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POINCARÉ-LEFSCHETZ DUALITY FOR THE HOMOLOGY CONLEY INDEX

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ABSTRACT. The Conley index for continuous dynamical systems is defined for (one-sided) semiflows. For (two-sided) flows, there are two indices defined: one for the forward flow; and one for the reverse flow. In general, the two indices give different information about the flow; but for flows on orientable manifolds, there is a duality isomorphism between the homology Conley indices of the forward and reverse flows. This duality preserves the algebraic structure of many of the constructions of the Conley index theory: sums and products; continuation; attractor-repeller sequences and connection matrices.

0. INTRODUCTION

The Conley index for continuous dynamical systems is defined for isolated invariant sets in local semiflows. A local semiflow on a Hausdorff space X consists of a neighborhood U of $X \times \{0\}$ in $X \times \mathbf{R}^+$ and a map $\mu: U \to X$ such that $\mu_0 = \mathrm{id}$ and $\mu_s \circ \mu_t = \mu_{s+t}$. If S is isolated in X, it has index pairs (N, L) defined, with the homotopy type $[N/L]$ well-defined independent of the index pair chosen. This defines the Conley index $I(X; S)$ of S in X, and after passing to homology, the homology Conley index $CH_*(X; S)$. The Conley index theory developed from these objects has proved to be a valuable tool for the study of dynamical systems in a number of settings (cf. [1–4, 8, 9]).

If the space has a (two-sided) flow $\phi: V \to X$, where V is a neighborhood of $X \times \{0\}$ in $X \times \mathbf{R}$, there are two local semiflows defined: $U_1 = V \cap (X \times \mathbf{R}^+)$, $\mu_1 = \phi | U_1$; and $U_2 = (\mathrm{id}, -\mathrm{id})(V \cap (X \times \mathbf{R}^-))$, $\mu_2 = \phi \circ (\mathrm{id}, -\mathrm{id})$. These will be referred to as the forward and reverse flows, respectively. Any isolated invariant set in X then has two Conley indices defined: its index in the forward flow, and its index in the reverse flow. These are not equal, yet in many cases, they provide the same information about the flow. In this paper, the differences and similarities between the homology and cohomology indices in the forward and reverse flows are examined.

The main result, Theorem 2.1, shows that for flows on an orientable manifold, the indices for the forward and reverse flows are related by a Poincaré-Lefschetz duality isomorphism. This duality is well-defined (i.e. independent of the index

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pairs chosen) and preserves the algebraic structure of many of the constructions of the Conley index theory: sums; products; continuation; attractor-repeller sequences; connection matrices and Morse decompositions. In particular, duality shows that any dynamical information obtained from these algebraic constructions can be obtained equally from the forward and reverse flows. This can be summarized as follows:

Given a property of flows on orientable manifolds which can be detected by the group structure of the (co)homology Conley indices of isolated invariant sets of a single flow or parameterized family of flows, the homology and cohomology indices for the forward and reverse flows all provide equivalent information about the property. When other classes of flows or other algebraic structures are considered, the four indices may provide distinct information.

Section 1 describes the main elements of the Conley index theory, and presents some examples which will be used to illustrate the results. In §2, the duality isomorphism is constructed and shown to be well defined. In §3, the isomorphism is shown to commute with sums, products, continuation, and attractor-repeller sequences, and to conjugate connection matrices and Morse inequalities. Some simple examples are considered to illustrate the properties of the index not preserved by duality. Finally, some of the dynamical applications of duality are considered in §4.

1. The Conley index

We begin with a brief survey of the index theory, using the notation of [10]. The reader is referred to [1, 6, 7, 13] for more complete developments of the theory.

We will assume that M is a locally compact manifold with boundary with a complete C^1 flow defined on it. A compact set $S \subseteq M$ is an *isolated invariant set* if there exists a compact neighborhood N of S such that S is the maximal invariant set of N $S = \{x \in N : x \cdot \mathbf{R} \subseteq N\}$. Such an N is an *isolating neighborhood* for S. The Conley index studies isolated invariant sets; the essential tool for this study being an *index pair for S in M*– a compact pair (N, L) satisfying the following axioms:

(i) $\overline{N \backslash L}$ is an isolating neighborhood for S.

(ii) L is positively invariant in N: if $x \in L$, $x \cdot [0, t] \subseteq N$, then $x \cdot t \in L$.

(iii) L is an exit set for N: if $x \in N$, $x \cdot \mathbf{R}^+ \not\subseteq N$, then there exists a $t > 0$ such that $x \cdot [0, t] \subseteq N$, $x \cdot t \in L$.

In [1, 4, 12, 13] the basic properties of index pairs are established. These are that index pairs exist for all isolated invariant sets in locally compact metric flows, and that all index pairs for S in M have homotopic quotient spaces N/L. As we will be interested in comparing the indices of S in the forward and reverse flows, it will be convenient to work with a special class of index pairs, similar to those developed in [12].

Given S, let N be an isolating neighborhood of S in M, $N_0 = N \cap \partial M$, N_1 the "immediate entrance set" $\{x \in \partial N : \forall \varepsilon > 0,\ x \cdot [\varepsilon, 0] \not\subseteq N\}$, N_2 the "immediate exit set" $\{x \in \partial N : \forall \varepsilon > 0,\ x \cdot [0, \varepsilon] \not\subseteq N\}$. The quadruple (N, N_0, N_1, N_2) can be chosen so that $\partial N = N_0 \cup N_1 \cup N_2$, and so that N, N_0, N_1, N_2 are all topological manifolds with boundary. Let $\mathscr{M}(M, \partial M; S)$ denote the set of such quadruples for S in M. Then (N, N_1) is an index pair for S in M in the reverse flow; (N, N_2) is an index pair for S in the forward flow.

An explicit formula for the homotopy equivalence of index pairs in terms of the flow is given in [13]. Given (N, N_0, N_1, N_2), $(N', N_0', N_1', N_2') \in \mathscr{M}(M, \partial M; S)$, the index pairs of the reverse flow are related by the homotopy equivalence $\phi_-: (N/N_1, N_0/N_1) \to (N'/N_1', N_0'/N_1')$, given by

$$\phi_-[x] = \begin{cases} [x \cdot -3T], & x \cdot [-2T, 0] \subseteq N \backslash N_1, \ x \cdot [-3T, -T] \subseteq N' \backslash N_1', \\ [N_1'], & \text{otherwise}, \end{cases}$$

for T sufficiently large. Likewise for the index pairs of the forward flow, there is a homotopy equivalence $\phi^+: (N/N_2, N_0/N_2) \to (N'/N_2', N_0'/N_2')$ given by

$$\phi^+[x] = \begin{cases} [x \cdot 3T], & x \cdot [0, 2T] \subseteq N \backslash N_2, \ x \cdot [T, 3T] \subseteq N' \backslash N_2', \\ [N_2'], & \text{otherwise}. \end{cases}$$

The quotient spaces N/N_1 for the reverse flow and N/N_2 for the forward flow, together with these maps, form categories $I^-(M; S)$ and $I^+(M; S)$ respectively, which define the Conley index of S in M for the reverse and forward flows (see [13] for details). The categories are also inverse systems, ordered by the inclusion $N' \subseteq N$. Applying one of the functors H^*, H_* to one of the categories I^+, I^- generates an inverse system of (isomorphic) groups, with the *(co)homology Conley index groups* $CH_+^*(M; S)$, $CH_-^*(M; S)$, $CH_*^+(M; S)$, $CH_*^-(M; S)$ the inverse limits of the systems.

As every neighborhood of S contains the total space N of an index quadruple, the subcategories of index pairs derived from index quadruples are cofinal with the total categories. Thus, for the purposes of this paper, the homology Conley index can be viewed as an inverse limit of (isomorphic) homology groups derived from index quadruples. That is, for any homology theory $\mathscr{h}_*$, the cohomology Conley index $\mathscr{Ch}_*^+(M; S)$ for the forward flow are represented by $\mathscr{h}_*(N, N_2)$; the index $\mathscr{Ch}_*^-(M; S)$ for the reverse flow is represented by $\mathscr{h}_*(N, N_1)$.

Example 1.1. Suppose M is an orientable n-manifold with a flow, S a fixed point or periodic orbit, or more generally a compact orientable invariant submanifold of M. Further, suppose S is normally hyperbolic in the flow (i.e. there is a splitting of the tangent bundle $T_S M \cong TS \oplus E^s \oplus E^u$ and constants C, $a > 0$ such that for all $t > 0$, $\|D\mu_t \mathbf{v}\| < Ce^{-at}\|\mathbf{v}\|$ for $\mathbf{v} \in E^s$ and $\|D\mu_t \mathbf{v}\| > C^{-1}e^{at}\|\mathbf{v}\|$ for $\mathbf{v} \in E^u$). If E^s has dimension s and E^u has dimension u, there is a tubular neighborhood E of S in M and fibration $\mathbf{R}^s \times \mathbf{R}^u \to E \to S$ with the following properties:

(i) E and S have flows inherited from M;

(ii) The projection map $E \to S$ is equivariant with respect to these flows;

(iii) The flow on E is linear on fibers, contracting on $\mathbf{R}^s$, expanding on $\mathbf{R}^u$.

An index pair for S in the forward flow is then a pair (N, N_2) with fibration

$$(D^s \times D^u, D^s \times S^{u-1}) \to (N, N_2) \to S.$$

Likewise, an index pair for S in the reverse flow is a pair (N, N_1) with fibration

$$(D^s \times D^u, S^{s-1} \times D^u) \to (N, N_1) \to S.$$

If S is a single critical point, the forward and reverse indices have the homotopy types of spheres: $I^+(M; S) = \Sigma^u := [S^u]$; $I^-(M; S) = \Sigma^s$. That is,

$$CH^k_-(M; S) = \begin{cases} \mathbf{Z}, & k = u, \\ 0, & k \neq u, \end{cases}$$

and

$$CH^+_k(M : S) = \begin{cases} \mathbf{Z}, & k = s = n - u, \\ 0, & k \neq s. \end{cases}$$

It is trivial then that $CH^k_-(M; S) \cong CH^+_{n-k}(M; S)$.

More generally, if S is a compact orientable manifold and one (hence both) of the fibrations above is orientable, the forward and reverse homology indices are just suspensions (of degree u and s respectively) of the homology of S: $CH^k_-(M; S) \cong H^{k-u}(S)$ and $CH^+_{n-k}(M; S) \cong H_{k-s}(S)$. As S is a compact orientable manifold of dimension $n-u-s$, we can apply Poincaré duality to the homology of S and obtain an isomorphism $H^{k-u}(S) \cong H_{n-k-s}(S)$. Composing these yields $CH^k_-(M; S) \cong CH^+_{n-k}(M; S)$. The goal of this paper will be to establish this duality in much greater generality. The following examples will be used in §3 to illustrate some of the limitations of this duality.

Example 1.2. Consider the flow on $\mathbf{R}^3$ given by

$$(x, y, z) \cdot t = (x + at, e^{bt}y, z + ct), \qquad a, b, c > 0,$$

and let $\mathbf{Z} \oplus \mathbf{Z}$ act on $\mathbf{R}^3$ by $(x, y, z) \cdot (n, m) = (x + n, (-1)^n y, z + m)$. The flow and the group action commute, so there is a flow induced on $M = \mathbf{R}^3/\mathbf{Z} \oplus \mathbf{Z}$. The group action is free, so M is a 3-manifold. Its only isolated invariant set is

$$S = (\mathbf{R} \times \{0\} \times \mathbf{R})/\mathbf{Z} \oplus \mathbf{Z},$$

which is a torus, and is a repeller in M. An index pair for S in the forward flow is $((\mathbf{R} \times D^1 \times \mathbf{R})/\mathbf{Z}\oplus\mathbf{Z}, (\mathbf{R} \times \{-1, 1\} \times \mathbf{R})/\mathbf{Z}\oplus\mathbf{Z})$; an index pair for the reverse flow is $((\mathbf{R} \times \{0\} \times \mathbf{R})/\mathbf{Z} \oplus \mathbf{Z}, \varnothing)$. The homology indices are $CH^+_*(M; S) \cong (0, \mathbf{Z}_2, \mathbf{Z}_2, 0, \dots)$, $CH^*_-(M; S) \cong (\mathbf{Z}, \mathbf{Z} \oplus \mathbf{Z}, \mathbf{Z}, 0, \dots)$. □

Example 1.3. Take flows on S^1 and S^2 so that each has a repelling fixed point. Form $X = S^1 \vee S^2$ by wedging the two spheres together at the repellers. The wedge point is then an isolated invariant set in X, with index pair $(D^1 \vee D^2, S^0 \vee S^1)$ in the forward flow and $(D^1 \vee D^2, \varnothing)$ in the reverse flow. Thus $CH^*_-(X; S) = (R, 0, 0, \dots)$, $CH^+_*(X; S) = (0, R \oplus R, R, 0, \dots)$. □

To establish a more general duality statement, manifolds with boundary will be considered. If M is a manifold with boundary, its boundary ∂M is closed and invariant under any flow on M. If S is isolated in M then $S \cap \partial M$ isolated in ∂M [10]. An index quadruple supplies index pairs $(N_0, N_0 \cap N_1)$ for $S \cap \partial M$ in ∂M in the reverse flow, $(N_0, N_0 \cap N_2)$ for $S \cap \partial M$ in ∂M in the forward flow. Similarly, there is an index for S in the pair $(M, \partial M)$, with $CH^+_*(M, \partial M; S)$ for the forward flow are represented by $H_*(N, N_0 \cup N_2)$; $CH^-_*(M, \partial M; S)$ for the reverse flow represented by $H_*(N, N_0 \cup N_1)$. The indices are related by the exact sequence

$$\to CH^\pm_*(\partial M, S \cap \partial M) \to CH^\pm_*(M; S) \to CH^\pm_*(M, \partial M; S) \to$$

Example 1.4. Take the upper half-plane $\mathbf{H}^2$ with flow $\phi(x, y, t) = (e^t x, e^t y)$. The origin is the only isolated invariant set, with index quadruple $N = \{(x, y) \in \mathbf{H}^+ : x^2 + y^2 \leq 1\}$, $N_0 = \{(x, 0): |x| \leq 1\}$, $N_1 = \varnothing$, $N_2 = \{(x, y) \in \mathbf{H}^+ : x^2 + y^2 = 1\}$. From these the homology indices are readily computed and seen to form the exact sequence above:

$$CH_*^+(\partial M; S) \cong (0, \mathbf{Z}, 0, \ldots), \qquad CH_*^-(\partial M; S) \cong (\mathbf{Z}, 0, 0, \ldots),$$
$$CH_*^+(M; S) \cong (0, 0, 0, \ldots), \qquad CH_*^-(M; S) \cong (\mathbf{Z}, 0, 0, \ldots),$$
$$CH_*^+(M, \partial M; S) \cong (0, 0, \mathbf{Z}, 0, \ldots), \qquad CH_*^-(M, \partial M; S) \cong (0, 0, 0, \ldots).$$

□

It is sometimes possible to simplify the computation of index pairs by decomposing the isolated invariant set. The simplest such decomposition is to recognize S as a sum or a product. If S is the disjoint union of two isolated invariant sets S_1 and S_2, then the Conley index of S is the wedge product of the indices of S_1 and S_2: $I(M; S) = I(M; S_1) \vee I(M; S_2)$. The homology index is then additive: $CH_*^\pm(M; S) = CH_*^\pm(M; S_1) \oplus CH_*^\pm(M; S_2)$.

Similarly, if M is a product $M_1 \times M_2$ with the flow preserving the product structure and $S = S_1 \times S_2$, then the Conley index of S is the smash product of the indices of S_1 and S_2: $I(M; S) = I(M_1; S_1) \wedge I(M_2; S_2)$. The homology index is given by the Künneth formula. Of course, in any manifold, if some neighborhood of S can be coordinatized so that the flow has such a product structure, the same product formula applies. This can be clearly seen in Examples 1.1 and 1.2.

One of the key properties of the Conley index is its stability under perturbation of the flow. This stability, referred to as continuation, is described as follows. Let $M \times \Lambda$ be a parameterized family of flows, and let M_λ represent M with the λ-flow. Let $\mathscr{S}(M) = \{(S_\lambda, \lambda): S_\lambda \text{ is isolated in } M_\lambda\}$. For every compact $N \subseteq M$, the set $\Lambda(N)$ of parameter values for which N is isolating is open. Topologize $\mathscr{S}(M)$ by taking as a subbasis sets of the form $\sigma_N(U) = \{(S_\lambda, \lambda): \lambda \in U,\ U \subseteq \Lambda(N) \text{ open},\ S_\lambda \text{ the maximal invariant set of } N \text{ in } M_\lambda\}$. Then (S_μ, μ) and (S_λ, λ) are *related by continuation* if they lie in the same path component of $\mathscr{S}(M)$. That is, if there exists a collection of sets $K_i \subseteq \Lambda$ covering a path from λ to μ, and sets $N_i \subset M$ such that $K_i \subseteq \Lambda(N_i)$ and $S_\lambda = \sigma_{N_1}(K_1)$, $S_\mu = \sigma_{N_n}(K_n)$.

The stability property of the Conley index is that if (S_λ, λ) and (S_μ, μ) are related by continuation, than $I(M_\lambda; S_\lambda)$ and $I(M_\mu; S_\mu)$ are isomorphic. This isomorphism is established by showing that the neighborhoods K_i above can be chosen sufficiently small that for each K and every $\kappa \in K$, the inclusion $(X_\kappa, S_\kappa) \to (X \times K, S(K))$ induces an index isomorphism. The composition isomorphism $F(\lambda, \mu): I(M_\lambda; S_\lambda) \to I(M_\mu; S_\mu)$ derived from these inclusions and their inverses depends only on the homotopy class of the path from λ to μ. For the homology index then, there is an isomorphism

$$F_*(\lambda, \mu): CH_*^+(M_\lambda; S_\lambda) \to CH_*^+(M_\mu; S_\mu).$$

Example 1.5. In [2], the index is used to prove the existence and stability of magnetohydrodynamic shock waves. In the system studied, there are four rest points u_0, u_1, u_2, u_3, and the shock waves correspond to connecting orbits between these points. The points are all hyperbolic, with u_i having index Σ^i. The main points of the argument developed in [2] are:

(i) The set of bounded solutions with positive volume and temperature S is isolated.

(ii) The flow is gradient-like [1], so S consists of the four rest points and connecting orbits between them. Thus if there are no connecting orbits, the homology index is the sum of the indices of the rest points u_i: $CH_*^+(M; S) \cong (\mathbf{Z}, \mathbf{Z}, \mathbf{Z}, \mathbf{Z}, 0, \dots)$.

(iii) As the energy of the system is increased, the flow continues to one with no rest points, so S is related by continuation to the empty set and $CH_*^+(M; S) = 0$. Thus connecting orbits (i.e. shock waves) exist. Further, these solutions are stable, as the continuation argument used is stable. □

The Conley index arose as a generalization of Morse theory. Example 1.1 shows this: a nondegenerate critical point with Morse index p is an isolated invariant set with Conley index Σ^p. The key elements of Morse theory—decomposing a manifold by unstable manifolds of critical points of a gradient flow, and relating the topology of the manifold to the numbers of critical points—are likewise extended in the Conley index theory. These generalizations are Morse decompositions and (generalized) Morse inequalities. Closely related is the algebraic machinery of attractor-repeller sequences and connection matrices.

If $(P, <)$ is a partially ordered indexing set, an *interval* in P is a subset $I \subset P$ such that, if $p < q < r$, p, $r \in I$, then $q \in I$. An *attracting interval* is an interval J such that, if $p \in J$, $q < p$, then $q \in J$. A collection of subinvariant sets (*Morse sets*) $\mathscr{S} = \{S_p\}_{p \in P}$ is a $(P, <)$-*ordered Morse decomposition of* S *in* M if each S_p is isolated in M and every orbit in $S \backslash (\bigcup_{p \in P} S_p)$ runs from S_q to S_p for some $p < q$.

Then for each interval $I \subseteq P$, the set $S(I) = \{x \in S : \omega(x) \subseteq S_p, \omega^*(p) \subseteq S_q$ for some $p, q \in I\}$ is isolated in M. Index pairs for all of the isolated invariant sets $S(I)$ are produced simultaneously by an *index filtration* [6]: a collection $\{N(J) : J$ attracting interval$\}$ such that $(N(J), N(\varnothing))$ is an index pair for $S(J)$ and $N(J_1) \cup N(J_2) = N(J_1 \cup J_2)$, $N(J_1) \cap N(J_2) = N(J_1 \cap J_2)$. Then for any interval I, there exist attracting intervals J, K so that $I = K \backslash J$, and for any such J, K, $(N(K), N(J))$ is an index pair for $S(I)$.

The simplest nontrivial case of a Morse decomposition is an attractor-repeller pair: $P = \{0, 1\}$ with $0 < 1$; $S_0 = A$ an attractor in S; $S_1 = A^*$ its dual repeller. In this case, the index filtration is a nested triple of compact spaces $N_0 \subseteq N_1 \subseteq N_2$. These may be chosen so that N_0 and N_1 are neighborhood retracts in N_2. Such a triple is called a *regular index triple*, and the exact sequence of the triple

$$\xrightarrow{\partial} H_k(N_1, N_0) \to H_k(N_2, N_0) \to H_k(N_2, N_1) \xrightarrow{\partial}$$

then defines the homology attractor-repeller sequence

$$\xrightarrow{\partial} CH_k(M; A) \to CH_k(M; S) \to CH_k(M; A^*) \xrightarrow{\partial}.$$

For a general Morse decomposition $\mathscr{S} = \{S_p\}_{p \in P}$, a pair of intervals (I, J) is said to be *adjacent* if $IJ := I \cup J$ is an interval and no $i \in I$, $j \in J$ has

$j < i$. Every adjacent pair of intervals generates an index triple for the attractor-repeller decomposition $(S(I), S(J))$ of $S(IJ)$, and so generates a homology attractor-repeller sequence. To understand how all of these induced homology sequences are related, we reformulate the sequence as follows:

In field coefficients, the sequence

$$\xrightarrow{\partial} CH_k(M; S(I)) \to CH_k(M; S(IJ)) \to CH_k(M; S(J)) \xrightarrow{\partial}$$

defines a matrix $\Delta = \left[\begin{smallmatrix} 0 & \partial \\ 0 & 0 \end{smallmatrix}\right]$ on the vector space $CH_*(M; S(I)) \oplus CH_*(M; S(J))$. As $\Delta^2 = 0$, $\{CH_*(M; S(I)) \oplus CH_*(M; S(J)), \Delta\}$ may be viewed as a chain complex. Further, if $CH_*(M; S(I))$ and $CH_*(M; S(J))$ are viewed as chain complexes with trivial boundary operators, then the sequence

$$0 \to CH_*(M; S(I)) \to CH_*(M; S(I)) \oplus CH_*(M; S(J)) \to CH_*(M; S(J)) \to 0$$

is a short exact sequence of chain complexes. Let $H_*\Delta(IJ)$ denote the homology of $\{CH_*(M; S(I)) \oplus CH_*(M; S(J)), \Delta\}$. There is an isomorphism of homology sequences

$$\begin{array}{ccccc} \xrightarrow{\partial} CH_k(M; S(I)) \to & & H_k\Delta(IJ) & & \to CH_k(M; S(J)) \xrightarrow{\partial} \\ \Big\downarrow \mathrm{id} & & \Big\downarrow \cong & & \Big\downarrow \mathrm{id} \\ \xrightarrow{\partial} CH_k(M; S(I)) \to & & CH_k(M; S(IJ)) & & \to CH_k(M; S(J)) \xrightarrow{\partial} \end{array}$$

so the matrix Δ computes the homology attractor-repeller sequence. Such a matrix is defined for every adjacent pair of intervals. In [7], Franzosa shows that all of these matrices and their induced homology sequences may be derived from a single matrix, known as a *connection matrix.*

To do so, fix a coefficient field for all homology groups. For every interval J, let $C_*\Delta(J) = \bigoplus_{p \in J} CH_*(M; S_p)$. If $\Delta\colon C_*\Delta(P) \to C_*\Delta(P)$ is a matrix, $\Delta_{p,q}$ is a linear map from $CH_*(M; S_q)$ to $CH_*(M; S_p)$. Δ is said to be *strictly upper triangular* if $\Delta_{p,q} = 0$ for all $p \not< q$; and is said to be a *boundary map* if each $\Delta_{p,q}$ is degree -1 and $\Delta^2 = 0$. If $\Delta\colon C_*\Delta(P) \to C_*\Delta(P)$ is a strictly upper triangular boundary map, I, J intervals in P, define $\Delta(J, I)\colon C_*\Delta(J) \to C_*\Delta(I)$ by $\Delta(J, I) = [\Delta_{p,q}]_{p \in I, q \in J}$, and denote $\Delta(J, J)$ by $\Delta(J)$. It is easily computed that $\Delta(J)$ is a strictly upper triangular boundary map on $C_*\Delta(J)$.

That is, Δ defines a chain complex $\{C_*\Delta(J), \Delta(J)\}$ for every interval, whose homology is denoted by $H_*\Delta(J)$. Note that for $p \in P$, $\{C_*\Delta(p), \Delta(p)\} = \{CH_*(M; S_p), 0\}$ so $H_*\Delta(p) = CH_*(M; S_p)$. Further, if (I, J) is an adjacent pair of intervals, then the inclusions and projections of factors give a short exact sequence of chain complexes

$$0 \to C_*\Delta(I) \to C_*\Delta(IJ) \to C_*\Delta(J) \to 0$$

and hence a long exact homology sequence

$$\cdots \xrightarrow{\partial(J,I)} H_*\Delta(I) \to H_*\Delta(IJ) \to H_*\Delta(J) \xrightarrow{\partial(J,I)} \cdots$$

where $\partial(J, I)$ is the induced map on homology of $\Delta(J, I)$.

A *connection matrix* on $C_*\Delta(P)$ is a strictly upper triangular boundary map $\Delta\colon C_*\Delta(P) \to C_*\Delta(P)$ and a collection of maps $\{\phi(J)\colon J \text{ an interval}\}$ with $\phi(J)\colon H_*\Delta(J) \to CH_*(M; S(J))$ such that $\phi(p) = \mathrm{id}$ for all $p \in P$, and such that, for every adjacent pair of intervals (I, J), the following diagram commutes:

$$\begin{array}{ccccccc} \xrightarrow{\partial(J,I)} & H_*\Delta(I) & \rightarrow & H_*\Delta(IJ) & \rightarrow & H_*\Delta(J) & \xrightarrow{\partial(J,I)} \\ & \downarrow \phi(I) & & \downarrow \phi(IJ) & & \downarrow \phi(J) & \\ \xrightarrow{\partial} & CH_k(M; S(I)) & \rightarrow & CH_k(M; S(IJ)) & \rightarrow & CH_k(M; S(J)) & \xrightarrow{\partial} \end{array}$$

A simple induction argument shows that all of the $\phi(J)$'s are in fact isomorphisms. Thus a connection matrix computes (up to isomorphism) all homology attractor-repeller sequences. Connection matrices exist for all Morse decompositions [7], but are not unique. We will denote the set of connection matrices defined for $\mathscr{S}$ with ordering $(P, <)$ by $CM_*^+(\mathscr{S}, <)$.

This construction is based on the homology index CH_*^+. Similar constructions can be made using CH_*^-, CH_-^*, CH_+^* (connection matrices based on cohomology will be degree $+1$ coboundary maps). The set of connection matrices defined on CH_*^- will be denoted $CM_*^-(\mathscr{S}, <)$; the set defined on CH_-^* is denoted $CM_-^*(\mathscr{S}, <)$, etc.

The Morse inequalities, as formulated by Conley [4], follow naturally from the existence of connection matrices. Namely, given a finite chain complex $\{C, d\}$ with field coefficients, $\text{rk}(C_k) = \text{rk}(Z_k) + \text{rk}(B_{k-1})$ and $\text{rk}(Z_k) = \text{rk}(H_k(C)) + \text{rk}(B_k)$. Thus $\text{rk}(C_k) = \text{rk}(H_k(C)) + \text{rk}(B_{k-1}) + \text{rk}(B_k)$. For each of the graded groups C_*, $H_*(C)$, B_*, there are Poincaré polynomials $P(C_*, t) = \text{rk}(C_k)t^k$, $P(H_*(C), t) = \text{rk}(H_k(C))t^k$, $P(B_*, t) = \text{rk}(B_k)t^k$, which then satisfy the equation $P(C_*, t) = P(H_*(C)) + (1+t)P(B_*)$. Applying this now to the chain complex $\{C_*\Delta(P), \Delta(P)\}$ generated by a connection matrix yields the Morse inequalities:

$$\sum_{p\in P} P(S_p, t) = P(S, t) + (1+t)P(\text{im}\,\Delta_*(P), t)$$

where $P(S, t) = P(CH_*(M; S), t)$ and $P(S_p, t) = P(CH_*(M : S_p), t)$.

Consider for instance the system in Example 1.5. As the system is gradient-like, the four rest points form a Morse decomposition for S, with ordering $0 < 1 < 2 < 3$. Just as S continues to the empty set for high enough energy, so too $S(01)$ and $S(23)$ continue to the empty set. The attractor-repeller sequences of u_0 and u_1 in $S(01)$, and of u_2 and u_3 in $S(23)$, can then be computed. In both cases, $CH_*(M; S(I))$ is zero, so the connection map $\partial: CH_p(M; u_i) \to CH_{p-1}(M; u_{i-1})$ is an isomorphism. Thus there exist connecting orbits from u_1 to u_0 and from u_3 to u_2. These maps also appear as entries in the connection matrix of the system, and are in fact enough to compute the entire matrix. Taking only the nonzero dimensions of each $CH_*(M; u_i)$, any connection matrix is a 4×4 matrix with only the first upper diagonal nonzero:

$$\Delta = \begin{bmatrix} 0 & \partial_{01} & 0 & 0 \\ 0 & 0 & \partial_{12} & 0 \\ 0 & 0 & 0 & \partial_{23} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The attractor-repeller sequences of $S(01)$ and $S(12)$ show that ∂_{01} and ∂_{23} are nonzero, while the requirement $\Delta^2 = 0$ forces $\partial_{12} = 0$. Thus the shock waves corresponding to the u_1-to-u_0 and u_3-to-u_2 connections exist for all parameter values, while connections from u_2 and u_1 are not guaranteed algebraically, and do not exist for all parameter values (cf. [8]).

In the Morse inequality description, $\sum_{p\in P} P(S_p, t) = 1 + t + t^2 + t^3$, while $P(S, t) = 0$, so $(1+t)P(\operatorname{im}\Delta_*(P), t) = 1 + t + t^2 + t^3$ and $P(\operatorname{im}\Delta_*(P), t) = 1 + t^2$.

2. Poincaré-Lefschetz duality

In Example 1.1, both the forward and reverse homology indices are suspensions of the homology of S, so the duality of the forward and reverse indices is trivial. We show now that duality holds quite independent of the topology of S and the orientability of the fibrations

$$(D^s \times D^u, D^s \times S^{u-1}) \to (N, N_2) \to S,$$
$$(D^s \times D^u, S^{s-1} \times D^u) \to (N, N_1) \to S.$$

However, as Examples 1.2 and 1.3 indicate, orientability of the ambient manifold and the possible intersection of the invariant set with the boundary of the manifold must be taken into account.

Theorem 2.1. *If M is an orientable n-manifold with boundary, $S \subseteq M$ an isolated invariant set for a C^1 flow on M, then there exist duality isomorphisms*

$$D\colon CH^*_-(M, \partial M; S) \to CH^+_*(M; S),$$
$$D\colon CH^*_-(M; S) \to CH^+_*(M, \partial M; S).$$

Further, if $S \cap \partial M = \varnothing$, both isomorphisms reduce to an isomorphism

$$D\colon CH^*_-(M; S) \to CH^+_*(M; S).$$

Proof. The proof consists of two steps, which are carried out in the lemmas below:

(i) For each quadruple $(N, N_0, N_1, N_2) \in \mathscr{M}(M, \partial M; S)$, there exist isomorphisms $\cap z_N\colon H^k(N, N_0 \cup N_1) \to H_{n-k}(N, N_2)$ and $\cap z_N\colon H^k(N, N_1) \to H_{n-k}(N, N_0 \cup N_2)$, where $z_N \in H_n(N, \partial N)$ is the fundamental class of $(N, \partial N)$.

(ii) The cap product isomorphisms commute with the bonding maps of the inverse system of index quadruples. Given (N, N_0, N_1, N_2), $(N', N'_0, N'_1, N'_2) \in \mathscr{M}(M, \partial M; S)$ with $N' \subseteq N$, the homotopy equivalences ϕ_- and ϕ^+ induce a commutative diagram

$$\text{(2.2)}\qquad \begin{array}{ccccccc}
H^k(N, N_0 \cup N_1) & \underset{\cong}{\longleftarrow} & \widetilde{H}^k(N/N_1, N_0/N_1) & \overset{\phi_-^*}{\longleftarrow} & \widetilde{H}^k(N'/N'_1, N'_0/N'_1) & \underset{\cong}{\longrightarrow} & H^k(N', N'_0 \cup N'_1) \\
\big\downarrow \cap z_N & & & & & & \big\downarrow \cap z_{N'} \\
H_{n-k}(N, N_2) & \underset{\cong}{\longrightarrow} & \widetilde{H}_{n-k}(N/N_2) & \overset{\phi_*^+}{\longrightarrow} & \widetilde{H}_{n-k}(N'/N'_2) & \underset{\cong}{\longleftarrow} & H_{n-k}(N', N'_2)
\end{array}$$

The isomorphism $D\colon CH^*_-(M, \partial M; S) \to CH^+_*(M; S)$ is then the inverse limit of the cap products $\cap z_N$.

Lemma 2.3. *If M is an orientable n-manifold, $S \subseteq M$ an isolated invariant* *set, and* $(N, N_0, N_1, N_2) \in \mathscr{M}(M, \partial M; S)$, *then the maps*

$$\cap z_N \colon H^k(N, N_0 \cup N_1) \to H_{n-k}(N, N_2),$$
$$\cap z_N \colon H^k(N, N_1) \to H_{n-k}(N, N_0 \cup N_2)$$

are isomorphisms.

Proof. As the two isomorphisms are analogous, we prove only the first. Let $\widetilde{N}$ be formed by identifying two copies of N (denoted N^+ and N^-) along N_2. For any subset U of N, form $\widetilde{U} \subseteq \widetilde{N}$ similarly. Note that $\widetilde{N}_0 \cup \widetilde{N}_1 = \partial\widetilde{N}$, and that $N_2 \cap (\widetilde{N}_0 \cup \widetilde{N}_1) = \partial N_2$. N is an orientable manifold with fundamental class z_N, so N^+ and N^- are orientable with fundamental classes z_+, z_-. Thus $\widetilde{N}$ is an orientable manifold, with fundamental class $\tilde{z}$ the image of (z_+, z_-) under the composition

$$H_n(N^+, \partial N^+) \oplus H_n(N^-, \partial N^-) \to H_n(\widetilde{N}, N_2 \cup \partial\widetilde{N}) \leftarrow H_n(\widetilde{N}, \partial\widetilde{N}).$$

Further, as N_2 is a neat $n-1$ submanifold with boundary of $\widetilde{N}$, with fundamental class $z_2 \in H_{n-1}(N_2, \partial N_2)$ is the image of $\tilde{z}$, z_+ and z_- under the appropriate boundary maps.

N^+, N^- and N_2 are all excisive in $\widetilde{N}$, so there are exact sequences

$$\to H^k(\widetilde{N}, \partial\widetilde{N}) \to H^k(N^+, N^+ \cap \partial\widetilde{N}) \oplus H^k(N^-, N^- \cap \partial\widetilde{N}) \to H^k(N_2, \partial N_2) \to$$

and

$$\to H_{n-k}(N_2) \to H_{n-k}(\widetilde{N}) \to H_{n-k}(N^+, N_2) \oplus H_{n-k}(N^-, N_2) \to .$$

The cap products link the two sequences, forming the diagram:

(2.4)

$$\begin{array}{ccccc}
\to H^k(\widetilde{N}, \partial\widetilde{N}) & \longrightarrow & H^k(N^+, N^+ \cap \partial\widetilde{N}) \oplus H^k(N^-, N^- \cap \partial\widetilde{N}) & \longrightarrow & H^k(N_2, \partial N_2) \to \\
\big\downarrow{\scriptstyle \cap\tilde{z}} & & \big\downarrow{\scriptstyle (\cup z_+, \cap z_+)} & & \big\downarrow{\scriptstyle \cap z_2} \\
\to H_{n-k}(\widetilde{N}) & \longrightarrow & H_{n-k}(N^*, N_2) \oplus H_{n-k}(N^-, N_2) & \longrightarrow & H_{n-k-1}(N_2) \to
\end{array}$$

The cap product $\cap\tilde{z}$ commutes with $(\cap z_+, \cap z_+)$, and $(\cap z_+, \cap z_+)$ commutes with $\cap z_2$, from the naturality of cap products and their commutativity with Mayer-Vietoris boundaries [14, 5.6.16, 5.6.20]. To show that $\cap\tilde{z}$ and $\cap z_2$ commute requires slightly more work. There are inclusion-induced diagrams

$$\begin{array}{ccccc}
H^k(N_2, \partial N_2) & \overset{\cong}{\longleftarrow} & H^k(\partial N^+, N^+ \cap \partial\widetilde{N}) & \longrightarrow & H^k(\partial N^+) \\
\big\downarrow{\scriptstyle \cap z_2} & & \big\downarrow{\scriptstyle \cap i_* z_2} & & \big\downarrow{\scriptstyle \cap \partial_* z_+} \\
H_{n-k-1}(N_2) & \longrightarrow & H_{n-k-1}(\partial N^+) & \overset{\text{id}}{\longleftarrow} & H_{n-k-1}(\partial N^+)
\end{array} \tag{2.5}$$

and

$$\begin{array}{ccccc}
H^{k+1}(N^+, \partial N^+) & \longleftarrow & H^{k+1}(\widetilde{N}, N^- \cup \partial\widetilde{N}) & \longrightarrow & H^{k+1}(\widetilde{N}, \partial\widetilde{N}) \\
\big\downarrow{\scriptstyle \cap z_+} & & \big\downarrow{\scriptstyle \cap \tilde{z}_+} & & \big\downarrow{\scriptstyle \cap \tilde{z}} \\
H_{n-k-1}(N^+) & \longrightarrow & H_{n-k-1}(\widetilde{N}) & \overset{\text{id}}{\longleftarrow} & H_{n-k-1}(\widetilde{N})
\end{array} \tag{2.6}$$

which are linked by the commutative diagram

$$\text{(2.7)}\qquad \begin{array}{ccc} H^k(\partial N^+) & \xrightarrow{\delta^*} & H^{k+1}(N^+, \partial N^+) \\ \big\downarrow{\scriptstyle \cap \partial_* z} & & \big\downarrow{\scriptstyle \cap z_+} \\ H_{n-k-1}(\partial N^+) & \longrightarrow & H_{n-k-1}(N^+) \end{array}$$

The composition of 2.5, 2.6 and 2.7 produces the required commutative square.

By Poincaré-Lefschetz duality, the cap products $\cap z_2$ and $\cap \tilde{z}$ are isomorphisms, so $\cap z_+$ and $\cap z_-$ are isomorphisms. But each of these is just the cap product $\cap z_N\colon H^k(N, N_0 \cup N_1) \to H_{n-k}(N, N_2)$. $\square$

This shows that each index quadruple induces a duality isomorphism. To show that all such isomorphisms are compatible, we first consider a special case of nested index pairs.

Lemma 2.8. *Suppose* (N, N_0, N_1, N_2), $(N', N_0', N_1', N_2') \in \mathscr{M}(M, \partial M; S)$ *with* $N' \subseteq \operatorname{int}_M N$ *such that, if* $x \cdot [t_0, t_1] \subseteq N$, *and* $x \cdot t_0$, $x \cdot t_1 \in N'$, *then* $x \cdot [t_0, t_1] \subseteq N'$. *Then diagram 2.2 commutes*: *for* $c' \in H^k(N', N_0' \cup N_1')$,

$$(\pi_*')^{-1} \circ \phi_* \circ \pi_*(\pi^* \circ \phi^* \circ (\pi'^*)^{-1}(c') \cap z_N) = c' \cap z_{N'}.$$

Proof. Let $P^- = \{x \in N\colon \exists t < 0 \text{ with } x \cdot [t, 0] \subseteq N,\ x \cdot t \in N'\}$, $P^+ = \{x \in N\colon \exists t > 0 \text{ with } x \cdot [0, t] \subseteq N, x \cdot t \in N'\}$. Then $P^+ \cap P^- = N'$, and $N = P^+ \cup P^- \cup \{x \in N\colon x \cdot [\tau_+(x), \tau_-(x)] \cap N' = \varnothing\}$, where $x \cdot [\tau_+(x), \tau_-(x)]$ is the maximal orbit segment of x contained in N. There are continuous maps $\theta^-\colon P^- \to \mathbf{R}$, $\theta^+\colon P^+ \to \mathbf{R}$ defined by

$$\theta^-(x) = \max\{t \le 0\colon x \cdot [t, 0] \subseteq N, x \cdot t \in N'\},$$
$$\theta^+(x) = \min\{t \ge 0\colon x \cdot [0, t] \subseteq N, x \cdot t \in N'\}.$$

There is then a map $\phi\colon P^+ \cup P^- \to N'$ defined by

$$\phi(x) = \begin{cases} x \cdot \theta^+(x), & x \in P^+, \\ x \cdot \theta^-(x), & x \in P^-. \end{cases}$$

On $P^+ \cap P^- = N'$, $\theta^+ = \theta^- = 0$, so ϕ is well defined, and is the identity map when restricted to N'. It maps $P^- \backslash N'$ to N_2', $P^+ \backslash N'$ to N_1', $(P^+ \cup P^-) \cap N_0$ to N_0', and $\partial_N(P^+ \cup P^-)$ to $N_1' \cap N_2'$. There is then a continuous map $\tilde{\phi}\colon N/((N_0 \cup N_1) \cap N_2) \to N'/((N_0' \cup N_1') \cap N_2')$ defined by

$$\tilde{\phi}(x) = \begin{cases} [\phi(x)], & x \in P^+ \cup P^-, \\ [N_1' \cap N_2'], & \text{otherwise}. \end{cases}$$

Let $\pi\colon N \to N/((N_0 \cup N_1) \cap N_2)$ and $\pi'\colon N' \to N'/((N_0' \cup N_1') \cap N_2')$ be the natural projections. Then the following diagram commutes:

$$\begin{array}{ccccc} H_n(N, \partial N) & \longrightarrow & H_n(N, N \backslash \operatorname{int} N') & \longleftarrow & H_n(N', \partial N') \\ \cong \big\downarrow{\scriptstyle \pi_*} & & & & \cong \big\downarrow{\scriptstyle \pi_*'} \\ \begin{array}{c} H_n(N/(N_0 \cup N_1) \cap N_2, \\ \partial N/(N_0 \cup N_1) \cap N_2) \end{array} & & \xrightarrow{\quad \tilde{\phi}_* \quad} & & \begin{array}{c} H_n(N'/(N_0' \cup N_1') \cap N_2', \\ \partial N'/(N_0' \cup N_1') \cap N_2'). \end{array} \end{array}$$

Thus $(\pi'_*)^{-1}\circ\tilde{\phi}_*\circ\pi_*(z)=z'$, and the maps π, $\tilde{\phi}$ and π' induce a commutive diagram of cap products

$$\begin{array}{ccc} H^k(N, N_0\cup N_1) & \xleftarrow{\pi^*\circ\tilde{\phi}^*\circ(\pi'^*)^{-1}} & H^k(N', N_0'\cup N_1') \\ \Big\downarrow \cap z & & \Big\downarrow \cap z' \\ H_{n-k}(N, N_2) & \xrightarrow{(\pi'_*)^{-1}\circ\tilde{\phi}_*\circ\pi_*} & H_{n-k}(N', N_2') \end{array}$$

To complete the proof, it suffices to show that $\phi_*^+=(\pi'_*)^{-1}\circ\tilde{\phi}_*\circ\pi_*$ and that $\phi_-^*=\pi^*\circ\tilde{\phi}^*\circ(\pi'^*)^{-1}$. To do so, we show that the compositions

$$(N/((N_0\cup N_1)\cap N_2), N_2/((N_0\cup N_1)\cap N_2)) \xrightarrow{\pi} (N/N_2, *) \xrightarrow{\phi} (N'/N_2', *)$$

and

$$(N/(N_0\cup N_1)\cap N_2, N_2/(N_0\cup N_1)\cap N_2)$$
$$\xrightarrow{\tilde{\phi}} (N'/(N_0'\cup N_1')\cap N_2',\ N_2'/(N_0'\cup N_1')\cap N_2') \xrightarrow{\pi} (N'/N_2', *)$$

are homotopic. Define $\Phi\colon (N/(N_0\cup N_1)\cap N_2)\times[0,1]\to N'/N_2'$ by

$$\Phi([x], s) = \begin{cases} [x\cdot(3sT+(1-s)\theta^+(x))], & x\in P^+, \\ & x\cdot[0, 3sT+(1-s)\theta^+(x)]\subseteq N, \\ & x\cdot((1-s)\theta^+(x))\cdot[sT, 3sT]\subseteq N', \\ [N_2'], & \text{otherwise.} \end{cases}$$

As every x with $x\cdot[t_0, t_1]\subseteq N$ and $x\cdot t_0$, $x\cdot t_1\in N'$ has $x\cdot[t_0, t_1]\subseteq N'$, the set $\{x\cdot[0, 3sT+(1-s)\theta^+(x)]\subseteq N,\ x\cdot((1-s)\theta^+(x))\cdot[sT, 3sT]\subseteq N'\}$ has its boundary contained in $\{x\colon x\cdot(3sT+(1-s)\theta^+(x))\in N_2'\}$, and so is mapped to $[N_2']$. Thus Φ is continuous, with $\Phi_0=\pi\circ\tilde{\phi}$, $\Phi_1=\phi^+\circ\pi$. There is a similar homotopy between $\pi\circ\tilde{\phi}$ and $\phi^-\circ\pi$. □

Lemma 2.9. *If* (N, N_0, N_1, N_2), $(N', N_0', N_1', N_2')\in\mathcal{M}(M, \partial M; S)$ *with* $N'\subseteq N$, *then the diagram* (2.2) *commutes.*

Proof. If $(N, N_0, N_1, N_2)\in\mathcal{M}(M, \partial M; S)$, $T>0$, let $N^T=\{x\in N\colon x\cdot[-T, T]\subseteq N\}$, and let $N_0^T=N_0\cap N^T$, $N_1^T=(N_1\cdot T)\cap N^T$, $N_2^T=(N_2\cdot(-T))\cap N^T$. Then $(N^T, N_0^T, N_1^T, N_2^T)\in\mathcal{M}(M, \partial M; S)$. Further, if $(N', N_0', N_1', N_2')\in\mathcal{M}(M, \partial M; S)$, there is a T such that $N^T\subseteq \operatorname{int}(N')$. Then $N'^T\subseteq N^T$, and each of the containments $N'^T\subseteq N'$, $N^T\subseteq N$, $N'^T\subseteq N^T$ satisfies the hypothesis of Lemma 2.8. Thus all of the maps ϕ^+, ϕ_- are defined, and the corresponding diagram commutes for each of these containments. Further, the maps ϕ^+ and ϕ_- form commutative diagrams

$$\begin{array}{ccc} H^k(N, N_0\cup N_1) & \longrightarrow & H^k(N^T, (N_0\cup N_1)^T) \\ \Big\downarrow & & \Big\downarrow \\ H^k(N', N_0'\cup N_1') & \longrightarrow & H^k(N'^T, (N_0'\cup N_1')^T) \end{array}$$

and

$$\begin{array}{ccc} H_{n-k}(N'^T, N'^T_2) & \longrightarrow & H_{n-k}(N', N'_2) \\ \downarrow & & \downarrow \\ H_{n-k}(N^T, N^T_2) & \longrightarrow & H_{n-k}(N, N_2) \end{array}$$

The composition of these then gives the required diagram. □

If $S \cap \partial M = \varnothing$, then $CH^p_-(\partial M; S \cap \partial M)$, $CH^+_p(\partial M; S \cap \partial M) = 0$ and the indices of S in M and in $(M, \partial M)$ are isomorphic, via the exact sequences

$$\to CH^p_-(M, \partial M; S) \to CH^p_-(M : S) \to CH^p_-(\partial M; S \cap \partial M) \to,$$
$$\to CH^+_p(\partial M; S \cap \partial M) \to CH^+_p(M; S) \to CH^+_p(M, \partial M; S) \to .$$

There is then an isomorphism D: $CH^*_-(M; S) \to CH^+_*(M; S)$. Further, since $\partial(\partial M) = \varnothing$, $CH^*_-(\partial M; S \cap \partial M)$ and $CH^+_*(\partial M; S \cap \partial M)$ are dual. The exact sequences connect these various duality isomorphisms.

Theorem 2.10. *If S is isolated in M, the following diagram commutes:*

$$\begin{array}{ccccccc} \to CH^p_-(M, \partial M; S) \to & & CH^p_-(M; S) & \to & CH^p_p(\partial M; S \cap \partial M) & \to \\ \downarrow D & & \downarrow D & & \downarrow D & \\ \to \quad CH^+_{n-p}(M; S) & & \to CH_{n-p}(M, \partial M; S) & \to & CH^+_{n-p-1}(\partial M; S \cap \partial M) & \to \end{array}$$

Proof. The sequences are represented by the exact sequences of the triples $(N, N_0 \cup N_1, N_1)$ and $(N, N_0 \cup N_2, N_2)$. If $z_0 \in H_{n-1}(N_0, \partial N_0)$ is the fundamental class of N_0, z_0 is the image of z under the Mayer-Vietoris boundary. Then the naturality of the cap product yields commutative diagram

$$\begin{array}{ccccccc} \to H^k(N, N_0 \cup N_1) \to & & H^k(N, N_1) & \to & H^k(N_0, N_0 \cap N_1) & \to \\ \downarrow \cap z & & \downarrow \cap z & & \downarrow \cap z_0 & \\ \to \quad H_{n-k}(N, N_2) & & \to H_{n-k}(N, N_0 \cup N_2) & \to & H_{n-k-1}(N_0, N_0 \cap N_2) & \to \end{array}$$

The index diagram is then the inverse limit of such diagrams. □

3. Naturality

Duality does more than just show that the homology indices are isomorphic as (ungraded) groups—it shows that for many of the constructions used in the Conley index theory, they carry the same information. To establish this, we show that the duality isomorphism is natural with respect to the basic properties of the Conley index: sums and products; continuation; attractor-repeller sequences and connection matrices. Examples show that other properties, such as ring structures and naturality with respect to semiconjugacies, are not preserved by duality.

Theorem 3.1. *The duality isomorphism is natural with respect to sums and products of isolated invariant sets. That is, suppose M and M' are manifolds with flows, $S \subseteq M$ and $S' \subseteq M'$ isolated invariant sets. Then*

(i) *If* $S_1, S_2 \subseteq M$ *are isolated so that* $S_1 \cup S_2 = S$, $S_1 \cap S_2 = \varnothing$, *then the following diagram commutes*:

$$\begin{array}{ccc} CH^k_-(M, \partial M; S_1) \oplus CH^k_-(M, \partial M; S_2) & \longleftarrow & CH^k(M, \partial M; S) \\ \downarrow{\scriptstyle D_1 \oplus D_2} & & \downarrow{\scriptstyle D} \\ CH^+_{n-k}(M; S_1) \oplus CH^+_{n-k}(M; S_2) & \longrightarrow & CH^+_{n-k}(M; S) \end{array}$$

(ii) *The following diagram commutes*:

$$\begin{array}{ccc} CH^k_-(M, \partial M; S) \oplus CH^{k'}_-(M', \partial M'; S') & \rightarrow & CH^{k+k'}_-(M \times M', \partial(M \times M'); S) \\ \downarrow{\scriptstyle D_1 \oplus D_2} & & \downarrow{\scriptstyle D} \\ CH^+_{n-k}(M; S) \oplus CH^+_{n'-k'}(M'; S') & \rightarrow & CH^+_{n+n'-k-k'}(M \times M', S \times S') \end{array}$$

Proof. (i) Choose $(N, N_0, N_1, N_2) \in M(M, \partial M; S_1)$, $(N', N'_0, N'_1, N'_2) \in \mathscr{M}(M, \partial M; S_2)$ such that $N \cap N' = \varnothing$. Then $(N \cup N', N_0 \cup N'_0, N_1 \cup N'_1, N_2 \cup N'_2) \in \mathscr{M}(M, \partial M; S)$, and there is commutative diagram

$$\begin{array}{ccc} H^k(N, N_0 \cup N_1) \oplus H^k(N', N'_0 \cup N'_1) & \longleftarrow & H^k(N \cup N', N_0 \cup N'_0 \cup N_1 \cup N'_1) \\ \downarrow{\scriptstyle (\cap z_1) \oplus (\cap z_2)} & & \downarrow{\scriptstyle \cap z} \\ H_{n-k}(N, N_2) \oplus H_{n-k}(N', N'_2) & \longrightarrow & H_{n-k}(N \cup N', N_2 \cup N'_2) \end{array}$$

(ii) Similarly, choose $(N, N_0, N_1, N_2) \in \mathscr{M}(M, \partial M; S)$, $(N', N'_0, N'_1, N'_2) \in \mathscr{M}(M', \partial M'; S')$. Then $(N \times N', (N \times N'_0) \cup (N_0 \times N'), (N \times N'_1) \cup (N_1 \times N'), (N \times N'_2) \cup (N_2 \times N')) \in \mathscr{M}(M \times M', \partial(M \times M'); S \times S')$, and there is commutative diagram

$$\begin{array}{ccc} H^k(N, N_0 \cup N_1) \oplus H^{k'}(N', N'_0 \cup N'_1) & \xrightarrow{\times} & H^{k+k'}(N \times N', (N \times (N'_0 \cup N'_1)) \cup ((N_0 \cup N_1) \times N')) \\ \downarrow{\scriptstyle D_1 \oplus D_2} & & \downarrow{\scriptstyle D} \\ H_{n-k}(N, N_2) \oplus H_{n'-k'}(N', N'_2) & \xrightarrow{\times} & H_{n+n'-k-k'}(N \times N', (N \times N'_0) \cup (N_0 \times N')) \end{array} \quad \square$$

Theorem 3.2. *The duality isomorphism is preserved under continuation. That is, if* $M \times \Lambda$ *is a parameterized family of flows with* (S_μ, μ) *and* (S_λ, λ) *related by continuation, then there exists commutative diagram of isomorphisms*:

$$\begin{array}{ccc} CH^p_-(M_\mu, \partial M_\mu; S_\mu) & \xrightarrow[F^*(\lambda, \mu)]{} & CH^p_-(M_\lambda, \partial M_\lambda; S_\lambda) \\ \downarrow{\scriptstyle D} & & \downarrow{\scriptstyle D} \\ CH^+_{n-p}(M_\mu; S_\mu) & \xrightarrow[F_*(\mu, \lambda)]{} & CH^+_{n-p}(M_\lambda; S_\lambda) \end{array}$$

Proof. Without loss, we may assume $\Lambda = [0, 1]$ and that for every $t \in [0, 1]$ $(X_t, S_t) \to (X \times \Lambda, S(\Lambda))$ is an index isomorphism. For any $U \subseteq X \times \Lambda$, $t \in \Lambda$, let $U_t = U \cap (X \times \{t\})$. Choose an index quadruple $(N(\Lambda), N_0(\Lambda), N_1(\Lambda), N_2(\Lambda))$ for $S(\Lambda)$. Then $(N_t, N_{0t} \cup N_{1t})$ is an index pair for $(X_t, \partial X_t; S_t)$ in the reverse flow; (N_t, N_{2t}) is an index pair for $(X_t; S_t)$ in the forward flow. The inclusions $(N_0, N_{00} \cup N_{10}, N_{20}) \to (N, N_0 \cup N_1, N_2)$ and $(N_1, N_{01} \cup$

$N_{11}, N_{21}) \to (N, N_0 \cup N_1, N_2)$ induce diagram

$$\begin{array}{ccccc} H^k(N_0, N_{00} \cup N_{10}) & \xleftarrow{\cong} & H^k(N, N_0 \cup N_1) & \xrightarrow{\cong} & H^k(N_1, N_{01} \cup N_{11}) \\ \downarrow \cap z_0 n & & \downarrow \cap z_\Lambda & & \downarrow \cap z_1 \\ H_{n-k}(N_0, N_{20}) & \xrightarrow{\cong} & H_{n-k}(N, N_2) & \xleftarrow{\cong} & H_{n-k}(N_1, N_{21}) \end{array}$$

where the inclusion $H_n(X, \partial X) \xrightarrow{\cong} H_n(X \times \Lambda, \partial X \times \Lambda)$ maps z_N to z_Λ. But the horizontal compositions define the isomorphisms $F_*(0, 1)$ and $F^*(1, 0)$. □

Theorem 3.3. *The duality isomorphism commutes with attractor-repeller sequences. If S is an isolated invariant set with attractor-repeller pair (A, A^*), then the following diagrams commute*:

$$\begin{array}{ccccccc} \to CH^p_-(M, \partial M; A) & \to & CH^p_-(M, \partial M; S) & \to & CH^p_-(M, \partial M; A^*) & \to \\ \downarrow D & & \downarrow D & & \downarrow D & \\ \to CH^+_{n-p}(M; A) & \to & CH^+_{n-p}(M; S) & \to & CH^+_{n-p}(M; A^*) & \to \end{array}$$

and

$$\begin{array}{ccccccc} \to CH^p_-(M; A) & \to & CH^p_p(M; S) & \to & CH^p_-(M; A^*) & \to \\ \downarrow D & & \downarrow D & & \downarrow D & \\ \to CH^+_{n-p}(M, \partial M; A) & \to & CH^+_{n-p}(M, \partial M; S) & \to & CH^+_{n-p}(M, \partial M; A^*) & \to \end{array}$$

Proof. We show that the first diagram commutes. Choose $(N, N_0, N_1, N_2) \in \mathscr{M}(M, \partial M; S)$. There exist neighborhoods L_0 of A and L_1 of A^* in N such that $L_0 \cup L_1 = N$, $\partial_N L_0 = L_0 \cap L_1 = \partial_N L_1$, and such that $L_0 \cap L_1$ lies in the exit set of L_1 and the entrance set of L_0 [12]. Then

$$(L_0, N_0 \cap L_0, (N_1 \cup L_1) \cap L_0, N_2 \cap L_0) \in \mathscr{M}(M, \partial M; A),$$
$$(L_1, N_0 \cap L_1, N_1 \cap L_1, (N_2 \cup L_0) \cap L_1) \in \mathscr{M}(M, \partial M; A^*).$$

Fundamental classes for N, L_0 and L_1 are related by the diagram

$$\begin{array}{ccccc} H_n(N, L_1 \cup \partial N) & \longleftarrow & H_n(N, \partial N) & \longrightarrow & H_n(N, L_0 \cup \partial N) \\ \uparrow & & & & \uparrow \\ H_n(L_0, \partial L_0) & & & & H_n(L_1, \partial L_1) \end{array}$$

The cap products of N, L_0 and L_1 link the exact sequences, forming diagram

$$\begin{array}{ccccc} \xrightarrow{\delta^*} \delta^* H^p(N_1, (N_0 \cup N_1 \cup L_1) \cap L_0) & \to H^p(N, N_0 \cup N_1) \to & H^p(L_1, (N_0 \cup N_1) \cup L_1) \xrightarrow{\delta^*} \\ \downarrow \cap z_{L_0} & \downarrow \cap z_N & \downarrow \cap z_{L_1} \\ \xrightarrow{\partial_*} \quad H_{n-p}(L_0, N_2 \cup L_0) & \to H_{n-p}(N, N_2) & \to H_{n-p}(L_1, (N_2 \cup L_0) \cap L_2) \xrightarrow{\partial_*} \end{array}$$

which represents the index diagram.

The commutativity of z_{L_0} and z_N, and of z_N and z_{L_1} follows from the naturality of the cap product. The commutativity of z_{L_1} and z_{L_0} is obtained by applying [5, VII, 12.25] with $X_1 = L_1 \cup N_0 N_1$, $X_2 = L_0 \cup N_2$, $A_1 = N_0 \cup N_1$, $A_2 = N_2$, and composing it with the diagrams induced by the inclusions

$$\begin{aligned}&(L_1, (N_0 \cup N_1) \cap L_1, (N_2 \cup L_0) \cap L_1)\\ &\quad \to (L_1 \cup N_0 \cup N_1, N_0 \cup N_1, ((N_0 \cup N_1) \cap L_0) \cup ((N_2 \cup L_0) \cap L_1)\end{aligned}$$

and

$$\begin{aligned}&(L_0, (L_1 \cup N_0 \cup N_1) \cap L_0, N_2 \cap L_0)\\ &\quad \to (L_0 \cup N_2, ((N_0 \cup N_1) \cap L_0) \cup ((N_2 \cup L_0) \cap L_1), N_2).\end{aligned}$$

The composition yields the commutative diagram

$$\begin{array}{ccc} H^p(L_1, (N_0 \cup N_1) \cap L_1) & \xrightarrow{\delta^*} & H^{p+1}(L_0, (N_0 \cup N_1 \cup L_1) \cap L_0) \\ \big\downarrow \cap z_{L_1} & & \big\downarrow \cap z_{L_0} \\ H_{n-p}(L_1, (N_2 \cup L_0) \cap L_1) & \xrightarrow{\partial_*} & H_{n-p-1}(L_0, N_2 \cap L_0). \end{array} \quad \square$$

Note that if field coefficients F are used, there is a natural isomorphism $CH^*_{\pm}(M; S) \to \mathrm{Hom}(CH^{\pm}_*(M; S), F)$, and a (noncanonical) isomorphism $CH^{\pm}_*(M; S) \cong \mathrm{Hom}(CH^{\pm}_*(M; S), F)$. That is, if $S \cap \partial M = \varnothing$, then the four attractor-repeller sequences (i.e. those of $CH^{\pm}_*$ and $CH^*_{\pm}$) all express the same information.

In a Morse decomposition $\mathscr{S} = \{S_p\}$ of S (assume now that $S \cap \partial M = \varnothing$), $CH^*_-(M; S_p)$ and $CH^+_*(M; S_p)$ are dual for each p, with duality isomorphism D_p. Thus $C^*_-\Delta(P) = \bigoplus_{p\in P} CH^*_-(M; S_p)$ and $C^+_*\Delta(P) = \bigoplus_{p \in P} CH^+_*(M; S_p)$ are dual, with duality isomorphism $D = \bigoplus_{p\in P} D_p$.

Theorem 3.4. *There is a bijection* $\widetilde{D}\colon CM^*_-(\mathscr{S}, <) \to CM^+_*(\mathscr{S}, <)$, *with* $\widetilde{D}(\{\Delta^*_-(P), \phi^*_-(I)\}) = \{D \circ \Delta^*_-(P) \circ D^{-1}, D(I) \circ \phi^*_-(I) \circ D(I)^{-1}\}$.

Proof. It suffices to show that $\widetilde{D}$ does indeed map into $CM^+_*(\mathscr{S}, <)$. If it does, then it has an inverse $\widetilde{D}^{-1}$ defined analogously from D^{-1}.

(i) $\Delta^+_*(P)$ is the composition

$$C\Delta^+_k(P) \xrightarrow{D^{-1}} C\Delta^{n-k}_-(P) \xrightarrow{\Delta} C\Delta^{n-k+1}_-(P) \xrightarrow{D} C\Delta^+_{k-1}(P),$$

so it is a degree -1 map with $\Delta^+_{p,q} = D \circ 0 \circ D^{-1} = 0$ for $p \not< q$ and with

$$\Delta^+_*(P) \circ \Delta^+_*(P) = D \circ \Delta^*_+(P) \circ D^{-1} \circ D \circ \Delta^*_+(P) \circ D^{-1} = D \circ \Delta^*_+(P) \circ \Delta^*_+(P) \circ D^{-1} = 0.$$

That is, it is a strictly upper triangular boundary map.

For any adjacent pair of intervals (I, J) (with respect to the forward flow, so that (J, I) is an adjacent interval in the reverse flow), there is then an isomorphism of short exact chain complex sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C\Delta^*_-(I) & \longrightarrow & C\Delta^*_-(JI) & \longrightarrow & C\Delta^*_-(J) & \longrightarrow & 0 \\ & & \big\downarrow D & & \big\downarrow D & & \big\downarrow D & & \\ 0 & \longrightarrow & C\Delta^+_*(I) & \longrightarrow & C\Delta^+_*(IJ) & \longrightarrow & C\Delta^+_*(J) & \longrightarrow & 0 \end{array}$$

which induces an isomorphism of long exact sequences

$$\begin{array}{ccccccc}
\to & H\Delta^+_{n-k}(I) & \to & H\Delta^+_{n-k}(IJ) & \to & H\Delta^+_{n-k}(J) & \to \\
 & \Big\downarrow{\scriptstyle D_*^{-1}} & & \Big\downarrow{\scriptstyle D_*^{-1}} & & \Big\downarrow{\scriptstyle D_*^{-1}} & \\
\to & H\Delta^k_-(I) & \to & H\Delta^k_-(JI) & \to & H\Delta^k_-(J) & \to \\
 & \Big\downarrow{\scriptstyle D} & & \Big\downarrow{\scriptstyle D} & & \Big\downarrow{\scriptstyle D} & \\
\to & CH^k_-(M\,;S(I)) & \to & CH^k_-(M\,;S(JI)) & \to & CH^k_-(M\,;S(J)) & \to \\
 & \Big\downarrow{\scriptstyle D} & & \Big\downarrow{\scriptstyle D} & & \Big\downarrow{\scriptstyle D} & \\
\to & CH^+_{n-k}(M\,;S(I)) & \to & CH^+_{n-k}(M, S(IJ)) & \to & CH^+_{n-k}(M\,;S(J)) & \to
\end{array}$$

Thus the maps $D_*^{-1} \circ \phi_+(I) \circ D$ make the diagram commute, with $D_*^{-1} \circ \phi_+(p) \circ D = D^{-1} \circ \mathrm{id} \circ D = \mathrm{id}$, as required. □

Duality divides the four collections of connection matrices into two pairs: $CM^*_-(\mathscr{S}, <) \cong CM^+_*(\mathscr{S}, <)$; $CM^*_+(\mathscr{S}, <) \cong CM^-_*(\mathscr{S}, <)$. As with attractor-repeller sequences, these are further related by the Hom functor. There are bijections $B^+: CM^+_*(\mathscr{S}, <) \to CM^*_+(\mathscr{S}, <)$, $B^-: CM^-_*(\mathscr{S}, <) \to CM^*_-(\mathscr{S}, <)$ defined by $B^{\pm}(\Delta^{\pm}_*(P), \phi^{\pm}_*(I)) = (h^{-1} \circ \mathrm{Hom}(\Delta^{\pm}_*(P)) \circ h, h^{-1} \circ \mathrm{Hom}(\phi^{\pm}_*(I)) \circ h)$, where h is the isomorphism of the cohomology universal coefficient theorem. As in Theorem 3.4, B^+ and B^- carry connection matrices to connection matrices. Thus the four collections of connection matrices are all naturally identified with each other, and the four homology index functors provide equivalent information about Morse decompositions.

In particular, they all generate equivalent Morse inequalities:

Corollary 3.5. *The Morse inequalities generated by CH^+_* and CH^*_+ are equal; the Morse inequalities generated by CH^-_* and CH^*_- are equal; the Morse inequalities generated by CH^+_* and CH^-_* are related by exchanging t^k and t^{n-k} in the Poincaré polynomials.*

While the duality isomorphism exists for a large class of flows, and commutes with most of the basic structures of the Conley index theory, it does not hold in complete generality, nor preserve all of the index structures. Some examples illustrate these limitations.

In Example 1.2, the manifold M is nonorientable, and the invariant set S has integer (co)homology indices

$$CH^+_*(M\,;S) \cong (0, \mathbf{Z}_2, \mathbf{Z}_2, 0, \dots), \quad CH^*_+(M\,;S) \cong (0, \mathbf{Z}_2, \mathbf{Z}_2, 0, \dots),$$
$$CH^-_*(M\,;S) \cong (\mathbf{Z}, \mathbf{Z} \oplus \mathbf{Z}, \mathbf{Z}, 0, \dots), \quad CH^*_-(M\,;S) \cong (\mathbf{Z}, \mathbf{Z} \oplus \mathbf{Z}, \mathbf{Z}, 0, \dots),$$

while the indices with $\mathbf{Z}_2$ coefficients are

$$CH^+_*(M\,;S) \cong CH^*_+(M\,;S) \cong (0, \mathbf{Z}_2, \mathbf{Z}_2 \oplus \mathbf{Z}_2, \mathbf{Z}_2, 0, \dots),$$
$$CH^-_*(M\,;S) \cong CH^*_-(M\,;S) \cong (\mathbf{Z}_2, \mathbf{Z}_2 \oplus \mathbf{Z}_2, \mathbf{Z}_2, 0, \dots).$$

Moreover, the ring structures with $\mathbf{Z}_2$ coefficients are

$$CH^+(M\,;S) \cong (0, \langle a \rangle, \langle a^2, b \rangle, \langle a^2 b \rangle, 0, \dots),$$
$$CH^-(M\,;S) \cong (\langle 1 \rangle, \langle \alpha, \beta \rangle, \langle \alpha\beta \rangle, 0, \dots).$$

This example illustrates several distinctions between the indices. First, the groups above are dual when $\mathbf{Z}_2$ coefficients are used, but not when $\mathbf{Z}$ coefficients are used, as M is orientable over $\mathbf{Z}_2$, but not over $\mathbf{Z}$.

Second, the indices are dual as groups, but not as rings. The cohomology groups have distinct nontrivial ring structures, while their dual homology groups have no natural ring structure. That is, as groups, all four indices are essentially the same. As rings, they are all distinct. In particular, the forward and reverse cohomology groups have different cuplengths, and so may give different information in Ljusternik-Schnirelmann arguments, such as that used in [3].

Third, the duality isomorphism does not conjugate the maps induced by semiconjugacies (cf. [10]). That is, if $f\colon M' \to M$ is a flow map, $S \subseteq M$ an isolated invariant set, the diagram may not commute:

$$\begin{array}{ccc} CH^*_-(M'; f^{-1}(S)) & \xleftarrow{f^*} & CH^*_-(M; S) \\ \downarrow D & & \downarrow D \\ CH_*^+(M'; f^{-1}(S)) & \xrightarrow{f_*} & CH_*^+(M; S) \end{array}$$

For example, in Example 1.2 take the inclusion $i\colon S \to M$. The homology Conley index of a set in itself is simply the homology of the set: $CH_*^\pm(S; S) = H_*(S)$. In the reverse flow in Example 1.2, S is an attractor, so its homology index is the Čech homology of the set: $CH_*^-(M; S) = \check{H}_*(S)$. The inclusion is a proper semiconjugacy with $i_*\colon CH_*^+(S; S) \to CH_*^+(M; S)$ the zero map and $i^*\colon CH^*_-(S; S) \to CH^*_-(M; S)$ an isomorphism.

Finally, in the wedge of spheres $X = S^1 \vee S^2$ in Example 1.3, the wedge point had homology indices $CH^*_-(X; S) = (R, 0, 0, \dots)$ and $CH_*^+(X; S) = (0, R \oplus R, R, 0, \dots)$. These are not dual, regardless of the ring R chosen. That is, duality holds only for flows on manifolds.

4. Applications

Theorem 2.1 is essentially a closed result, establishing the hypotheses needed for the duality observed in Example 1.1 to hold for all isolated invariant sets. However, by adding another element to the Conley index theory which is both canonical and natural, the theorem raises the question of applications: does the duality isomorphism provide any dynamical information? Two modes of application are indicated in this section.

The first use of duality is that it can simplify computation of homology index groups. For example, if S is isolated in M and $CH_*^\pm(\partial M; S \cap \partial M)$ is nonzero, then $S \cap \partial M$ is nonempty, and remains so under continuation. Duality allows this to be observed without computing $CH_*^+(\partial M; S \cap \partial M)$ directly.

Corollary 4.1. *If $CH^*_-(M; S)$ and $CH_*^+(M; S)$ are not dual, then every isolated invariant set related by continuation to S intersects ∂M, and has*

$$CH_*^\pm(\partial M; S \cap \partial M)$$

nonzero. Further, if $CH_*^+(M; S)$ is nonzero, then $W^u(S)$ is not contained in ∂M; if $CH_*^-(M; S)$ is nonzero, then $W^s(S)$ is not contained in ∂M.

In Examples 1.4, $CH_*^+(M; S) = 0$, while $CH_*^-(M; S) \cong (\mathbf{Z}, 0, 0, \dots)$. Thus $CH_1^+(\partial M; S \cap \partial M) \cong CH_2^+(M, \partial M; S) \cong \mathbf{Z}$, and $S \cap \partial M$ remains nonempty under continuation. Further, as $CH_*^+(M; S)$ is nonzero, $W^s(S)$ is not contained in ∂M and remains so under continuation. That is, S and everything related to it by continuation must intersect the boundary and attract orbits from the interior.

A similar application of duality relates the stable and unstable sets of S. The stable and unstable sets of S are $W^s(S) = \{x \in M: \omega(x) \subseteq S\}$ and $W^u(S) = \{x \in M: \omega^*(x) \subseteq S\}$ respectively. If S is hyperbolic, these are immersed manifolds whose tangent bundles along S are $TS \oplus E^s$ and $TS \oplus E^u$ respectively. If σ^s and σ^u are sections of $W^s(S)\backslash S$ and $W^u(S)\backslash S$ respectively, let $W_\sigma^s(S) = \{x \cdot \mathbf{R}^-: x \in \sigma^s\}$, $W_\sigma^u(S) = \{x \cdot \mathbf{R}^+: x \in \sigma^u\}$. For any two sections σ^s, $\tilde{\sigma}^s$, there exists a flow-defined homeomorphism between $(W_\sigma^s(S), \sigma^s)$ and $(W_{\tilde{\sigma}}^s(S), \tilde{\sigma}^s)$, so there are well-defined groups $H_*(W_\sigma^s(S), \sigma^s)$ and $H_*(W_\sigma^u(S), \sigma^u)$ independent of the section chosen. These groups find repeated use in relating the dynamics of the flow to the topology of the manifold. For instance, in Morse theory, the unstable sets form a CW decomposition of the manifold and these groups form the associated (co)chain complex.

If $(N, \varnothing, N_1, N_2)$ is an index quadruple, then $\sigma^S = N_1 \cap W^s(S)$ and $\sigma^u = N_2 \cap W^u(S)$ are such sections. Further, sequences of quadruples $(N_\alpha, \varnothing, N_{1\alpha}, N_{2\alpha})$ can be chosen such that $N_{1\alpha} \cap W^s(S) = N_{1\beta} \cap W^s(S)$ and $\cap_\alpha N_\alpha = W_\sigma^s(S)$. Thus $\check{H}_*(W_\sigma^u(S), \sigma^u) \cong CH_*^+(M; S)$ and $\check{H}_*(W_\sigma^s(S), \sigma^s) \cong CH_*^-(M; S)$, where $\check{H}$ denotes Čech (co)homology. All of this is standard, and is used in the development of the Churchill sequence [1]. The new feature is that these groups are dual:

Corollary 4.2. *There exists a duality isomorphism*

$$D: \check{H}^k(W_\sigma^u(S), \sigma^u) \to \check{H}_{n-k}(W_\sigma^s(S), \sigma^s)$$

which is independent of the sections σ^s *and* σ^u.

Note that this duality does not hold in general if singular homology is used. For example, there exists flow on $\mathbf{R}^2$ which have a Warsaw circle S as an attractor. Then $(W_\sigma^u(S), \sigma^u) = (S, \varnothing)$, while $(W_\sigma^s(S), \sigma^s) = (A, \partial A)$, where A is an annulus. Clearly, $H^1(W_\sigma^u(S), \sigma^u)$ and $H_{2-1}(W_\sigma^s(S), \sigma^s)$ are not dual.

These applications use only the fact that the duality map is an isomorphism, without using the precise form of the map. Another aspect of duality is to discover what information can be obtained by knowing which elements of $CH_-^p(M; S)$ and $CH_{n-p}^+(M; S)$ are identified by the map. Note that 2.1 and the naturality results help make the map computable: if it can be computed for one index quadruple, it is then computed for all others; if the set can be recognized as a sum or product, the map can be computed on the factors; if the map can be computed for one isolated invariant set, it is computed for all others related by continuation. However, the map is not "flow-defined." That is, the flow generates the maps ϕ_+, ϕ^-, θ used to show that D is well defined, and the map $F_{\lambda,\mu}$ used to show that it is continuation-invariant, but the flow does not generate the map D itself.

One way of using the map D is to define from it a nonsingular pairing of $CH_*^-(M; S)$ and $CH_*^+(M; S)$, which can be thought of as an intersection pairing:

$$CH_p^-(M; S) \otimes CH_{n-p}^+(M; S) \to \check{H}_0(S)$$

Naturally related to it is an intersection pairing of $\check{H}_*(W^u_\sigma(S), \sigma^u)$ and $\check{H}_*(W^s_\sigma(S), \sigma^s)$. In [9], Kurland independently defines such intersection pairings and uses them to prove the existence of layers of solutions to certain boundary value problems.

Another intersection pairing related to duality involves attractor-repeller pairs. If (A, A^*) is an attractor-repeller pair for S, then the set $C(A^*, A; S)$ of connecting orbits from A^* to A is the intersection $W^u(A^*) \cap W^s(A) \cap S$. In [11], this intersection is measured by an intersection pairing $CH^-_p(M; A) \otimes CH^+_q(M; A^*) \to \check{H}_{p+q-n-1}(C)$ where C is a section of $C(A^*, A; S)$. The duality isomorphism is used to show that this intersection pairing is related to the connection map $\partial: CH^+_q(M; A^*) \to CH^+_{q-1}(M; A)$. Namely, in the appropriate dimensions, an element $a \otimes b$ has nonzero intersection pairing if and only if ∂b is a nonzero multiple of Da. Thus the connection map can be used to partially compute the topology of the connecting orbit set.

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