

ALTERNATING SEQUENCES AND INDUCED OPERATORS

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ABSTRACT. We show that when a positive L_p contraction is equipped with a norming function having full support, then it is related in a natural way to an operator on any other L_p space, $1 < p < \infty$. This construction is used to generalize a theorem of Rota concerning the convergence of alternating sequences.

1. INTRODUCTION

Let L_p be the usual Banach space of complex-valued functions. Denote by L_p^+ the class of L_p functions taking nonnegative values. An L_p operator T is positive if $TL_p^+ \subseteq L_p^+$. It is a contraction if $\|Tf\|_p \leq \|f\|_p$ for every $f \in L_p$. We say u is *semi-invariant* for a positive L_p contraction T if both u and Tu have full support and $\|Tu\|_p = \|u\|_p$.

(1.1) **Theorem.** Suppose $1 < p < \infty$ and $1 < r < \infty$. If T is a positive L_p contraction with a semi-invariant function u , then the formula

$$T_r f = (Tu)^{p/r-1} T(u^{1-p/r} f),$$

where $f \in L_r$, defines a positive L_r contraction. This operator is independent of the choice of semi-invariant function. We call T_r the L_r operator **induced** by T .

We apply this notion of induced operators to the question of convergence of alternating sequences. For simplicity of notation, the following theorem is stated for L_p only. The analogous result is proved for all of L_p . T^* denotes the adjoint of T ; it is an operator on L_q where $q = p(p-1)^{-1}$. Whenever u is semi-invariant for an L_p operator T , then $(Tu)^{p-1}$ is semi-invariant for T^* .

(1.2) **Theorem.** Suppose $1 < p < \infty$ and $1 < r < \infty$. Let $\langle T_n \rangle_{n=1}^\infty$ be a sequence of positive L_p contractions with semi-invariant functions defined over a σ -finite

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Lebesgue space. Then

$$(T_1^*) \cdots (T_n^*) (T_n \cdots T_1 f)^{p/r}$$

converges a.e. for every $f \in L_p^+$.

This theorem generalizes Rota's theorem of the alternating procedure [Rt]. We say an operator is bistochastic if $T\mathbf{1} = T^*\mathbf{1} = \mathbf{1}$, where $\mathbf{1}$ is the function taking the value 1 everywhere.

(1.3) **Theorem (Rota).** *If $\langle T_n \rangle_{n=1}^\infty$ is a sequence of positive bistochastic operators over a probability space, then*

$$(1.4) \quad T_1^* \cdots T_n^* T_n \cdots T_1 f$$

converges a.e. for every $f \in L_p$, where $1 < p < \infty$.

A positive bistochastic operator is a contraction of every L_p , where $1 \leq p \leq \infty$; thus the expression (1.4) is well defined for every p . A positive L_p contraction with a semi-invariant function does not necessarily have this property, but we may use the operator induced by T^* to define a "pseudo-adjoint" of T which operates on L_p .

In the finite measure case, $\mathbf{1}$ is semi-invariant for any bistochastic operator and for its adjoint. Furthermore, $T_r^* = T^*$ for any r , $1 < r < \infty$. Thus, Rota's theorem is a consequence of (1.1) with $r = p$.

2. PRELIMINARIES

(2.1) **Definitions.** For any σ -finite measure space $(X, \mathcal{F}, \mu)$, let $\mathcal{M}(d\mu)$ be the vector space of $\mathcal{F}$ -measurable complex-valued functions defined on X . Let $\mathcal{M}^+(d\mu)$ be the class of functions in $\mathcal{M}(d\mu)$ whose ranges are subsets of $\mathbb{R}^+ = [0, \infty)$. Let $\overline{\mathcal{M}}^+(d\mu)$ be the set of $\mathcal{F}$ -measurable functions on X with values in the extended nonnegative reals, $[0, \infty]$.

The usual Banach space of functions in $\mathcal{M}(d\mu)$ for which $\int_X |f|^p d\mu < \infty$ is denoted by $L_p(d\mu)$, where $1 \leq p < \infty$, while $L_\infty(d\mu)$ denotes the space of essentially bounded functions $\mathcal{M}(d\mu)$. We also use $L_p^+(d\mu) = L_p(d\mu) \cap \mathcal{M}^+(d\mu)$. All of the relations between the functions in these classes are in the μ -a.e. sense, even when this is not made explicit. With the convention $0 \cdot \infty = 0$, functions in $\overline{\mathcal{M}}^+(d\mu)$ may be multiplied pointwise.

Let $(Y, \mathcal{G}, \nu)$ be another σ -finite measure space. Consider the class of all mappings

$$T: \overline{\mathcal{M}}^+(d\mu) \rightarrow \overline{\mathcal{M}}^+(d\nu)$$

which satisfy the following two conditions:

(2.2) T is "positive-linear"; that is, if $\alpha, \beta \in \mathbb{R}^+$ and $f, g \in \overline{\mathcal{M}}^+(d\mu)$, then

$$T(\alpha f + \beta g) = \alpha T f + \beta T g.$$

(2.3) T is "order-continuous" in the sense that $Tf_n \uparrow Tf$ ν -a.e. whenever $f_n \uparrow f$ μ -a.e. (the arrows indicate monotone nondecreasing pointwise convergence in $\mathbb{R}^+$).

If T is such a mapping, then its restriction to $\mathcal{M}^+(d\mu)$ need not be extendable linearly to $\mathcal{M}(d\mu)$. Thus, these mappings should not necessarily be associated with the usual class of linear operators. Nonetheless, it is convenient to make the following definition.

(2.4) **Definition** A mapping satisfying (2.2) and (2.3) will be called a *positive operator* on $\overline{\mathcal{M}}^+(d\mu)$ (or from $\overline{\mathcal{M}}^+(d\mu)$ to $\overline{\mathcal{M}}^+(d\nu)$).

(2.5) **Lemma.** Given a positive operator $T: \overline{\mathcal{M}}^+(d\mu) \rightarrow \overline{\mathcal{M}}^+(d\nu)$ there exists a unique positive operator $T^*: \overline{\mathcal{M}}^+(d\nu) \rightarrow \overline{\mathcal{M}}^+(d\mu)$ such that

$$\int_X f T^* g \, d\mu = \int_Y T f \cdot g \, d\nu$$

for every $f \in \overline{\mathcal{M}}^+(d\mu)$ and $g \in \overline{\mathcal{M}}^+(d\nu)$.

Proof. Given $g \in \overline{\mathcal{M}}^+(d\nu)$, the mapping

$$f \in \overline{\mathcal{M}}^+(d\mu) \mapsto \int_Y T f \cdot g \, d\nu \in \mathbb{R}^+$$

is integration with respect to some measure on $(X, \mathcal{F})$ which is absolutely continuous with respect to μ . This measure may be represented as $\rho \, d\mu$ for some $\rho \in \overline{\mathcal{M}}^+(d\mu)$. Define T^* by $T^* g = \rho$. $\square$

(2.6) **Definition.** The operator T^* defined above is called the *adjoint* of T .

If $T: L_p(d\mu) \rightarrow L_p(d\nu)$ is a positive operator in the usual sense, then its restriction to $L_p^+(d\mu)$ can be extended to a positive operator on $\overline{\mathcal{M}}^+(d\mu)$, which will also be called T . It is unique by the requirement that it satisfy (2.3). If a positive operator on $\overline{\mathcal{M}}^+(d\mu)$ in the sense of (2.4) can be obtained in this way, then we will call it a positive L_p operator on $\overline{\mathcal{M}}^+(d\mu)$. The following definition states this in a different way.

(2.7) **Definition.** A positive operator T on $\overline{\mathcal{M}}^+(d\mu)$ is said to be a positive L_p operator if

$$\|T\|_p^p = \sup \left\{ \int (Tf)^p \, d\nu \mid f \in \overline{\mathcal{M}}^+(d\mu) \text{ and } \int f^p \, d\mu \leq 1 \right\}$$

is finite. If, furthermore, $\|T\|_p \leq 1$, then T is called a positive L_p contraction.

Throughout this paper, whenever a number p with $1 < p < \infty$ is understood, then q denotes the adjoint index; that is, the number $p(p-1)^{-1}$. Note that T is a positive L_p operator if and only if T^* is a positive L_q operator. In this case, the definition of the adjoint operator agrees with the usual definition in the Banach space sense.

The following theorem is a standard result. Under the hypothesis one easily shows that the operator is a contraction of both L_1 and L_∞ . The conclusion then follows by the Riesz convexity theorem.

(2.8) **Theorem.** Let T be a positive operator such that $T\mathbf{1} \leq \mathbf{1}$ and $T^*\mathbf{1} \leq \mathbf{1}$. Then T is a positive L_p contraction for all p , $1 \leq p \leq \infty$.

(2.9) **Definition.** If T is a positive L_p operator and $u \in L_p$ is a function satisfying $\|Tu\|_p = \|T\|_p \|u\|_p$, we say that u is a *norming function* for T . We say that u is *semi-invariant* for T if $\|Tu\|_p = \|u\|_p$ and both u and Tu are strictly positive a.e. A semi-invariant function for a contraction is clearly a norming function.

(2.10) **Lemma.** If u is a norming function for a positive L_p operator T , then

$$T^*(Tu)^{p-1} = \|T\|_p^p u^{p-1}.$$

Consequently, if u is semi-invariant for a positive contraction T , then $(Tu)^{p-1}$ is semi-invariant for T^* .

Proof.

$$\begin{aligned} \|Tu\|_p^p &= \int (Tu)(Tu)^{p-1} d\nu = \int u T^*(Tu)^{p-1} d\mu \\ &\leq \|u\|_p \|T^*(Tu)^{p-1}\|_q \leq \|u\|_p \|T^*\|_q \|(Tu)^{p-1}\|_q \\ &= \|u\|_p \|T\|_p \|Tu\|_p^{p-1} = \|Tu\|_p^p, \end{aligned}$$

where the first inequality follows from Hölder's inequality. Thus, we have equality in Hölder's inequality, and so $T^*(Tu)^{p-1}$ is a constant multiple of u^{p-1} . $\square$

(2.11) **Definition.** Suppose T is a positive operator on $\overline{\mathcal{M}}^+(d\mu)$. A set $E \in \mathcal{F}$ is called a *reducing set* for T if $T(\chi_E) \cdot T(\mathbf{1} - \chi_E) = 0$, where χ_E is the characteristic function of the set E .

(2.12) **Lemma.** The support of a norming function is a reducing set.

Proof. Let u be a norming function for T , and E be the support of u . Then

$$\begin{aligned} \int (Tu)^{p-1} T(\mathbf{1} - \chi_E) d\nu &= \int T^*(Tu)^{p-1} (\mathbf{1} - \chi_E) d\mu \\ &= \|T\|_p^p \int u^{p-1} (\mathbf{1} - \chi_E) d\mu = 0. \end{aligned}$$

Hence $(Tu)^{p-1} T(\mathbf{1} - \chi_E) = 0$, and so $(Tu)T(\mathbf{1} - \chi_E) = 0$. Now approximate $\frac{1}{u}\chi_E$ from below by simple functions. Conclude by (2.3) and positivity that $T(\chi_E)T(\mathbf{1} - \chi_E) = 0$. $\square$

The following lemma concerning functions of a real variable is needed. Observe that the conclusion of the lemma remains valid if we replace t' in the hypothesis by any differentiable function which is strictly monotone almost everywhere.

(2.13) **Lemma.** Let $\phi, \theta: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be measurable functions satisfying

$$\begin{aligned} \int_0^\infty \phi(t) dt &= \int_0^\infty \theta(t) dt < \infty, \\ \int_0^\alpha \phi(t) dt &\leq \int_0^\alpha \theta(t) dt, \end{aligned} \tag{2.14}$$

and

$$(2.15) \quad \int_0^\infty t^r \phi(t) dt = \int_0^\infty t^r \theta(t) dt$$

for every $\alpha \geq 0$ and some $r > 0$. Then $\phi = \theta$ a.e.

Proof.

$$\begin{aligned} \int_0^\infty t^r \phi(t) dt &= \int_0^\infty rs^{r-1} \left(\int_s^\infty \phi(t) dt \right) ds \\ &\geq \int_0^\infty rs^{r-1} \left(\int_s^\infty \theta(t) dt \right) ds = \int_0^\infty t^r \theta(t) dt. \end{aligned}$$

By (2.15), we have equality. Thus, the set of points at which inequality (2.14) is strict has measure zero. Since

$$\int_0^\alpha \phi(t) dt = \int_0^\alpha \theta(t) dt$$

for a.a. α , and ϕ and θ are positive functions, it follows that $\phi = \theta$ a.e., as desired. $\square$

(2.16) **Definition.** A point transformation $\tau: X \rightarrow X$ is called an *automorphism* if it is invertible and both τ and τ^{-1} are measurable and nonsingular. An automorphism induces two measures, $\mu \circ \tau^{-1}$ and $\mu \circ \tau$, both absolutely continuous with respect to μ . Let ρ denote the Radon-Nikodým derivative of $\mu \circ \tau^{-1}$ with respect to μ . If $1 \leq p < \infty$, then define $Q: L_p \rightarrow L_p$ by

$$Qf = \rho^{1/p} (f \circ \tau^{-1})$$

for $f \in L_p$. We call Q the L_p isometry induced by τ .

(2.17) **Lemma.** If Q is the L_p isometry induced by an automorphism τ , then Q^{-1} is the L_p isometry induced by τ^{-1} and Q^* is the L_q -isometry induced by τ^{-1} .

Proof. This follows immediately from the definitions if one observes that when ρ is the Radon-Nikodým derivatives of $\mu \circ \tau^{-1}$ with respect to μ , then the Radon-Nikodým derivatives of $\mu \circ \tau$ with respect to μ is $1/(\rho \circ \tau)$. $\square$

(2.18) **Definition.** Suppose $1 \leq p < \infty$ and $1 \leq r < \infty$. Define $\psi_{p,r}: L_p \rightarrow L_r$ by means of the equation

$$[\psi_{p,r}(f)](x) = \text{sign}(f(x)) |f(x)|^{p/r},$$

where $\text{sign}(z)$ is the complex number of unit modulus having the same argument as z . When p and r are understood, we refer to this embedding simply as ψ . Usually f^* is used to represent $\psi_{p,q}f$. Perhaps the most important property of $\psi_{p,r}$ is that when $f \in L_p$, then $\|\psi_{p,r}f\|_r = \|f\|_p^{p/r}$.

(2.19) **Lemma.** Let $1 \leq p < \infty$ and $1 \leq r < \infty$. Suppose Q_p and Q_r are, respectively, the L_p and L_r isometries induced by an automorphism τ . If $\psi = \psi_{p,r}$ and $f \in L_p$, then

$$Q_r \psi f = \psi Q_p f.$$

Proof.

$$\begin{aligned} Q_r \psi f &= \rho^{1/r} [\text{sign}(f) |f|^{p/r}] \circ \tau^{-1} \\ &= \text{sign}(f \circ \tau^{-1}) \rho^{1/r} |f \circ \tau^{-1}|^{p/r} \\ &= \text{sign}[\rho^{1/p} (f \circ \tau^{-1})] |\rho^{1/p} (f \circ \tau^{-1})|^{p/r} \\ &= \psi Q_p f. \quad \square \end{aligned}$$

(2.20) **Definition.** When $(X, \mathcal{F}, \mu)$ is a measure space and $\mathcal{F}'$ is a sub- σ -algebra of $\mathcal{F}$, then $E(\cdot | \mathcal{F}')$ denotes the conditional expectation operator with respect to $\mathcal{F}'$. We adopt the convention that $E(f | \mathcal{F}')$ is 0 on any atom of $\mathcal{F}'$ of infinite measure.

(2.21) **Theorem** (Martingale convergence theorem for finite σ -algebras). Let $(X, \mathcal{F}, \mu)$ be a σ -finite measure space. For each $k \geq 1$, suppose $\mathcal{G}_k$ is a finite sub- σ -algebra of $\mathcal{F}$ and $\mathcal{G}_k \subseteq \mathcal{G}_{k+1}$. Let $\mathcal{G}_\infty = \sigma(\bigcup_{k=1}^\infty \mathcal{G}_k)$, the smallest σ -algebra containing the algebra $\bigcup_{k=1}^\infty \mathcal{G}_k$. Suppose $1 \leq p < \infty$ and $f \in L_p(d\mu)$. Let $f_k = E(f | \mathcal{G}_k)$ for $1 \leq k \leq \infty$. Then $f_k \rightarrow f$ a.e. and in L_p norm.

If $p > 1$, then the f_k 's have a maximal function; more precisely, there is a function $g \in L_p^+$ with $|f_k| \leq g$ for every $k \geq 1$, and $\|g\|_p \leq q \|f\|_p$.

Proof. See any reference on martingales, e.g. [S, pp. 89-94].

(2.22) **Lemma.** Let $\langle \mathcal{G}_k \rangle_{k=1}^\infty$ be as in the previous theorem and suppose $\langle \mathcal{H}_k \rangle_{k=1}^\infty$ is another monotone sequence of finite sub- σ -algebras of $\mathcal{F}$. Let

$$\mathcal{H}_\infty = \sigma \left(\bigcup_{k=1}^\infty \mathcal{H}_k \right).$$

Let $f \in L_p^+(d\mu)$, where $1 < p < \infty$, and $f_k = E(f | \mathcal{G}_k)$. Then

$$E(f_k^p | \mathcal{H}_k) \rightarrow E(f_\infty^p | \mathcal{H}_\infty)$$

a.e. and in L_1 norm.

Proof. Let $\phi_k = f_k^p$ for each $k \geq 1$. Then $g = \sup f_k \in L_p$ by the martingale convergence theorem. Thus $0 \leq \phi_k \leq \theta = g^p \in L_1$, and $\phi_k \rightarrow \phi_\infty$ a.e. The proof is then completed by the following more general lemma.

(2.23) **Lemma.** Let $0 \leq \phi_k \leq \theta \in L_1$ for $k \geq 1$, and let $\phi_k \rightarrow \phi_\infty$ a.e. Then $E(\phi_k | \mathcal{G}_k) \rightarrow E(\phi_\infty | \mathcal{G}_\infty)$ a.e. and in L_1 norm.

Proof. Let

$$\xi_k = \inf_{n \geq k} \phi_n \quad \text{and} \quad \eta_k = \sup_{n \geq k} \phi_n.$$

Then $(\eta_k - \xi_k) \downarrow 0$ a.e. and in L_1 norm, by the dominated convergence theorem. We have, for any $n \geq k$,

$$\begin{aligned} E(\xi_k | \mathcal{G}_n) &\leq E(\xi_n | \mathcal{G}_n) \leq E(\phi_n | \mathcal{G}_n) \\ &\leq E(\eta_n | \mathcal{G}_n) \leq E(\eta_k | \mathcal{G}_n). \end{aligned}$$

If $n \rightarrow \infty$ with k fixed, then

$$E(\xi_k | \mathcal{G}_\infty) \leq \underline{\lim} E(\phi_n | \mathcal{G}_n) \leq \overline{\lim} E(\phi_n | \mathcal{G}_n) \leq E(\eta_k | \mathcal{G}_\infty).$$

Thus

$$\begin{aligned} &\| \overline{\lim} E(\phi_n | \mathcal{G}_n) - \underline{\lim} E(\phi_n | \mathcal{G}_n) \|_1 \\ &\leq \| E(\eta_k | \mathcal{G}_\infty) - E(\xi_k | \mathcal{G}_\infty) \|_1 \leq \| \eta_k - \xi_k \|_1 \end{aligned}$$

which can be made arbitrarily small. This completes the proof. $\square$

We will need the following four lemmas from [AS2], where they are numbered (2.2), (2.3), (2.5), and (2.8) respectively. L_p always refers to the case $1 < p < \infty$ over a σ -finite measure space.

(2.24) **Lemma.** Let $f_k \in L_p$ for every k , $1 \leq k \leq n$. If $V: L_p \rightarrow L_p$ is a positive bounded linear operator, then

$$\max_{1 \leq k \leq n} |V f_k| \leq V \left(\max_{1 \leq k \leq n} |f_k| \right)$$

and, consequently,

$$\left\| \max_{1 \leq k \leq n} |V f_k| \right\|_p \leq \|V\| \cdot \left\| \max_{1 \leq k \leq n} |f_k| \right\|_p.$$

(2.25) **Lemma.** For each $\varepsilon > 0$ there is a $\delta > 0$ such that if $E: L_p \rightarrow L_p$ is a conditional expectation operator, $f \in L_p$, and $\|f\|_p - \|Ef\|_p < \delta \|f\|_p$, then $\|f - Ef\|_p < \varepsilon \|f\|_p$.

(2.26) **Lemma.** Let $f_{km} \in L_p$ for every $m \geq 0$ and every k , $1 \leq k \leq n$. If $\lim_{m \geq 0} \|f_{km} - f_m\|_p = 0$ for each k , then

$$\lim_{m \geq 0} \left\| \max_{1 \leq k \leq n} |f_{km}| - \max_{1 \leq k \leq n} |f_k| \right\|_p = 0.$$

(2.27) **Lemma.** Let $\langle f_n \rangle_{n=0}^\infty$ be a sequence of functions in L_p such that $(\sup_{n \geq 0} |f_n|) \in L_p$. Then $\langle f_n \rangle_{n=0}^\infty$ converges a.e. if and only if

$$\lim_{n \geq 0} \left\| \sup_{k \geq n} |f_k - f_n| \right\|_p = 0.$$

The following are analogous to Lemmas (2.6) and (2.7) in [AS2]. The first one follows from a result of Mazur [M], since the mapping $\psi_{p,r}$ may be regarded as a composition of his map F from L_1 to L_r and his map G from L_p to L_1 , both uniformly continuous on the unit ball.

(2.28) **Lemma** (Uniform continuity of $\psi_{p,r}$). *Let $1 \leq p < \infty$ and $1 \leq r < \infty$. Given $\varepsilon > 0$ and $M > 0$, there is a $\delta > 0$ depending only on ε , M , p , and r such that $\|\psi f - \psi g\|_r < \varepsilon$ whenever $\|f\|_p \leq M$, $\|g\|_p \leq M$, and $\|f - g\|_p < \delta$.*

(2.29) **Lemma**. *Given $\varepsilon > 0$ and $M > 0$ there is a $\delta > 0$ depending only on ε , M , p , and r such that if $\langle f_k \rangle_{k=0}^\infty$ is a sequence in L_p with $\|\sup_{k \geq 0} |f_k|\|_p \leq M$ and $\|\sup_{k \geq 0} |f_k - f_0|\|_p < \delta$ then*

$$\left\| \sup_{k \geq 0} |\psi f_k - \psi f_0| \right\|_r < \varepsilon.$$

Proof. Let δ be as given in the uniform continuity of ψ corresponding to $\varepsilon/2$, M , p , and r . Let $n \geq 1$ be given. Fix a partition $\{A_1, \dots, A_n\}$ of X such that

$$\max_{0 \leq k \leq n} |\psi f_k - \psi f_0| = \sum_{m=1}^n |\psi f_m - \psi f_0| \chi_{A_m}.$$

Let $f = \sum_{m=1}^n f_m \chi_{A_m}$, so that

$$\max_{0 \leq k \leq n} |\psi f_k - \psi f_0| = |\psi f - \psi f_0|.$$

We have $\|f\|_p \leq M$, $\|f_0\|_p \leq M$, and $\|f - f_0\| \leq \|\sup_{k \geq 0} |f_k - f_0|\|_p$. Therefore, if this last norm is less than δ , the uniform continuity of ψ implies that $\|\psi f - \psi f_0\|_r < \varepsilon/2$. This completes the proof. $\square$

We also need the following, which is an immediate consequence of

$$\|T_n f_n - T f\|_p \leq \|T_n\| \cdot \|f_n - f\|_p + \|T_n f - T f\|_p.$$

(2.30) **Lemma**. *Suppose $\langle T_n \rangle_{n=1}^\infty$ and T are L_p contractions and*

$$\lim_{n \geq 1} \|T_n f - T f\|_p = 0$$

whenever $f \in L_p$. If $f_n \rightarrow f$ in L_p norm, then

$$\lim_{n \geq 1} \|T_n f_n - T f\|_p = 0.$$

3. INDUCED OPERATORS

In this section, we will be interested primarily in positive L_p operators with strictly positive norming functions. We begin, however, with two more general lemmas.

(3.1) **Lemma**. *Let T be a positive operator on $\overline{\mathcal{M}}^+(d\mu)$. Suppose $u \in \mathcal{M}^+(d\mu)$ is strictly positive. If there is a $\lambda \in \mathbb{R}^+$ such that*

$$(3.2) \quad T^*(Tu)^{p-1} \leq \lambda^p u^{p-1},$$

then T is a positive L_p operator with $\|T\|_p \leq \lambda$.

(3.3) **Remarks**. In the Borel case, this follows from a result in [AS1] concerning dilations. The general case was considered in [K1]. We have included the following short proof to make this paper more self-contained.

Proof. If $\lambda = 0$, it is easy to see that $T = 0$, since $\int (Tu)^{p-1} (Tf) d\mu = 0$ for every $f \in \overline{\mathcal{M}}^+(d\mu)$.

Suppose $\lambda > 0$ and let $v = Tu$. Because of (3.2), the σ -finiteness of μ and the fact that u is finite a.e., one argues that v is finite a.e. (The proof is essentially contained in [AS1, p. 391].)

Let $d\mu' = u^p d\mu$ and $d\nu' = (v/\lambda)^p d\nu$. Define an operator $R: \overline{\mathcal{M}}^+(d\mu') \rightarrow \overline{\mathcal{M}}^+(d\nu')$ by $Rf = \chi_{G \setminus v} \frac{1}{v} T(uf)$ for $f \in \overline{\mathcal{M}}^+(d\mu')$, where G is the support of v . This is clearly a positive operator in the sense of (2.4). A routine computation shows that the adjoint, $R^*: \overline{\mathcal{M}}^+(d\nu') \rightarrow \overline{\mathcal{M}}^+(d\mu')$, is given by

$$R^*g = \frac{1}{\lambda^p u^{p-1}} T^*(v^{p-1}g)$$

for $g \in \overline{\mathcal{M}}^+(d\nu')$. Thus $R1 \leq 1$ and $R^*1 \leq 1$, so by Theorem (2.8), R is an L_p contraction. This means that if $f \in \overline{\mathcal{M}}^+(d\mu')$, then

$$\int (Rf)^p d\nu' \leq \int f^p d\mu'.$$

If $f \in \overline{\mathcal{M}}^+(d\mu)$, then $f = u\tilde{f}$ for some $\tilde{f} \in \overline{\mathcal{M}}^+(d\mu')$. Hence

$$\begin{aligned} \int (Tf)^p d\nu &= \int [T(u\tilde{f})]^p d\nu = \lambda^p \int (R\tilde{f})^p d\nu' \\ &\leq \lambda^p \int \tilde{f}^p d\mu' = \lambda^p \int f^p d\mu. \end{aligned}$$

This shows that T is an L_p operator with $\|T\|_p \leq \lambda$. $\square$

(3.4) Lemma. Let T be a positive operator on $\overline{\mathcal{M}}^+(d\mu)$. Suppose $u \in \mathcal{M}^+(d\mu)$ is strictly positive, and that there is a $\lambda \in \mathbb{R}^+$ such that

$$T^*(Tu)^{p-1} \leq \lambda^p u^{p-1}.$$

Let $v = Tu$ and let G be the support of v . Let r be any exponent, $1 < r < \infty$. Then

$$Sf = \chi_G \left(\frac{v}{\lambda} \right)^{p/r-1} T(u^{1-p/r} f),$$

for $f \in \overline{\mathcal{M}}^+(d\mu)$, defines a positive L_r operator $S: \overline{\mathcal{M}}^+(d\mu) \rightarrow \overline{\mathcal{M}}^+(d\nu)$ with $\|S\|_r \leq \lambda$.

Proof. $S^*: \overline{\mathcal{M}}^+(d\nu) \rightarrow \overline{\mathcal{M}}^+(d\mu)$ is easily calculated; one sees that for $g \in \overline{\mathcal{M}}^+(d\nu)$,

$$S^*g = (\lambda u)^{1-p/r} T^*(v^{p/r-1} \chi_G g).$$

Let $\tilde{u} = u^{p/r}$. Then $\tilde{u}$ is strictly positive a.e., and $S^*(Su)^{r-1} \leq \lambda^r \tilde{u}^{r-1}$. Thus, Lemma (3.1) completes the proof. $\square$

(3.5) Lemma. Suppose u_1 and u_2 are strictly positive norming functions for a positive L_p operator T on $\overline{\mathcal{M}}^+(d\mu)$. For any $\alpha \in \mathbb{R}^+$, the set

$$E_\alpha = \left\{ x \in X \mid \frac{u_2(x)}{u_1(x)} > \alpha \right\}$$

is a reducing set for T .

Proof. As in the proof of Lemma (3.1), let $d\mu' = u_1^p d\mu$ and $d\nu' = (v_1/\lambda)^p d\nu$, where $v_1 = Tu_1$ and $\lambda = \|T\|_p$. Observe that even if v_1 is not strictly positive a.e., its support is equal to the support of $v_2 = Tu_2$ a.e. Without loss of generality then, we may replace the set Y with this common support. Define $R: \overline{\mathcal{M}}^+(d\mu') \rightarrow \overline{\mathcal{M}}^+(d\nu')$ for $f \in \overline{\mathcal{M}}^+(d\mu')$ by $Rf = T(u_1 f)/v_1$.

$R\mathbf{1} = R^*\mathbf{1} = \mathbf{1}$, so R is an L_p contraction. $\mathbf{1}$ is a norming function for R ; we now show that $u = u_2/u_1$ is another. One may verify that $R^*(Ru)^{p-1} = u^{p-1}$, from which $\|Ru\|_p = \|u\|_p$ easily follows. Let $v = Ru$.

Let $\alpha \geq 0$ be arbitrary. Let $u_\alpha = u \wedge \alpha$, the function u truncated at the value α . Observe that E_α is the support of $u - u_\alpha$. Also note that $Ru_\alpha \leq v_\alpha = v \wedge \alpha$, hence

$$(3.6) \quad \int u_\alpha d\mu' = \int Ru_\alpha d\nu' \leq \int v_\alpha d\nu'.$$

Let $\phi: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be the distribution of u ; that is, $\phi(t) = \mu'\{x: u(x) \geq t\}$. Let θ be the distribution of v , similarly defined with respect to ν' . Inequality (3.6) has the equivalent form

$$(3.7) \quad \int_0^\alpha \phi(t) dt \leq \int_0^\alpha \theta(t) dt.$$

Since $\|u\|_p = \|v\|_p$, we have

$$(3.8) \quad \int_0^\infty t^{p-1} \phi(t) dt = \int_0^\infty t^{p-1} \theta(t) dt.$$

Finally, $u \in L_1(d\mu')$, since $p > 1$ and μ' is a finite measure. Since $\|u\|_1 = \|v\|_1$, we have

$$(3.9) \quad \int_0^\infty \phi(t) dt = \int_0^\infty \theta(t) dt < \infty.$$

Conditions (3.7)–(3.9) allow us to invoke Lemma (2.13) and conclude that $\phi = \theta$ a.e. in Lebesgue measure. Since

$$\|u - u_\alpha\|_p^p = p \int_\alpha^\infty t^{p-1} \phi(t) dt,$$

we have

$$\|u - u_\alpha\|_p = \|v - v_\alpha\|_p \leq \|R(u - u_\alpha)\|_p,$$

where the inequality follows because $Ru_\alpha \leq v_\alpha$. As R is a contraction, we conclude that the norms are in fact equal. Thus, $u - u_\alpha$ is a norming function. By Lemma (2.12), then, its support is a reducing set for R . It easily follows that E_α also reduces T . $\square$

(3.10) *Remarks.* One may replace the “less than” in the definition of E_α by any other inequality, simply by considering complements or reversing the roles of u_1 and u_2 . The complement of a reducing set is a reducing set; it is also easy

to show that the intersection of reducing sets is a reducing set. In fact, the class of reducing sets of a bounded L_p operator is a sub- σ -algebra of the underlying measure space. This is shown in [K2], which also includes a different proof of the above lemma.

(3.11) **Theorem.** Suppose T is a positive L_p operator on $\overline{\mathcal{M}}^+(d\mu)$, and u_1 and u_2 are strictly positive norming functions for T . Let $v_i = Tu_i$ for $i = 1, 2$ and let G be the support of the v_i 's. Let $1 < r < \infty$, and define positive operators S_1 and S_2 on $\overline{\mathcal{M}}^+(d\mu)$ by

$$S_i f = \chi_G \|T\|_p^{1-p/r} v_i^{p/r-1} T(u_i^{1-p/r} f)$$

for $f \in \overline{\mathcal{M}}^+(d\mu)$ and $i = 1, 2$. Then $S_1 f = S_2 f$ a.e. for every $f \in \overline{\mathcal{M}}^+(d\mu)$.

Proof. By (2.3), it suffices to consider $f \in \mathcal{M}^+(d\mu)$.

Let $s = p/r - 1$. If $s = 0$, there is nothing to prove. Otherwise, let $\varepsilon > 0$ be given, and choose a positive integer $N > 1/\varepsilon$.

For each $n \geq 1$, let

$$E_n = \left\{ x \in X \mid \frac{N+n-1}{N} < \frac{u_2(x)}{u_1(x)} \leq \frac{N+n}{N} \right\}$$

and

$$E_{-n} = \left\{ x \in X \mid \frac{N+n-1}{N} < \frac{u_1(x)}{u_2(x)} \leq \frac{N+n}{N} \right\}.$$

Also, let E_0 be the set of points in A where $u_1(x) = u_2(x)$. Then $\{E_n | n \in \mathbb{Z}\}$ is a partition of X into reducing sets.

Let $f \in \overline{\mathcal{M}}^+(d\mu)$ be given and let $f_n = f\chi_{E_n}$ for every $n \in \mathbb{Z}$. The f_n 's have disjoint support, as do the functions $T(u_1^{-s} f_n)$ and $T(u_2^{-s} f_n)$.

Now suppose $n \geq 1$ and $s > 0$. Since T is positive, we have

$$(3.12) \quad \left(\frac{N}{N+n} \right)^s T \left(\frac{f_n}{u_1^s} \right) \leq T \left(\frac{f_n}{u_2^s} \right) \leq \left(\frac{N}{N+n-1} \right)^s T \left(\frac{f_n}{u_1^s} \right).$$

Let $u_{in} = u_i \chi_{E_n}$ and $v_{in} = T(u_{in})$ for every $n \in \mathbb{Z}$ and $i = 1, 2$. The functions $T(u_i^{-s} f_n)$ and v_{mi} will have disjoint supports unless $m = n$; thus $S_i f_n$ depends only on $T(u_i^{-s} f_n)$ and v_{ni}^s . We have

$$(3.13) \quad \left(\frac{N+n-1}{N} \right)^s v_{n1}^s \leq v_{n2}^s \leq \left(\frac{N+n}{N} \right)^s v_{n1}^s.$$

Therefore,

$$(3.14) \quad \left(\frac{N+n-1}{N+n} \right)^s S_1 f_n \leq S_2 f_n \leq \left(\frac{N+n}{N+n-1} \right)^s S_1 f_n.$$

If $(S_1 f_n)(x) = 0$, then $(S_2 f_n)(x)$ must be zero as well. Otherwise,

$$(3.15) \quad \left| \left(\frac{(S_2 f_n)(x)}{(S_1 f_n)(x)} \right)^{1/s} - 1 \right| \leq \frac{1}{N+n-1} < \varepsilon.$$

If $s < 0$, then the order of the terms in (3.14) is reversed, but (3.15) remains valid.

If $n \leq -1$, the argument is symmetric, with the conclusion

$$\left| \left(\frac{(S_1 f_n)(x)}{(S_2 f_n)(x)} \right)^{1/s} - 1 \right| \leq \frac{1}{N + n - 1} < \varepsilon.$$

It is clear that $S_1 f_0 = S_2 f_0$. Since $\varepsilon > 0$ is arbitrary, we conclude that $S_1 f_n = S_2 f_n$ a.e. for each $n \in \mathbb{Z}$. Thus $S_1 f = S_2 f$ a.e., as desired. $\square$

(3.16) **Theorem.** Suppose $1 < p < \infty$ and $1 < r < \infty$. Let T be a positive L_p operator with a strictly positive norming function u . Let $v = Tu$ and let G be the support of v . Then

$$T_r f = \chi_G \|T\|^{1-p/r} v^{p/r-1} T(u^{1-p/r} f),$$

for $f \in \overline{\mathcal{M}}^+(d\mu)$, defines a positive L_r operator $T_r: \overline{\mathcal{M}}^+(d\mu) \rightarrow \overline{\mathcal{M}}^+(d\nu)$ such that $\|T_r\|_r = \|T\|_p$. This operator, called the L_r operator induced by T , is independent of the choice of u .

Proof. Whether T is given as an L_p operator in the Banach space sense or in the sense of Definition (2.4), it is clear that T_r is a positive operator in the sense of (2.4). Lemmas (2.10) and (3.4) combine to show that T_r is in fact an L_r operator with norm bounded by $\|T\|_p$. To see that this norm is actually achieved, let $f = u^{p/r}$. Theorem (3.11) demonstrates that T_r does not depend on the choice of norming function. $\square$

(3.17) **Corollary.** Suppose T is an L_p contraction with a semi-invariant function where $1 < p < \infty$. For every r , $1 < r < \infty$,

$$T_r f = v^{p/r-1} T(u^{1-p/r} f)$$

defines a positive contraction of L_r .

(3.18) **Remarks.** If T is an L_p isometry induced by an automorphism τ (as in (2.16)), then T_r is simply the L_r isometry induced by τ . When the underlying space has finite measure, we may take $u = \mathbf{1}$ and $v = \rho^{1/p}$. The general σ -finite case is not much harder to check.

A larger and more important class of operators has the form EQE , where Q is an L_p isometry induced by an automorphism and E is a conditional expectation operator of finite rank. Such operators were crucial to the proof of the pointwise ergodic theorem for positive L_p contractions (see [A]). Thus, the following lemma is of some general interest as well as being necessary for §5 of this paper.

(3.19) **Lemma.** Suppose $1 < p < \infty$, $1 < r < \infty$, and that Q_p and Q_r are, respectively, the L_p and L_r isometries induced by an automorphism τ over a measure space $(X, \mathcal{F}, \mu)$. Let $\widetilde{\mathcal{F}}$ be a sub- σ -algebra of $\mathcal{F}$ and let $\bar{\mu}$ be the

restriction of μ to $\widetilde{\mathcal{F}}$. Let E be conditional expectation with respect to $\widetilde{\mathcal{F}}$ and suppose

$$T: L_p(X, \widetilde{\mathcal{F}}, \tilde{\mu}) \rightarrow L_p(X, \widetilde{\mathcal{F}}, \tilde{\mu})$$

is given by $T = EQ_p E$. If T has a semi-invariant function u , then $T_r = EQ_r E$.

Proof. Let $v = Tu$. For $f \in L_r(X, \widetilde{\mathcal{F}}, \tilde{\mu})$, we have

$$\begin{aligned} T_r f &= v^{p/r-1} T(u^{1-p/r} f) \\ &= v^{p/r-1} E(\rho^{1/p} (u \circ \tau^{-1})^{1-p/r} [(Ef) \circ \tau^{-1}]) \\ &= v^{p/r-1} E[\rho^{1/p} (u \circ \tau^{-1})^{1-p/r} (f \circ \tau^{-1})], \end{aligned}$$

where the third line follows because f is already $\widetilde{\mathcal{F}}$ -measurable. Because $\|v\|_p = \|u\|_p$, Q is an isometry and $p > 1$, we conclude that $Q_p u$ must already be $\widetilde{\mathcal{F}}$ -measurable, lest some norm be lost in taking the conditional expectation. Thus $v = \rho^{1/p} (u \circ \tau^{-1})$ and

$$\begin{aligned} T_r f &= E[v^{p/r-1} \rho^{1/p} (u \circ \tau^{-1})^{1-p/r} (f \circ \tau^{-1})] \\ &= E[(\rho^{1/p})^{p/r-1} (u \circ \tau^{-1})^{p/r-1} \rho^{1/p} (u \circ \tau^{-1})^{1-p/r} (f \circ \tau^{-1})] \\ &= E[\rho^{1/r} (f \circ \tau^{-1})] = EQ_r f = EQ_r E f. \quad \square \end{aligned}$$

(3.20) **Lemma.** Let $1 < p < \infty$, $1 < r < \infty$, and let Q be the L_p isometry induced by an automorphism τ . Let $T = EQE$ for some conditional expectation operator E . If T has a semi-invariant function and $R = R_r$ is the L_r isometry induced by τ^{-1} , then $(T^*)_r = ERE$.

Proof. $(T^*)_r = (EQ^* E)_r = (ER_q E)_r = ERE$. We have used the self-adjointness of E and Lemmas (3.19) and (2.17) for the fact that Q^* is the L_r isometry induced by τ^{-1} . $\square$

4. FINITE-DIMENSIONAL APPROXIMATION

In [AK], it was shown that all positive contractions over the unit interval are induced by a point mapping of some type, followed by a conditional expectation. For positive contractions with semi-invariant functions, the argument is easier and does not require the underlying space to be interval. However, we will want to extract a point mapping from a set mapping, so we will require our measure spaces to be Lebesgue spaces. That is, a measure space $(X, \mathcal{F}, \mu)$ where X is a complete metric space and $\mathcal{F}$ is the Borel σ -algebra. We allow the space to have σ -finite measure. Since a separable metric space is second countable, the σ -algebra of measurable sets in a Lebesgue space can always be generated by a countable algebra of sets.

The details of the construction give us a family of finite-dimensional operators $\langle T^n \rangle_{n=1}^\infty$ (these are ordinary superscripts, not powers), each with a semi-invariant function u_n , where $u_n \rightarrow u$ a.e. Furthermore, these operators have

the property that $(T^n)_r f \rightarrow T_r f$ a.e. and in L_r norm for every $f \in L_r$. These finite-dimensional approximations to the induced operator provide the key to the proof of the Theorem (1.2).

(4.1) **Definitions.** Let $\mathbf{X} = (X, \mathcal{F}, \mu)$ be a σ -finite Lebesgue space and suppose $T: L_p(d\mu) \rightarrow L_p(d\mu)$ has a semi-invariant function u . Let $\mathbf{I} = (I, \mathcal{B}, m)$ be the usual Lebesgue space of the unit interval. Let $\mathbf{W} = (W, \mathcal{H}, \omega) = \mathbf{X} \times \mathbf{I}$.

Let $\mathcal{F} = \{F \times I | F \in \mathcal{F}\}$, the "vertical" sub- σ -algebra of $\mathcal{H}$, and let v be the $\mathcal{F}$ -measurable function given by $v(x, y) = (Tu)(x)$ for every y in the unit interval.

Suppose $\langle \mathcal{F}_n \rangle_{n=1}^\infty$ is an increasing sequence of finite sub- σ -algebras of $\mathcal{F}$ such that $\sigma(\bigcup_{n=1}^\infty \mathcal{F}_n) = \mathcal{F}$. That is, $\mathcal{F}$ is the smallest σ -algebra containing all the $\mathcal{F}_n$'s. Let $\mathcal{F}_n = \{F \times I | F \in \mathcal{F}_n\}$.

For each $n \geq 1$, fix an enumeration $\{F_{n,i}\}_{i=1}^{k_n}$ of the atoms of $\mathcal{F}_n$. Let $\gamma_{n,0} = 0$, and for each i , $1 \leq i \leq k_n$, let

$$\gamma_{n,i} = T \left(u \sum_{j=1}^i \chi_{F_{n,j}} \right),$$

and

$$H_{n,i} = \left\{ (x, y) \in W \mid \frac{\gamma_{n,i-1}(x)}{(Tu)(x)} < y \leq \frac{\gamma_{n,i}(x)}{(Tu)(x)} \right\}.$$

Let $\mathcal{H}_n$ be the finite sub- σ -algebra of $\mathcal{H}$ generated by the partition $\{H_{n,i}\}_{i=1}^{k_n}$ of W . Let Π_n be the set mapping from $\mathcal{F}_n$ to $\mathcal{H}_n$ determined by $\Pi_n F_{n,i} = H_{n,i}$ for each i , $1 \leq i \leq k_n$.

(4.2) **Lemma.** *There is a point mapping $\pi: W \rightarrow X$ such that $\pi^{-1} F_{n,i} = H_{n,i}$ for every $n \geq 1$ and every i , $1 \leq i \leq k_n$.*

Proof. The family of set mappings Π_n determines a unique set mapping of the algebra $\bigcup_{n=1}^\infty \mathcal{F}_n$, because of $\mathcal{F}_n$'s form a monotone sequence. This mapping preserves unions and complements, and it extends to a homomorphism of the measure algebras of $(X, \mathcal{F})$ and $(W, \mathcal{H})$. Since the sets underlying both spaces are complete metric spaces, there is a point mapping π defined from almost all of W onto almost all of X which induces the set mapping (see [Ry, p. 329]). Thus if $\Pi F = H$, then $\pi^{-1} F = H$. Since $\Pi F_{n,i} = \Pi_n F_{n,i}$, the desired result follows. $\square$

(4.3) **Lemma.** *For every $F \in \mathcal{F}$, $\int_F u^p d\mu = \int_{\pi^{-1}F} v^p d\omega$.*

Proof. If $F = F_{n,i} \in \mathcal{F}$ for some $n \geq 1$ and some i , $1 \leq i \leq k_n$, then

$$\begin{aligned} \int_{\pi^{-1}F} v^p d\omega &= \int_X (Tu)^p \left[\frac{\gamma_{n,i} - \gamma_{n,i-1}}{Tu} \right] d\mu \\ &= \int_X (Tu)^{p-1} T(u\chi_F) d\mu \\ &= \int_F uT^*(Tu)^{p-1} d\mu = \int_F u^p d\mu. \end{aligned}$$

The lemma is true for a generating subalgebra of $\mathcal{F}$. The proof is easily completed. $\square$

(4.4) **Lemma.** Suppose ϕ is an $\mathcal{F}$ -measurable function and θ is a $\mathcal{X}$ -measurable function with $\phi > 0$ μ -a.e. and $\theta > 0$ ω -a.e. such that

$$\int_F \phi d\mu = \int_{\pi^{-1}F} \theta d\omega$$

for every $F \in \mathcal{F}$. Then, if $1 \leq p < \infty$,

$$Sf = \left(\frac{\theta}{\phi \circ \pi} \right)^{1/p} (f \circ \pi),$$

for $f \in L_p(d\mu)$, defines an isometry $S: L_p(d\mu) \rightarrow L_p(d\omega)$.

Proof. First suppose $f = \phi^{1/p} \chi_F$ for some $F \in \mathcal{F}$. Then

$$\begin{aligned} \|Sf\|_p^p &= \int_Z \frac{\theta}{\phi \circ \pi} (\phi^{1/p} \chi_F)^p \circ \pi d\omega \\ &= \int_{\pi^{-1}F} \theta d\omega = \int_F \phi d\mu = \|f\|_p^p. \end{aligned}$$

In the general case, approximate $f\phi^{-1/p}$ by $\mathcal{F}$ -simple functions. $\square$

This isometry yields a result analogous to the theorem of Akcoglu and Koop [AK].

(4.5) **Theorem.** Define $Q: L_p(d\mu) \rightarrow L_p(d\omega)$ by

$$Q = \frac{v}{u \circ \pi} (f \circ \pi) \quad \text{for } f \in L_p(d\mu).$$

If ω' is ω restricted to $\mathcal{F}$, and we identify $\mathbf{X}$ with $(W, \mathcal{F}, \omega')$, then $Tf = E(Qf|\mathcal{F})$ for every $F \in L_p(d\mu)$.

Proof. By the two previous lemmas, we see that Q is an isometry of the indicated spaces. Suppose $f = u\chi_F$ for some $F \in \mathcal{F}$. Then

$$\begin{aligned} [E(Qf|\mathcal{F})](x) &= \int_0^1 (Qf)(x, y) dy = \int_0^1 v(x, y) \chi_{\pi^{-1}F}(x, y) dy \\ &= (Tu)(x) \left(\frac{T(u\chi_F)(x)}{(Tu)(x)} \right) = (Tf)(x). \end{aligned}$$

For a general $\mathcal{F}$ -measurable f , approximate fu^{-1} by $\mathcal{F}$ -simple functions. $\square$

(4.6) **Lemma.** Suppose $\widetilde{\mathcal{F}}$ is a finite sub- σ -algebra of $\mathcal{F}$ and $\tilde{\mu}$ is the restriction of μ to $\widetilde{\mathcal{F}}$. If $\tilde{v} \in L_p(d\omega)$ satisfies $\tilde{v} > 0$ a.e., then there is a unique $\widetilde{\mathcal{F}}$ -measurable function $\tilde{u}$ such that, for every $F \in \widetilde{\mathcal{F}}$,

$$\int_F \tilde{u}^p d\mu = \int_{\pi^{-1}F} \tilde{v}^p d\omega.$$

Furthermore, the mapping

$$f \in L_p(X, \widetilde{\mathcal{F}}, \tilde{\mu}) \mapsto \frac{\tilde{v}}{\tilde{u} \circ \pi}(f \circ \pi) \in L_p(d\omega)$$

is an isometry which transforms $\tilde{u}$ to $\tilde{v}$.

Proof. Let $\{F_i\}_{i=1}^k$ be an enumeration of the atoms of $\widetilde{\mathcal{F}}$. Let $H_i = \pi^{-1}F_i$ for each i , $1 \leq i \leq k$. Then

$$\tilde{u} = \sum_{i=1}^k \left(\frac{1}{\mu F_i} \int_{H_i} \tilde{v}^p d\omega \right)^{1/p} \chi_{F_i}.$$

If $f = \sum_{i=1}^k c_i \chi_{F_i} \in L_p(X, \widetilde{\mathcal{F}}, \tilde{\mu})$, then

$$\left\| \frac{\tilde{v}}{\tilde{u} \circ \pi}(f \circ \pi) \right\|_p^p = \sum_{i=1}^k c_i^p \int_{\pi^{-1}F_i} \frac{\tilde{v}^p}{(\tilde{u} \circ \pi)^p} d\omega = \sum_{i=1}^k c_i^p \mu F_i = \|f\|_p^p,$$

as desired. $\square$

(4.7) **Theorem.** For each $n \geq 1$, let $v_n = E(v|\mathcal{F}_n)$. Let u_n be the corresponding $\mathcal{F}_n$ -measurable functions as given by Lemma (4.6). Then $u_n \rightarrow u$ μ -a.e.

Proof. Let $u_n = \sum_{i=1}^{k_n} u_{n,i} \chi_{F_{n,i}}$. Then

$$\begin{aligned} u_{n,i}^p &= \frac{1}{\mu F_{n,i}} \int_{H_{n,i}} v_n^p d\omega \\ &= \left(\frac{1}{\mu F_{n,i}} \int_{F_{n,i}} u^p d\mu \right) \frac{(\omega H_{n,i})^{-1} \int_{H_{n,i}} v_n^p d\omega}{(\omega H_{n,i})^{-1} \int_{H_{n,i}} v^p d\omega}. \end{aligned}$$

Thus

$$u_n^p \circ \pi = \frac{E(v_n^p|\mathcal{H}_n)}{E(v^p|\mathcal{H}_n)} [E(u^p|\mathcal{F}_n) \circ \pi].$$

By the martingale convergence theorem, with $p = 1$, we have $E(u^p|\mathcal{F}_n) \rightarrow u^p$ μ -a.e., and $E(v^p|\mathcal{H}_n) \rightarrow E(v^p|\mathcal{H})$ ω -a.e. By Lemma (2.22), we also have $E(v_n^p|\mathcal{H}_n) \rightarrow E(v^p|\mathcal{H})$ ω -a.e. Therefore

$$u_n^p \circ \pi \rightarrow E(u^p|\mathcal{F}_n) \circ \pi,$$

and so $u_n^p \rightarrow u^p$ μ -a.e., by the martingale convergence theorem. This completes the proof. $\square$

(4.8) **Definition.** For each $n \geq 1$, let u_n and v_n be as defined in the hypothesis of the previous lemma. Define

$$Q^n: L_p(X, \mathcal{F}_n, \mu_n) \rightarrow L_p(d\omega),$$

where μ_n is the restriction of μ to $\mathcal{F}_n$, by

$$Q^n f = \frac{v_n}{u_n \circ \pi} (f \circ \pi)$$

for $f \in L_p(X, \mathcal{F}_n, \mu_n)$.

By Lemma (4.6), this is an isometry. If ω_n is the restriction of ω to $\mathcal{F}_n$, and we make the obvious identification of $(W, \mathcal{F}_n, \omega_n)$ with $(X, \mathcal{F}_n, \mu_n)$, then define $T^n: L_p(d\mu) \rightarrow L_p(X, \mathcal{F}_n, \mu_n)$ by

$$T^n f = E(Q^n E(f|\mathcal{F}_n)|\mathcal{F}_n)$$

for $f \in L_p(X)$. Each T^n is a positive contraction, and it is easy to see that if $f \in L_p(d\mu)$, then $T^n f \rightarrow T f$ μ -a.e.

Observe that u_n is a semi-invariant function for each T^n ; the reason is that v_n is already $\mathcal{F}_n$ -measurable. (In fact, it is easy to see that u_n is the only normalized semi-invariant function for T^n .) Thus, the induced operator $(T^n)_r$ is defined for any r , $1 < r < \infty$. For brevity, denote it R_n .

(4.9) **Theorem.** $\|R_n f - T_r f\|_r \rightarrow 0$ as $n \rightarrow \infty$, for every $f \in L_r(d\mu)$.

Proof. If f is $\mathcal{F}_n$ -measurable, then

$$\begin{aligned} R_n f &= v_n^{p/r-1} T^n(v_n^{1-p/r} f) \\ &= v_n^{p/r-1} E \left[\frac{v_n}{u_n \circ \pi} (u_n^{1-p/r} \circ \pi) (f \circ \pi) | \mathcal{F}_n \right] \\ &= E \left[\left(\frac{v_n}{u_n \circ \pi} \right)^{p/r} (f \circ \pi) | \mathcal{F}_n \right], \end{aligned}$$

since v_n is $\mathcal{F}_n$ -measurable. Whether or not f is $\mathcal{F}_n$ -measurable, define

$$\phi_n = \left(\frac{v_n}{u_n \circ \pi} \right)^{p/r} [E(f|\mathcal{F}_n) \circ \pi].$$

Then $R_n f = E(\phi_n | \mathcal{F}_n)$ for any $f \in L_r(d\mu)$. Similarly, if $\phi = (v/u \circ \pi)^{p/r} (f \circ \pi)$, then $T_r f = E(\phi | \mathcal{F})$.

Clearly $\phi_n \rightarrow \phi$ a.e.; if we can show that $\|\phi_n\|_r \rightarrow \|\phi\|_r$, we may conclude that $\phi_n \rightarrow \phi$ in L_r norm (see [Ry, p. 118]):

$$\begin{aligned} \|\phi_n\|_r^r &= \|Q^n([E(f|\mathcal{F}_n)]^{r/p})\|_p^p \\ &= \|[E(f|\mathcal{F}_n)]^{r/p}\|_p^p = \|E(f|\mathcal{F}_n)\|_r^r \rightarrow \|f\|_r^r, \end{aligned}$$

as $E(f|\mathcal{F}_n)$ is an L_r martingale. The second line follows because Q^n is an isometry. Also $\|\phi\|_r = \|f\|_r$ by a similar calculation. This tells us that

$$\|\phi_n - \phi\|_r \rightarrow 0.$$

To conclude the proof, observe that

$$\|R_n f - T_r f\|_r \leq \|E(\phi_n|\mathcal{F}_n) - E(\phi_n|\mathcal{F})\|_r + \|E(\phi_n|\mathcal{F}) - E(\phi|\mathcal{F})\|_r.$$

The first term tends to zero by the martingale convergence theorem and the second term is dominated by $\|\phi_n - \phi\|_r$. $\square$

5. CONVERGENCE OF THE ALTERNATING SEQUENCE

This section is in many ways analogous to §§3 and 4 of [AS2], and so the reader will often be referred there for details. Where we follow [AS2] closely, every effort is made to keep the notation consistent.

(5.1) **Definitions.** Suppose $1 < p < \infty$, $1 < r < \infty$, and let $\psi = \psi_{p,r}$. Let $\langle T_n \rangle_{n=1}^\infty$ be a sequence of positive linear contractions with semi-invariant functions operating on the L_p space of a σ -infinite Lebesgue space. Call such a sequence of operators a *norming sequence*. Call a norming sequence *special* if all operators are finite dimensional.

Let V_0 and U_0 be the identities on L_p and L_r respectively, and make the following definitions for each $n \geq 1$:

$$V_n = T_n \cdots T_1, \quad U_n = (T_1^*)_r \cdots (T_n^*)_r.$$

For a given $f \in L_p$ and an $n \geq 0$, let $g_n = U_n \psi(V_n f)$. Observe that $g_0 = \psi f$ and that $\|g_0\|_r = \|f\|_p^{p/r}$.

We say that *Estimate A* holds for a norming sequence $\langle T_n \rangle_{n=1}^\infty$ if

$$\left\| \sup_{n \geq 0} |g_n| \right\|_r \leq (q \|f\|_p)^{p/r} (= q^{p/r} \|g_0\|_r)$$

for every $f \in L_p$.

We say that *Estimate B* holds for a norming sequence $\langle T_n \rangle_{n=1}^\infty$ if for every $\varepsilon > 0$ there is a $\delta > 0$, depending only on ε , p , and r , such that

$$\left\| \sup_{n \geq 0} |g_n - g_0| \right\|_r < \varepsilon \|f\|_p^{p/r} \quad (= \varepsilon \|g_0\|_r)$$

whenever $f \in L_p$ is such that

$$\|f\|_p - \lim_{n \geq 0} \|V_n f\|_p < \delta \|f\|_p.$$

Given a norming sequence $\langle T_n \rangle_{n=1}^\infty$, a fixed $n \geq 1$, and a function $f \in L_p$, let $\tilde{f} = V_n f$. For every $k \geq 1$, let $\tilde{T}_k = T_{n+k}$. Let $\tilde{V}_0$ and $\tilde{U}_0$ be the identities on L_p and L_r respectively. For each $k \geq 1$, let

$$\tilde{V}_k = \tilde{T}_k \cdots \tilde{T}_1, \quad \tilde{U}_k = (\tilde{T}_1^*)_r \cdots (\tilde{T}_k^*)_r,$$

and for each $k \geq 0$, let $\tilde{g}_k = \tilde{U}_k \psi(\tilde{V}_k \tilde{f})$. Observe that $g_{n+k} = U_n \tilde{g}_k$ for every $k \geq 0$.

(5.2) **Theorem.** Suppose Estimates A and B are satisfied for every norming sequence. Then given a norming sequence $\langle T_n \rangle_{n=1}^\infty$ and an $f \in L_p$, $\langle g_n \rangle_{n=0}^\infty$ converges a.e.

Proof. Because of Estimate A, $(\sup_{n \geq 0} |g_n|) \in L_r$. Therefore, by Lemma (2.27), it suffices to show that

$$\lim_{n \geq 0} \left\| \sup_{k \geq n} |g_{n+k} - g_n| \right\|_r = 0.$$

Let $\beta = \lim_{n \geq 0} \|V_n f\|_p$ and distinguish two cases:

Case 1: $\beta = 0$. Given $\varepsilon > 0$, find $n_0 \geq 1$ such that

$$\|V_{n_0} f\|_p < \frac{1}{q} \left(\frac{\varepsilon}{2} \right)^{r/p}.$$

Fix $n \geq n_0$ and define $\tilde{f}$ and $\tilde{g}_k$ as above. Observe that $\|\tilde{f}\|_p \leq \|V_{n_0} f\|_p$. We have

$$\begin{aligned} \left\| \sup_{k \geq 0} |g_{n+k} - g_n| \right\|_r &= \left\| \sup_{k \geq 0} |U_n \tilde{g}_k - U_n \tilde{g}_0| \right\|_r \\ &\leq \left\| \sup_{k \geq 0} |\tilde{g}_k - \tilde{g}_0| \right\|_r \leq 2 \left\| \sup_{k \geq 0} |\tilde{g}_k| \right\|_r \\ &\leq 2(q\|\tilde{f}\|_p)^{p/r} < \varepsilon, \end{aligned}$$

where the first inequality follows from Lemma (2.24) and the third follows from Estimate A for the sequence $\langle \tilde{T}_k \rangle_{k=1}^\infty$.

Case 2: $\beta > 0$. Given $\varepsilon > 0$, choose $\delta > 0$ as given by Estimate B, corresponding to $\varepsilon/\|f\|_p^{p/r}$. Choose $n_0 \geq 1$ such that $\|V_{n_0} f\|_p < (1 + \delta)\beta$. Fix $n \geq n_0$ and define $\tilde{f}$ and $\tilde{g}_k$ as above. Observe that $\beta \leq \|\tilde{f}\|_p$, since the $\|V_n f\|_p$'s form a monotone sequence. We have

$$\|\tilde{f}\|_p - \lim_{k \geq 0} \|\tilde{V}_k \tilde{f}\|_p = \|V_n f\|_p - \beta < (1 + \delta)\beta - \beta \leq \delta \|\tilde{f}\|_p.$$

Now apply Estimate B for $\langle \tilde{T}_k \rangle_{k=1}^\infty$; we conclude

$$\left\| \sup_{k \geq 0} |g_{n+k} - g_k| \right\|_r \leq \left\| \sup_{k \geq 0} |\tilde{g}_k - \tilde{g}_0| \right\|_r < \left(\frac{\varepsilon}{\|f\|_p^{p/r}} \right) \|\tilde{f}\|_p^{p/r} \leq \varepsilon,$$

where the first inequality follows as in Case 1. $\square$

(5.3) **Lemma.** If Estimate A holds for every special norming sequence, then it holds for every norming sequence.

Proof. Suppose $\langle T_n \rangle_{n=1}^\infty$ is a uniform norming sequence for which Estimate A fails. Then there is a function $f \in L_p$ and an $n \geq 1$ such that

$$\left\| \max_{0 \leq k \leq n} |g_k| \right\|_r > q^{p/r} \|g_0\|_r.$$

Suppose $\langle \mathcal{F}_m \rangle_{m=1}^\infty$ is a monotone sequence of finite sub- σ -algebras of $\mathcal{F}$ with $\mathcal{F} = \sigma(\bigcup_{m=1}^\infty \mathcal{F}_m)$, the smallest σ -algebra containing the algebra $\bigcup_{m=1}^\infty \mathcal{F}_m$. For each k and m , $1 \leq k \leq n$ and $m \geq 1$, let T_k^m be the finite-dimensional operator as defined in (4.8). Let $f^m = E(f|\mathcal{F}_m)$.

Let $m \geq 1$ be arbitrary. Let V_0^m and U_0^m be $E(\cdot|\mathcal{F}_m)$ operating on L_p and L_r respectively. For each k , $1 \leq k \leq n$, let

$$V_k^m = T_k^m \cdots T_1^m, \quad U_k^m = ((T_1^*)^m)_r \cdots ((T_k^*)^m)_r.$$

For $f \in L_p$, $m \geq 1$, and each k , $0 \leq k \leq n$, let

$$g_{km} = U_k^m \psi(V_k^m f) = U_k^m \psi(V_k^m f^m).$$

By the martingale convergence theorem, $\lim_{m \geq 1} \|f - f^m\|_p = 0$. We will show that

$$(5.4) \quad \lim_{m \geq 1} \|g_{km} - g_k\|_r = 0$$

as well. Therefore, by applying Lemma (2.26),

$$\lim_{m \geq 1} \left\| \max_{0 \leq k \leq n} |g_{km}| - \max_{0 \leq k \leq n} |g_k| \right\|_r = 0.$$

Thus, for a suitably large integer m_0 ,

$$\left\| \max_{0 \leq k \leq n} |g_{km_0}| \right\|_r > (q \|f^{m_0}\|_p)^{p/r},$$

since the same inequality holds for f and the g_k 's. Because $\mathcal{F}_{m_0}$ is finite, the operators $\langle T_1^{m_0}, \dots, T_n^{m_0} \rangle$ are essentially finite dimensional. Therefore, they form the initial portion of a special norming sequence for which Estimate A fails, contradicting the hypothesis of the lemma.

To prove (5.4), we first prove

$$(5.5) \quad \lim_{m \geq 1} \|V_k^m f - V_k f\|_p = 0$$

for every k , $0 \leq k \leq n$. When $k = 0$, this is simply the martingale convergence theorem. For the inductive step, observe that

$$\|V_{k+1}^m f - V_{k+1} f\|_p = \|T_{k+1}^m V_k^m f - T_{k+1} V_k f\|_p,$$

where $\lim_{m \geq 1} \|V_k^m f - V_k f\|_p = 0$ by the inductive hypothesis. We apply Theorem (4.9) with $r = p$ and Lemma (2.30) to conclude that

$$\lim_{m \geq 1} \|V_{k+1}^m f - V_{k+1} f\|_p = 0,$$

completing the induction.

Because of the uniform continuity of ψ ,

$$\lim_{m \geq 1} \|\psi V_k^m f - \psi V_k f\|_p = 0$$

for each k , $0 \leq k \leq n$.

We now perform another induction similar to the proof of (5.5) to show that when $g \in L_r$,

$$\lim_{m \geq 1} \|U_k^m g - U_k g\|_r = 0,$$

for each k , $0 \leq k \leq n$. This completes the proof. $\square$

(5.6) **Lemma.** Suppose that for every $\xi > 0$, there is an $\eta > 0$ depending only on ξ , p , and r such that

$$\left\| \max_{0 \leq k \leq n} |g'_k - g'_0| \right\|_r < \xi \|f'\|_p^{p/r}$$

whenever $\langle T'_n \rangle_{n=1}^\infty$ is a special norming sequence, $n \geq 1$, and $f' \in L_p$ is such that $\|f'\|_p - \|V'_n f'\|_p < \eta \|f'\|_p$, where V'_n and g'_n are defined exactly as V_n and g_n in (5.1), relative to $\langle T'_n \rangle_{n=1}^\infty$. Then Estimate B holds for every norming sequence.

Proof. Let $\langle T_n \rangle_{n=1}^\infty$ be a norming sequence and suppose $\xi > 0$ is given. Choose $\eta > 0$ from the hypothesis of the lemma, corresponding to $\xi/2$. If Estimate B fails for $\langle T_n \rangle_{n=1}^\infty$, then there is a function $f \in L_p$ with $\|f\|_p - \|V_n f\|_p < \eta \|f\|_p$, but for which

$$\left\| \max_{0 \leq k \leq n} |g_k - g_0| \right\|_r > \frac{\xi}{2} \|f\|_p^{p/r}.$$

As in the proof of the previous lemma, we approximate the operators T_k with the operators T_k^m from (4.8). Define g_{km} as before, for each $m \geq 1$ and each k , $0 \leq k \leq n$, and let $h_k = g_k - g_0$ and $h_{km} = g_{km} - g_{0m}$ for the same set of indices. Then

$$\|h_{km} - h_k\|_r \leq \|g_{km} - g_k\|_r + \|g_{0m} - g_0\|_r,$$

and we have seen that both of these terms tend to zero as m increases. Thus $\lim_{m \geq 1} \|h_{km} - h_k\|_r = 0$, and we may apply Lemma (2.26) to conclude

$$\lim_{m \geq 1} \left\| \max_{0 \leq k \leq n} |g_{km} - g_{0m}| - \max_{0 \leq k \leq n} |g_k - g_0| \right\|_r = 0.$$

At the same time, we have

$$\lim_{m \geq 1} \|f - f^m\|_p = 0 \quad \text{and} \quad \lim_{m \geq 1} \|V_n f - V_n^m f^m\|_p = 0.$$

Thus, we may choose an m_0 sufficiently large that we maintain the relations

$$\|f^{m_0}\|_p - \|V_n^{m_0}\|_p < \eta \|f^{m_0}\|_p$$

and

$$\left\| \max_{0 \leq k \leq n} |g_{km_0} - g_{0m_0}| \right\|_r > \frac{\xi}{2} \|f^{m_0}\|_p^{p/r}.$$

As $\mathcal{F}_m$ is finite, $\langle T_1^{m_0}, \dots, T_n^{m_0} \rangle$ form the initial portion of a special norming sequence for which the hypothesis of the lemma fails. $\square$

We have reduced the proof of Theorem (1.2) to verifying that finitary versions of Estimates A and B hold for every special norming sequence. In order to show that this is true, we introduce a dilation of these operators similar to the one given in [A].

(5.7) **Definitions.** Let $(X, \mathcal{F}, \mu)$ be a measure space in which $\mathcal{F}$ is a finite set. Let $\{F_i\}_{i=1}^d$ be an enumeration of the atoms of $\mathcal{F}$ of positive measure. Let the indices i and j range through the integers $\{1, \dots, d\}$. If T is a positive operator with a semi-invariant function u , let $u = \sum_i \alpha_i \chi_{F_i}$ and $Tu = \sum_i \beta_i \chi_{F_i}$. We have $\alpha_i > 0$ and $\beta_i > 0$ for each i . Let $m_i = \mu(F_i)$ and let $a_{ij} = \omega[\pi^{-1}F_i \cap (F_j \times [0, 1])]$, with π and ω as given in §4. Observe that $\sum_i a_{ij} = m_j$ for each j , and that for each i ,

$$\alpha_i^p m_i = \int_{F_i} u^p d\mu = \int_{\pi^{-1}F_i} v^p d\omega = \sum_j \beta_j^p a_{ij}.$$

Let

$$b_{ij} = \left(\frac{\beta_j}{\alpha_i}\right)^p \frac{a_{ij}}{m_i}.$$

It is easy to verify that $\sum_j b_{ij} = 1$. Observe also that $a_{ij} = 0$ if and only if $b_{ij} = 0$.

We are going to construct a set Z in the coordinate plane $\mathbb{R}^2$ and an isometry of its L_p space. The construction is virtually identical to the one given in [A] and used in [AS2], except that some of the subrectangles may have measure zero. However, because of the last observation, this will cause no problems.

Let $\langle I_i \rangle_{i=1}^d$ be disjoint intervals on the x -axis of the coordinate plane, each of length m_i . Let $\langle J_i \rangle_{i=1}^d$ be disjoint intervals on the y -axis, each of unit length. Let $P_i = I_i \times J_i$ and $Z = \bigcup_i P_i$. Let $\mathbb{Z} = (Z, \mathcal{B}, \lambda)$, where $\mathcal{B}$ is the Borel σ -algebra of Z and λ is the restriction of Lebesgue measure on $\mathbb{R}^2$ to Z . Let L_p denote $L_p(\mathbb{Z})$, and let $\mathcal{P}$ be the partition $\{P_i\}_{i=1}^d$ of Z . Let $E = E(\cdot | \mathcal{P})$ and let $l_p = \bar{E}L_p$.

Define a further partitioning of Z as follows. Each I_i is partitioned into d subintervals $\langle I_{ij} \rangle_{j=1}^d$, each of length a_{ij} . Each J_i is partitioned into d subintervals $\langle J_{ij} \rangle_{j=1}^d$, each of length b_{ij} . Let $R_{ij} = I_i \times J_{ij}$, a horizontal strip of P_i , and $S_{ij} = I_{ij} \times J_j$, a vertical strip of P_j .

Define a point transformation $\tau: Z \rightarrow Z$ by mapping each R_{ij} of nonzero measure to the corresponding S_{ij} , in such a way that the Radon-Nikodým derivative for the mapping of these rectangles is constant. Thus, τ "squeezes" the width of R_{ij} from m_i to a_{ij} and "stretches" its height from b_{ij} to 1; this deformation determines the constant value of

$$\rho = \frac{d(\lambda \circ \tau^{-1})}{d\lambda}$$

on S_{ij} .

$\lambda(R_{ij}) = 0$ if and only if $\lambda(S_{ij}) = 0$, because of the corresponding property of a_{ij} and b_{ij} , and so τ is an automorphism of Z . An automorphism of Z determined in this manner by any pair of sequences of a_{ij} 's and b_{ij} 's satisfying $\sum_i a_{ij} = m_j$, $\sum_j b_{ij} = 1$, and $a_{ij} = 0$ if and only if $b_{ij} = 0$, is called an *admissible* automorphism. Each admissible automorphism induces an *admissible* L_p isometry Q in the usual manner by $Qf = \rho^{1/p}(f \circ \tau^{-1})$.

(5.8) **Theorem.** *The action of EQ on l_p is isomorphic to the action of the original operator T on $L_p(\mathbf{X})$.*

Proof. Let i range through $\{1, \dots, d\}$. Let Φ be given by

$$\sum_i c_i \chi_{P_i} \in l_p \mapsto \sum_i c_i \chi_{F_i} \in L_p(\mathbf{X}).$$

This is an isometric isomorphism since $\lambda(P_i) = \mu(F_i) = m_i$.

Let $\mathbf{W} = (W, \mathcal{K}, \omega)$, π , $\mathcal{J}$, and v be as given in Theorem (4.5). According to that theorem, if we define $R: L_p(d\mu) \rightarrow L_p(d\omega)$ by

$$Rg = \frac{v}{u \circ \pi}(g \circ \pi),$$

then $Tg = E(Rg|\mathcal{J})$ for every $g \in L_p(d\mu)$. Since $u = \sum_i \alpha_i \chi_{F_i}$ and $Tu = \sum_i \beta_i \chi_{F_i}$, we have $Rg = (\beta_j/\alpha_i)c_i$ on each $\pi^{-1}F_i \cap (F_j \times [0, 1]) \subseteq F_j \times I$.

When $f \in l_p$, then $Qf = \rho_{ij}^{1/p} c_i$ on each $S_{ij} \subseteq P_j$, where ρ_{ij} is the constant value of the Radon-Nikodým derivatives ρ on the rectangle S_{ij} . Observe that

$$\rho_{ij} = \frac{\lambda(R_{ij})}{\lambda(S_{ij})} = \frac{m_i b_{ij}}{a_{ij}} = \left(\frac{\beta_j}{\alpha_i} \right)^p.$$

We also have $\omega[\pi^{-1}F_i \cap (F_j \times [0, 1])] = \lambda(S_{ij}) = a_{ij}$. This means that Qf and Rg are simple functions taking the same range of values over sets of identical measure. Therefore, $Tg = E(Rg|\mathcal{J}) = \Phi(EQf)$ as desired. $\square$

The proof of the convergence of the alternating sequence is now reduced to an examination of the actions of admissible isometries of $\mathbf{Z}$, intertwined with the conditional expectation operator with respect to $\mathcal{P}$.

(5.9) **Definitions.** Let G be a subset $\mathbb{R}^2$. A subset F of G is called a *vertical subset* of G if

$$F = (F' \times \mathbb{R}) \cap G$$

for some subset F' of the x -axis. Similarly, if

$$H = (\mathbb{R} \times H') \cap G$$

for some subset H' of the y -axis, then H is called a *horizontal subset* of G .

We say that a function f is *constant on vertical lines* if $f(x_1, y_1) = f(x_2, y_2)$ whenever $x_1 = x_2$. We say that f is *constant on horizontal lines* if $f(x_1, y_1) = f(x_2, y_2)$ whenever $y_1 = y_2$.

The following is a summary of Lemmas (4.5) through (4.12) from [AS2].

(5.10) **Lemma.** Let τ be an admissible automorphism, and let Q be the induced L_p isometry.

- (a) Suppose $\mathcal{G}$ is a finite partition of Z in which each atom is a vertical subset of some P_i . Let f be an L_p function which is constant on vertical lines. Then

$$QE(f|\mathcal{G}) = E(Qf|\mathcal{P} \vee \tau\mathcal{G}).$$

- (a') Suppose $\mathcal{H}$ is a finite partition of Z in which each atom is a horizontal subset of some P_i . Let f be an L_p function which is constant on horizontal lines. Then

$$Q^{-1}E(f|\mathcal{H}) = E(Q^{-1}f|\mathcal{P} \vee \tau\mathcal{H}).$$

- (b) If f_1 and f_2 are L_p functions that are constant on vertical lines and $Ef_1 = Ef_2$, then also $EQf_1 = EQf_2$.
- (b') If f_1 and f_2 are L_p functions that are constant on horizontal lines and $Ef_1 = Ef_2$, then also $EQ^{-1}f_1 = EQ^{-1}f_2$.
- (c) If f is constant on vertical lines, then Qf is constant on vertical lines.
- (c') If f is constant on horizontal lines, then $Q^{-1}f$ is constant on horizontal lines.

(5.11) **Definitions.** Let n be a fixed integer, $n \geq 1$, and let k range through $\{0, 1, \dots, n\}$. If $1 \leq k \leq n$, let τ_k be an admissible isometry of Z , let Q_k be the L_p isometry induced by τ_k , and let R_k be the L_r isometry induced by τ_k^{-1} . Let Q_0 and R_0 be the identities on L_p and L_r , respectively. Let

$$\begin{aligned} T_k &= EQ_k E, & V_k &= T_k \cdots T_0, & W_k &= Q_k \cdots Q_0, \\ S_k &= ER_k E, & U_k &= S_0 \cdots S_k, & D_k &= R_0 \cdots R_k. \end{aligned}$$

Observe that $S_k = (T_k^*)_r$ by Lemma (3.20).

Let f be a fixed but arbitrary function in L_p . Let $g_k = U_k \psi(V_k f)$ and $\phi_k = W_k^{-1} E W_k E f$. Observe that $g_0 = \psi \phi_0 = \psi E f$.

(5.12) **Lemma.** For any $f \in L_p$, $V_k f = E W_k E f$.

Proof. This is Lemma (4.14) of [AS2]. When $k = 0$ this is immediate from the definitions. The inductive step is given by Lemma (5.10)(b) and (c). $\square$

(5.13) **Lemma.** For any $g \in L_r$, $U_k g = E D_k E g$.

Proof. We will show that

$$(5.14) \quad S_i \cdots S_j g = E R_i \cdots R_j E g$$

for every pair i, j with $0 \leq i \leq j \leq n$. This will prove the lemma, since the desired identity is (5.14) with $i = 0$ and $j = k$. The proof is by induction on $j - i$. When $i = j$, (5.14) is simply the definition of $S_i g$.

Now suppose (5.14) holds for some pair $i + 1, j + 1$ with $0 \leq i \leq j < n$. We have

$$ER_{i+1} \cdots R_{j+1} E g = ES_{i+1} \cdots S_{j+1} g,$$

by the inductive hypothesis and the idempotence of E , the outermost operator in $S_{i+1} \cdots R_{i+1} \cdots R_{j+1} E g$ is constant on horizontal lines, by repeated application of Lemma (5.10)(c'). Thus, by Lemma (5.10)(b'), we have

$$ER_i R_{i+1} \cdots R_{j+1} E g = ER_i S_{i+1} \cdots S_{j+1} = S_i \cdots S_{j+1} g.$$

This completes the induction. $\square$

(5.15) **Lemma.** $g_k = E\psi(\phi_k)$.

Proof.

$$\begin{aligned} g_k &= U_k \psi(V_k f) = ED_k E\psi(EW_k E f) \\ &= ED_k \psi(EW_k E f) = E\psi[(R_0)^p \cdots (R_k)^p EW_k E f]. \end{aligned}$$

The second line follows from the two previous lemmas. The third line follows because ψ maps $\mathcal{P}$ -measurable functions to $\mathcal{P}$ -measurable functions. For the fourth line, we use $(R_i)^p$ to denote the L_p isometry induced by τ_i^{-1} . Thus, this line follows by an application of Lemma (2.19). By Lemma (2.17), that isometry is Q_i^{-1} . Thus

$$g_k = E\psi(W_k^{-1} R W_k E f) = E\psi(\phi_k),$$

as desired. $\square$

(5.16) **Lemma.** *There exists a monotone sequence $\mathcal{G}_n \subseteq \mathcal{G}_{n-1} \subseteq \cdots \subseteq \mathcal{G}_0$ of finite σ -algebras such that*

$$\phi_k = W_n^{-1} E(W_n E f | \mathcal{G}_k).$$

Proof. This is Lemma (4.16) of [AS2]. We may take $\mathcal{G}_n = \mathcal{P}$. Lemma (5.10)(a) provides the induction step needed to show that we may take

$$\mathcal{G}_{n-k} = \mathcal{P} \vee \tau_n \mathcal{P} \vee \cdots \vee \tau_n \cdots \tau_{n-k+1} \mathcal{P}$$

when $1 \leq k \leq n$. $\square$

(5.17) **Definition.** Let $u_k = E(W_n E f | \mathcal{G}_k)$, where the $\mathcal{G}_k$'s are as in the previous lemma. Observe that $\phi_k = W_n^{-1} u_k$.

(5.18) **Theorem.** *The sequence $\langle u_0, \dots, u_n \rangle$ is an L_p martingale. Furthermore,*

$$\left\| \max_{0 \leq k \leq n} |u_k| \right\|_p \leq q \|u_0\|_p$$

and

$$\left\| \max_{0 \leq k \leq n} |u_k - u_n| \right\|_p \leq q \|u_0 - u_n\|_p.$$

Proof.

$$u_k = E(W_n E f | \mathcal{G}_k) = E(E(W_n E f | \mathcal{G}_0) | \mathcal{G}_k) = E(u_0 | \mathcal{G}_k),$$

since $\mathcal{G}_k \subseteq \mathcal{G}_0$ for every k , $0 \leq k \leq n$. As well,

$$u_k - u_n = E(u_0 | \mathcal{G}_k) - E(u_n | \mathcal{G}_k) = E(u_0 - u_n | \mathcal{G}_k).$$

In the first case, this follows from the above computation. In the second case, $u_n = E(u_n | \mathcal{G}_k)$ because u_n is already constant on the atoms of $\mathcal{G}_k$.

The lemma now follows by an application of the martingale convergence theorem for L_p . $\square$

(5.19) **Theorem.** $\| \max_{0 \leq k \leq n} |g_k| \|_r \leq (q \|f\|_p)^{p/r}$.

Proof. Since $\phi_k = W_n^{-1} u_k$ and W_n^{-1} is a positive isometry, we have $|\phi_k| = W_n^{-1} |u_k|$ and $\max_{0 \leq k \leq n} |\phi_k| = W_n^{-1} (\max_{0 \leq k \leq n} |u_k|)$ and so

$$(5.20) \quad \left\| \max_{0 \leq k \leq n} |\phi_k| \right\|_p = \left\| \max_{0 \leq k \leq n} |u_k| \right\|_p \leq q \|u_0\|_p \leq q \|f\|_p.$$

The inequalities follow by an application of Theorem (5.18) and the fact that $\|u_0\|_p = \|Ef\|_p$.

Since $g_k = E\psi(\phi_k)$, we have

$$\max_{0 \leq k \leq n} |g_k| \leq E \left(\max_{0 \leq k \leq n} |\psi(\phi_k)| \right) = E\psi \left(\max_{0 \leq k \leq n} |\phi_k| \right),$$

where Lemma (2.24) was used for the inequality. Thus

$$\begin{aligned} \left\| \max_{0 \leq k \leq n} |g_k| \right\|_r &\leq \left\| \psi \left(\max_{0 \leq k \leq n} |\phi_k| \right) \right\|_r \\ &= \left\| \max_{0 \leq k \leq n} |\phi_k| \right\|_p^{p/r} \leq (q \|f\|_p)^{p/r}. \quad \square \end{aligned}$$

(5.21) **Theorem.** For any $\xi > 0$ there is an $\eta > 0$ depending only on ξ , p , and r such that

$$\left\| \max_{0 \leq k \leq n} |g_k - g_0| \right\|_r < \xi \|f\|_p^{p/r}$$

whenever $\|f\|_p - \|V_n f\|_p < \eta \|Ef\|_p$.

Proof. Since $u_n = E(u_0 | \mathcal{G}_n)$, we may apply Lemma (2.25) to choose an $\eta > 0$, depending only on δ (which will be specified later) and p so that

$$\|u_0\|_p - \|u_n\|_p < \eta \|u_0\|_p$$

implies

$$\|u_0 - u_n\|_p < \frac{\delta}{2q} \|u_0\|_p.$$

We have already observed that $\|u_0\|_p = \|Ef\|_p$. As well, we note that $\|u_n\|_p = \|V_n f\|_p$. Thus, if $\|f\|_p - \|V_n f\|_p < \eta \|Ef\|_p$, we have

$$\begin{aligned} \left\| \max_{0 \leq k \leq n} |u_k - u_0| \right\|_p &\leq 2 \left\| \max_{0 \leq k \leq n} |u_k - u_n| \right\|_p \\ &\leq 2q \|u_0 - u_n\|_p < \xi \|u_0\|_p, \end{aligned}$$

where the second inequality follows from Theorem (5.18).

As in the proof of the previous theorem, we deduce

$$\max_{0 \leq k \leq n} |\phi_k - \phi_0| = W_n^{-1} \left(\max_{0 \leq k \leq n} |u_k - u_0| \right),$$

and so $\|\max_{0 \leq k \leq n} |\phi_k - \phi_0|\|_p \leq \delta \|Ef\|_p$.

Since the inequality $\|\max_{0 \leq k \leq n} |\phi_k|\|_p \leq q \|Ef\|_p$ is simply a restatement of (5.20), we are in a position to apply Lemma (2.29). Choose δ from that lemma corresponding to ξ , q (which depends only on p), p and r , and conclude that

$$\left\| \max_{0 \leq k \leq n} |\psi(\phi_k) - \psi(\phi_0)| \right\|_r < \xi \|Ef\|_p^{p/r}$$

whenever $\|f\|_p - \|V_n f\|_p < \eta \|Ef\|_p$.

Now apply Lemma (2.24):

$$\begin{aligned} \left\| \max_{0 \leq k \leq n} |g_k - g_0| \right\|_r &\leq \left\| E \left(\max_{0 \leq k \leq n} |\psi(\phi_k) - \psi(\phi_0)| \right) \right\|_r \\ &< \xi \|Ef\|_p^{p/r}. \end{aligned}$$

This completes the proof of this theorem, and hence of Theorem (1.2). $\square$

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