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THE CLASSIFICATION OF COMPLETE MINIMAL SURFACES WITH TOTAL CURVATURE GREATER THAN -12π

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ABSTRACT. We classify complete orientable minimal surfaces with finite total curvature -8π.

INTRODUCTION

The classification of complete minimal surfaces with finite total curvature in $\mathbf{R}^3$ has been an important problem in the classical differential geometry.

Some basic properties of these surfaces were studied by R. Osserman (see [8, 9]), who showed the first nontrivial result about this subject.

Concretely, he characterized the catenoid and Enneper surface as the unique complete orientable minimal surfaces of total curvature -4π.

However until recent years no more relevant results have been obtained.

W. H. Meeks [6] gave the classification of nonorientable complete minimal surfaces with total curvature greater than -8π.

This paper is concerned with the total classification of orientable complete minimal surfaces with total curvature -8π.

Chen and Gackstatter [1] discovered the first example of a complete minimal surface properly of genus 1 (see Theorem 1). The picture of Chen-Gackstatter surface is obtained by joining a handle on Enneper's surface. This genus one minimal surface has total curvature -8π, and no other examples of such surfaces were found.

So, it is expected that no other genus one orientable minimal surface of total curvature -8π does exist.

In this paper we give a proof of this fact. More precisely, we prove that

"*Chen-Gackstatter surface is the only genus one orientable complete minimal surface with finite total curvature* -8π."

Of course, it is not difficult to find genus zero minimal surfaces with total curvature -8π.

A geometrically interesting example, described by Jorge and Meeks [5], is the trinoid.

This surface has three embedded catenoid ends. Moreover its normal vectors at these ends are placed symmetrically in an equator of $\mathbf{S}^2$.

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To find all the genus zero orientable complete minimal surfaces of total curvature -8π is merely an elementary algebraic problem (see §3).

These facts together with the above Osserman result complete our classification.

Finally, in §5, we prove a modified version of the Osserman-Mo Theorem [7] for complete minimal surfaces.

It is interesting to notice that recently C. Costa [2] has obtained the classification of genus one embedded complete minimal surfaces in $\mathbf{R}^3$ of finite total curvature -12π. All these surfaces lie in a smooth one-parameter family of tori punctured in three points, which correspond to three catenoid ends, except Costa's surface which has two catenoid ends and a planar end. This family was described by Hoffman and Meeks (see [3 and 4]).

On the other hand, Chen and Gackstatter [1] constructed an example of a genus two orientable minimal surface punctured in a point with total curvature -12π.

So, to exhibit the total classification of such surfaces could be an interesting open problem.

The editor sent me two Ph.D. theses by E. L. Barbanel [12] and Y. Fang [13]. The first one develops the classification and study of genus zero complete minimal surfaces of total curvature -8π, and the second one includes a partial version of this classification in the genus one case.

1. Preliminaries

In this section we expose some basic results about orientable complete minimal surfaces of finite total curvature. For more details, see [9, Chapter 9] and [8].

Let $x: M \to \mathbf{R}^3$ be an orientable complete minimal surface of finite total curvature in the Euclidean space $\mathbf{R}^3$.

Denote by g, ω the meromorphic function and the holomorphic 1-form determined by the Weierstrass representation of x [9].

It is well known that, modulo natural identifications, g is the Gauss map of M. Moreover, $g\omega$ and $g^2\omega$ are holomorphic 1-forms and

$$x = \mathrm{Re}\int(\phi_1, \phi_2, \phi_3) \tag{1}$$

where $\phi_1 = (1-g^2)\omega/2$, $\phi_2 = i(1+g^2)w/2$ and $\phi_3 = g\omega$.

Osserman proved (see [9]) that M is conformally equivalent to a compact Riemann surface $\overline{M}$ punctured in a finite set of points $\{P_1, \dots, P_k\} : M = \overline{M} - \{P_1, \dots, P_k\}$.

The points $P_1, \dots, P_k$, correspond to the ends of M, and g, ω extend meromorphically to $\overline{M}$. Then ϕ_i, $i = 1, 2, 3$, have poles of order $\mathrm{Ord}_{p_j}\phi_i$ at P_j, $j = 1, \dots, k$.

If we put

$$n = \mathrm{Degree}(g), \quad \gamma = \mathrm{Genus}(M),$$

$$I_j + 1 = \max\{\mathrm{Ord}_{P_j}\phi_i, i = 1, 2, 3\}, \qquad j = 1, \dots, k,$$

Jorge-Meeks formula gives (see [5])

$$2n = 2\gamma - 2 + \sum_{j=1}^{k}(I_j + 1). \tag{2}$$

In the following, we will assume that $n = 2$, i.e., M has total curvature -8π.

Using (2) and taking into account that $I_j \geq 1$, $j = 1, \dots, k$, we deduce $\gamma \leq 1$ (observe that the case $\gamma = 2$, $k = 1$, $I_1 = 1$ is impossible).

We describe the distinct possibilities in the following table:

TABLE 1

$n=2$	$k=1$	$k=2$	$k=3$
$\gamma=0$	$I_1=5$	$I_1=I_2=2$ $I_1=1, I_2=3$	$I_1=I_2=I_3=1$
$\gamma=1$	$I_1=3$		

Schoen has characterized the catenoid as the unique surface with $k = 2$ and $I_1 = I_2 = 1$ (see [10]). From (2), the case $k = 3$, $\gamma = 1$ is impossible.

We will classify all the surfaces which correspond to the other possibilities in the above table.

2. Genus one minimal surfaces of total curvature -8π

Throughout this section, we assume $\gamma = 1$, $n = 2$ and therefore, $k = 1$, $I_1 = 3$. We write $M = \overline{M} - \{P\}$, where $\overline{M}$ is a genus one compact Riemann surface and P is the end of M.

It is clear that $b_g(P) \leq 1$, where $b_g(P)$ is the branch number of g at P.

We will discuss separately the case when $b_g(P) = 0$ and $b_g(P) = 1$.

2.1. *First case.* $b_g(P) = 0$. Consider the following initial value problems of linear differential equations in the complex domain:

$$(\mathrm{I}_i) \qquad \begin{cases} f_i'(a) = \dfrac{-a}{2(1-a^2)} f_i(a) + \dfrac{3}{4(1-a^2)} g_i(a), \\[2ex] g_i'(a) = \dfrac{1}{(1-a^2)} f_i(a) - \dfrac{3a}{2(1-a^2)} g_i(a), \end{cases} \qquad i = 1, 2,$$

$$f_1(0) = \int_{-1}^{0} \sqrt{\frac{x}{x^2-1}}\, dx, \qquad g_1(0) = \int_{-1}^{0} \sqrt{\frac{x^2-1}{x}}\, dx,$$

$$f_2(0) = \int_{0}^{1} \sqrt{\frac{x}{x^2-1}}\, dx, \qquad g_2(0) = \int_{0}^{1} \sqrt{\frac{x^2-1}{x}}\, dx,$$

where we have fixed in each case the branch of $\sqrt{(x^2-1)/x}$ such that $f_1(0)$, $g_1(0) < 0$ and $f_2(0) = i f_1(0)$, $g_2(0) = -i g_2(0)$.

Take a, $b \in \mathbf{C}$, and define $[a, b] = \{ta + (1-t)b | t \in [0, 1] \subset \mathbf{R}\}$. We will write $\int_a^b h(x)\, dx$ instead of $\int_{[a,b]} h(x)\, dx$, for each function h defined on $[a, b]$.

Lemma 1. *The initial value problems* (I_i), $i=1,2$, *have well-defined solutions* f_i, g_i, $i=1,2$, *on the domain* $\mathbf{C}-\{y\in\mathbf{R}||y|>1\}$.

In fact, the solutions of (I_i), $i=1,2$, *are the functions*

$$(3)\qquad \begin{aligned} f_1(a)&=\int_{-1}^{a}\sqrt{\frac{x-a}{x^2-1}}\,dx, & g_1(a)&=\int_{-1}^{a}\sqrt{\frac{x^2-1}{x-a}}\,dx,\\ f_2(a)&=\int_{a}^{1}\sqrt{\frac{x-a}{x^2-1}}\,dx, & g_2(a)&=\int_{a}^{1}\sqrt{\frac{x^2-1}{x-a}}\,dx,\end{aligned}$$

respectively, for a suitable single-valued branch of $\sqrt{(x^2-1)/(x-a)}$, $a\in\mathbf{C}-\{y\in\mathbf{R}||y|>1\}$, $x\in[-1,a]\cup[a,1]$.

Moreover,

$$(4)\qquad \begin{gathered} f_2(a)=if_1(-a),\quad g_2(a)=-ig_1(-a),\\ f_1(\overline{a})=\overline{f}_1(a)\quad\text{and}\quad g_1(\overline{a})=\overline{g}_1(a).\end{gathered}$$

Proof. The function $F:[0,1]\times(\mathbf{C}-\{y\in\mathbf{R}|y\geq 1\})\to\mathbf{C}$ defined by

$$F(t,a)=(a+1)t-2$$

has rank $\mathbf{C}-\{y\in\mathbf{R}|y>0\}$.

Then, $\sqrt{F}$ has a single-valued branch on $[0,1]\times(\mathbf{C}-\{y\in\mathbf{R}|y>1\})$, and so $\sqrt{(t-1)/((a+1)t^2-2t)}$.

Since

$$(5)\qquad \begin{aligned} f_1(a)&=(a+1)\int_0^1\sqrt{\frac{t-1}{(a+1)t^2-2t}}\,dt,\\ g_1(a)&=(a+1)\int_0^1\sqrt{\frac{(a+1)t^2-2t}{t-1}}\,dt,\end{aligned}$$

we deduce that f_1, g_1 are well defined and holomorphic on $\mathbf{C}-\{y\in\mathbf{R}|y\geq 1\}$, and $f_1(0)$, $g_1(0)<0$ for a suitable election of the above branch.

Analogously, f_2, g_2 are holomorphic functions on $\mathbf{C}-\{y\in\mathbb{R}/y\leq -1\}$, and can be chosen such that $f_2(0)=if_1(0)$, $g_2(0)=-ig_1(0)$.

On the other hand

$$f_1'(a)=\int_0^1\sqrt{\frac{t-1}{(a+1)t^2-2t}}\,dt-\frac{a+1}{2}\int_0^1\sqrt{\frac{(t-1)t}{((a+1)t-2)^3}}\,dt$$

and integrating by parts

$$f_1'(a)=\frac{a}{2(a-1)}\int_0^1\sqrt{\frac{t-1}{(a+1)t^2-2t}}\,dt-\frac{3}{4(a-1)}\int_0^1\sqrt{\frac{(a+1)t^2-2t}{t-1}}\,dt.$$

Analogously

$$g_1'(a) = \frac{-1}{a-1}\int_0^1 \sqrt{\frac{t-1}{(a+1)t^2-2t}}\,dt + \frac{3a}{2(a-1)}\int_0^1 \sqrt{\frac{(a+1)t^2-2t}{t-1}}\,dt.$$

Hence, f_1, g_1 verify (I_1).

In a similar way, f_2, g_2 verify (I_2).

At last, observe that $if_1(-a)$, $-ig_1(-a)$ satisfy the initial value problem (I_2), and by the uniqueness of solutions, $if_1(-a) = f_2(a)$, $-ig_1(-a) = g_2(a)$.

Taking into account that $g_1(a), f_1(a) \in \mathbf{R}$ if $a \in [-1, 1]$, and the analyticity of the solutions, (4) holds and the proof is complete. Q.E.D.

Remark 1. Note that f_i, g_i, $i = 1, 2$, are also the solutions of the following initial value problems

$$\begin{aligned} &f_i''(a) = \frac{1}{4(1-a^2)}f_i(a), && g_i''(a) = \frac{-3}{4(1-a^2)}g_i(a), && i = 1, 2,\\ (6)\quad &f_1(0) = \int_{-1}^0 \sqrt{\frac{x}{x^2-1}}\,dx, && g_1(0) = \int_{-1}^0 \sqrt{\frac{x^2-1}{x}}\,dx, &&\\ &f_2(0) = if_1(0), && g_2(0) = -ig_1(0), &&\\ &f_i'(0) = \frac{3}{4}g_i(0), && g_i'(0) = f_i(0), && i = 1, 2. \end{aligned}$$

Remark 2. The functions f_1, g_1 are defined and holomorphic on $\mathbf{C} - \{y \in \mathbf{R} | y \geq 1\}$ and have both a simple zero at -1. Analogously, f_2, g_2 are defined and holomorphic on $\mathbf{C} - \{y \in \mathbf{R} | y \leq -1\}$ and have both a simple zero at 1.

Consider the Riemann sphere $\mathbf{C} \cup \{\infty\}$, and draw a straight line l from 1 to -1 passing by ∞ along the real axis. Then cut and open along l.

The closure Ω of the resulting domain has two copies of the line l. Call one of them l_1 and the other one l_2 (see Figure 1).

We will put a_1, a_2, the two points corresponding to $a \in l$, in l_1 and l_2 respectively.

Observe that $\infty_1 \neq \infty_2$ and $1_1 = 1_2$, $-1_1 = -1_2$.

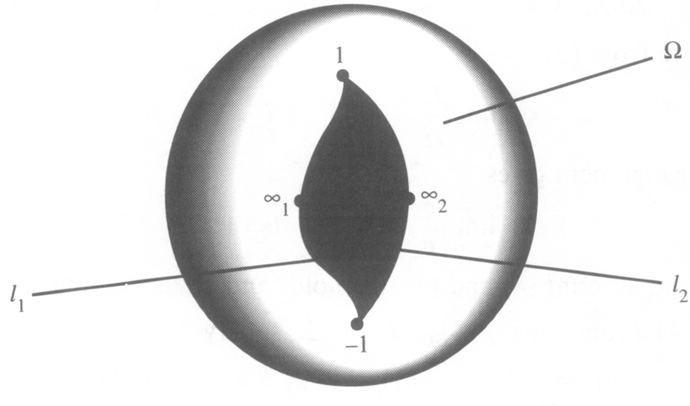

FIGURE 1

Lemma 2. *The functions* f_i, g_i, $i = 1, 2$, *extend continuously to* $\Omega - \{\infty_1, \infty_2\}$, *satisfying*

(7)

$$f_1(a), g_1(a) \neq 0, \quad a \neq \infty_1, \infty_2, -1, \quad f_1(1) = -2\sqrt{2}, \quad g_1(1) = -\frac{4\sqrt{2}}{3},$$

$$f_1'(-1) = \int_0^1 \sqrt{\frac{1-t}{2t}}\,dt < 0, \quad g_1'(-1) = 2f_1'(-1),$$

$$f_2(a), g_2(a) \neq 0, \quad a \neq \infty_1, \infty_2, 1, \quad f_2(-1) = if_1(1), \quad g_2(-1) = -ig_1(1),$$

$$f_2'(1) = -if_1'(-1), \quad g_2'(1) = ig_1'(-1)$$

and

(8)

$$\lim_{a\to\infty}\left|\frac{\sqrt{a}}{f_i}\right| = 0, \quad \lim_{a\to\infty_j}\left|\frac{a\sqrt{a}}{g_i}\right| \neq 0, \infty, \quad \lim_{a\to\infty_j}\left|\frac{a^\varepsilon\sqrt{a}}{f_i}\right| = \infty, \quad \varepsilon > 0,$$

$$\lim_{a\to 1}\frac{f_1 g_2}{f_2 g_1} = -3, \quad \lim_{a\to -1}\frac{f_1 g_2}{f_2 g_1} = -\frac{1}{3} \quad \lim_{a\to\infty_j}\frac{f_1 g_2}{f_2 g_1} = 1$$

where $i = 1, 2$ *and* $j = 1, 2$.

Proof. Using (3), it is straightforward to check that f_i, g_i, $i = 1, 2$, extend continuously to $\Omega - \{\infty_1, \infty_2, 1, -1\}$ and $f_1(a)$, $g_1(a) \neq 0$ if $\mathrm{Im}(a) = 0$, $a \neq \infty_1, \infty_2, -1$.

Notice that if $\mathrm{Im}(a) \neq 0$, $\mathrm{Im}\sqrt{2-(a+1)t} \neq 0$, where $t \in]0, 1]$. Thus, from (5), $f_1(a) \neq 0$, and in the same way, $g_1(a) \neq 0$. So, from (4), we have $f_2(a)$, $g_2(a) \neq 0$, $a \neq \infty_1, \infty_2, 1$.

Using now (4), (5) again, it is not hard to prove (7). On the other hand, (5) yields

$$\lim_{a\to\infty_i}\left|\frac{\sqrt{a}}{f_1}\right| = \left|\int_0^1 \sqrt{\frac{1-t}{t^2}}\,dt\right|^{-1} = 0, \qquad \lim_{a\to\infty_i}\left|\frac{a\sqrt{a}}{g_1}\right| = \left|\int_0^1 \sqrt{\frac{t^2}{1-t}}\,dt\right| \neq 0,$$

$$\lim_{a\to\infty_i}\left|\frac{a^\varepsilon\sqrt{a}}{f_1}\right| = \infty$$

where $i = 1, 2$, and the same holds for f_2, g_2 (see (4)).

Moreover

$$\lim_{a\to\infty}\left|\frac{1}{\sqrt{a}}\int_0^1 \sqrt{\frac{x-a}{x^2-1}}\,dx\right|, \qquad \lim_{a\to\infty}\left|\frac{1}{\sqrt{a}}\int_{-1}^0 \sqrt{\frac{x-a}{x^2-1}}\,dx\right| \neq 0$$

and therefore from (3),

$$\lim_{a\to\infty_i}\frac{f_1}{f_2} = -1, \qquad i = 1, 2.$$

A similar argument gives

$$\lim_{a\to\infty_i}\frac{g_1}{g_2} = 1, \qquad i = 1, 2.$$

Taking into account (4) and (7), (8) holds and so the lemma. Q.E.D.

Remark 3. The functions f_i, g_i, $i = 1, 2$, satisfy

$$f_i(a_1) = (-1)^{i-1}\overline{f}_i(a_2), \qquad g_i(a_1) = (-1)^{i-1}\overline{g}_i(a_2)$$

where $a_i \in l_i$, $i = 1, 2$, $a \neq \infty$.

Remark 4. Observe that $f_1(a)$, $g_1(a) \in \mathbf{R}$, $a \in]-\infty, 1]$, $f_2(a)/i$, $g_2(a)/i \in \mathbf{R}$, $a \in [-1, +\infty[$. Moreover

$$f_1(a), g_1(a) > 0, \quad a \in]-\infty, -1[, \qquad f_1(a), g_1(a) < 0, \quad a \in]-1, 1]$$

$$\frac{1}{i} f_2(a), \qquad -\frac{1}{i} g_2(a) < 0, \qquad a \in]-1, 1[,$$

$$\operatorname{Re}(f_1(a_i)), \qquad \operatorname{Re}(g_1(a_i)) < 0,$$

$$(-1)^i \operatorname{Im}(f_1(a_i)) = \frac{1}{i} f_2(a_1) = \frac{1}{i} f_2(a_2) > 0,$$

$$(-1)^i \operatorname{Im}(g_1(a_i)) = \frac{1}{i} g_2(a_1) = \frac{1}{i} g_2(a_2) < 0,$$

where $a_i \in l_i - \{\infty_i\}$, $a_i > 1$, $i = 1, 2$.

Remark 5. The functions f_i, g_i satisfy (I_1), $i = 1, 2$, on

$$(l_1 \cup l_2) - \{\infty_1, \infty_2, 1, -1\}.$$

Lemma 3. *If* $a \in \Omega - \{\infty_1, \infty_2\}$,

$$(f_1 g_2 - f_2 g_1)(a) = \frac{4\pi i}{3}(1 - a^2).$$

Proof. Consider the genus one compact Riemann surface

$$\overline{M}_a = \{(z, w) \in (\mathbf{C} \cup \{\infty\})^2 | w^2 = (z - a)(z^2 - 1)\}, \text{ where } a \in \mathbf{C} - \{1, -1\}.$$

We can construct a "concrete" representation of $\overline{M}_a$ as a two-sheeted covering of the sphere $\mathbf{C} \cup \{\infty\}$.

Picture two copies of the sphere, and label these two copies sheet I and sheet II. On each sheet, cut along two smooth curves joining -1 to "a" and 1 to ∞, in such way that these cuts do not intersect.

Each "cut" has two banks; an N-bank and an S-bank.

Joining every S-bank on sheet I to the N-bank of the corresponding "cut" on sheet II, and then joining the corresponding S-bank on sheet II to the N-bank of the corresponding "cut" on sheet I, we have the desired representation of $\overline{M}_a$ (see Figure 2).

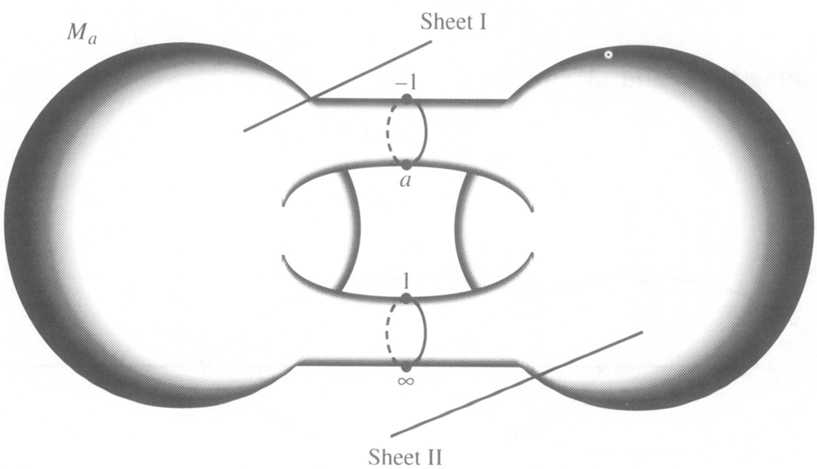

FIGURE 2

We can construct a canonical homology basis for $\overline{M}_a$ drawing simple smooth curves γ_i, $i = 1, 2$; γ_1 is given winding once around the "cut" from -1 to "a" in one sheet of $\overline{M}_a$, and γ_2 starting from a point on "cut" from -1 to "a" going on the first sheet to a point on "cut" from 1 to ∞, and returning on the second sheet (indicated in Figure 3 below by dotted lines) to the original point.

The orientation of γ_1, γ_2 is illustrated in Figure 3.

Define now the meromorphic 1-forms:

$$\tau_1^a = \frac{z-a}{w}\,dz, \quad \tau_2^a = \frac{w}{z-a}\,dz,$$

and observe that

$$2f_i(a) = \int_{\gamma_i} \tau_1^a, \quad 2g_i(a) = \int_{\gamma_i} \tau_2^a, \qquad i = 1, 2.$$

Using standard bilinear relations (see [11]), we conclude the proof. Q.E.D.

Lemma 4. *If* $a_i \in l_i - \{\infty_i\}$, $i = 1, 2$, *then*

$$|f_1 g_2 / f_2 g_1|(a_i) \neq 1.$$

Proof. Remark 3 involves $|f_1g_2/f_2g_1|(a_1) = |f_1g_2/f_2g_1|(a_2)$. So we need only to prove the lemma for $a_1 \in l_1 - \{\infty\}$.

Suppose $a_1 > 1$.

From Remark 4 and Lemma 3

$$\left|\frac{f_1 g_2}{f_2 g_1}\right|(a_1) = 1 \Leftrightarrow \left(\frac{f_2}{\mathrm{Re}(f_1)} + \frac{g_2}{\mathrm{Re}(g_1)}\right)(a_1) = 0. \tag{9}$$

Let $t\colon]1, +\infty[\to \mathbf{R}$ the function defined by

$$t(a_1) = (2\,\mathrm{Re}(f_1) - \sqrt{3}\,\mathrm{Re}(g_1))(a_1).$$

Remarks 4, 5 and 1 yields

$$t'(a_1) = \frac{1}{a_1^2 - 1}\left((a_1 + \sqrt{3})\,\mathrm{Re}(f_1)(a_1) - \frac{3}{2}(1 + \sqrt{3}a_1)\,\mathrm{Re}(g_1)(a_1)\right),$$

$$t''(a_1) = \frac{1}{2(1 - a_1^2)}\left(\mathrm{Re}(f_1)(a_1) + \frac{3\sqrt{3}}{2}\,\mathrm{Re}(g_1)(a_1)\right) > 0.$$

Looking at (8) and (3), we have

$$\lim_{a_1 \to \infty} t'(a_1) = 0$$

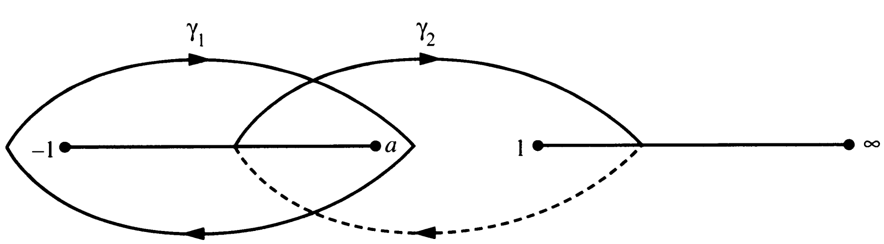

FIGURE 3

and therefore

$$t'(a_1) < 0, \quad \text{for each } a_1 \in]1, +\infty[.$$

But from (7), $t(1) < 0$ and then

$$t(a_1) < 0, \quad \text{for each } a_1 \in [1, +\infty[. \tag{10}$$

If $|f_1 g_2/f_2 g_1|(a_1') = 1$, $a_1' > 1$, from (9) and Remark 2,

$$\left(\frac{f_2}{\mathrm{Re}(f_1)} + \frac{g_2}{\mathrm{Re}(g_1)}\right)(a_1') = \left(\frac{f_2}{\mathrm{Re}(f_1)} + \frac{g_2}{\mathrm{Re}(g_1)}\right)(1) = 0$$

and then there exists $b \in]1, a_1'[$ such that

$$\left(\frac{f_2}{\mathrm{Re}(f_1)} + \frac{g_2}{\mathrm{Re}(g_1)}\right)'(b) = 0.$$

Taking into account Remarks 5, 4 and Lemma 3: $t(b) = 0$. So, from (10), we get a contradiction and thus

$$|f_1 g_2/f_2 g_1|(a_1) \neq 1, \quad \text{for each } a_1 \in]1, +\infty[.$$

Using (4) and (7), the same holds for $a_1 \in]-\infty, -1[$ and $a_1 = 1, -1$. Q.E.D.

Remark 6. Observe that from the proof of Lemma 4 we can deduce

$$\frac{1}{i}(\mathrm{Re}(f_1)g_2 + \mathrm{Re}(g_1)f_2)(a_1) > 0 \quad \text{if } a_1 \in]1, +\infty_1[\subset l_1.$$

Lemma 5.

$$(f_1\overline{g}_2/f_2\overline{g}_1)(a) \neq 1, \quad \textit{for each } a \in i\mathbf{R} - \{0\}.$$

Proof. Write $t_i = \mathrm{Re}(f_i)$, $s_i = \mathrm{Im}(f_i)$, $u_i = \mathrm{Re}(g_i)$, $v_i = \mathrm{Im}(g_i)$, $i = 1, 2$.

Since $a \in i\mathbf{R}$, it is easy to check from (4) that

$$\frac{f_1\overline{g}_2}{f_2\overline{g}_1}(a) = 1 \Leftrightarrow \mathrm{Im}(f_1 g_1)(a) = 0 \Leftrightarrow (t_1 v_1 + s_1 u_1)(a) = 0. \tag{11}$$

In the following, we put $a = iy$, $y \in \mathbf{R}$ and write simply $t_i(y)$, $s_i(y)$, $u_i(y)$ and $v_i(y)$ instead of $t_i(iy)$, $s_i(iy)$, $u_i(iy)$ and $v_i(iy)$, $i = 1, 2$. Also t_i', t_i'' will mean dt_i/dy, d^2t_i/dy^2 respectively, and the same for s_i, u_i and v_i, $i = 1, 2$.

Using Lemma 1 and Remark 1, we deduce that the above functions verify the differential equations

$$\begin{cases} t_i' = \dfrac{y}{2(1+y^2)}t_i - \dfrac{3}{4(1+y^2)}v_i, \\ s_i' = \dfrac{y}{2(1+y^2)}s_i + \dfrac{3}{4(1+y^2)}u_i, \end{cases} \qquad \begin{cases} u_i' = \dfrac{-1}{1+y^2}s_i + \dfrac{3y}{2(1+y^2)}u_i, \\ v_i' = \dfrac{1}{1+y^2}t_i + \dfrac{3y}{2(1+y^2)}v_i, \end{cases}$$
$$\begin{cases} t_i'' = \dfrac{-1}{4(1+y^2)}t_i, \\ s_i'' = \dfrac{-1}{4(1+y^2)}s_i, \end{cases} \qquad \begin{cases} u_i'' = \dfrac{3}{4(1+y^2)}u_i, \\ v_i'' = \dfrac{3}{4(1+y^2)}v_i, \end{cases} \tag{12}$$

where $i = 1, 2$.

If $i = 1$, the initial values are

$$\begin{aligned} &t_1(0) = f_1(0), \quad t_1'(0) = 0, \qquad u_1(0) = g_1(0), \quad u_1'(0) = 0, \\ &s_1(0) = 0, \quad s_1'(0) = \frac{3}{4} g_1(0), \quad v_1(0) = 0, \quad v_1'(0) = f_1(0), \end{aligned} \tag{13}$$

and analogously if $i = 2$.

Observe that (4) involves

$$\frac{f_1 \overline{g}_2}{f_2 \overline{g}_1}(iy) = 1 \Leftrightarrow \frac{f_1 \bar{g}_2}{f_2 \overline{g}_1}(-iy) = 1.$$

Then from (11), it is sufficient to prove that

$$(t_1 v_1 + s_1 u_1)(y) \neq 0, \quad \text{for each } y > 0.$$

First, we will show that

$$\begin{gathered} u_1(y),\ v_1(y),\ s_1(y) < 0, \quad \forall y > 0, \\ \exists y_0 > 0 \text{ such that } t_1(y_0) = 0, \quad t_1(y) < 0 \text{ if } 0 < y < y_0, \\ t_1(y) > 0 \text{ if } y_0 < y < +\infty, \\ \lim_{y \to +\infty} \frac{s_1}{t_1} = -1, \qquad \lim_{y \to +\infty} \frac{v_1}{u_1} = 1. \end{gathered} \tag{14}$$

If u_1 vanishes at some point $y' > 0$, take $y_1 > 0$ the first point such that $u_1(y_1) = 0$. Since $g_1(0) < 0$, (12) and (13) yield

$$u_1''(y),\ u_1'(y),\ u_1(y) < 0, \qquad y \in]0, y_1],$$

a contradiction.

In a similar way, $v_1(y) < 0$, for each $y > 0$.

Suppose s_1 vanishes at $y' > 0$, and take as before $y_1 > 0$ the first point such that $s_1(y_1) = 0$.

Using that $u_1(y_1) < 0$ and (12), we have $s_1'(y_1) < 0$, which contradicts (13).

We know that $v_1(y) < 0$, $y > 0$. So, from (12), if $t_1(y_0) = 0$, $y_0 > 0$, then $t_1'(y_0) > 0$. Since $t_1(0) < 0$ (see (13)), t_1 vanishes at most at one point $y_0 > 0$.

On the other hand, from (3),

$$t_1(y) = \mathrm{Re} \int_{-1}^{0} \sqrt{\frac{x - iy}{x^2 - 1}}\, dx + \mathrm{Re}\left(i \int_0^y \sqrt{i \frac{y - r}{r^2 + 1}}\, dr \right)$$

and therefore

$$\lim_{y \to +\infty} t_1(y) = +\infty, \qquad \lim_{y \to +\infty} \frac{s_1}{t_1}(y) = -1.$$

It is now easy to deduce that t_1 vanishes at only one point $y_0 > 0$.

By similar arguments

$$\lim_{y \to +\infty} \frac{v_1}{u_1}(y) = 1$$

and (14) holds.

To finish the lemma, from (11) and (14) it is sufficient to show

$$\left(\frac{s_1}{t_1} + \frac{v_1}{u_1} \right)(y) \neq 0, \quad \text{for each } y > 0,\ y \neq y_0,$$

and note that from (14), $(s_1 u_1 + v_1 t_1)(y_0) = (s_1 v_1)(y_0) \neq 0$.

Suppose now

$$\left(\frac{s_1}{t_1}+\frac{v_1}{u_1}\right)(y_1)=0, \qquad y_1>0.$$

If $y_1<y_0$, since $(s_1/t_1+v_1/u_1)(0)=0$, we have

$$\left(\frac{s_1}{t_1}+\frac{v_1}{u_1}\right)'(y_2)=0, \quad \text{where } y_2\in]0, y_1[.$$

From (12)

$$\left(\frac{s_1}{t_1}+\frac{v_1}{u_1}\right)'(y_2)=\left(\left(\frac{u_1t_1+v_1s_1}{1+y^2}\right)\left(\frac{3}{4t_1^2}+\frac{1}{u_1^2}\right)\right)(y_2).$$

Thus $(u_1t_1+v_1s_1)(y_2)=0$, that is, $\operatorname{Re}(f_1\overline{g}_1)(iy_2)=0$.
But Lemma 3 and (4) give

$$\operatorname{Re}(f_1\overline{g}_1)(-y)=-\frac{2\pi}{3}(1+y^2)\neq 0, \quad \text{for each } y\in \mathbf{R},$$

which gets a contradiction.

If $y_0<y_1$, observe that from (14)

$$\lim_{y\to+\infty}\left(\frac{s_1}{t_1}+\frac{v_1}{u_1}\right)=0.$$

Reasoning as before, this case is also impossible. Q.E.D.

Remark 7. Note that

$$(u_1s_1+v_1t_1)(y)>0, \quad \text{for each } y>0,$$
$$(u_1s_1+v_1t_1)(y)<0, \quad \text{for each } y<0.$$

We can now state the main theorem of this section.

Theorem 1. *Let $x: M\to\mathbf{R}^3$ be a orientable complete minimal surface of finite total curvature -8π and genus one.*

Suppose that its Gauss map is regular at the unique end of M.

Then up to homothety and rigid motion, x is given by

$$x\colon \overline{M}-\{(\infty,\infty)\}\to\mathbf{R}^3, \qquad x=\operatorname{Re}\int(\phi_1,\phi_2,\phi_3)$$

where

$$\overline{M}=\{(z,w)\in(\mathbf{C}\cup\{\infty\})^2 \mid w^2=z(z^2-1)\},$$
$$\phi_1=\frac{1}{2}\left(\frac{z}{w}-A^2\frac{w}{z}\right)dz, \quad \phi_2=\frac{i}{2}\left(\frac{z}{w}+A^2\frac{w}{z}\right)dz,$$
$$\phi_3=A\,dz \quad \textit{and} \quad A^2=\frac{f_1}{g_1}(0).$$

Proof. We know that M is conformally equivalent to $\overline{M}-\{P\}$, where $\overline{M}$ is a compact Riemann surface and P is a point of $\overline{M}$.

If g is the Gauss map of M, after a rotation in $\mathbf{R}^3$, we can suppose that $g(P)=\infty$. Since $b_g(P)=0$, there exist four points P_i, $i=1,2,3,4$, in $\overline{M}$ such that $b_g(P_i)=1$, $i=1,2,3,4$. We call $a_i=g(P_i)$, $i=1,2,3,4$, and observe that $a_i\neq a_j$, $i\neq j$.

Therefore $\overline{M}$ is conformally equivalent to

$$\{(z_1, w_1) \in (\mathbf{C} \cup \{\infty\})^2 | w_1^2 = (z_1 - a_1)(z_1 - a_2)(z_1 - a_3)(z_1 - a_4)\}$$

where $g = z_1$.

After a suitable change of parameter

$$\overline{M} = \{(z_2, w_2) \in (\mathbf{C} \cup \{\infty\})^2 | w_2^2 = (z_2^2 - 1)(z_2 - c_1)(z_2 - d_1)\}, \qquad g = A_1 z_2 + B_1,$$

where $c_1, d_1, A_1, B_1 \in \mathbf{C}$, $c_1 \neq d_1$, $c_1, d_1 \neq 1, -1$, $A_1 \neq 0$.

As $z_2(P) = g(P) = \infty$ and $b_g(P) = 0$, there exists $Q \in \overline{M}$ such that $z_2(Q) = g(Q) = \infty$.

Without loss of generality, we can suppose that the meromorphic function on $\overline{M} : S$, defined by

$$S(z_2, w_2) = w_2 + z_2^2 - \frac{c_1 + d_1}{2} z_2 + \left(\frac{c_1 d_1 - 1}{2} - \frac{(c_1 + d_2)^2}{8} \right) \tag{15}$$

satisfy:

$$S(P) = \infty, \qquad b_s(P) = 1, \qquad S(Q) = 0, \qquad \text{Degree}(S) = 2.$$

From (1), (2), the holomorphic 1-form ω determined by the Weierstrass representation of x has a pole of order two at P, a zero of order two at Q and no other zeroes and poles.

So, ω/S is holomorphic, and then $b_s(Q) = 1$.

This fact and (15) yield

$$(c_1 + d_1)((c_1 - d_1)^2 - 4) = 0. \tag{16}$$

Taking into account (16) and after a suitable change of parameter

$$\overline{M} = \{(z_3, w_3) \in (\mathbf{C} \cup \{\infty\})^2 | w_3^2 = (z_3^2 - 1)(z_3^2 - c^2)\},$$
$$g = A_2 z_3 + B_2, \qquad \omega = C_1 \left(w_3 + z_3^2 - \frac{c^2 + 1}{2} \right) \frac{dz_3}{w_3},$$

where $c, A_2, B_2, C_1 \in \mathbf{C}$, $c \neq 1, -1, 0$, $A_2, C_1 \neq 0$.

Taking $z = w_3 + z_3^2 - (c^2 + 1)/2$, we have $\text{Degree}(x) = 2$ and

$$\overline{M} = \{(z, w) \in (\mathbf{C} \cup \{\infty\})^2 | w^2 = (z - a)(z^2 - 1)\},$$
$$g = A \frac{w}{z - a} + B, \qquad \omega = C \frac{z - a}{w} dx,$$

where $a, A, B, C \in \mathbf{C}$, $a \neq 1, -1, 0$, $A, C \neq 0$.

As the 1-forms ϕ_i, $i = 1, 2, 3$, do not have real periods, (3) and the proof of Lemma 3 give

$$\begin{array}{ll} (1) & BCf_i(a) \in \mathbf{R}, \\ (2) & Cf_i(a) = \overline{C}\,\overline{B}^2 \overline{f}_i(a) + \overline{C}\,\overline{A}^2 \overline{g}_i(a), \end{array} \qquad i = 1, 2. \tag{17}$$

If $B \neq 0$, from (1) and (2)

$$C(1 - |B|^2) f_i(a) = \frac{\overline{B} C A^2}{B} g_i(a), \qquad i = 1, 2.$$

This fact contradicts Lemma 3.

Therefore $B = 0$, and looking at (17), x is well defined if and only if

$$(f_1\overline{g}_2 - f_2\overline{g}_1)(a) = 0. \tag{18}$$

Using Lemma 2, $f_i(a)$, $g_i(a) \neq 0$, if $a \in \mathbf{C} - \{1, -1\}$, $i = 1, 2$, and then (18) involves

$$\log\left|\frac{f_1 g_2}{f_2 g_1}\right|(a) = 0.$$

Consider $h : \Omega \to \mathbf{R}^3$ defined by

$$h(a) = \log\left|\frac{f_1 g_2}{f_2 g_1}\right|(a).$$

From Lemma 2, h is a continuous function on Ω, harmonic in $\overset{\circ}{\Omega}$. Using (4), $h(a) = 0$ if $a \in i\mathbf{R} \cup \{\infty_1, \infty_2\}$, and if $h(a) = 0$ then $h(\overline{a}) = h(-a) = 0$.

On the other hand, Lemma 4 shows that $h(a_i) \neq 0$, where $a_i \in l_i$, $a_i \neq \infty_i$, $i = 1, 2$.

Maximum principle for harmonic functions yields

$$h(a) = 0 \quad \text{if and only if} \quad a \in i\mathbf{R} \cup \{\infty_1, \infty_2\}.$$

But if $a \in i\mathbf{R}$, Lemma 5 tells us that (18) holds if and only if $a = 0$. This completes the proof of the theorem. Q.E.D.

The surface in Theorem 1 was discovered by Chen and Gackstatter, and we label it as a Chen-Gackstatter surface.

2.2. *Second case.* $b_g(P) = 1$. As in the first case, we need some previous results.

Lemma 6. *If* $a_i \in l_i - \{1, -1, \infty_i\}$, $i = 1, 2$,

$$\operatorname{Im}(3g_1g_2 - 4f_1f_2 + 2a(g_1f_2 + g_2f_1))(a_i) \neq 0.$$

Proof. Define

$$j(a_i) = \operatorname{Im}(3g_1g_2 - 4f_1f_2 + 2a(g_1f_2 + g_2f_1))(a_i), \qquad a_i \in l_i - \{\infty_i\}, \ i = 1, 2.$$

Using (4) and Remark 4, it is sufficient to prove

$$j(a_1) = (3\operatorname{Re}(g_1)g_2 - 4\operatorname{Re}(f_1)f_2 + 2a(\operatorname{Re}(g_1)f_2 + \operatorname{Re}(f_1)g_2))(a_1) \neq 0$$

if $a_1 \in]1, +\infty_1[\subset l_1$.

From Remark 5,

$$j'(a) = \frac{-2a}{1 - a^2}\left(j(a) + \frac{a^2 - 1}{a}(\operatorname{Re}(g_1)f_2 + \operatorname{Re}(f_1)g_2)(a)\right) \tag{19}$$

and looking at (7),

$$\lim_{a_1 \to 1} -i\frac{j(a_1)}{a_1 - 1} > 0. \tag{20}$$

If j vanishes at some point $a_1' \in]1, \infty_1[$, take a_0 the first such point, $a_0 > 1$. Remark 6 and (19) gives $-ij'(a_0) > 0$, and therefore $-ij(a_1) < 0$, for each $a_1 \in]1, a_0[$, which is contrary to (20). Q.E.D.

Lemma 7. *If* $a \in i\mathbf{R}$, $|a| \geq \sqrt{3}$,

$$\mathrm{Re}(3g_1^2(a) - 4f_1^2(a) + 4af_1(a)g_1(a)) \neq 0.$$

Proof. Fixing the same notation as Lemma 5, if $iy \in i\mathbf{R}$, define

$$\begin{aligned} k(y) &= \mathrm{Re}(3g_1^2(iy) - 4f_1^2(iy) + 4iyf_1(iy)g_1(iy)) \\ &= 3(u_1^2(y) - v_1^2(y)) + 4(s_1^2(y) - t_1^2(y)) - 4y(u_1(y)s_1(y) + t_1(y)v_1(y)) \end{aligned}$$

where $y \in \mathbf{R}$.

From (12),

$$k'(y) = \frac{2y}{1+y^2}\left(k(y) - 2\frac{1+y^2}{y}(u_1(y)s_1(y) + t_1(y)v_1(y))\right). \tag{21}$$

On the other hand, the function $s:]-\infty, 0] \to \mathbf{R}$ defined by

$$s(a) = (\sqrt{3}g_1 - 2f_1)(a)$$

satisfy (see (7) and Remark 5)

$$s'(a) = \frac{1}{1-a^2}\left((a+\sqrt{3})f_1(a) - \frac{3}{2}(1+\sqrt{3}a)g_1(a)\right),$$

$$s''(a) = \frac{1}{2(a^2-1)}f_1(a) + \frac{3\sqrt{3}}{4(a^2-1)}g_1(a), \qquad s'(-1) < 0, \ s(-1) = 0.$$

Thus, Remark 4 yields

$$s''(a) < 0 \quad \text{for each } a \in]-1, 1[, \qquad s'(0)s(0) < 0,$$

and then, $s(0) < 0$.

Thus

$$k(0) = s(0)(\sqrt{3}g_1(0) + 2f_1(0)) > 0. \tag{22}$$

Observe now

$$k(\sqrt{3}) = (\sqrt{3}g_1 + 2if_1)^2(i\sqrt{3}).$$

If we write

$$H(x) = \frac{ix + 1/\sqrt{3}}{\sqrt{(x - i\sqrt{3})(x^2-1)}}$$

from (3) and integrating by parts

$$(\sqrt{3}g_1 + 2if_1)(i\sqrt{3}) = 4\int_{-1}^{i\sqrt{3}} H(x)\,dx$$

for a suitable choice of the above branch of $\sqrt{(x - i\sqrt{3})(x^2-1)}$.

If $x \in]-1, 0[$, it is straightforward to check

$$0 < \mathrm{Re}(H)(x) < -\mathrm{Im}(H)(x). \tag{23}$$

After a suitable change of parameter

$$\int_0^{i\sqrt{3}} H(x)\,dx = 4(1+i)3^{-1/4}\int_{-1}^{1/2} \frac{u}{\sqrt{u^3+1}}\,du$$

and therefore

$$\text{(24)} \qquad \operatorname{Re}\int_0^{i\sqrt{3}} H(x)\,dx = \operatorname{Im}\int_0^{i\sqrt{3}} H(x)\,dx < 0.$$

So, (23) and (24) yield

$$\text{(25)} \qquad k(\sqrt{3}) < 0.$$

Remark 7 and (21) involve that k vanishes only at one point $y_0 > 0$.
But (25) and (22) give $k(0)k(\sqrt{3}) < 0$. Therefore $y_0 \in]0, \sqrt{3}[$ and $k(y) \neq 0$, $y \geq \sqrt{3}$. From (4), also $k(y) \neq 0$, $y \leq -\sqrt{3}$. Q.E.D.

Lemma 8. *If* $a \in]-\infty, 1[, a \neq -1$,

$$\frac{3g_1^2(a) - 4f_1^2(a) + 4af_1g_1(a)}{1 - a^2} > 0.$$

Proof. By Remark 2, $k = 3g_1^2 - 4f_1^2 + 4af_1g_1$ is well defined on $]-\infty, 1[$.
Remark 5 gives

$$\text{(26)} \qquad k'(a) = \frac{2a}{a^2 - 1}\left(k(a) + \frac{2(a^2 - 1)}{a} f_1(a)g_1(a)\right).$$

Moreover, from (5) and (7)

$$\left(\frac{k}{(a+1)^2}\right)(-1) = 0, \qquad \left(\frac{k}{(a+1)^2}\right)'(-1) > 0,$$

and then

$$\text{(27)} \qquad k(-1-\varepsilon) < 0, \qquad k(-1+\varepsilon) > 0,$$

$\varepsilon > 0$ small enough.
Suppose k vanishes at a point $a' < -1$, and take $a_0 < -1$ the nearest point to -1 such that $k(a_0) = 0$.
From (26), $k'(a_0) > 0$ and then $k(a) > 0$, $a \in]a_0, -1[$, which is contrary to (27).
So, $k(a) < 0$ for each $a \in]-\infty, -1[$.
Analogously, $k(a) > 0$, $a \in]-1, 1[$, and the lemma holds. Q.E.D.

Lemma 9. *For each* $a \in \Omega - \{\infty_1, \infty_2, 1, -1\}$,

$$(3g_1^2(a) - 4f_i^2(a) + 4af_i(a)g_i(a)) \neq 0.$$

Proof. Notice that from (4), we can suppose $i = 1$.
Define now $\Omega'_+ = \{a \in \Omega \mid \operatorname{Im}(a) > 0\}$ and $\Omega_+ = \overline{\Omega'_+}$.
Using again (4), we need only to prove the lemma for $a \in \Omega_+$, $a \neq \infty_1$, 1, -1.
First, we will show that $\gamma : \partial\Omega_+ \to \mathbf{C}$ defined by $\gamma(a) = f_1(a)/g_1(a)$ is a single closed curve.
For, we must have

$$\text{(28)} \qquad \gamma(b) \neq \gamma(c), \quad b, c \in \partial\Omega_+, \quad b \neq c.$$

Note that (see (8))

$$\lim_{|a| \to +\infty} \left|\frac{f_1}{g_1}\right|(a) = 0.$$

Use Lemma 1 to obtain

(29) $$\left(\frac{f_1}{g_1}\right)' = \frac{3g_1^2 - 4f_1^2 + 4ag_1f_1}{4(1-a^2)g_1^2}.$$

Hence, it follows from Remark 5 and Lemma 8 that γ is injective on $[-\infty, 1]$.
Observe now that Remark 4 and Lemma 3 imply

$$\mathrm{Im}\left(\frac{f_1}{g_1}\right)(a_1) \neq 0, \quad \text{for each } a_1 \in [1, +\infty_1[.$$

Since $\gamma(a) \in \mathbf{R}$, $a \in [-\infty, 1]$, it remains only to check (28) for $b, c \in]1, +\infty_1[$.

If $a_1 \in]1, +\infty_1[$, Remark 4 yields

$$\mathrm{Arg}\left(\frac{f_1}{g_1}\right)(a_1) = \mathrm{Arg}\left(\frac{\mathrm{Re}(f_1)\,\mathrm{Re}(g_1) - |f_2||g_2|}{\mathrm{Re}(f_1)|g_2| + \mathrm{Re}(g_1)|f_2|} - i\right)(a_1)$$

for each $a_1 \in]1, +\infty_1[$.

Let b denote the function

$$b:]1, +\infty_1[\to \mathbf{R},$$
$$b(a_1) = \frac{\mathrm{Re}(f_1)\,\mathrm{Re}(g_1) - |f_2||g_2|}{\mathrm{Re}(f_1)|g_2| + \mathrm{Re}(g_1)|f_2|}(a_1).$$

By Lemma 3

$$(\mathrm{Re}(f_1)|g_2| + \mathrm{Re}(g_1)|f_2|)(a_1) = -\frac{4\pi}{3}(a_1^2 - 1), \qquad a_1 \in]1, +\infty_1[,$$

and thus using Remarks 4 and 5

$$b'(a_1) = \frac{3}{4\pi}\frac{(3/4)(\mathrm{Re}(g_1)^2 + |g_2|^2)(a_1) + (\mathrm{Re}(f_1)^2 + |f_2|^2)(a_1)}{(a_1^2-1)^2} > 0.$$

Therefore b is injective, then $\mathrm{Arg}(f_1/g_1)$ so is and (28) holds.

To conclude the lemma, notice that by (29)

$$3g_1^2(a) - 4f_1^2(a) + 4ag_1(a)f_1(a) = 0 \quad \text{if and only if} \quad \left(\frac{f_1}{g_1}\right)'(a) = 0$$

for each $a \in \Omega_+ - \{\infty_1, 1, -1\}$.

Suppose $(f_1/g_1)'(a_0) = 0$, $a_0 \in \mathring{\Omega}_+$ and write $\alpha_0 = (f_1/g_1)(a_0)$. Then

$$2 \leq \frac{1}{2\pi i}\int_{\partial\Omega_+} \frac{(f_1/g_1)'(a)}{(f_1/g_1(a) - \alpha_0)} da = n(\gamma(\partial\Omega_+), \alpha_0)$$

where $n(\gamma(\partial\Omega_+), \alpha_0)$ is the winding number of $\gamma(\partial\Omega_+)$ around α_0.

But (28) involves $n(\gamma(\partial\Omega_+), f_1(a)/g_1) = 1$ for each $a \in \mathring{\Omega}_+$, which is contrary to our assumption.

Taking into account Remark 4 and Lemmas 6, 8, it is easy to obtain

$$3g_1^2(a) - 4f_1^2(a) + 4af_1(a)g_1(a) \neq 0$$

for $a \in \partial\Omega_+ - \{\infty_1, 1, -1\}$.

This concludes the proof. Q.E.D.

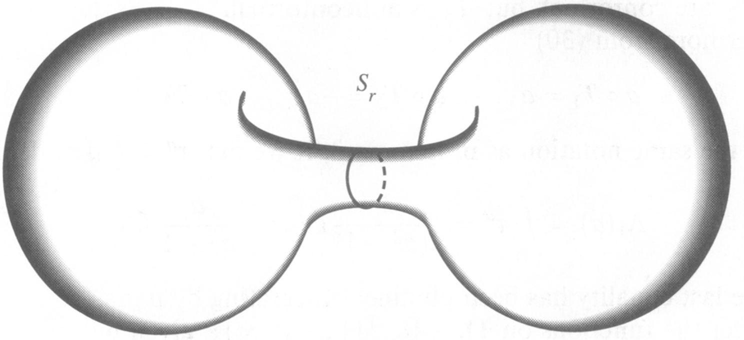

FIGURE 4

Take $r \in \mathbf{R}$ and consider the genus zero Riemann surface

$$S_r = \{(\alpha, \beta) \in (\mathbf{C} \cup \{\infty\})^2 | \alpha^2 = -3\beta^2 + 2r\beta + 3\}.$$

In a way similar to that in Lemma 3, a "concrete" representation of S_r is given by cutting and joining two copies of the sphere.

In this case, we can cut each sheet along a smooth curve joining $\frac{r}{3} - \sqrt{1 + (\frac{r}{3})^2}$ and $\frac{r}{3} + \sqrt{1 + (\frac{r}{3})^2}$, for instance, the real interval

$$\left[\frac{r}{3} - \sqrt{1 + \left(\frac{r}{3}\right)^2}, \frac{r}{3} + \sqrt{1 + \left(\frac{r}{3}\right)^2}\right].$$

See Figure 4.

Let a denote the meromorphic function on S_r defined by

$$a : S_r \to \mathbf{C}, \qquad a(\alpha, \beta) = \frac{r - 3\beta}{\alpha}. \tag{30}$$

Observe that $\text{Degree}(a) = 2$ and denote by Ω_r the closure of $a^{-1}(\mathring{\Omega})$.

Concretely, cut S_r along the two lines l^1, l^2 contained in $a^{-1}\{z \in \mathbf{R} \cup \{\infty\} | |z| \geq 1\}$:

$$l^1 = \left\{(\alpha, \beta) \in S_r | \beta \in \left[\frac{r}{3} + \frac{1}{2}\sqrt{1 + \left(\frac{r}{3}\right)^2}, \frac{r}{3} + \sqrt{1 + \left(\frac{r}{3}\right)^2}\right]\right\},$$

$$l^2 = \left\{(\alpha, \beta) \in S_r | \beta \in \left[\frac{r}{3} - \sqrt{1 + \left(\frac{r}{3}\right)^2}, \frac{r}{3} - \frac{1}{2}\sqrt{1 + \left(\frac{r}{3}\right)^2}\right]\right\}.$$

Then, open along l^1, l^2 and Ω_r is the closure of the resulting domain.

In Ω_r we have two copies of each line l^1, l^2. We will denote these copies by: l_j^1, l_j^2, $j = 1, 2$, respectively, and write $(\alpha, \beta)_1^i$, $(\alpha, \beta)_2^i$ the two points corresponding to $(\alpha, \beta) \in l^i$, $i = 1, 2$.

Denote by T_i, $i = 1, 2, 3$, the following automorphisms of S_r:

$$T_1(\alpha, \beta) = \left(-\alpha, -\beta + \frac{2r}{3}\right), \quad T_2(\alpha, \beta) = (-\alpha, \beta), \quad T_3(\alpha, \beta) = (\overline{\alpha}, \overline{\beta})$$

for each $(\alpha, \beta) \in S_r$.

T_1, T_2 are conformal, but T_3 is anticonformal.
Furthermore from (30)

$$a \circ T_1 = a, \qquad a \circ T_2 = -a, \qquad a \circ T_3 = \overline{a}. \tag{31}$$

Using the same notation as in Lemma 3, if we put $\tau^a = \frac{1}{w} dz$, define

$$\Delta_1(a) = \int_{\gamma_i} \tau^a = \frac{3}{2(a^2-1)} g_i(a) - \frac{a}{a^2-1} f_i(a) \tag{32}$$

where the last equality has been obtained integrating by parts.
Consider the functions on $\Omega_r - \{a^{-1}\{1, -1, \infty\}\}$ given by

$$\theta_i(\alpha, \beta) = \left(\left(-\sqrt{1+\left(\frac{r}{3}\right)^2} + 2\left(\beta - \frac{r}{3}\right)\right)\Delta_i(a) - \alpha f_i(a)\right), \qquad i = 1, 2,$$

$$\sigma_i(\alpha, \beta) = \left(\left(\sqrt{1+\left(\frac{r}{3}\right)^2} + 2\left(\beta - \frac{r}{3}\right)\right)\Delta_i(a) - \alpha f_i(a)\right), \qquad i = 1, 2.$$

Let $B : \Omega_r - \{a^{-1}\{1, -1\}\} \to \mathbf{R}$ defined by

$$B(\alpha, \beta) = \log\left|\frac{\theta_1\sigma_2}{\theta_2\sigma_1}\right|(\alpha, \beta). \tag{33}$$

Lemma 10. *B is a continuous function on $\Omega_r - \{a^{-1}\{1, -1\}\}$, harmonic on $\mathring{\Omega}_r$, verifying*

$$B \circ T_1 = -B, \qquad B \circ T_2 = -B, \qquad B \circ T_3 = B, \tag{34}$$

$$\lim_{|a| \to 1} |B| = +\infty. \tag{35}$$

Proof. First, note that from (32):

$$\theta_1\sigma_1(\alpha, \beta) = \frac{\alpha^2}{4(a^2-1)}(3g_i^2(a) - 4f_i^2(a) + 4af_i(a)g_i(a)), \qquad i = 1, 2.$$

By Lemma 9 and (33), B is well defined if $a(\alpha, \beta) \neq 1, -1, \infty$.
On the other hand, (30) gives

$$a^{-1}\{\infty\} = \left\{\left(0, \frac{r}{3} \pm \sqrt{1+\left(\frac{r}{3}\right)^2}\right)\right\},$$

$$a^{-1}\{1, -1\} = \left\{\left(\frac{3}{2}\sqrt{1+\left(\frac{r}{3}\right)^2}, \frac{r}{3} \pm \frac{1}{2}\sqrt{1+\left(\frac{r}{3}\right)^2}\right), \left(-\frac{3}{2}\sqrt{1+\left(\frac{r}{3}\right)^2}, \frac{r}{3} \pm \frac{1}{2}\sqrt{1+\left(\frac{r}{3}\right)^2}\right)\right\}. \tag{36}$$

Taking into account (8), (32) and (33)

$$\lim_{|a| \to \infty} B = 1$$

and B is well defined if $a = \infty$.

Define

$$\nu_i(\alpha, \beta) = \alpha g_i(a) - 2\left(\beta - \frac{r}{3} + \sqrt{1 + \left(\frac{r}{3}\right)^2}\right) f_i(a), \qquad i = 1, 2,$$

$$\mu_i(\alpha, \beta) = \alpha g_i(a) - 2\left(\beta - \frac{r}{3} - \sqrt{1 + \left(\frac{r}{3}\right)^2}\right) f_i(a), \qquad i = 1, 2.$$

Then, using (32) and (33)

$$B = \log\left|\frac{\nu_1\mu_2}{\nu_2\mu_1}\right|$$

and therefore by (7), (4) and (36), (35) holds.

To deduce (34), use (4) and (31).

It is straightforward that $\Delta B = 0$. Q.E.D.

Theorem 2. *There does not exist any orientable complete genus one minimal surface in $\mathbf{R}^3$, of finite total curvature -8π which Gauss map is singular at its unique end.*

Proof. Suppose $x : M \to \mathbf{R}^3$ is such a surface.

As usual, g, ω will denote the Weierstrass representation of x.

We know M is conformally equivalent to $\overline{M} - \{P\}$, where $\overline{M}$ is a compact genus one Riemann surface.

Since $b_g(P) = 1$, there exist three points P_1, P_2, P_3, $P \neq P_i$, $i = 1, 2, 3$, such that $b_g(P) = 1$.

Denote $c_i = g(P_i)$, $I = 1, 2, 3$. It is clear that $c_i \neq c_j$, $i \neq j$.

Therefore

$$\overline{M} = \{(z, w) \in (\mathbf{C} \cup \{\infty\})^2 | w^2 = (z - c_1)(z - c_2)(z - c_3)\}$$

and $g = z$, $\omega = A \cdot \frac{1}{w} dz$ (see (2)).

Up to a rigid motion in $\mathbf{R}^3$, we will suppose $r = c_1 + c_2 + c_3 \in \mathbf{R}$.

For γ_i, $i = 1, 2$, homology basis of $\overline{M}$, put

$$d_i = \int_{\gamma_i} \frac{1}{w} dz, \quad e_i = \int_{\gamma_i} \frac{z}{w} dz, \qquad i = 1, 2.$$

Integrating by parts

$$\int_{\gamma_i} \frac{z^2}{w} dz = \frac{2r}{3} e_i - \frac{s}{3} d_i, \qquad i = 1, 2, \tag{37}$$

where $s = c_1c_2 + c_1c_3 + c_2c_3$.

As x is well defined, from (1) and (37):

$$\overline{A}\,\overline{d}_i = A\left(\frac{2r}{3} e_i - \frac{s}{r} d_i\right), \qquad Ae_i \in i\mathbf{R}. \tag{38}$$

Since $\operatorname{Im}(d_2/d_1) \neq 0$ (see [11]), (38) yields $s = -3$, and (38) gets

$$\overline{A}\,\overline{d}_i = A\left(\frac{2r}{3} e_i + d_i\right), \qquad Ae_i \in i\mathbf{R}. \tag{39}$$

Consider $y : M \to \mathbf{R}^3$ defined by

$$y = \left(x_3 - \frac{r}{3} x_1, \sqrt{1 + \left(\frac{r}{3}\right)^2} x_2, x_1 + \frac{r}{3} x_3\right).$$

It is clear that $y = R \circ H \circ x$, where R is a rigid motion and H is a homothety.

Furthermore, the Weierstrass representation (ψ_1, ψ_2, ψ_3) of y is given by (see (37) and (1)):

$$\psi_1 = A\left(1+\left(\frac{r}{3}\right)^2\right)\frac{z}{w}\,dz + dk_1,$$
$$\psi_2 = iA\sqrt{1+\left(\frac{r}{3}\right)^2}\left(\frac{1}{w}+\frac{r}{3}\frac{z}{w}\right)dz + dk_2,$$
$$\psi_3 = dk_3,$$

for suitable k_i, $i = 1, 2, 3$, meromorphic functions on $\overline{M}$.

Thus, y is well defined if and only if

$$\begin{aligned}(40)\quad &\left(\overline{d}_1 + \left(\sqrt{1+\left(\frac{r}{3}\right)^2}+\frac{r}{3}\right)\overline{e}_1\right)\left(\overline{d}_2 + \left(-\sqrt{1+\left(\frac{r}{3}\right)^2}+\frac{r}{3}\right)e_2\right) \\ &= \left(\overline{d}_2 + \left(\sqrt{1+\left(\frac{r}{3}\right)^2}+\frac{r}{3}\right)\overline{e}_2\right)\left(d_1 + \left(-\sqrt{1+\left(\frac{r}{3}\right)^2}+\frac{r}{3}\right)e_1\right).\end{aligned}$$

Put $c_0 = -c_1c_2c_3$ and take $(\alpha_0, \beta_0) \in S_r$ such that

$$\alpha_0^2(r - 3\beta_0) = \beta_0^3 - r\beta_0^2 - 3\beta_0 + c_0.$$

Then, consider the following change of parameter:

$$(41)\qquad x = \frac{1}{\alpha_0}z - \frac{\beta_0}{\alpha_0}.$$

Hence, using (3), (30) and (41), (40) becomes

$$(42)\qquad (\theta_1\overline{\sigma}_2 - \theta_2\overline{\sigma}_1)(\alpha_0, \beta_0) = 0.$$

Note that $a_0 = a(\alpha_0, \beta_0) \neq 1, -1, \infty$ because of $c_i \neq c_j$, $i \neq j$.

By (32) and (30)

$$\theta_i\sigma_i(\alpha_0, \beta_0) = \frac{\alpha_0^2}{4(a_0^2-1)}(3g_i^2(a_0) - 4f_i^2(a_0) + 4a_0f_i(a_0)g_i(a_0)), \qquad i = 1, 2,$$

and therefore Lemma 9 gives $\theta_i(\alpha_0, \beta_0)$, $\sigma_i(\alpha_0, \beta_0) \neq 0$, $i = 1, 2$.

So, (42) involves $B(\alpha_0, \beta_0) = 0$.

On the other hand, define

$$\Gamma = \left\{(\alpha, \beta) \in \Omega_r | \beta \in \left(\mathbf{R} - \left]\frac{r}{3} - \sqrt{1+\left(\frac{r}{3}\right)^2}, \frac{r}{3} + \sqrt{1+\left(\frac{r}{3}\right)^2}\right[\cup \{\infty\}\right)\right\}.$$

We observe (see (30)) that $\Gamma = a^{-1}(\{z \in i\mathbf{R} \cup \{\infty\} | |a| \geq \sqrt{3}\})$. Using (34)

$$B(\alpha, \beta) = 0 \quad \text{if } (\alpha, \beta) \in \Gamma.$$

We will show that B vanishes only on Γ.

If $(\alpha, \beta) \in l_j^i$, $i = 1, 2$, $j = 1, 2$, and $\alpha \neq 0$, we have $\alpha, \beta \in \mathbf{R}$. Suppose $a(\alpha, \beta) > 1$. Using Remark 4, (32) and Lemma 3, $B(\alpha, \beta)_1^1 = 0$ if and only if

$$\begin{aligned}&(3g_2\,\mathrm{Re}(g_1) - 4f_2\,\mathrm{Re}(f_1))(a_1) \\ &\qquad - 2a_1(\mathrm{Re}(g_1)f_2 + \mathrm{Re}(f_1)g_2)(a_1) = 0.\end{aligned}$$

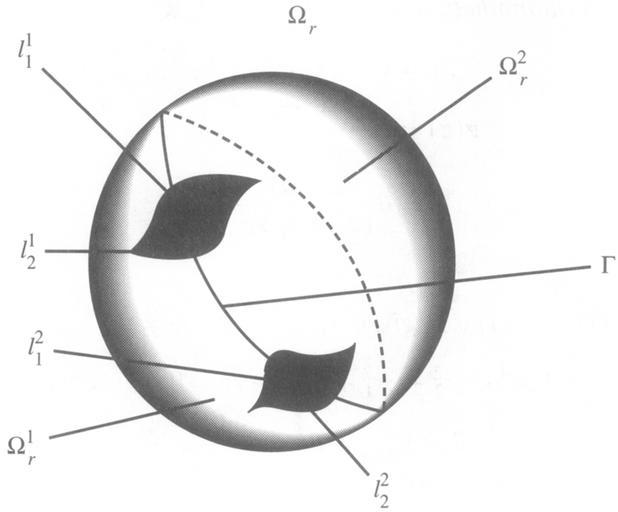

FIGURE 5

So, (34) and Lemma 6 get $B(\alpha, \beta) \neq 0$, $(\alpha, \beta) \in l_j^i$, $i = 1, 2$, $j = 1, 2$, $\alpha \neq 0$.

Let Ω_r^1, Ω_r^2 denote the two components of $\Omega_r - \Gamma$, and note that $\partial\Omega_r^j \subset \Gamma \cup l_1^1 \cup l_1^2 \cup l_2^1 \cup l_2^2$ (see Figure 5).

If $B(\alpha, \beta) = 0$, $(\alpha, \beta) \notin \Gamma$, we have seen $(\alpha, \beta) \notin l_j^i$, $i = 1, 2$, $j = 1, 2$, and therefore (α, β) is an interior point of Ω_r^1 or Ω_r^2.

The symmetries in (34), together with the Maximum Principle for harmonic functions yields a contradiction.

So, if $(\alpha, \beta) \in \Omega_r$,

$$B(\alpha, \beta) = 0 \quad \text{if and only if} \quad (\alpha, \beta) \in \Gamma. \tag{43}$$

Thus $(\alpha_0, \beta_0) \in \Gamma$, and obviously $\alpha_0 \neq 0$.

Since $\alpha_0 \in i\mathbf{R}$, $\beta_0 \in \mathbf{R}$, then $a_0 \in i\mathbf{R}$, and using (4), (32) and (42) we get

$$\operatorname{Re}(3f_1^2(a_0) - 4g_1^2(a_0) + 4a_0 f_1(a_0) g_1(a_0)) = 0 \tag{44}$$

which is contrary to Lemma 7.

This fact completes the proof. Q.E.D.

3. GENUS ZERO MINIMAL SURFACES OF TOTAL CURVATURE -8π

In the following, we will suppose $\gamma = \text{Genus}(M) = 0$.

Looking at Table 1, we can distinguish three different cases: $k = 1$, $k = 2$ and $k = 3$.

First case. $k = 3$. Jorge-Meeks formula (2) yields $I_1 = I_2 = I_3 = 1$.

After a suitable change of parameter in $\overline{M} = \mathbf{C} \cup \{\infty\}$, suppose that $1/\sqrt{3}$, $-1/\sqrt{3}$ and ∞ are the three ends of M, and up to rigid motion in $\mathbf{R}^3$, $g(\infty) = \infty$, $b_g(\infty) = 0$.

Theorem 3. *Up to homothety and rigid motion in* $\mathbf{R}^3$:

$$(45)\qquad \begin{gathered} M = \mathbf{C} - \{1/\sqrt{3}, -1\sqrt{3}\}, \\ g(z) = B\frac{z^2+cz+d}{z+a}, \\ \omega = \theta\frac{(z+a)^2}{(z^2-1/3)^2}\,dz \end{gathered}$$

where

(1) *if* $a \neq 1/\sqrt{3},\ -1/\sqrt{3}$, *given* $r_1,\ r_2 \in \mathbf{R},\ r_2 \neq 0$,

$$(46)\qquad \begin{gathered} c = 0, \qquad 12a^4 - (r_2^2 + 3r_1^2 + 4)a^2 - r_1^2 = 0, \qquad a^2(1-3d)^2 = r_1^2, \\ \theta = 1, \qquad B^2 = \frac{3|3a^2-1|^2}{r_2^2}, \end{gathered}$$

(2)

$$(47)\qquad c = 0, \quad a = 1/\sqrt{3}, -1/\sqrt{3}, \quad d = 1, \quad \theta = 1, \quad B \in \mathbf{R} - \{0\},$$

Proof. Up to rigid motion and homothety, we can assume (45), where $B \in \mathbf{R} - \{0\}$ and $|\theta| = 1$.

Looking at (1), since x is well defined (that is, ϕ_k, $k = 1, 2, 3$, do not have real periods):

$$(48)\qquad \begin{gathered} c = 0, \qquad 3\overline{a}^2 - 1 = \theta^2 B^2(1+3d)(1-d), \\ \mathrm{Im}(\theta) = 0, \qquad \mathrm{Im}(\theta a(1-3d)) = 0. \end{gathered}$$

Thus, we can suppose $\theta = 1$.

If $a \neq 1/\sqrt{3}, -1/\sqrt{3}$, then $B^2 = (3\overline{a}^2 - 1)/((1+3d)(1-d))$.

Writing $r_1 = a(1-3d)$ and $r_2^2 = 3(3a^2-1)(1+3d)(1-d)$, (46) holds.

If $a = 1/\sqrt{3}$, $-1/\sqrt{3}$, (48) gives $d = 1$ or $d = -1/3$.

Since $\mathrm{Degree}(g) = 2$, $d \neq -1/3$ (see (45)). By (48), it is easy now to conclude the lemma. Q.E.D.

Definition 1. Denote by $\mathscr{F}$ the family of surfaces given by (45)–(46) and (47) satisfying $(3d-1)^2 \neq 12a^2$.

Geometrically, $\mathscr{F}$ is the family of genus zero orientable complete minimal surfaces of finite total curvature -8π and three catenoid ends. Here, a catenoid end means an embedded end asymptotic to a catenoid. An embedded end P_i is a catenoid end when the Gauss map g is regular at P_i (see [5]).

The Jorge-Meeks surface of degree 3 (the trinoid) is the first interesting example in $\mathscr{F}$.

3.2. *Second case.* $k = 2$. We will suppose $M = \mathbf{C} - \{0\}$, that this, $P_1 = \infty$ and $P_2 = 0$ are the two ends of M. Moreover, we can assume $g(\infty) = \infty$.

By formula (2) again, we have two possibilities: $I_1 = I_2 = 2$ or $I_1 = 1$, $I_2 = 3$.

The following theorems are consequences of similar arguments that are given in Theorem 3.

Theorem 4. *Suppose* $I_1 = I_2 = 2$. *Then, up to change of parameter in* $\mathbf{C} \cup \{\infty\}$, *homothety and rigid motion in* $\mathbf{R}^3$:

(1) *If* g *has a regular end* (*without loss of generality, assume* ∞ *is such an end*), $g(0) \neq \infty$,

$$M = \mathbf{C} - \{0\}, \quad g(z) = B\frac{z^2 + cz + d}{z+1}, \quad \omega = \theta\frac{(z+1)^2}{z^3}\,dz,$$

where

$$|\theta| = 1, \quad B \in \mathbf{R} - \{0\}, \quad -1 = \theta^2 B^2(c^2 + 2d), \quad \theta(1+c) \in \mathbf{R}, \quad 1 - c + d \neq 0,$$

and if $b_g(\infty) = 0$, $g(0) = \infty$,

$$M = \mathbf{C} - \{0\}, \quad g(z) = B\frac{z^2 + cz + 1}{z}, \quad \omega = \theta\frac{1}{z}\,dz,$$

where

$$\theta = i, \quad B \in \mathbf{R} - \{0\}, \quad -1 = B^2(c^2 + 2).$$

(2) *If* $b_g(\infty) = b_g(0) = 1$,

$$M = \mathbf{C} - \{0\}, \quad g(z) = Bz^2, \quad \omega = \theta\frac{1}{z^3}\,dz,$$

where $B \in \mathbf{R} - \{0\}$, $\theta = 1$.

Theorem 5. *Suppose* $I_1 = 1$ *and* $I_2 = 3$. *Then, up to change of parameter in* $\mathbf{C} \cup \{\infty\}$, *homothety and rigid motion in* $\mathbf{R}^3$:

(1) *If* $b_g(\infty) = 0$, $g(0) \neq \infty$,

$$M = \mathbf{C} - \{0\}, \quad g(z) = B\frac{z^2 + d}{z+1}, \quad \omega = \theta\frac{(z+1)^2}{z^4}\,dz$$

where $B \in \mathbf{R} - \{0\}$, $\theta = 1$, $d \in \mathbf{C} - \{-1\}$, *and if* $b_g(\infty) = 0$, $g(0) = \infty$,

$$M = \mathbf{C} - \{0\}, \quad g(z) = B\frac{z^2 + 1}{z}, \quad \omega = \theta\frac{1}{z^2}\,dz$$

where $B \in \mathbf{R} - \{0\}$, $\theta = 1$.

(2) *If* $b_g(\infty) = 1$, $g(0) \neq 0$,

$$M = \mathbf{C} - \{0\}, \quad g(z) = B(z^2 + 1), \quad \omega = \theta\frac{1}{z^4}\,dz$$

where $B \in \mathbf{R} - \{0\}$, $\theta = 1$ *and if* $b_g(\infty) = 1$, $g(0) = 0$,

$$M = \mathbf{C} - \{0\}, \quad g(z) = Bz^2, \quad \omega = \theta\frac{1}{z^4}\,dz$$

where $B \in \mathbf{R} - \{0\}$, $\theta = 1$.

3.3. *Third case.* $k = 1$. In this case, (2) involves $I_1 = 5$.

Assume that ∞ is the unique end of M, and up to rotation in $\mathbf{R}^3$, $g(\infty) = \infty$. As before, the following theorem holds.

Theorem 6. *Up to change of parameter in* $\mathbf{C}\cup\{\infty\}$, *homothety and rigid motion in* $\mathbf{R}^3$:

(1) *If* $b_g(\infty)=0$,

$$M=\mathbf{C}, \quad g(z)=B\frac{z^2+cz+1}{z}, \quad \omega=\theta z^2\,dz$$

where

$$B\in\mathbf{R}-\{0\}, \quad c, \theta\in\mathbf{C}, \qquad |\theta|=1.$$

(2) *If* $b_g(\infty)=1$,

$$M=\mathbf{C}, \quad g(z)=B(z^2+c), \quad \omega=\theta\,dz$$

where

$$B\in\mathbf{R}-\{0\}, \quad c, \theta\in\mathbf{C}, \quad |\theta|=1.$$

4. Statement of result

An Osserman classical result (see [9]) classifies the catenoid and Enneper surface as the unique complete minimal surfaces with total curvature -4π.

Thus, Theorems 1, 2, 3, 4, 5, and 6 imply our main result:

Corollary 1. *Let* M *be an orientable complete minimal surface in* $\mathbf{R}^3$ *of finite total curvature greater than* -12π.

Then, M *is one of the following surfaces: a plane, a catenoid, Enneper surface, a surface described in Theorems* 3, 4, 5, 6, *or Chen-Gackstatter surface.*

5. A geometric consequence

Finally, we prove the following fact, which is related to Osserman-Mo theorem in [7].

Corollary 2. *Let* M *be a orientable complete minimal surface in* $\mathbf{R}^3$.

If the Gauss map g *takes on five distinct values* (*without counting multiplicities*) *at most once, then* M *is one of the following surfaces: the plane, the catenoid, Enneper surface, a surface in the family* $\mathcal{F}$ *or Chen-Gackstatter surface.*

Proof. First, by Osserman-Mo theorem in [7], M has finite total curvature.

Since $M=\overline{M}-\{P_1,\dots,P_k\}$ where $\overline{M}$ is a compact Riemann surface, and g extends meromorphically to $\overline{M}$, we can define the total branching number V of g by

$$V=\sum_{P\in\overline{M}} b_g(P).$$

It is well known (see [11]) that

$$V=2n+2\gamma-2 \tag{52}$$

where, as usual, $\gamma=\text{Genus}(\overline{M})$ and $n=\text{Degree}(g)$.

Write by b_i, $i=1,\dots,5$, the five points in $\mathbf{S}^2$ such that

$$\text{Card}((g^{-1}\{b_i\})\cap M)\leq 1, \tag{53}$$

$i=1,\dots,5$, where "Card" means cardinality.

By (53),

(54) $$k+5 \geq \text{Card}(g^{-1}\{b_1, \ldots, b_5\}) \geq 5n - V.$$

Then (54) and (2) yield

(55) $$2k+\gamma \leq 6.$$

If $\gamma \geq 5$, (55) gives $k=0$, a contradiction. Assume $\gamma = 3, 4$. In this case, (55) involves $k=1$. Looking at (53), we deduce

$$V \geq (n-2) + 4(n-1),$$

that is (see (52)),

(56) $$2n \leq 2\gamma + 4.$$

Hence, using (2) and (56), we conclude $I_1 = 1$, a contradiction.

In a similar way, if $\gamma = 2$, then $k = 2$ and $I_1 = I_2 = 1$ or $k = 1$ and $I_1 = 1$. Schoen (see [10]) characterized first surfaces as catenoids and pairs of planes. So, this case is also impossible.

Suppose now $\gamma = 1$.

From (54) and (2), $k = 2$ and $I_1 = I_2 = 1$ or $k = 1$ and $I_1 = 3$.

As before, the first case gets a contradiction. Taking into account Theorems 1 and 2 and (2), the second case corresponds to Chen-Gackstatter surface.

Finally, $\gamma = 0$ involves (see (54) and (2)):

(1) $k = 1$ and $I_1 = 3$, which gives Enneper's surface (see [9]).
(2) $k = 2$ and $I_1 = I_2 = 1$, that is, a catenoid (see [9]).
(3) $k = 3$ and $I_1 = I_2 = I_3 = 1$. Observe that (2) yields $n = 2$, and (53) implies $b_g(P_i) = 0$, $i = 1, 2, 3$. These are the surfaces in $\mathscr{F}$.

This fact completes the proof. Q.E.D.

Note added in proof. The author has learned that D. Bloss has obtained Theorems 1 and 2 using different techniques (D. Bloss, *Elliptische Funktionen und Vollständige Minimalflächen*, Ph.D. thesis, Freien Universität Berlin, Berlin, November, 1989.)

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