \rangle$ (with respect to dV), we have

$$V^* f(z) = \lambda_\Omega \int_\Omega \frac{K(z, w)^2}{K(w, w)} f(w) dV(w) = Qf(z),$$

where Q is the operator in Lemma 1.

Proposition 5. For any $\phi \in L^2(\Omega, dV)$, we have

- (1) $h_{\overline{\phi}} = h_{\overline{P\phi}}$;
- (2) $h_{\overline{\phi}} = h_{\overline{V\phi}}$;
- (3) $\overline{V\phi(z)} = \lambda_\Omega \langle h_{\overline{\phi}} k_z, \overline{k}_z \rangle$.

Proof. (1) and (3) follow directly from the definitions of $h_{\overline{\phi}}$ and V . (2) is a consequence of (1) and Proposition 2:

$$h_{\overline{\phi}} = h_{\overline{P\phi}} = h_{\overline{PV\phi}} = h_{\overline{V\phi}}.$$

Proposition 6. $V : \mathbb{C}(\overline{\Omega}) \rightarrow \mathbb{C}_0(\Omega)$, where $\mathbb{C}(\overline{\Omega})$ is the space of complex functions on Ω which extend to $\overline{\Omega}$ (the Euclidean closure of Ω) continuously.

Proof. We will obtain this in §4 as a corollary. We give an alternative proof here in the special case $\Omega = B_n$. Let $\{\phi_z : z \in \Omega\}$ be the collection of auto-

morphisms defined in [8], then a change of variable gives

$$\begin{aligned} Vf(z) &= \lambda_\Omega \int_\Omega f(w) \overline{k_z(w)}^2 dV(w) \\ &= \lambda_\Omega \int_\Omega f \circ \phi_z(w) \overline{k_z}^2(\phi_z(w) |k_z(w)|^2) dV(w) \\ &= \lambda_\Omega \int_\Omega f \circ \phi_z(w) \frac{|k_z(w)|^2}{k_z(w)^2} dV(w) \\ &= \lambda_\Omega \int_\Omega f \circ \phi_z(w) \frac{K(w, z)}{K(z, w)} dV(w). \end{aligned}$$

If $f \in C(\overline{\Omega})$ and $z_0 \in \partial\Omega$, then

$$\lim_{z \rightarrow z_0} f \circ \phi_z(w) = f(z_0)$$

for each w since

$$\lim_{z \rightarrow z_0} \phi_z(w) = z_0$$

uniformly on compact sets. Therefore, by dominated convergence, we have

$$\lim_{z \rightarrow z_0} Vf(z) = \lambda_\Omega f(z_0) \int_\Omega \frac{K(w, z_0)}{K(z_0, w)} dV(w).$$

The reproducing property of K now gives

$$\begin{aligned} \int_\Omega \frac{K(w, z_0)}{K(z_0, w)} dV(w) &= \lim_{r \rightarrow 1^-} \int_\Omega \frac{K(w, rz_0)}{K(rz_0, w)} dV(w) \\ &= \lim_{r \rightarrow 1^-} \frac{1}{K(rz_0, rz_0)} = 0. \end{aligned}$$

3. BOUNDEDNESS OF $h_{\overline{\phi}}$

Proposition 7. *If $\phi \in L^2(\Omega, dV)$, then $h_{\overline{\phi}}$ is bounded if and only if $V\phi \in BC$. In this case, we have*

$$\|h_{\overline{\phi}}\| \leq \|V\phi\|_\infty \leq \lambda_\Omega \|h_{\overline{\phi}}\|.$$

Proof. Since

$$\overline{V\phi(z)} = \lambda_\Omega \langle h_{\overline{\phi}} k_z, \overline{k_z} \rangle$$

and k_z is a unit vector for each z , we must have $|V\phi(z)| \leq \lambda_\Omega \|h_{\overline{\phi}}\|$ for all z , and hence $\|V\phi\|_\infty \leq \lambda_\Omega \|h_{\overline{\phi}}\|$. On the other hand, we have by Proposition 5 $\|h_{\overline{\phi}}\| = \|h_{\overline{V\phi}}\|$. It is easy to see that $\|h_\psi\| \leq \|\psi\|_\infty$ for all $\psi \in L^\infty$. Thus we also have $\|h_{\overline{\phi}}\| \leq \|V\phi\|_\infty$.

Theorem 8 [10]. *If ϕ is holomorphic and in $L^2(\Omega, dV)$, then the following are equivalent:*

- (1) $h_{\overline{\phi}}$ is bounded;

- (2) $V\phi \in BC(\Omega)$;
- (3) $\phi \in PL^\infty(\Omega)$;
- (4) $|\int_\Omega f(z)\overline{\phi(z)} dV(z)| \leq C \int_\Omega |f(z)| dV(z)$ for some constant $C > 0$ and all $f \in H^\infty(\Omega)$.

Moreover, the following quantities are equivalent: $\|h_{\overline{\phi}}\|, \|V\phi\|_\infty, \inf\{\|\psi\|_\infty : \phi = P\psi\}, \sup\{|\int_\Omega f(z)\overline{\phi(z)} dV(z)| : \|f\|_1 \leq 1, f \in H^\infty\}$, where $\|f\|_1 = \int_\Omega |f(z)| dV(z)$.

Proof. The equivalence of (1) and (2) was proved in Proposition 7 along with the inequalities $\|h_{\overline{\phi}}\| \leq \|V\phi\|_\infty \leq \lambda_\Omega \|h_{\overline{\phi}}\|$.

By Proposition 2, $\phi \in L^2(\Omega, dV)$ is holomorphic implies that $\phi = PV\phi$. Therefore $\inf\{\|\psi\|_\infty : \phi = P\psi\} \leq \|V\phi\|_\infty$.

If $\phi = P\psi$ with $\psi \in L^\infty$, then $\|h_{\overline{\phi}}\| = \|h_{\overline{P\psi}}\| = \|h_{\overline{\psi}}\| \leq \|\psi\|_\infty$. This implies that $\|h_{\overline{\phi}}\| \leq \inf\{\|\psi\|_\infty : \phi = P\psi\}$.

It is well-known [2] that for each $z \in \Omega$, the function $k_z(w)$ is holomorphic in a neighbourhood of Ω . In particular, $k_z \in H^\infty$ for all $z \in \Omega$. Thus

$$\begin{aligned} \|V\phi\|_\infty &= \sup\{|V\phi(z)| : z \in \Omega\} \\ &= \lambda_\Omega \sup \left\{ \left| \int_\Omega \phi(w)\overline{k_z^2(w)} dV(w) \right| : z \in \Omega \right\} \\ &\leq \lambda_\Omega \sup \left\{ \left| \int_\Omega \phi(w)\overline{f(w)} dV(w) \right| : \|f\|_1 \leq 1, f \in H^\infty \right\}. \end{aligned}$$

Finally, if $f \in H^\infty$ and $\|f\|_1 \leq 1$, then

$$\begin{aligned} \int_\omega f(z)\overline{\phi(z)} dV(z) &= \int_\Omega Qf(z)\overline{\phi(z)} dV(z) \\ &= \int_\Omega f(z)\overline{V\phi(z)} dV(z) \end{aligned}$$

by Remark 2 in the last section. It follows that

$$\left| \int_\Omega f(z)\overline{\phi(z)} dV(z) \right| \leq \|V\phi\|_\infty \|f\|_1 \leq \|V\phi\|_\infty,$$

completing the proof of Theorem 8.

Corollary 9. Let $X = PL^\infty(\Omega)$ be equipped with the norm $\|\phi\|_* = \|h_{\overline{\phi}}\|$, then $L_a^1(\Omega)^* = X$ with equivalent norms.

Proof. Suppose $\xi \in L_a^1(\Omega)^*$, then the Hahn-Banach extension theorem implies that ξ can be extended to $L^1(\Omega, dV)$ without increasing the norm. Since $L^1(\Omega, dV)^* = L^\infty(\Omega)$, there must exist a function $\psi \in L^\infty(\Omega)$ such that

$\|\xi\| = \|\psi\|_\infty$ and $\xi(f) = \int_\Omega f(z)\overline{\psi(z)} dV(z)$ for all $f \in L^1_a(\Omega)$. Let $\phi = P\psi \in X$, then for all $f \in H^\infty$ (dense in $L^1_a(\Omega)$), we have

$$\begin{aligned} \xi(f) &= \int_\Omega f(z)\overline{\psi(z)} dV(z) \\ &= \int_\Omega Pf(z)\overline{\psi(z)} dV(z) \\ &= \int_\Omega f(z)\overline{P\psi(z)} dV(z) \\ &= \int_\Omega f(z)\overline{\phi(z)} dV(z). \end{aligned}$$

Moreover,

$$\|\phi\|_* \leq \sup \left\{ \left| \int_\Omega f(z)\overline{\phi(z)} dV(z) \right| : f \in H^\infty, \|f\|_1 \leq 1 \right\} = \|\xi\|.$$

Conversely, if $\phi \in X$, then ϕ induces a bounded linear functional ξ_ϕ on $L^1_a(\Omega)$ as follows:

$$\xi_\phi(f) = \int_\Omega f(z)\overline{\phi(z)} dV(z), \quad f \in H^\infty \text{ (dense in } L^1_a(\Omega)\text{)}.$$

Moreover, we have

$$\|\xi_\phi\| = \sup \left\{ \left| \int_\Omega f(z)\overline{\phi(z)} dV(z) \right| : f \in H^\infty, \|f\|_1 \leq 1 \right\} \leq \lambda_\Omega \|\phi\|_*.$$

Remark. Since the dual of any Banach space is still a Banach space, thus X is complete in the norm $\|\cdot\|_*$. Another way of seeing this is to consider X as a quotient space:

$$P: L^\infty(\Omega) \rightarrow X.$$

$\|\cdot\|_*$ is equivalent to the quotient norm. The quotient space of any Banach space is still a Banach space.

4. COMPACTNESS OF $h_{\overline{\phi}}$

Proposition 10. *Suppose $\phi \in L^2(\Omega, dV)$ with $V\phi \in BC$, then $h_{\overline{\phi}}$ is compact if and only if $V\phi \in C_0$ if and only if $V\phi \in C(\overline{\Omega})$.*

Proof. Since $\overline{V\phi(z)} = \lambda_\Omega \langle h_{\overline{\phi}}, k_z, \overline{k}_z \rangle$ and $k_z \rightarrow 0$ weakly as $z \rightarrow \partial\Omega$, thus $h_{\overline{\phi}}$ is compact implies that $V\phi \in C_0$ which in turn implies that $V\phi \in C(\overline{\Omega})$.

On the other hand, if $V\phi \in C(\overline{\Omega})$, then $V\phi$ can be approximated in $L^\infty(\Omega)$ by finite linear combinations of $z^\alpha \overline{z}^\beta$ (Stone-Weierstrass). Since each $h_{z^\alpha \overline{z}^\beta}$ is a finite rank operator and $\psi \mapsto h_\psi$ is norm continuous, $h_{\overline{V\phi}}$ can be approximated in norm by finite linear combinations of finite rank operators. Hence $h_{\overline{\phi}} = h_{\overline{V\phi}}$ is compact.

Corollary 11. $V : C(\overline{\Omega}) \rightarrow C_0(\Omega)$.

Proof. Suppose $\phi \in C(\overline{\Omega})$, then $h_{\overline{\phi}}$ is compact by the proof of Proposition 10. Thus $V \in C_0(\Omega)$ by the proposition.

Theorem 12 [10]. *If ϕ is holomorphic in Ω with $h_{\bar{\phi}}$ bounded, then the following are equivalent:*

- (1) $h_{\bar{\phi}}$ is compact;
- (2) $V\phi \in \mathbb{C}_0(\Omega)$;
- (3) $V\phi \in \mathbb{C}(\bar{\Omega})$;
- (4) $\phi \in PC_0(\Omega)$;
- (5) $\phi \in PC(\bar{\Omega})$.

Proof. The equivalences of (1), (2), and (3) are in Proposition 10. $\phi = PV\phi$ gives the implication (2) \Rightarrow (4). (4) \Rightarrow (5) is trivial. If $\phi = P\psi$ with $\psi \in \mathbb{C}(\bar{\Omega})$, then $h_{\bar{\phi}} = h_{\bar{\psi}}$ is compact by the Stone-Weierstrass argument used in the proof of Proposition 10, completing the proof of Theorem 12.

Recall that $X = PL^\infty(\Omega)$ is the Banach space with norm $\|f\|_* = \|h_{\bar{f}}\|$. Let X_0 be the closed subspace of X consisting of f such that $h_{\bar{f}}$ is compact. It is easy to see that the polynomials are dense in X_0 [10], thus X_0 is separable (X is not!).

Corollary 13. $X_0^* = L_a^1(\Omega)$ with equivalent norms.

Proof. Suppose $f \in L_a^1(\Omega)$, then f induces a bounded linear functional ξ_f on X :

$$\xi_f(g) = \int_{\Omega} g(z) \overline{f(z)} dV(z), \quad g \in \mathcal{P},$$

where \mathcal{P} is the set of polynomials (dense in X_0). By Corollary 9, $\|\xi_f\| \leq \lambda_{\Omega} \|f\|_1$.

Conversely, suppose that $\xi \in X_0^*$, we wish to find a function $f \in L_a^1(\Omega)$ such that $\xi = \xi_f$. Consider the mapping

$$V: X_0 \rightarrow \mathbb{C}_0(\Omega).$$

By Proposition 7, V is an imbedding, thus the image VX_0 is a closed subspace of $\mathbb{C}_0(\Omega)$. Moreover,

$$\xi \circ V^{-1}: VX_0 \rightarrow \mathbb{C}$$

is a bounded linear functional on VX_0 with norm $\leq \|V^{-1}\| \|\xi\|$. Extend $\xi \circ V^{-1}$ to the space $\mathbb{C}_0(\Omega)$ by the Hahn-Banach theorem. The Riesz representation theorem then produces a finite complex Borel measure μ on Ω such that

$$\xi \circ V^{-1}(f) = \int_{\Omega} f(z) d\mu(z)$$

for all f in VX_0 . It follows that

$$\xi(g) = \int_{\Omega} Vg(z) d\mu(z)$$

for all g in X_0 . If $g \in X_0$ is a polynomial, then Fubini's theorem gives

$$\begin{aligned} \xi(g) &= \int_{\Omega} d\mu(z)\lambda_{\Omega} \int_{\Omega} \frac{K(z, w)^2}{K(z, z)} g(w) dV(w) \\ &= \int_{\Omega} g(w) dV(w)\lambda_{\Omega} \int_{\Omega} \frac{K(z, w)^2}{K(z, z)} d\mu(z). \end{aligned}$$

Let

$$f(w) = \lambda_{\Omega} \int_{\Omega} \frac{K(w, z)^2}{K(z, z)} d\bar{\mu}(z),$$

then $f \in L^1_a(\Omega)$ with

$$\begin{aligned} \|f\|_1 &\leq \lambda_{\Omega} \int_{\Omega} dV(w) \int_{\Omega} \frac{|K(w, z)|^2}{K(z, z)} d|\mu|(z) \\ &= \lambda_{\Omega} \int_{\Omega} d|\mu|(z) \int_{\Omega} \frac{|K(w, z)|^2}{K(z, z)} dV(w) \\ &= \lambda_{\Omega} \int d|\mu|(z) = \lambda_{\Omega} \|\mu\| \leq \lambda_{\Omega} \|\xi\| \|V^{-1}\|. \end{aligned}$$

Moreover, we have

$$\xi(g) = \int_{\Omega} g(w)\overline{f(w)} dV(w) = \xi_f(g)$$

for all g in \mathcal{P} (dense in X_0). This completes the proof of Corollary 13.

Remark. The results in this section as well as the last section are mostly taken from [10]. The approach here is improved.

5. SCHATTEN-VON NEUMANN CLASS HANKEL OPERATORS

Suppose A is a bounded operator from $L^2_a(\Omega)$ into $\overline{L^2_a(\Omega)}$, then A^*A is a positive operator on the Hilbert space $L^2_a(\Omega)$. We say that A is in the Schatten-Von Neumann p -ideal S_p if $\text{tr}((A^*A)^{p/2})$ is finite. We will be mainly concerned with the case $1 \leq p < \infty$. In this case, each S_p is a Banach space. We will write S_{∞} for the set of all bounded operators. Note that some authors write S_{∞} for the set of all compact operators. [5] contains a fair amount of information about the ideals S_p . We will need the interpolating property:

$$[S_{p_0}, S_{p_1}]_{\theta} = S_p$$

if $1 \leq p_0 < p_1 \leq +\infty$ and $1/p = (1 - \theta)/p_0 + \theta/p_1$ for $\theta \in (0, 1)$, where $[\ , \]_{\theta}$ is the functor of complex interpolation. The L^p -spaces also have the same interpolating properties:

$$[L^{p_0}(d\mu), L^{p_1}(d\mu)]_{\theta} = L^p(d\mu)$$

if $1 \leq p_0 < p_1 \leq +\infty$ and $1/p = (1 - \theta)/p_0 + \theta/p_1$ with $\theta \in (0, 1)$. Any book on interpolation of Banach spaces should contain the above results.

Lemma 14. For $1 \leq p \leq +\infty$, we have

$$\phi \in L^p(d\lambda) \Rightarrow h_\phi \in S_p.$$

Proof. The case $p = +\infty$ is clear. The case $p = 2$ is also easy. Recall that

$$h_\phi f(z) = \overline{P}(\phi f)(z) = \int_{\Omega} K(w, z)\phi(w)f(w) dV(w).$$

Thus h_ϕ is an integral operator with kernel $K(w, z)\phi(w)$ satisfying

$$\begin{aligned} & \int_{\Omega} \int_{\Omega} |K(w, z)\phi(w)|^2 dV(z) dV(w) \\ &= \int_{\Omega} |\phi(w)|^2 dV(w) \int_{\Omega} |K(z, w)|^2 dV(z) \\ &= \int_{\Omega} |\phi(w)|^2 K(w, w) dV(w) \\ &= \|\phi\|_{L^2(d\lambda)}^2. \end{aligned}$$

Therefore, $\phi \in L^2(d\lambda)$ implies $h_\phi \in S_2$. By complex interpolation, we have $\phi \in L^p(d\lambda)$ implies that $h_\phi \in S_p$ for all $2 < p < +\infty$. The case $1 \leq p \leq 2$ is a little more difficult. Again we will use complex interpolation. It suffices to check the endpoint $p = 1$. The method below is contained in [6].

First observe that

$$h_\phi = \int_{\Omega} \phi(w)h_{\overline{K}_w} dV(w),$$

where $K_w(z) = K(z, w)$. In fact, if $f \in L^2_a(\Omega)$, then

$$\begin{aligned} h_{\overline{K}_w} f(z) &= \overline{P}(\overline{K}_w f)(z) \\ &= \int_{\Omega} K(u, z)\overline{K}_w(u)f(u) dV(u) \\ &= \int_{\Omega} K(u, z)K(w, u)f(u) dV(u) \\ &= K(w, z)f(w) = f(w)\overline{K}_w(z), \end{aligned}$$

and

$$\begin{aligned} \int_{\Omega} \phi(w)h_{\overline{K}_w} f(z) dV(w) &= \int_{\Omega} \phi(w)f(w)\overline{K}_w(z) dV(w) \\ &= \int_{\Omega} K(w, z)\phi(w)f(w) dV(w) \\ &= h_\phi f(z). \end{aligned}$$

Therefore, we have

$$h_\phi = \int_{\Omega} \phi(w)h_{\overline{K}_w} dV(w)$$

with each $h_{\overline{K}_w}$ being a rank 1 operator.

Suppose $\phi \in L^1(\Omega, d\lambda)$, we wish to show that $h_\phi \in S_1$. It suffices to show that the above integral converges in S_1 (because S_1 is complete). Since $h_{\bar{K}_w}$ is a rank 1 operator, we have

$$\|h_{\bar{K}_w}\|_{S_1} = \|h_{\bar{K}_w}\|.$$

Note that

$$h_{\bar{K}_w} f = f(w)\bar{K}_w,$$

thus

$$\begin{aligned} \|h_{\bar{K}_w}\| &= \sup\{|f(w)| \|\bar{K}_w\|_2 : \|f\|_2 = 1 \text{ in } L^2_a(\Omega)\} \\ &= \sqrt{K(w, w)} \sup\{|f(w)| : \|f\|_2 = 1 \text{ in } L^2_a(\Omega)\} \\ &= K(w, w). \end{aligned}$$

Therefore

$$\begin{aligned} \int_\Omega \|\phi(w)h_{\bar{K}_w}\|_{S_1} dV(w) &= \int_\Omega |\phi(w)|K(w, w) dV(w) \\ &= \int_\Omega |\phi(w)| d\lambda(w) < +\infty \end{aligned}$$

if $\phi \in L^1(d\lambda)$. This completes the proof of Lemma 14.

Theorem 15. Suppose $1 \leq p \leq \infty$ and $V\phi \in BC$, then

$$h_\phi \in S_p \Leftrightarrow V\phi \in L^p(d\lambda).$$

Proof. The case $p = +\infty$ was settled in §3. Since $h_\phi = h_{V\phi}$, Lemma 14 gives the implication $V\phi \in L^p(d\lambda) \Rightarrow h_\phi \in S_p$.

Suppose $h_\phi \in S_p$, then $(h_\phi^*h_\phi)^{p/2} \in S_1$. By [1] or [11],

$$\text{tr}[(h_\phi^*h_\phi)^{p/2}] = \int_\Omega \langle (h_\phi^*h_\phi)^{p/2}k_z, k_z \rangle K(z, z) dV(z).$$

If $p \geq 2$, then the generalized Hölder inequality in [1] gives

$$\begin{aligned} \text{tr}[(h_\phi^*h_\phi)^{p/2}] &\geq \int_\Omega \langle h_\phi^*h_\phi k_z, k_z \rangle^{p/2} K(z, z) dV(z) \\ &= \int_\Omega \|h_\phi k_z\|^p K(z, z) dV(z). \end{aligned}$$

Since k_z is a unit vector, the Cauchy-Schwarz inequality gives

$$\|h_\phi k_z\| \geq |\langle h_\phi k_z, \bar{k}_z \rangle| = \lambda_\Omega^{-1} |V\phi(z)|.$$

Thus we get

$$\lambda_\Omega \text{tr}[(h_\phi^*h_\phi)^{p/2}] \geq \int_\Omega |V\phi(z)|^p d\lambda(z)$$

for all $p \geq 2$. This gives the implication $h_\phi \in S_p \Rightarrow V\phi \in L^p(d\lambda)$ for all $p \geq 2$.

The case $1 \leq p \leq 2$ is once again more difficult. We outline a method (used in [7]) to prove the implication $h_{\bar{\phi}} \in S_p \Rightarrow Vf \in L^p(d\lambda)$ for all $1 \leq p < +\infty$. (At least for the case of the open unit ball or the poly-disc in \mathbb{C}^n —added in proof.)

Fix a sequence of points $\{a_n\}$ in Ω such that

(1) $\Omega = \bigcup_{n=1}^{\infty} E(a_n)$, where $E(a_n)$ is the Bergman metric ball with center a_n and radius r (some fixed positive number);

(2) There exists a constant $C > 0$ such that every function f in $L_a^2(\Omega)$ can be written as

$$f(z) = \sum_{n=1}^{\infty} \lambda_n k_{a_n}(z)$$

with

$$\|f\|_2 \leq C \inf \left\{ \sqrt{\sum_{n=1}^{\infty} |\lambda_n|^2} : f = \sum_{n=1}^{\infty} \lambda_n k_{a_n} \right\}.$$

For the construction of such a sequence $\{a_n\}$, see [4].

Define an operator A on $L_a^2(\Omega)$ by letting

$$Ae_n = k_{a_n}, \quad n = 1, 2, \dots,$$

where $\{e_n\}$ is a fixed orthonormal basis of $L_a^2(\Omega)$. If $f \in L_a^2(\Omega)$ with $f = \sum_{n=1}^{\infty} f_n e_n$, then

$$Af = \sum_{n=1}^{\infty} f_n k_{a_n}$$

and by (2) above,

$$\begin{aligned} \|Af\| &\leq C \inf \left\{ \sqrt{\sum_{n=1}^{\infty} |\lambda_n|^2} : Af = \sum_{n=1}^{\infty} \lambda_n k_{a_n} \right\} \\ &\leq C \sqrt{\sum_{n=1}^{\infty} |f_n|^2} = C \|f\|. \end{aligned}$$

Thus A is a bounded linear operator. Let \bar{A} be the operator on $\overline{L_a^2(\Omega)}$ defined by $\bar{A}\bar{e}_n = \bar{k}_{a_n}$, then \bar{A} is also bounded.

Suppose $h_{\bar{\phi}} \in S_p$ with $p \geq 1$, then we also have $\bar{A}^* h_{\bar{\phi}} A \in S_p$. This implies that

$$\sum_{n=1}^{\infty} |\langle \bar{A}^* h_{\bar{\phi}} A e_n, \bar{e}_n \rangle|^p < +\infty,$$

or

$$\sum_{n=1}^{\infty} |\langle h_{\bar{\phi}} k_{a_n}, \bar{k}_{a_n} \rangle|^p < +\infty,$$

or

$$\sum_{n=1}^{\infty} |V\phi(a_n)|^p < +\infty.$$

It is not difficult to show that [11] $V\phi(z)$ behaves like $V\phi(a_n)$ for $z \in E(a_n)$. Also [11] $K(z, z)$ behaves like $K(a_n, a_n) \cong \frac{1}{E(a_n)}$ for z in $E(a_n)$. It follows that

$$\begin{aligned} \int_{\Omega} |V\phi(z)|^p d\lambda(z) &\leq \sum_{n=1}^{\infty} \int_{E(a_n)} |V\phi(z)|^p K(z, z) dV(z) \\ &\leq C_1 \sum_{n=1}^{\infty} \frac{1}{|E(a_n)|} \int_{E(a_n)} |V\phi(z)|^p dV(z) \\ &\leq C_2 \sum_{n=1}^{\infty} \frac{1}{|E(a_n)|} \int_{E(a_n)} |V\phi(a_n)|^p dV(z) \\ &= C_2 \sum_{n=1}^{\infty} |V\phi(a_n)|^p < +\infty, \end{aligned}$$

completing the proof of Theorem 15.

Corollary 16. *Let $\{a_n\}$ be as in the proof of the above theorem, then for all $\phi \in L^2(\Omega, dV)$ we have*

- (1) $h_{\bar{\phi}}$ is bounded $\Leftrightarrow \{V\phi(a_n)\} \in l^\infty$;
- (2) $h_{\bar{\phi}}$ is compact $\Leftrightarrow \{V\phi(a_n)\} \in c_0$;
- (3) $h_{\bar{\phi}}$ is in S_p $\Leftrightarrow \{V\phi(a_n)\} \in l^p$, ($p \geq 1$).

Corollary 17. *If ϕ is holomorphic in Ω with $V\phi \in BC$, then for all $p \geq 1$, we have $h_{\bar{\phi}} \in S_p$ if and only if $\phi \in PL^p(d\lambda)$.*

Proof. If $\phi = P\psi$ with $\psi \in L^p(d\lambda)$, then

$$h_{\bar{\phi}} = h_{\overline{P\psi}} = h_{\bar{\psi}} \in S_p$$

by Lemma 14. Conversely, if $h_{\bar{\phi}} \in S_p$, then $V\phi \in L^p(d\lambda)$ by Theorem 15. It follows that $\phi = PV\phi$ is in $PL^p(d\lambda)$.

6. THE OPEN UNIT BALL B_n

In this section, we specialize to the special case $\Omega = B_n$, the open unit ball in \mathbb{C}^n . We show that if f is holomorphic in B_n , then Vf behaves like the following functions:

$$(1 - |z|^2)^{n+1} \frac{\partial^{n+1} f}{\partial z^\alpha}(z) \quad (|\alpha| = n + 1).$$

To see why this is true, we first observe that for $f \in L^2_a(\Omega)$,

$$f(z) = \int_{B_n} \frac{f(w) dV(w)}{(1 - \langle z, w \rangle)^{n+1}},$$

$$(1 - |z|^2)^{n+1} \frac{\partial^{n+1} f}{\partial z^\alpha}(z) = \frac{(2n + 1)!}{n!} \int_{B_n} \frac{(1 - |z|^2)^{n+1} \bar{w}^\alpha f(w)}{(1 - \langle z, w \rangle)^{2n+2}}.$$

On the other hand,

$$Vf(z) = \lambda_{B_n} \int_{B_n} \frac{(1 - |z|^2)^{n+1} f(w)}{(1 - \langle z, w \rangle)^{2n+2}} dV(w).$$

Therefore, the only difference between $Vf(z)$ and $(1 - |z|^2)^{n+1} \partial^{n+1} f(z) / \partial z^\alpha$ ($|\alpha| = n + 1$) is the factor \bar{w}^α in the integrand. It suggests that $Vf(z)$ should behave like the derivatives $(1 - |z|^2)^{n+1} \partial^{n+1} f(z) / \partial z^\alpha$ ($|\alpha| = n + 1$). The following theorem shows that this is indeed the case. Let

$$T_\alpha f(z) = (1 - |z|^2)^{n+1} \frac{\partial^{n+1} f}{\partial z^\alpha}(z) \quad (|\alpha| = n + 1).$$

Theorem 18. *If $f \in L^p_a(B_n)$ and $p \geq 1$, then*

- (1) $Vf \in BC \Leftrightarrow T_\alpha f \in BC$ for all $|\alpha| = n + 1$;
- (2) $Vf \in C_0 \Leftrightarrow T_\alpha f \in C_0$ for all $|\alpha| = n + 1$;
- (3) $Vf \in L^p(d\lambda) \Leftrightarrow T_\alpha f \in L^p(d\lambda)$ for all $|\alpha| = n + 1$.

Remark. Since $Vf \in BC$ iff $f \in PL^\infty(\Omega)$ and $Vf \in C_0$ iff $f \in PC_0(\Omega)$, (1) and (2) above follow from the results in [12]. Our idea of proving (3) above is still the same as that in [12]. Although Theorem 18 is conceivably obvious in view of the observation proceeding it, we still need some technical lemmas to make the proof complete.

Recall that the fundamental theorem of calculus gives

$$f(z) - f(0) = \sum_{i=1}^n z_i \int_0^1 \frac{\partial f}{\partial z_i}(tz) dt.$$

Let

$$A_i f(z) = \int_0^1 \frac{\partial f}{\partial z_i}(tz) dt, \quad 1 \leq i \leq n.$$

For any n -tuple $\alpha = (\alpha_1, \dots, \alpha_n)$ of nonnegative integers, let

$$A_\alpha = A_1^{\alpha_1} \cdots A_n^{\alpha_n},$$

acting on holomorphic functions (the A_i 's commute with each other).

Lemma 19. *For all $\sigma \leq 0$ and $p \geq 1$, the operators A_α are bounded on $L^p_a(B_n, K(z, z)^\sigma dV(z))$.*

This is basically a Gleason problem [8]. It can either be proved by the method in [12], or be proved by using Cauchy integrals. Either way, it suffices to consider the operators A_1, \dots, A_n . The boundedness of A_α follows from the boundedness of A_1, \dots, A_n . We omit the details here.

Corollary 20. *There exists a constant $C > 0$ such that for all holomorphic f with $\partial^{|\alpha|} f(0)/\partial z^\alpha = 0$ (for all $|\alpha| \leq n$) we have*

$$f(z) = \sum_{|\alpha|=n+1} z^\alpha A_\alpha f(z).$$

Moreover, $\|K^{-1}A_\alpha f\|_{L^p(d\lambda)} \leq C\|K^{-1}f\|_{L^p(d\lambda)}$ for all $|\alpha| = n + 1$, where $K^{-1}f(z) = K(z, z)^{-1}f(z) = (1 - |z|^2)^{n+1}f(z)$.

We can now prove Theorem 18. Only (3) needs to be checked.

Suppose $Vf \in L^p(d\lambda)$. The equality $f = PVf$ gives

$$f(z) = \int_{B_n} \frac{Vf(w)}{(1 - \langle z, w \rangle)^{n+1}} dV(w).$$

Differentiating under the integral sign leads to

$$(1 - |z|^2)^{n+1} \frac{\partial^{n+1} f}{\partial z^\alpha}(z) = \frac{(2n + 1)!}{n!} \int_{B_n} \frac{(1 - |z|^2)^{n+1} \bar{w}^\alpha Vf(w)}{(1 - \langle z, w \rangle)^{2n+2}} dV(w).$$

Let M_α be the operator defined by

$$M_\alpha f(z) = \frac{(2n + 1)!}{n! \lambda_{B_n}} \bar{z}^\alpha f(z) = (n + 1)! \bar{z}^\alpha f(z),$$

then the above equality becomes

$$T_\alpha f = VM_\alpha Vf.$$

Clearly, M_α is bounded on $L^p(d\lambda)$. Also by Proposition 3, V is bounded on $L^p(d\lambda)$. Thus $Vf \in L^p(d\lambda)$ implies that $T_\alpha f = VM_\alpha(Vf) \in L^p(d\lambda)$ for all $|\alpha| = n + 1$.

Conversely, suppose $T_\alpha f(z) \in L^p(d\lambda)$ for all $|\alpha| = n + 1$. We wish to show that $Vf \in L^p(d\lambda)$ ($p \geq 1$). Since $Vg \in L^p(d\lambda)$ for all polynomials g . We may assume that $\partial^{|\alpha|} f(0)/\partial z^\alpha = 0$ for all $|\alpha| \leq n$. Thus

$$f(z) = \sum_{|\alpha|=n+1} z^\alpha f_\alpha(z),$$

where $f_\alpha(z) = A_\alpha f(z)$. In order to prove $Vf \in L^p(d\lambda)$, it suffices to show that there exists a constant $C > 0$ such that

$$|\langle Vf, g \rangle_\lambda| \leq C\|g\|_{L^q(d\lambda)}$$

for all $g \in L^q(d\lambda)$ with $1/p + 1/q = 1$. By the proof of Proposition 4,

$$\langle Vf, g \rangle_\lambda = \langle f, Vg \rangle_\lambda = \langle Vf, Vg \rangle_\lambda.$$

Write $Vg(z) = K(z, z)^{-1}\tilde{g}(z)$ with \tilde{g} holomorphic. It suffices to produce a constant $C > 0$ such that

$$|\langle Vf, K^{-1}\tilde{g} \rangle_\lambda| \leq C\|K^{-1}\tilde{g}\|_{L^q(d\lambda)}.$$

Since $\partial^{|\alpha|} f(0)/\partial z^\alpha = 0$ ($|\alpha| \leq n$), $K(z, z)^{-1} = (1 - |z|^2)^{n+1}$ is unitarily invariant, we must have

$$\langle f, K^{-1} \tilde{g} \rangle_\lambda = \langle f, K^{-1} \tilde{g}_1 \rangle_\lambda$$

where $\tilde{g}(z) = \tilde{g}_0(z) + \tilde{g}_1(z)$ with

$$\tilde{g}_1(z) = \sum_{|\alpha|=n+1} z^\alpha A_\alpha \tilde{g}(z).$$

Let $\tilde{g}_\alpha = A_\alpha \tilde{g}$, then

$$\begin{aligned} \langle f, K^{-1} \tilde{g}_1 \rangle_\lambda &= \sum_{|\alpha|=n+1} \int_{\Omega} f(z) \bar{z}^\alpha \tilde{g}_\alpha(z) dV(z) \\ &= \frac{1}{(n+1)!} \sum_{|\alpha|=n+1} \langle f, M_\alpha^* g_\alpha \rangle \\ &= \frac{1}{(n+1)!} \sum_{|\alpha|=n+1} \langle M_\alpha f, \tilde{g}_\alpha \rangle \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ is the usual pairing with respect to dV and $M_\alpha^* f(z) = z^\alpha f(z)$. Going back to the pairing $\langle \cdot, \cdot \rangle_\lambda$, we get

$$\langle f, K^{-1} \tilde{g} \rangle_\lambda = \frac{1}{(n+1)!} \sum_{|\alpha|=n+1} \langle M_\alpha f, K^{-1} \tilde{g}_\alpha \rangle_\lambda$$

Since $K^{-1} \tilde{g}$ is in the image of V and V is a projection in the pairing $\langle \cdot, \cdot \rangle_\lambda$, we have

$$\begin{aligned} \langle f, K^{-1} \tilde{g} \rangle_\lambda &= \frac{1}{(n+1)!} \sum_{|\alpha|=n+1} \langle VM_\alpha f, K^{-1} \tilde{g}_\alpha \rangle_\lambda \\ &= \frac{1}{(n+1)!} \sum_{|\alpha|=n+1} \langle T_\alpha f, K^{-1} \tilde{g}_\alpha \rangle_\lambda. \end{aligned}$$

It follows that

$$|\langle f, K^{-1} \tilde{g} \rangle_\lambda| \leq \frac{1}{(n+1)!} \sum_{|\alpha|=n+1} \|T_\alpha f\|_{L^p(d\lambda)} \|K^{-1} \tilde{g}_\alpha\|_{L^q(d\lambda)}.$$

By Corollary 20, we have

$$|\langle f, K^{-1} \tilde{g} \rangle_\lambda| \leq C \sum_{|\alpha|=n+1} \|T_\alpha f\|_{L^p(d\lambda)} \|K^{-1} \tilde{g}\|_{L^q(d\lambda)},$$

completing the proof of Theorem 18.

Corollary 21. Suppose $p \geq 1$ and ϕ is holomorphic on B_n , then

- (1) $h_{\bar{\phi}}$ is bounded $\Leftrightarrow T_\alpha \phi \in B\mathbb{C}$ for all $|\alpha| = n+1$;
- (2) $h_{\bar{\phi}}$ is compact $\Leftrightarrow T_\alpha \phi \in \mathbb{C}_0$ for all $|\alpha| = n+1$;
- (3) $h_{\bar{\phi}}$ is in S_p $\Leftrightarrow T_\alpha \phi \in L^p(d\lambda)$ for all $|\alpha| = n+1$,

where

$$T_\alpha \phi(z) = (1 - |z|^2)^{n+1} \frac{\partial^{n+1} \phi}{\partial z^\alpha}(z).$$

7. THE POLY-DISC \mathbb{D}^n

In this section we compute Vf for holomorphic functions f on the polydisc \mathbb{D}^n in \mathbb{C}^n . For simplicity of notation we restrict our computation to the bidisc \mathbb{D}^2 in \mathbb{C}^2 . \mathbb{D} will denote the open unit disc in \mathbb{C} .

Recall that for f holomorphic in \mathbb{D}^2 we have

$$\begin{aligned} T_1 f(z_1) &= (1 - |z_1|^2)^2 \frac{\partial^2 f}{\partial z_1^2}(z, 0), & T_2 f(z_2) &= (1 - |z_2|^2)^2 \frac{\partial^2 f}{\partial z_2^2}(0, z_2), \\ T_3 f(z_1) &= (1 - |z_1|^2)^2 \frac{\partial^3 f}{\partial z_1^2 \partial z_2}(z_1, 0), & T_4 f(z_2) &= (1 - |z_2|^2)^2 \frac{\partial^3 f}{\partial z_1 \partial z_2^2}(0, z_2), \\ T f(z_1, z_2) &= (1 - |z_1|^2)^2 (1 - |z_2|^2)^2 \frac{\partial^4 f}{\partial z_1^2 \partial z_2^2}(z_1, z_2). \end{aligned}$$

We will show that the above five functions describe the behaviour of Vf . As a motivation, we first compare Vf with Tf . Writing $f = Pf$ and differentiating under integral sign, we easily get

$$Tf(z) = 36(1 - |z_1|^2)^2 (1 - |z_2|^2)^2 \int_{\mathbb{D}} \frac{\bar{w}_1^2 \bar{w}_2^2 f(w_1, w_2)}{(1 - z_1 \bar{w}_1)^4 (1 - z_2 \bar{w}_2)^4} dV(w).$$

By definition, we also have

$$Vf(z) = \lambda_{\mathbb{D}^2} (1 - |z_1|^2)^2 (1 - |z_2|^2)^2 \int_{\mathbb{D}^2} \frac{f(w_1, w_2)}{(1 - z_1 \bar{w}_1)^4 (1 - z_2 \bar{w}_2)^4} dV(w),$$

where $\lambda_{\mathbb{D}^2} = 3^2 = 9$. Thus the only difference between Tf and Vf in the integral representation is the factor $\bar{w}_1^2 \bar{w}_2^2$ in the integrand. Therefore, Tf and Vf are really closely related. On the other hand, Tf does not contain all the properties of Vf . For example, if $f(z_1, z_2)$ only depends on one variable, say, $f(z_1, z_2) = g(z_1)$, then $Tf = 0$, but

$$Vf(z_1, z_2) = 9(1 - |z_1|^2)^2 (1 - |z_2|^2) \int_{\mathbb{D}} \frac{g(w_1)}{(1 - z_1 \bar{w}_1)^4} dV(w_1).$$

If g is not in the Bloch space of \mathbb{D} , then $Vf(z_1, z_2)$ is not even bounded, although $Tf \equiv 0$. This explains why we need the functions $T_i f$ ($1 \leq i \leq 4$).

We point out that if we are only concerned with boundedness, compactness, or membership in S_p with $p > 1$, then it is sufficient to consider the following three lower order derivatives [10, 12]:

$$\begin{aligned} (1 - |z_1|^2) \frac{\partial f}{\partial z_1}(z_1, 0), & \quad (1 - |z_1|^2) \frac{\partial f}{\partial z_2}(0, z_2), \\ (1 - |z_1|^2)(1 - |z_2|^2) \frac{\partial^2 f}{\partial z_1 \partial z_2}(z_1, z_2). \end{aligned}$$

In order to cover the endpoint $p = 1$, we need all the five functions: $T_i f$ ($1 \leq i \leq 4$) and Tf .

Theorem 22. *Suppose $p \geq 1$ and f is holomorphic in \mathbb{D}^2 , we have*

- (1) $Vf \in BC \Leftrightarrow T_i f \in BC(\mathbb{D})$ ($1 \leq i \leq 4$), $Tf \in BC(\mathbb{D}^2)$;
- (2) $Vf \in \mathbb{C}_0 \Leftrightarrow T_i f \in \mathbb{C}_0(\mathbb{D})$ ($1 \leq i \leq 4$), $Tf \in \mathbb{C}_0(\mathbb{D}^2)$;
- (3) $Vf \in L^p(d\lambda) \Leftrightarrow T_i f \in L^p(\mathbb{D}, d\lambda)$ ($1 \leq i \leq 4$), $Tf \in L^p(\mathbb{D}^2, d\lambda)$.

Combining Theorems 8, 12, 15 and 22, we get

Corollary 23. *Suppose $p \geq 1$ and ϕ is holomorphic in \mathbb{D}^2 , then we have*

- (1) $h_{\bar{\phi}}$ is bounded $\Leftrightarrow T_i \phi \in BC(\mathbb{D})$ ($1 \leq i \leq 4$), $T\phi \in BC(\mathbb{D}^2)$;
- (2) $h_{\bar{\phi}}$ is compact $\Leftrightarrow T_i \phi \in \mathbb{C}_0(\mathbb{D})$ ($1 \leq i \leq 4$), $T\phi \in \mathbb{C}_0(\mathbb{D}^2)$;
- (3) $h_{\bar{\phi}}$ is in $S_p \Leftrightarrow T_i \phi \in L^p(\mathbb{D}, d\lambda)$ ($1 \leq i \leq 4$), $T\phi \in L^p(\mathbb{D}^2, d\lambda)$.

The proof of Theorem 22 is similar to that of Theorem 18.

Once again the three implications “ \Rightarrow ” are easier to prove. Given f holomorphic in \mathbb{D}^2 with $Vf \in BC$. Write $f = PVf$ and differentiate under integral sign, then

$$Tf(z) = 36(1 - |z_1|^2)^2(1 - |z_2|^2)^2 \int_{\mathbb{D}^2} \frac{\bar{w}_1^2 \bar{w}_2^2 Vf(w_1, w_2)}{(1 - z_1 \bar{w}_1)^4 (1 - z_2 \bar{w}_2)^4} dV(w).$$

Let M be the operator defined by

$$Mf(z) = 4\bar{z}_1^2 \bar{z}_2^2 f(z)$$

then we have

$$Tf = VMVf.$$

M maps BC into BC , \mathbb{C}_0 into \mathbb{C}_0 , and $L^p(d\lambda)$ into $L^p(d\lambda)$, and so does V , thus $Vf \in BC$ implies $Tf \in BC$, $Vf \in \mathbb{C}_0$ implies $Tf \in \mathbb{C}_0$, and $Vf \in L^p(d\lambda)$ implies $Tf \in L^p(d\lambda)$.

To get the same conclusions about $T_1 f$ and $T_2 f$, notice that $f = PVf$ gives

$$T_1 f(z_1) = 6 \int_{\mathbb{D}^2} \frac{(1 - |z_1|^2)^2 \bar{w}_1^2 Vf(w_1, w_2)}{(1 - z_1 \bar{w}_1)^4} dV(w).$$

Let

$$\phi_1(w_1) = \bar{w}_1^2 \int_{\mathbb{D}} Vf(w_1, w_2) dV(w_2),$$

then

$$T_1 f(z_1) = 6(1 - |z_1|^2)^2 \int_{\mathbb{D}} \frac{\phi_1(w_1) dV(w_1)}{(1 - z_1 \bar{w}_1)^4}$$

with

$$|\phi_1(z_1)|^p \leq \int_{\mathbb{D}} |Vf(z_1, z_2)|^p dV(z_2) \leq \int_{\mathbb{D}} \frac{|Vf(z_1, z_2)|^p}{(1 - |z_2|^2)^2} dV(z_2)$$

and

$$\int_{\mathbb{D}} \frac{|\phi_1(z_1)|^p}{(1 - |z_1|^2)^2} dV(z_1) \leq \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{|Vf(z_1, z_2)|^p dV(z)}{(1 - |z_1|^2)^2(1 - |z_2|^2)^2}$$

or

$$\|\phi_1\|_{L^p(\mathbb{D}, d\lambda)} \leq \|Vf\|_{L^p(\mathbb{D}^2, d\lambda)}.$$

Thus the result on $\mathbb{D} = B_1$ implies that $T_1f \in L^p(\mathbb{D}, d\lambda)$ if $Vf \in L^p(\mathbb{D}^2, d\lambda)$. Similarly, $T_1f \in C_0(\mathbb{D})$ if $Vf \in C_0(\mathbb{D}^2)$, $T_1f \in BC(D)$ if $Vf \in BC(\mathbb{D}^2)$. The case T_2f can be treated similarly.

To get the results for T_3f and T_4f , we write $f = PVf$ and differentiate under integral sign, then we get

$$T_3f(z_1) = 12(1 - |z_1|^2)^2 \int_{\mathbb{D}^2} \frac{\bar{w}_1^2 \bar{w}_2 Vf(w_1, w_2)}{(1 - z_1 \bar{w}_1)^4} dV(w).$$

Let

$$\psi_1(w_1) = \bar{w}_1^2 \int_{\mathbb{D}} \bar{w}_2 Vf(w_1, w_2) dV(w_2),$$

then

$$T_3f(z_1) = 12(1 - |z_1|^2)^2 \int_{\mathbb{D}} \frac{\psi_1(w_1)}{(1 - z_1 \bar{w}_1)^4} dV(w_1)$$

and ψ_1 satisfies the same estimate we did for ϕ_1 , thus the result on $\mathbb{D} = B_1$ gives the implication that $Vf \in BC(\mathbb{D}^2) \Leftrightarrow T_3f \in BC(\mathbb{D})$, $Vf \in C_0(\mathbb{D}^2) \Leftrightarrow T_3f \in C_0(\mathbb{D})$, and $Vf \in L^p(\mathbb{D}^2, \lambda) \Leftrightarrow T_3f \in L^p(\mathbb{D}, d\lambda)$. The same proof works for T_4f . We have completed the proof for the three implications “ \Rightarrow ” in Theorem 22.

We will prove the converse only for (3). The other two statements can be proved similarly and details can be found in [10, 12].

Given f holomorphic in \mathbb{D}^2 with $f(0) = 0$, we can always write

$$f(z_1, z_2) = h_1(z_1) + h_2(z_2) + z_1 z_2 (h_3(z_1) + h_4(z_2)) + z_1^2 z_2^2 h(z_1, z_2).$$

Moreover, the decomposition is unique if $h_4(0) = 0$. Since $f(0) = 0$, we have $h_1(z_1) = f(z_1, 0)$, $h_2(z_2) = f(0, z_2)$. Therefore,

$$\begin{aligned} Vh_1(z_1, z_2) &= 9(1 - |z_1|^2)^2(1 - |z_2|^2)^2 \int_{\mathbb{D}^2} \frac{h_1(w_1) dV(w)}{(1 - z_1 \bar{w}_1)^4(1 - z_2 \bar{w}_2)^4} \\ &= 9(1 - |z_2|^2)^2(1 - |z_1|^2)^2 \int_{\mathbb{D}} \frac{f_1(w_1, 0) dV(w_1)}{(1 - z_1 \bar{w}_1)^4}. \end{aligned}$$

The assumption that $T_1f \in L^p(\mathbb{D}, d\lambda)$ together with the results for $\mathbb{D} = B_1$ now gives $Vh_1 \in L^p(\mathbb{D}^2, d\lambda)$. Similarly, $T_2f \in L^p(\mathbb{D}, d\lambda) \Leftrightarrow Vh_2 \in L^p(\mathbb{D}^2, d\lambda)$.

It is easy to see that

$$\frac{\partial f}{\partial z_2}(z_1, 0) = h_2'(0) + z_1 h_3(z_1)$$

or

$$z_2 \frac{\partial f}{\partial z_2}(z_1, 0) = h'_2(0)z_2 + z_1 z_2 h_3(z_1).$$

It follows that

$$V(z_1 z_2 h_3(z_1))(z_1, z_2) = V\left(z_2 \frac{\partial f}{\partial z_2}(z_1, 0)\right)(z_1, z_2) - h'_2(0)V(z_2)(z_1, z_2).$$

Clearly, $V(z_2)(z_1, z_2) \in L^p(d\lambda)$. Also we have

$$\begin{aligned} & V\left(z_2 \frac{\partial f}{\partial z_2}(z_1, 0)\right)(z_1, z_2) \\ &= 9(1 - |z_1|^2)^2(1 - |z_2|^2)^2 \int_{\mathbb{D}^2} \frac{\frac{\partial f}{\partial w_2}(w_1, 0)w_2 dV(w)}{(1 - z_1 \bar{w}_1)^4(1 - z_2 \bar{w}_2)^4} \\ &= 18z_2(1 - |z_2|^2)^2(1 - |z_1|^2)^2 \int_{\mathbb{D}^2} \frac{\frac{\partial f}{\partial w_2}(w_1, 0)}{(1 - z_1 \bar{w}_1)^4} dV(w_1). \end{aligned}$$

So $T_3 f \in L^p(\mathbb{D}, d\lambda)$ together with the results on $\mathbb{D} = B_1$ gives

$$V\left(z_2 \frac{\partial f}{\partial z_2}(z_1, 0)\right)(z_1, z_2) \in L^p(\mathbb{D}^2, d\lambda).$$

Therefore,

$$V(z_1 z_2 h_3(z_1)) \in L^p(\mathbb{D}^2, d\lambda).$$

Similarly,

$$V(z_1 z_2 h_4(z_2)) \in L^p(\mathbb{D}^2, d\lambda).$$

It remains to show that

$$V(z_1^2 z_2^2 h(z_1, z_2)) \in L^p(\mathbb{D}^2, d\lambda).$$

In doing this we may as well assume that

$$f(z_1, z_2) = z_1^2 z_2^2 h(z_1, z_2).$$

Given $g \in L^q(d\lambda)$ with $1/p + 1/q = 1$, then

$$\langle Vf, g \rangle_\lambda = \langle f, Vg \rangle_\lambda.$$

Write $Vg(z) = K(z, z)^{-1} \tilde{g}(z)$ with

$$\tilde{g}(z) = \tilde{g}(0) + \tilde{h}_1(z_1) + \tilde{h}_2(z_2) + z_1 z_2 (\tilde{h}_3(z_1) + \tilde{h}_4(z_2)) + z_1^2 z_2^2 \tilde{h}(z_1, z_2).$$

The unitary invariance of K^{-1} and $d\lambda$ now implies that

$$\begin{aligned} \langle f, Vg \rangle_\lambda &= \langle f(z), K(z, z)^{-1} z_1^2 z_2^2 \tilde{h}(z_1, z_2) \rangle_\lambda \\ &= \langle \bar{z}_1^2 \bar{z}_2^2 f(z), K(z, z)^{-1} \tilde{h}(z) \rangle_\lambda. \end{aligned}$$

Because $K^{-1} \tilde{h}$ is in the image of V and V is a projection in the pairing $\langle \cdot, \cdot \rangle_\lambda$

thus

$$\begin{aligned} \langle f, Vg \rangle_\lambda &= \langle V(\bar{z}_1^2 \bar{z}_2^2 f(z)), K^{-1}(z, z)\tilde{h}(z) \rangle_\lambda \\ &= \frac{1}{4} \langle Tf, K^{-1}\tilde{h} \rangle_\lambda. \end{aligned}$$

It follows that

$$|\langle f, Vg \rangle_\lambda| \leq \frac{1}{4} \|Tf\|_{L^p(d\lambda)} \|K^{-1}\tilde{h}\|_{L^q(d\lambda)}.$$

The desired result now follows from the following.

Lemma 24. *Suppose g is holomorphic in \mathbb{D}^2 with $g(0) = 0$ and*

$$g(z_1 z_2) = h_1(z_1) + h_2(z_2) + z_1 z_2 (h_3(z_1) + h_4(z_2)) + z_1^2 z_2^2 h(z_1 z_2),$$

then

$$\|K^{-1}h\|_{L^p(d\lambda)} \leq C \|K^{-1}g\|_{L^p(d\lambda)}$$

with C independent of g .

Proof. Since $h_1(z_1) = g(z_1, 0)$, thus

$$\|K^{-1}h_1\|_{L^p(d\lambda)} \leq \|K^{-1}g\|_{L^p(d\lambda)}.$$

Similarly, $\|K^{-1}h_2\|_{L^p(d\lambda)} \leq \|K^{-1}g\|_{L^p(d\lambda)}$. Next write

$$\tilde{g}(z_1, z_2) = \frac{g(z_1, z_2) - h_1(z_1) - h_2(z_2)}{z_1 z_2} = h_3(z_1) + h_4(z_2) + z_1 z_2 h(z_1 z_2).$$

Using Lemma 15 in [10] (or a similar version of it), we can get

$$\|K^{-1}\tilde{g}\|_{L^p(d\lambda)} \leq C_1 \|K^{-1}g\|_{L^p(d\lambda)}.$$

Repeating the above process one more time, we get

$$\|K^{-1}h\|_{L^p(d\lambda)} \leq C_2 \|K^{-1}g\|_{L^p(d\lambda)}.$$

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