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# UNIVERSAL SPACES FOR R-TREES

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ABSTRACT. **R**-trees arise naturally in the study of groups of isometries of hyperbolic space. An **R**-tree is a uniquely arcwise connected metric space in which each arc is isometric to a subarc of the reals. It follows that an **R**-tree is locally arcwise connected, contractible, and one-dimensional. Unique and local arcwise connectivity characterize **R**-trees among metric spaces. A universal **R**-tree would be of interest in attempting to classify the actions of groups of isometries on **R**-trees. It is easy to see that there is no universal **R**-tree. However, we show that there is a universal separable **R**-tree $T_{\aleph_0}$. Moreover, for each cardinal $\alpha$, $3 \leq \alpha \leq \aleph_0$, there is a space $T_\alpha \subset T_{\aleph_0}$, universal for separable **R**-trees, whose order of ramification is at most $\alpha$. We construct a universal smooth dendroid $D$ such that each separable **R**-tree embeds in $D$; thus, has a smooth dendroid compactification. For nonseparable **R**-trees, we show that there is an **R**-tree $X_\alpha$, such that each **R**-tree of order of ramification at most $\alpha$ embeds isometrically into $X_\alpha$. We also show that each **R**-tree has a compactification into a smooth arboroid (a nonmetric dendroid). We conclude with several examples that show that the characterization of **R**-trees among metric spaces, rather than, say, among first countable spaces, is the best that can be expected.

## 1. INTRODUCTION

1.1. **R-trees.** An **R**-*tree* $(X, d)$ is a uniquely arcwise connected metric space in which each arc is isometric to a subarc of the reals. [1] **R**-trees arise naturally in the study of groups of isometries of hyperbolic space. Actions on **R**-trees can be seen as ideal points in the compactification of groups of isometries [Mr, Be, MrS]. The purpose of this paper, and a preceding paper of two of the authors [MO], is to better understand **R**-trees topologically.

We call the metric $d$ on an **R**-tree an **R**-*tree metric.* A metric topologically equivalent to $d$ need not have the property that each arc is isometric to a subarc of the reals. We also call the metric $d$ on an **R**-tree a *convex* metric since $d(x, y) = d(x, t) + d(t, y)$ for all $x, y \in X$ and all $t \in [x, y]$.

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[1] Ed Tymchatyn has observed that "uniquely" is superfluous in this definition.

Morgan [Mr, Proposition 1.5] showed that an $\mathbf{R}$-tree is uniquely arcwise connected, locally arcwise connected, contractible, and one-dimensional. In [MO] it is shown that a metric space has an equivalent metric for which it is an $\mathbf{R}$-tree iff it is locally arcwise connected and uniquely arcwise connected, thus characterizing $\mathbf{R}$-trees among metric spaces. With the ultimate goal of classifying group actions by isometries on $\mathbf{R}$-trees, one would like to know if there is a universal $\mathbf{R}$-tree. (See §1.3 below for definitions.)

1.2. **Main results.** We show in Theorem 1.8 that, because of the possibility of unlimited branching, there is no universal $\mathbf{R}$-tree. We provide a proof for Theorem 1.11 [AB, B]: each $\mathbf{R}$-tree admits an isometric completion to an $\mathbf{R}$-tree, so that the proof of Theorem 2.3 may be self-contained in our paper. In §2, we restrict ourselves to separable metric spaces. A separable $\mathbf{R}$-tree has order of ramification at most $\aleph_0$, i.e., countably infinite (Proposition 1.7). In Theorem 2.3, we show that for each cardinal $\alpha$, $3 \leq \alpha \leq \aleph_0$, there is an $\mathbf{R}$-tree $T_\alpha$ universal for separable $\mathbf{R}$-trees whose order of ramification is at most $\alpha$. In particular (Corollary 2.4), there is a universal separable $\mathbf{R}$-tree $T_{\aleph_0}$ which contains $T_\alpha$ for all $\alpha$, $3 \leq \alpha \leq \aleph_0$, isometrically.[2] In general, the embedding of an arbitrary separable $\mathbf{R}$-tree $X$ into the universal separable $\mathbf{R}$-tree $T_{\aleph_0}$ is not isometric.

In §3, we drop the separability assumption, and show in Theorem 3.4 that for each cardinal number $\alpha$, there is an $\mathbf{R}$-tree $X_\alpha$ universal for all $\mathbf{R}$-trees of order of ramification at most $\alpha$. The embedding we construct of an arbitrary $\mathbf{R}$-tree $X$ of order at most $\alpha$ into $X_\alpha$ is isometric. The results in this section follow directly from the work of Nikiel [N]. Indeed, the theorems of §2 could have been proved using the results in [N]. However, we felt that less general constructions and proofs would be welcome in the separable case.

The results in §§2 and 3 can be extended to $\mathbf{A}$-trees, where $\mathbf{A}$ is a subgroup of $\mathbf{R}$ (e.g., $\mathbf{Z}$ or $\mathbf{Q}$). By saying $X$ is an $\mathbf{A}$-tree we mean $X$ is an $\mathbf{R}$-tree with branching allowed only at those points $x \in X$ at $\mathbf{A}$-levels. (See the definitions of branch point and level function in §§1.3 and 1.10.) For example, there is a universal (separable) $\mathbf{Z}$-tree and a universal (separable) $\mathbf{Q}$-tree. If $\mathbf{A}$ is countable, then the embedding of an arbitrary separable $\mathbf{A}$-tree into the universal separable $\mathbf{A}$-tree can be chosen to be an isometry.

In §4, we consider compactifications of $\mathbf{R}$-trees. Our construction of the universal smooth dendroid $D$ was inspired by the similar construction of a universal smooth dendroid by Mohler and Nikiel [MoN]. We show that $T_{\aleph_0} \subset D$. Consequently, each separable $\mathbf{R}$-tree $X$ embeds in $D$; thus, has a compactification into a smooth dendroid. The embedding of $X$ into $D$ is generally not an isometry. Since $D$ is not locally connected, it is not a universal space for separable $\mathbf{R}$-trees, merely a containing space for them. For the nonseparable case, we show that each $\mathbf{R}$-tree admits a one-dimensional compactification into a smooth arboroid, which, however, is generally not metric.

In §5, we construct a number of examples, mostly of spaces which are not $\mathbf{R}$-trees. In [Mr] Morgan asked if every first countable space $X$ having the properties that it was uniquely arcwise connected, locally arcwise connected, contractible, and one-dimensional admited a metric making it an $\mathbf{R}$-tree, and if not, what additional properties must be assumed? Our object in this section is

[2] The question of the existence of a universal separable $\mathbf{R}$-tree was raised by John J. Walsh.

to show that the Characterization Theorem [MO] (Theorem 1.5 below), characterizing **R**-trees among metric spaces, rather than among first countable spaces, is the best that can be expected.

An important problem not resolved in this paper is to determine to what extent group actions on the universal separable **R**-tree $T_{\aleph_0}$ classify group actions on separable **R**-trees.

1.3. **Definitions and properties of R-trees.** As usual, $\mathbf{R}$, $\mathbf{Q}$, and $\mathbf{Z}$ denote the reals, rationals, and integers, respectively. By $\mathbf{R}^+$ we mean $(0, \infty)$, and similarly for $\mathbf{Q}^+$ and $\mathbf{Z}^+$. By $\mathbf{R}^*$ we mean $[0, \infty)$, and similarly for $\mathbf{Q}^*$ and $\mathbf{Z}^*$. By $\aleph_0$ we denote the first infinite cardinal.

A topological space $X$ is a *universal space* for a class $\mathscr{C}$ of spaces iff $X \in \mathscr{C}$ and every member of $\mathscr{C}$ embeds in $X$.

Suppose $X$ is an arcwise connected metric space. Let $p \in X$. By $\mathrm{ram}(p)$ we denote the *order of ramification* of $X$ at $p$; that is, the maximal cardinality of a family of arcs in $X$ disjoint except for their common endpoint $p$. By $\mathrm{Br}(X)$ we denote the *branch set* of $X$, the set of all points $p$ such that $\mathrm{ram}(p) \geq 3$. By $\mathrm{End}(X)$ we denote the *endpoint set* of $X$, the set of all points $p$ such that $\mathrm{ram}(p) = 1$. By $\mathrm{ram}(X)$ we denote $\sup\{\mathrm{ram}(p) \mid p \in X\}$.

In a uniquely arcwise connected space $X$, if $x, y \in X$, by $[x, y]$ we denote the unique arc in $X$ whose endpoints are $x$ and $y$. A *ray* with endpoint $p$ is the union of a nested family $\mathscr{A}$ of arcs in $X$ such that each $A \in \mathscr{A}$ has $p$ as an endpoint and no $A \in \mathscr{A}$ contains $\bigcup \mathscr{A}$. If $X$ is an **R**-tree, then each ray is isometric to $[0, r)$ for some $r \in \mathbf{R}^+ \cup \{\infty\}$. A *maximal ray* is the union of a maximal (with respect to inclusion) such nested family of arcs in $X$.

1.4. **Theorem** (Proposition 1.5 of [Mr]). *Let $(X, d)$ be an* **R**-*tree. Then $X$ is locally arcwise connected, uniquely arcwise connected, contractible, and 1-dimensional.*

1.5. **Theorem** (Characterization Theorem; Theorem 5.1 of [MO]). *Let $(X, \rho)$ be a metric space. Then the following are equivalent*:

(1) *$X$ admits an equivalent metric $d$ such that $(X, d)$ is an* **R**-*tree.*
(2) *$X$ is locally arcwise connected and uniquely arcwise connected.*

1.6. **Proposition.** *Let $X$ be a separable* **R**-*tree and $v \in X$. Then there exists a countable collection $\mathscr{J}$ of rays from $v$ such that $\bigcup \mathscr{J} = \{v\} \cup (X - \mathrm{End}(X))$.*

*Proof*. Since $X$ is separable and $X - \mathrm{End}(X)$ is dense in $X$, let $\{d_i\}_{i=1}^{\infty}$ be a dense subset of $X$ missing $\mathrm{End}(X) \cup \{v\}$. The collection $\{[v, d_i)\}_{i=1}^{\infty}$ is the required collection of rays. □

1.7. **Proposition.** *Let $X$ be a separable* **R**-*tree. Then* $\mathrm{Br}(X)$ *is countable, and* $\mathrm{ram}(X) \leq \aleph_0$.

*Proof*. Since $X$ is separable, each ray of the countable collection $\mathscr{J}$ in Proposition 1.6 contains only countably many branch points, and each branch point has only countably many branches. □

1.8. **Theorem.** *There is no universal* **R**-*tree.*

*Proof*. By way of contradiction, suppose $X$ is a universal **R**-tree. Then $\mathrm{weight}(X) = \alpha$, for some cardinal $\alpha$. Let $Y$ be a discrete space such that card$(Y) > \alpha$. Then $C(Y)$, the cone over $Y$, with the appropriate metric topology is an **R**-tree such that weight$(C(Y)) > \alpha$. Hence, $C(Y)$ cannot be embedded in $X$, a contradiction. □

1.9. **Proposition.** *Suppose $X$ is an arcwise connected, locally arcwise connected, separable metric space. Then the following are equivalent:*

(1) *$X$ is an* **R**-*tree.*
(2) *Every nonendpoint of $X$ separates $X$.*

*Proof.* Suppose $X$ is an arcwise connected, locally arcwise connected, and separable metric space. Then every nonendpoint of $X$ separates $X$ iff $X$ is uniquely arcwise connected iff, by the Characterization Theorem (Theorem 1.5), $X$ is an **R**-tree. □

1.10. **Cut point order, level, and meet.** Let $(X, d)$ be an **R**-tree, and let $p \in X$. We define the *cut point order* $\leq$ on $X$ (with respect to $p$) by $x \leq y \Leftrightarrow x \in [p, y]$. Thus, $[p, x] = \{y \in X \mid y \leq x\}$. Since $X$ is uniquely arcwise connected, for $x, y \in X$, $[p, x] \cap [p, y] = [p, z]$ for some $z \in X$. Define a *meet* function $\wedge : X \times X \to X$ by $x \wedge y = z$, where $z$ is defined as above. Define the *level function* $f : X \to [0, \infty)$ (with respect to $p$) by $f(x) = d(p, x)$. Note that $d$ and $f$ are related by the "railroad track" equation: $d(x, y) = f(x) + f(y) - 2f(x \wedge y)$.

1.11. **Theorem** (Completion Theorem [AB, B]). *Let $(X, d)$ be an* **R**-*tree. Then $X$ embeds isometrically into a complete* **R**-*tree $cX$ such that $cX - X \subset$* End$(cX)$, Br$(cX) =$ Br$(X)$, *and for all $x \in X$, the order of ramification of $x$ is unchanged in $cX$.*

*Proof.* Choose a point $p \in X$. Define the level function $f: X \to [0, \infty)$ associated to $p$ and $d$ by $f(x) = d(p, x)$. Let $\{C_\beta\}_{\beta \in \mathscr{B}}$ be the collection of all maximal rays from $p$ of finite length (i.e., $f(C_\beta) < \infty$), such that $\mathrm{Cl}(C_\beta) = C_\beta$ for every $\beta \in \mathscr{B}$. (Hence, each $C_\beta$ is a half-open arc.) For each $\beta \in \mathscr{B}$, choose a different point $x_\beta$. Let $cX$ be the disjoint union of $X$ and $\{x_\beta\}_{\beta \in \mathscr{B}}$. We will topologize $cX$ by defining a metric $D$ on it which will make $\{x_\beta\}_{\beta \in \mathscr{B}}$ a set of endpoints. Define $F : cX \to [0, \infty)$ by

$$F(x) = \begin{cases} f(x), & \text{if } x \in X, \\ \sup\{f(z) \mid z \in C_\beta\}, & \text{if } x = x_\beta. \end{cases}$$

Then $F$ is an extension of $f$ to $cX$, and defines a radial metric on $cX$ from $p$ as a vertex. For $x, y \in cX$, the *meet* function is extended to $cX$ by

$$x \wedge y = \begin{cases} u, & \text{if } x = x_\beta,\ \beta \in \mathscr{B},\ y \in X, \text{ and } C_\beta \cap [p, y] = [p, u], \\ v, & \text{if } x \in X,\ y = y_\delta,\ \delta \in \mathscr{B}, \text{ and } [p, x] \cap C_\delta = [p, v], \\ w, & \text{if } x = x_\beta \ \&\ y = y_\delta,\ \beta, \delta \in \mathscr{B}, \text{ and } C_\beta \cap C_\delta = [p, w]. \end{cases}$$

We take the railroad track extension of $F$ defined by $D(x, y) = F(x) + F(y) - 2F(x \wedge y)$ to be the metric on $cX$.

With the metric $D$ thus defined, $cX$ is an **R**-tree containing $X$ isometrically as a subspace. It is easy to verify that Br$(cX) =$ Br$(X)$, $cX - X \subset$ End$(cX)$, and the order of ramification of each point of $X$ is unchanged.

*Claim.* $cX$ is complete.

*Proof of Claim.* Suppose $\{x_i\}_{i=1}^{\infty}$ is a nonconstant Cauchy sequence in $cX$. Using the cut point order, it is easy to check that for each $n \in \mathbf{Z}^+$,

$$\bigcap_{i=n}^{\infty}[p, x_i] = [p, v_n]$$

for some $v_n \in X$, and that

$$J = \bigcup_{n=1}^{\infty}[p, v_n]$$

is an arc in $X$, possibly half-open, one of whose endpoints is $p$.

Let $T_n$ be the closure of the connected set in $cX$ minimal with respect to containing $\{x_i\}_{i=n}^{\infty}$. Then $T_n \subset T_{n+1}$ for all $n$. Since $\{x_i\}_{i=1}^{\infty}$ is Cauchy, and $cX$ is locally arcwise connected, $\lim \operatorname{diam}(T_n) = 0$. We claim that $v_n \in T_n$. To see this, note that $v_n \leq z$, in the cut point order, for every $z \in T_n$. Let $z \in T_n$ and extend $[p, v_n]$ to $[p, z]$. Since $T_n$ is closed, let $u_n$ be the first point of $[p, z]$ in $T_n$. Then $[p, v_n] \subset [p, u_n]$. Note that $u_n$ does not depend on $z$. It follows that

$$[p, u_n] \subset \bigcap_{i=n}^{\infty}[p, x_i] = [p, v_n].$$

Hence, $v_n = u_n$.

Since $v_n \in T_n$ and $\lim \operatorname{diam}(T_n) = 0$, $\{v_n\}_{n=1}^{\infty}$ is a Cauchy sequence. But $\{v_n\}_{n=1}^{\infty}$ all lie on the (possibly half-open) arc $J$ from $p$ with $J \subset X$. Since $\{v_n\}_{n=1}^{\infty}$ is Cauchy, $f(J) < \infty$. All such arcs have endpoints in $cX$ by construction. Hence, $v = \lim v_n$ exists. (It is the unique point in $J \cap F^{-1}(\lim F(v_n))$.) Since $v_n, x_n \in T_n$ and $\lim \operatorname{diam}(T_n) = 0$, $v = \lim x_n$ as well. Therefore, $cX$ is complete, as claimed. □

## 2. Universal separable R-trees

We construct for each cardinal $\alpha$, $3 \leq \alpha \leq \aleph_0$, a separable **R**-tree $T_\alpha$ such that $T_\alpha$ is a universal space for separable **R**-trees whose order of ramification is at most $\alpha$. It will take two steps to construct $T_\alpha$: Theorem 2.1 to obtain something with enough branches, and Theorem 2.2 to add enough endpoints via the Completion Theorem (1.11). In the Embedding Theorem (2.3) we show that if $X$ is a separable **R**-tree whose order of ramification is at most $\alpha$, then $X$ embeds in $T_\alpha$. We draw as Corollary 2.4 that $T_{\aleph_0}$ is a universal space for all separable **R**-trees. It will be clear from the construction that $T_{\aleph_0}$ contains $T_\alpha$ isometrically for each $\alpha$ with $3 \leq \alpha < \aleph_0$. In Theorem 2.5, we show that $T_\alpha$ is unique.

2.1. **Theorem.** *For each cardinal $\alpha$, $3 \leq \alpha \leq \aleph_0$, there exists a separable* **R**-*tree $S_\alpha$ with metric $d$ such that for each point $p \in \operatorname{Br}(S_\alpha)$, $\operatorname{ram}(p) = \alpha$, and if $f(x) = d(x, p)$, then*

(∗) *for all $q \in \mathbf{Q}^+$ and for all $y \in f^{-1}(q)$, there exist rays $\{C_i\}_{i \in (\alpha-1)}$ such that for all $i \neq j \in (\alpha - 1)$, $f(C_i) = [q, \infty)$ and $C_i \cap C_j = \{y\}$.*

*Proof.* Enumerate $\mathbf{Q}^+ = \{q_1, q_2, \dots\}$. We construct $S_\alpha$ inductively as an increasing union $S_\alpha = \bigcup_{n=0}^{\infty} X_n$, where each $X_n$ has $\operatorname{ram}(X_n) = \alpha$. The function

$f : S_\alpha \to [0, \infty)$ is defined by $f(x) = f_n(x)$, where $f_n : X_n \to [0, \infty)$ and satisfies condition $(*)$ for all $q \in \{q_1, q_2, \ldots, q_n\}$. We then use $f$ as a level function to define a metric $d$ on $S_\alpha$ by the railroad track equation.

Let $Y_0 = [0, \infty) \times \alpha$. Let $X_0 = Y_0/\sim$, where $\sim$ is the equivalence relation on $Y_0$ defined by $(a, b) \sim (c, d) \Leftrightarrow a = c = 0$. That is, $X_0$ is the one-point union at 0 of $\alpha$ many copies of $[0, \infty)$. (For simplicity, we will use the same notation $(a, b)$ for the ordered pair $(a, b) \in Y_0$ and the equivalence class of $(a, b)$ in $X_0$.) Define $f_0 : X_0 \to [0, \infty)$ by $f_0(a, b) = a$. Let $p_\alpha$ be the point $f_0^{-1}(0)$ in $X_0$. (Note that $X_0$ will not carry the quotient topology, but rather the **R**-tree topology generated by the metric $d$ defined below.)

Suppose that for all $i$, $0 \le i \le n$, $X_i$ and $f_i$ have been constructed satisfying condition $(*)$ for all $q \in \{q_1, q_2, \ldots, q_i\}$.

Let $Y_{n+1} = X_n \times (\alpha - 1)$. Let $X_{n+1} = Y_{n+1}/\sim$, where $\sim$ is the equivalence relation on $Y_{n+1}$ defined by $(a, b) \sim (c, d) \Leftrightarrow (a = c \;\&\; f_n(a) \le q_{n+1})$. Define $f_{n+1} : X_{n+1} \to [0, \infty)$ by $f_{n+1}(a, b) = f_n(a)$. We assume $X_n \subset X_{n+1}$, by making the natural identification of $X_n$ with $X_n \times \{0\} \subset Y_{n+1}$, and consequently, via the equivalence relation, with a subset of $X_{n+1}$.

Then $X_n \subset X_{n+1}$ and $f_n$ are defined for all $n$. Let $S_\alpha = \bigcup_{n=0}^\infty X_n$ and define $f : S_\alpha \to [0, \infty)$ by $f(x) = f_n(x)$ if $x \in X_n$.

Note that $S_\alpha$ is a countable union of distinct rays $\{J_i\}_{i=1}^\infty$ with a common endpoint $p_\alpha = f^{-1}(0)$ such that

(1) for all $i$, $f(J_i) = [0, \infty)$, and
(2) for all $x, y \in S_\alpha$, if $x \in J_j$ and $y \in J_k$ with $j \ne k$, then there is a unique arc $L = [p_\alpha, v] = J_j \cap J_k$ with $f(v) = q$ for some $q \in \mathbf{Q}^*$. (Possibly, $L = \{p_\alpha\}$.)

Suppose $x, y \in S_\alpha$. Then $x \in J_j$ and $y \in J_k$ for some $j, k \in \mathbf{Z}^+$. Define the *meet* $x \wedge y$ of $x$ and $y$ to be that point $z \in \{x, y\}$ such that $f(z) = \min\{f(x), f(y)\}$ if $j = k$, and define $x \wedge y = v$, where $v$ is as defined above in (2), if $j \ne k$.

We define the **R**-tree metric $d$ on $S_\alpha$ by taking the railroad track extension of the radial metric from $p_\alpha$ as a vertex induced by $f$; that is, for $x, y \in S_\alpha$, define $d(x, y) = f(x) + f(y) - 2f(x \wedge y)$. Then $(S_\alpha, d)$ is the required **R**-tree. It is not difficult to verify that this defines a metric; for example, see [MO, Theorem 4.9]. Observe that condition $(*)$ is satisfied for every $p \in \mathrm{Br}(S_\alpha)$, not merely for $p_\alpha$. □

The **R**-tree $S_\alpha$ is not the universal **R**-tree of ramification $\alpha$ that we are seeking, since it cannot contain an **R**-tree $X$ for which $\mathrm{End}(X)$ contains a Cantor set. (Recall that $f^{-1}(t)$ is countable for each $t \in [0, \infty)$.) Thus, for example, $S_3$ does not contain the Cantor dendrite (see Example 5.2) as a subspace, though the Cantor dendrite is an **R**-tree with ramification at most 3.

In Theorem 2.2 below, we add endpoints to $S_\alpha$ to produce a complete **R**-tree $T_\alpha$, which we show is universal in Theorem 2.3 and unique in Theorem 2.5.

2.2. **Theorem** (Existence Theorem). *There exists a complete separable* **R**-*tree* $T_\alpha \supset S_\alpha$ *such that* $\mathrm{Br}(T_\alpha) = \mathrm{Br}(S_\alpha)$, $T_\alpha - S_\alpha = \mathrm{End}(T_\alpha)$, *and* $\mathrm{ram}(x) = \alpha$ *for all* $x \in \mathrm{Br}(T_\alpha)$.

*Proof*. Let $S_\alpha$ be as in Theorem 2.1, and let $T_\alpha = cS_\alpha$, the completion defined in Theorem 1.11. It is easy to check that $T_\alpha$ has the required properties. □

2.2.1. **Definition.** We call the point $p_\alpha \in T_\alpha$ the *vertex* of $T_\alpha$. Actually though, as the Uniqueness Theorem (Theorem 2.5 below) indicates, there is nothing to distinguish $p_\alpha$ topologically or metrically from any other point of $\mathrm{Br}(T_\alpha)$.

2.2.2. **Theorem.** $\mathrm{End}(T_\alpha)$ *is uncountable, totally disconnected, and dense in* $T_\alpha$. *Moreover,* $\mathrm{End}(T_\alpha)$ *is one-dimensional, and* $\infty$ *is an explosion point for* $\mathrm{End}(T_\alpha)$.

*Proof*. As we observed in the proof of Theorem 2.1, $S_\alpha = \bigcup_{n=1}^\infty B_n$, where each $B_n$ is a ray from $p_\alpha$ isometric to $[0, \infty)$. That $\mathrm{End}(T_\alpha)$ is uncountable and dense in $T_\alpha$ is easy to see from the construction of the completion $T_\alpha = cS_\alpha$, where it follows that $T_\alpha - \mathrm{End}(T_\alpha) = S_\alpha$. We show that $\mathrm{End}(T_\alpha)$ is one-dimensional, and that $\mathrm{End}(T_\alpha)$ is totally disconnected, while $\mathrm{End}(T_\alpha) \cup \{\infty\}$ is connected.

To see that $\mathrm{End}(T_\alpha)$ is one-dimensional, suppose that $e \in \mathrm{End}(T_\alpha)$ and that $U$ is a bounded neighborhood of $e$ in $T_\alpha$. We will find a point $e_0 \in \mathrm{Bd}(U) \cap \mathrm{End}(T_\alpha)$. (This suffices since $\mathrm{End}(T_\alpha)$ is dense in $T_\alpha$.) We proceed inductively to construct infinite sequences $\{e_n\}_{n=1}^\infty$ of endpoints of $T_\alpha$ in $T_\alpha - U$ and $\{x_n\}_{n=1}^\infty$ of points in $U$ which will jointly converge to the desired point $e_0$.

Choose $x_1 \in (p_\alpha, e) \cap U$ and $e_1 \in \mathrm{End}(T_\alpha) - U$ such that

(1.1) $(x_1, e_1) \cap B_1 = \varnothing$, and

(1.2) $d(e_1, U) \le 2^{-1}$.

Choose $x_2 \in (x_1, e_1) \cap U$ and $e_2 \in \mathrm{End}(T_\alpha) - U$ such that

(2.1) $(x_2, e_2) \cap (B_1 \cup B_2) = \varnothing$, and

(2.2) $d(e_2, x_2) \le 2^{-2}$.

In general, choose $x_n \in (x_{n-1}, e_{n-1}) \cap U$ and $e_n \in \mathrm{End}(T_\alpha) - U$ such that

($n$.1) $(x_n, e_n) \cap (B_1 \cup \cdots \cup B_n) = \varnothing$, and

($n$.2) $d(e_n, x_n) \le 2^{-n}$

(see Figure 2.1).

Since $x_n \in (x_{n-1}, e_{n-1})$ and $d(e_n, x_n) \le 2^{-n}$, $\{e_n\}_{n=1}^\infty$ and $\{x_n\}_{n=1}^\infty$ are Cauchy. Since $T_\alpha$ is complete, there is an $e_0$ such that $e_n, x_n \to e_0$. By condition ($n$.1), $e_0 \notin \bigcup_{n=1}^\infty B_n$; hence, $e_0 \in \mathrm{End}(T_\alpha)$.

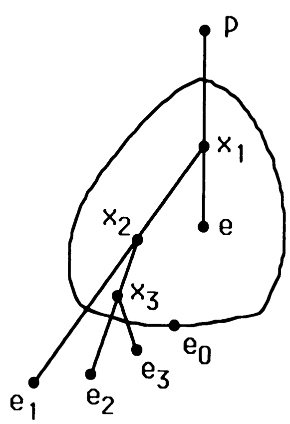


FIGURE 2.1

To see that $\mathrm{End}(T_\alpha)$ is totally disconnected, suppose that $x \neq y \in \mathrm{End}(T_\alpha)$. Let $z \in [x \wedge y, x]$. Since $z$ separates $T_\alpha$ (Proposition 1.9), let $U$ be the component of $T_\alpha - \{z\}$ containing $x$. Then the set $U \cap \mathrm{End}(T_\alpha)$ is an closed-open subset of $\mathrm{End}(T_\alpha)$ missing $y$.

Observe, however, that $U$ meets rays extending to $\infty$. If we now add $\infty$ as the common endpoint of all the rays of infinite length from $p_\alpha$, then in the resulting space $T_\alpha \cup \{\infty\}$ (no longer an **R**-tree), $\mathrm{End}(T_\alpha) \cup \{\infty\}$ is connected, since the closed-open subsets of $\mathrm{End}(T_\alpha) \cup \{\infty\}$ all extend to $\infty$. (We have shown that no bounded open set in $\mathrm{End}(T_\alpha)$ is closed.) Thus, $\infty$ is an explosion point for $\mathrm{End}(T_\alpha)$. □

2.3. **Theorem** (Embedding Theorem). *Let* $(X, r)$ *be a separable* **R**-*tree such that* $\mathrm{ram}(X) \leq \alpha$, $3 \leq \alpha \leq \aleph_0$, *and let* $v \in X$. *Let* $T_\alpha$ *with vertex* $p_\alpha$ *be the* **R**-*tree constructed in Theorem* 2.2. *Then there exists an embedding* $\phi : X \to T_\alpha$ *such that* $\phi(v) = p_\alpha$.

*Proof*. First assume that $v \notin \mathrm{End}(X)$. Define $g' : X \to [0, \infty)$ by $g'(x) = r(x, v)$. By Proposition 1.7, $\mathrm{Br}(X)$ is countable. Hence, $R = g'(\mathrm{Br}(X))$ is a countable subset of $[0, \infty)$. Consequently, there exists a homeomorphism $h : [0, \infty) \to [0, \infty)$ such that $h(R) \subset \mathbf{Q}^*$. Put $g = h \circ g'$. Then $g(v) = 0$. Note that the railroad track extension of $g$ gives an **R**-tree metric $\rho$ on $X$ equivalent to $r$. Replace $r$ by $\rho$, and henceforth consider the **R**-tree $(X, \rho)$.

By Proposition 1.6, there exists a countable collection $\mathscr{B}$ of rays from $v$ such that $\bigcup \mathscr{B} = X - \mathrm{End}(X)$. We lose no generality by supposing $\mathscr{B} = \{B_n\}_{n=1}^\infty$. We inductively construct an increasing sequence $\{L_n\}_{n=1}^\infty$ of sub-**R**-trees of $X$ such that $L = \bigcup_{n=1}^\infty L_n = X - \mathrm{End}(X)$.

Let $\mathscr{B}_1$ be a maximal subcollection of $\mathscr{B}$ such that $B_1 \in \mathscr{B}_1$ and $B \cap C = \{v\}$ for all $B \neq C \in \mathscr{B}_1$. Put $L_1 = \bigcup \mathscr{B}_1$. Since $v \notin \mathrm{End}(X)$, $L_1$ is the one-point union of at least two, and at most $\alpha$ many, rays, i.e., a fan of rays with vertex $v$.

Let $\mathscr{B}_2$ be a maximal subcollection of $\mathscr{B} - \mathscr{B}_1$ such that $B_2 \in \mathscr{B}_1 \cup \mathscr{B}_2$ and $B \cap C \subset L_1$ for all $B \neq C \in \mathscr{B}_2$. Put $L_2 = (\bigcup \mathscr{B}_2) \cup L_1$. Note that every component of $L_2 - L_1$ is an open arc with one endpoint in $L_1$.

Inductively, let $\mathscr{B}_{n+1}$ be a maximal subcollection of $\mathscr{B} - (\mathscr{B}_1 \cup \cdots \cup \mathscr{B}_n)$ such that $B_{n+1} \in \mathscr{B}_1 \cup \cdots \cup \mathscr{B}_{n+1}$ and $B \cap C \subset L_n$ for all $B \neq C \in \mathscr{B}_{n+1}$. Put $L_{n+1} = (\bigcup \mathscr{B}_{n+1}) \cup L_n$. (We lose no generality in supposing this process continues for each $n \in \mathbf{Z}^+$, though it might actually terminate at some finite step by exhausting $\mathscr{B}$.)

Let $L = \bigcup_{n=1}^\infty L_n$. Since $B_n \in \mathscr{B}_1 \cup \cdots \cup \mathscr{B}_n$, it follows that $L = X - \mathrm{End}(X)$.

For each $n \in \mathbf{Z}^+$, since $\mathrm{ram}(L_n) \leq \alpha$ and $g(\mathrm{Br}(L_n)) \subset \mathbf{Q}^* = f(\mathrm{Br}(T_\alpha))$, there is an isometric embedding $\psi_n : L_n \to T_\alpha$ such that $\psi_{n+1}$ extends $\psi_n$. Note that for $x \in L_n$, $f \circ \psi_n(x) = g(x)$, where $f$ is the level function of $T_\alpha$ with respect to $p_\alpha$. Let $\psi : L \to T_\alpha$ be the common extension of the $\psi_n$'s. Then $\psi$ is also an isometric embedding. Since $T_\alpha$ is complete, there is a unique natural extension $\phi : X \to T_\alpha$ of $\psi$ to the endpoint set $\mathrm{End}(X)$. Note that $\phi$ is an isometry with respect to the new metric $\rho$ on $X$, but not necessarily with respect to the original metric $r$.

The proof in case $v \in \mathrm{End}(X)$ is essentially identical to the above. The proofs differ mainly in what $\mathscr{B}_1$ looks like; for then $\mathscr{B}_1 = \{B_1\}$ and $L = \{v\} \cup (X - \mathrm{End}(X))$. $\square$

2.4. **Corollary** (Universal separable **R**-tree). *There is a universal separable* **R**-*tree* $T_{\aleph_0}$.

*Proof*. The order of ramification of a separable **R**-tree is at most $\aleph_0$ (Proposition 1.7). Consequently, by Theorem 2.3, each separable **R**-tree $X$ embeds in the universal **R**-tree $T_{\aleph_0}$. $\square$

2.4.1. *Remark.* It is clear that $T_{\aleph_0}$ contains isometrically each of the universal spaces $T_\alpha$ of order of ramification $\alpha$, $3 \leq \alpha < \aleph_0$.

2.5. **Theorem** (Uniqueness Theorem). *Let* $(X, r)$ *be a complete separable* **R**-*tree such that* $\mathrm{ram}(x) = \alpha$ *for all* $x \in \mathrm{Br}(X)$, *and let* $v \in \mathrm{Br}(X)$. *Put* $g(x) = r(v, x)$ *and assume that*

(1) $g(\mathrm{Br}(X)) = \mathbf{Q}^*$, *and*

(2) *for all* $q \in \mathbf{Q}^+$ *and for all* $y \in g^{-1}(q)$, $\mathrm{ram}(y) = \alpha$ *and there exist rays* $\{C_i\}_{i\in(\alpha-1)}$ *such that for all* $i \neq j \in (\alpha - 1)$, $f(C_i) = [q, \infty)$ *and* $C_i \cap C_j = \{y\}$.

*Then there is an isometry* $\phi : X \to T_\alpha$ *onto* $T_\alpha$ *such that* $\phi(v) = p_\alpha$.

*Proof*. Using the argument and notation of the proof of the Embedding Theorem (Theorem 2.3), we can obtain an embedding $\phi : X \to T_\alpha$. To complete the proof, note that if $X$ satisfies conditions (1) and (2) above, then $g(\mathrm{Br}(X)) = \mathbf{Q}^*$ with $g(v) = 0$. Hence, $r = \rho$, and we obtain $\phi$ as an isometric embedding of $X$ into $T_\alpha$ without any change of metric.

We need to make a further adjustment in the proof to get $\phi$ to be onto. We can represent $T_\alpha - \mathrm{End}(T_\alpha)$ as an increasing union of sub-**R**-trees $K = \bigcup_{n=1}^{\infty} K_n$ just as we did $X - \mathrm{End}(X)$ above. Now conditions (1) and (2) force each branch point $b$ of $X$ to have $\mathrm{ram}(b) = \alpha$ and each point $b \in g^{-1}(q)$ to be a branch point for every $q \in \mathbf{Q}^*$. Thus, we may suppose that the function $\psi_n : L_n \to T_\alpha$ carries $L_n$ isometrically onto $K_n$. Consequently, $\psi$ carries $L$ isometrically onto $K$. The fact that $X$ is complete then implies that $\phi$ is an isometry of $X$ onto $T_\alpha$. $\square$

2.6. **A-trees.** Let **A** be a subgroup of **R**. By $X$ is an **A**-tree, we mean that $X$ is an **R**-tree in which branch points occur only at points in $f^{-1}(\mathbf{A})$, where $f$ is the level function for $X$. What we have called an **A**-tree is called in [AB, B] the geometric realization of an **A**-tree. The general notion of a $\mathbf{\Lambda}$-tree is defined in [AB, B] for any totally ordered, abelian group $\mathbf{\Lambda}$. The proofs of Theorems 2.1–2.3 can be adapted to prove the following theorem.

2.6.1. **Theorem** (Universal separable **A**-tree). *For each* **A** *a subgroup of* **R**, *there is a universal separable* **A**-*tree. Moreover, if* **A** *is countable, then the embedding of an arbitrary separable* **A**-*tree into the universal separable* **A**-*tree can be taken to be an isometry. In particular, there are isometrically universal separable* **Z**- *and* **Q**-*trees.*

## 3. Universal R-tree of order $\alpha$

We construct for each cardinal number $\alpha > 2$ an **R**-tree $X_\alpha$ universal for all **R**-trees of order of ramification at most $\alpha$. Moreover, the embedding of an arbitrary **R**-tree of order at most $\alpha$ into $X_\alpha$ is isometric. We make use of the notation, definitions, and theorems of [N]. The following theorems, with appropriate modifications, could have been used to assert the existence of universal separable **R**-trees of orders $3 \leq \alpha \leq \aleph_0$ as well, but we felt that independent proofs in the separable case would be useful.

Let $(X, \leq)$ be a partially ordered set. For $x \in X$, let $l(x) = \{y \in X \mid y < x\}$ and $L(x) = \{y \in X \mid y \leq x\} = l(x) \cup \{x\}$. We say that $X$ is a *semilattice* iff for every $x, y \in X$, there exists a $z \in X$ such that $L(x) \cap L(y) = L(z)$. Such a point $z$ is denoted by $z = x \wedge y$, and is called the *meet* of $x$ and $y$, as in §1.10. We say that $X$ is a *pseudotree* iff $l(x)$ is linearly ordered by $\leq$, for every $x \in X$. We call a linearly ordered set in $(X, \leq)$ a *chain.* A maximal chain (with respect to inclusion) is a *branch* of $(X, \leq)$.

Let $(X, \leq)$ be a pseudotree and semilattice. For $x \in X$, let $\mathscr{A}(x)$ be a maximal family of branches such that $A_1 \cap A_2 = L(x)$ for all $A_1 \neq A_2 \in \mathscr{A}(x)$. Let $r'(x)$ be defined to be the cardinality of $\mathscr{A}(x)$. It can be shown that $r'(x)$ is a well-defined cardinal invariant of $x$ in $(X, \leq)$ [N]. Define

$$r(x) = \begin{cases} r'(x) & \text{if } x \text{ is the least element of } X \text{ (i.e., } l(x) = \varnothing), \\ r'(x) + 1 & \text{if there exists } y \in X \text{ such that } y < x \text{ (i.e., } l(x) \neq \varnothing). \end{cases}$$

On **R**-trees with the cut point order, $r(x)$ coincides with the notion $\operatorname{ram}(x)$ defined in §1.3.

The following theorems are in the spirit of the theorems of §2. For proofs and related results, refer to [N].

3.1. **Theorem** (Existence Theorem). *Suppose $\alpha > 2$ is a cardinal number. Then there exists a partially ordered set $(X_\alpha, \leq)$ such that the following conditions are satisfied*:

(1) *$X_\alpha$ is both a semilattice and a pseudotree.*
(2) *$X_\alpha$ has a least element $x_\alpha$.*
(3) *$r(x) = \alpha$ for each $x \in X_\alpha$.*
(4) *There is a level function $f : X_\alpha \to [0, \infty)$ such that, for every $x \in X_\alpha$, $f(l(x)) = [0, f(x))$ with $f^{-1}(0) = \{x_\alpha\}$, and $f(x) < f(x')$ whenever $x < x'$.*
(5) *$f(C) = [0, \infty)$ for every maximal linearly ordered subset $C$ of $X_\alpha$.*

3.1.1. **Remark.** Let $\alpha > 2$ be a cardinal number. For any points $x, y \in X_\alpha$, let $d(x, y) = f(x) + f(y) - 2f(x \wedge y)$. Then $d$ is a metric on $X_\alpha$ such that $(X_\alpha, d)$ is an **R**-tree and $\operatorname{ram}(x) = r(x) = \alpha$ for every $x \in X_\alpha$.

Theorem 3.1 is a particular case of [N, Theorem 7.6, p. 70]; in the notation of [N], our $X_\alpha$ is the same as $G(([0, \infty), \leq), \alpha)$. General versions of Theorems 3.2 and 3.3 below can be found in [N, Theorem 7.7, p. 72].

3.2. **Theorem** (Order Embedding Theorem). *Suppose $(Y, \leq)$ is a partially ordered set such that*

(1) *$Y$ is both a semilattice and a pseudotree,*
(2) *$Y$ has a least element,*
(3) *$r(y) \leq \alpha$ for each $y \in Y$, and*
(4) *there is a function $g \colon Y \to [0, \infty)$ such that, for every $y \in Y$, $g(l(y)) = [0, g(y))$.*

*Then there exists an order monomorphism* $i : Y \to X_\alpha$ *such that* $g = f \circ i$ *and* $i(y \wedge y') = i(y) \wedge i(y')$ *for every* $y, y' \in Y$.

3.3. **Theorem** (Order Uniqueness Theorem). *If* $(Y, \leq)$ *is a partially ordered set satisfying* (1), (2), *and* (4) *of Theorem* 3.2, *and moreover, satisfying*

(3′) $r(y) = \alpha$ *for each* $y \in Y$, *and*

(5) $g(D) = [0, \infty)$ *for every maximally linearly ordered subset* $D$ *of* $Y$,

*then there exists an order isomorphism* $j : Y \to X_\alpha$ *onto* $X_\alpha$ *such that* $g = f \circ j$.

3.4. **Theorem** (Embedding Theorem). *Suppose* $\alpha > 2$ *is a cardinal number. If* $(Y, \rho)$ *is an* **R**-*tree such that* $\operatorname{ram}(y) \leq \alpha$, *for each* $y \in Y$, *then* $(Y, \rho)$ *is isometric to a subset of* $(X_\alpha, d)$.

*Proof.* Let $y_0 \in Y$. For each $y \in Y$, define the level function $g : Y \to [0, \infty)$ by $g(y) = \rho(y_0, y)$. For $y, y' \in Y$, let $y \leq y'$ iff $y \in [y_0, y']$. Observe that $(Y, \leq)$ is both a semilattice and a pseudotree, and $y_0$ is the least element of $Y$. Moreover, for every $y \in Y$, $r(y) = \operatorname{ram}(y) \leq \alpha$ and $g(l(y)) = [0, g(y))$. Consequently, by Theorem 3.2, there is an order monomorphism $i : Y \to X_\alpha$ such that $g = f \circ i$. Note that for all $y, y' \in Y$, $\rho(y, y') = g(y) + g(y') - 2g(y \wedge y')$. Hence,

$$\begin{aligned} d(i(y), i(y')) &= f(i(y)) + f(i(y')) - 2f(i(y) \wedge i(y')) \\ &= f(i(y)) + f(i(y')) - 2f(i(y \wedge y')) \\ &= g(y) + g(y') - 2g(y \wedge y') = \rho(y, y'). \end{aligned}$$

Thus, $i : Y \to X_\alpha$ is an isometric embedding. □

3.5. **Theorem** (Uniqueness Theorem). *Suppose* $\alpha > 2$ *is a cardinal number. If* $(Y, \rho)$ *is an* **R**-*tree,* $p \in Y$, $\operatorname{ram}(y) = \alpha$ *for each* $y \in Y$, *and* $g(D) = [0, \infty)$ *for each maximal ray* $D$ *from* $p$, *where* $g(y) = \rho(p, y)$ *is a level function, then* $(Y, \rho)$ *is isometric to* $(X_\alpha, d)$.

3.6. *Remark.* As in §2, the results of this section can be extended to **A**-trees, for **A** a subgroup of **R**.

## 4. Compactification of R-trees

Using the results and techniques of §2, we show that each separable **R**-tree embeds in a universal smooth dendroid $D$ similar to that constructed in [MoN]. Thus, each separable **R**-tree has a compactification into a smooth dendroid, and so has a particularly nice one-dimensional metric compactification. Subsequently, we abandon the separability assumption and show that each **R**-tree has a compactification into a smooth arboroid; the compactification is, thus, one-dimensional, but generally not metric.

4.1. **Definitions.** A *continuum* is a compact, connected metric space. A *Hausdorff continuum* is a compact, connected Hausdorff space. A *Hausdorff arc* is a Hausdorff continuum with exactly two non-cut-points. (In comparison, a (metric) arc is characterized by the statement that it is a metric continuum with exactly two non-cut-points.) A Hausdorff space $X$ is *hereditarily unicoherent* iff for every pair of Hausdorff continua $K, M \subset X$, $K \cap M$ is a Hausdorff continuum. A *dendroid* is an arcwise connected, hereditarily unicoherent (metric) continuum. A *dendrite* is a locally arcwise connected dendroid.

Suppose $X$ is a dendroid. For each $x, y \in X$, let $[x, y]$ denote the unique arc in $X$ with endpoints $x$ and $y$. Fix a point $p \in X$. We define the *weak cut-point order* $\leq$ on $X$ (with respect to $p$) by $x \leq y \Leftrightarrow x \in [p, y]$. If $\{(x, y) \in X \times X \mid x \leq y\}$ is a closed subset of $X \times X$, we say that $X$ is *smooth* with respect to the point $p$. Equivalently, a dendroid $X$ is smooth with respect to a point $p \in X$ iff whenever $\{x_i\}_{i=1}^{\infty}$ is a sequence in $X$ and $\lim x_i = x$, then $\mathrm{Ls}[p, x_i] = [p, x]$ [CE]. Note that a dendrite is smooth with respect to every point, while a dendroid need not be smooth with respect to any point. However, if a dendroid $X$ is smooth with respect to a point $p$, then $X$ is contractible to $p$ [CE].

4.2. **Construction of the universal smooth dendroid $D$.** We construct a universal smooth dendroid $D$ with vertex $d$ (with respect to which $D$ is smooth). Let $D^* = D - \mathrm{End}(D)$. We show that each universal separable $\mathbf{R}$-tree $T_\alpha$ constructed in §2 is a subspace of $D^*$ with the vertex $p_\alpha$ of $T_\alpha$ corresponding to the vertex $d$ of $D^*$. We construct $D$ as an inverse limit of finite trees $D_i$ with bonding maps $f_i : D_{i+1} \to D_i$ which are level-preserving retractions.

Let $\{q_1, q_2, q_3, \ldots\}$ be an enumeration of $\mathbf{Q}^+$. Let $D_0 = [0, 1]$. Define a level function $g_0 : D_0 \to [0, 1]$ by $g_0 = \mathrm{id}_{[0,1]}$. Let $D_1' = D_0 \times \{0, 1\}$, and let $D_1 = D_1'/\sim$, where $\sim$ is the equivalence relation on $D_1'$ defined by $(a, b) \sim (c, d) \Leftrightarrow (a = c \ \& \ g_0(a) \leq q_1)$. (For simplicity, we will use the same notation $(a, b)$ for the ordered pair $(a, b) \in D_1'$ and the equivalence class of $(a, b)$ in $D_1$.) Define the bonding map $f_0 : D_1 \to D_0$ by $f_0(a, b) = a$. Then $f_0$ is a retraction. Define the level function $g_1 : D_1 \to [0, 1]$ by $g_1(a, b) = g_0(a)$. Note that $f_0$ preserves levels.

Suppose that for all $i$, $0 < i \leq n$, $D_i$, bonding maps $f_{i-1} : D_i \to D_{i-1}$ (which are retractions and preserve levels), and level functions $g_i : D_i \to [0, 1]$ (defined by $g_i(a, b) = g_{i-1}(a)$) have been constructed.

Let $D_{n+1}' = D_n \times \{0, 1\}$. Let $D_{n+1} = D_{n+1}'/\sim$, where $\sim$ is the equivalence relation on $D_{n+1}'$ defined by $(a, b) \sim (c, d) \Leftrightarrow (a = c \text{ and } g_n(a) \leq q_{n+1})$. Define $f_n : D_{n+1} \to D_n$ by $f_n(a, b) = a$. Define $g_{n+1} : D_{n+1} \to [0, 1]$ by $g_{n+1}(a, b) = g_n(a)$. We assume $D_n \subset D_{n+1}$ by making the natural identification of $D_n$ with $D_n \times \{0\} \subset D_{n+1}'$, and consequently, via the equivalence relation, with a subset of $D_{n+1}$.

Then $D_i \subset D_{i+1}$, level functions $g_i : D_i \to [0, 1]$, and retractions $f_i : D_{i+1} \to D_i$ are defined for all $i$. Let

$$D = \varprojlim \{D_i, f_i\}_{i=0}^{\infty}.$$

Note that the $g_i$'s induce a level function $g : D \to [0, 1]$.

Then $D$ is a version of the "simpler" construction of a universal smooth dendroid mentioned in [MoN, §5]. A direct proof that $D$ is a universal space for smooth dendroids is complicated. However, its universality also follows from a more general theorem to appear in a forthcoming paper by Mohler and Nikiel. For our purposes, it is only necessary that $D$ be a smooth dendroid into which each separable $\mathbf{R}$-tree embeds.

Let $d$ be the unique point in $D$ at level 0. We call $d$ the *vertex* of $D$. Note $d \in D^* = D - \mathrm{End}(D)$. As in [MoN], it can be shown that for each $x \in D^*$, $\mathrm{ram}(x) = c$, where $c$ denotes the cardinality of $\mathbf{R}$, and that $D$ is smooth with respect to $d$. Moreover, in $D$, each maximal arc from $d$ reaches all the way to level 1. Therefore, $\mathrm{End}(D) = g^{-1}(1)$ and is a closed set. Define the function $j : [0, 1) \to [0, \infty)$ by $j(x) = \frac{x}{1-x}$. Define a new level function $h : D^* \to [0, \infty)$ by $h = j \circ g$.

4.3. **Lemma.** *The universal separable* **R**-*tree* $T_{\aleph_0}$ *embeds by a level-preserving function into* $D^*$ (*with level function* $h$, *above*).

*Proof.* In the proof of the Uniqueness Theorem (Theorem 2.5), we observed that

$$S_{\aleph_0} = T_{\aleph_0} - \mathrm{End}(T_{\aleph_0}) = K = \bigcup_{n=1}^{\infty} K_n,$$

an increasing union of sub-**R**-trees. Since each point of $D^*$ has order of ramification $c$, there exists, for each $n$, a level-preserving embedding $\phi_n : K_n \to D^*$ such that $\phi_{n+1}|K_n = \phi_n$. The common extension $\phi$ of the $\phi_n$'s is a one-to-one continuous level-preserving function from $S_{\aleph_0}$ into $D^*$.

We will show that we may choose the $\phi_n$'s so that in addition $\phi^{-1}$ is continuous. It suffices to construct the $\phi_n$'s so that the following holds:

($*$) if $y_n = \phi_n(x_n)$, where $x_n \in K_n - K_{n-1}$ and $y_n \to y = \phi(x)$, then $x_n \wedge x \to x$.

That ($*$) suffices follows from the fact that $\phi$ is level-preserving. To see this, let $f$ denote the level function of $S_{\aleph_0}$. Suppose ($*$) holds and $y_m = \phi(x_m)$. If infinitely many $x_m$'s are in some $K_n$, then, since $K_n \subset K_j$ for all $j \geq n$, without loss of generality we may suppose $y_m = \phi_n(x_m) \to y = \phi_n(x)$. Since $\phi_n$ is an embedding, $x_n \to x$. So we may suppose that at most finitely many $x_m$'s lie in any one $K_n$. Hence, without loss of generality, there is an increasing sequence of indices $\{j_m\}_{m=1}^{\infty}$ such that $x_m \in K_{j_m} - K_{j_m - 1}$. Again without loss of generality, we may suppose $x_m \in K_m - K_{m-1}$. Since $\phi$ is level-preserving, $y_m \to y$ implies $f(x_m) \to f(x)$. By the railroad track equation,

$$d(x, x_m) = (f(x) - f(x_m \wedge x)) + (f(x_m) - f(x_m \wedge x)).$$

By ($*$) and level preserving, $f(x) - f(x_m \wedge x) \to 0$ and $f(x_m) - f(x_m \wedge x) \to 0$; hence, $x_m \to x$.

We construct the $\phi_n$'s so as to satisfy ($*$) by induction. Recalling the inverse limit construction of $D$, we may represent the point $x$ in $D^*$ (not uniquely) by $x = (t, z_1, z_2, \ldots)$, where $t = h(x) \in [0, \infty)$ is the level of $x$ and $z_i \in \{0, 1\}$ for all $i \in \mathbf{Z}^+$. Let $\pi_i$ denote the projection of $D$ to the $i$th factor space. Let $K_0 = \varnothing$.

Choose the embedding $\phi_1 : K_1 \to D^*$ so that there exist disjoint sequences $N_1$ and $D_1$ in $\mathbf{Z}^+$ such that

(1.1) $\{q_i \mid i \in N_1\}$ and $\{q_i \mid i \in D_1\}$ are each dense in $[0, \infty)$,

(1.2) $\pi_i \circ \phi_1(x) = 0$ for all $i \in N_1$, for all $x$,

(1.3) $\pi_i \circ \phi_1(x) = 1$ for all $i \in D_1$, for all $x \in K_1 - K_0$, and

(1.4) $\phi_1$ is level-preserving.

Suppose that for all $n < k$ embeddings $\phi_n : K_n \to D^*$ have been chosen so that there exist disjoint sequences $N_n$ and $D_n$ in $N_{n-1}$ such that

($n$.1) $\{q_i \mid i \in N_n\}$ and $\{q_i \mid i \in D_n\}$ are each dense in $[0, \infty)$,

($n$.2) $\pi_i \circ \phi_n(x) = 0$ for all $i \in N_n$, for all $x$,

($n$.3) $\pi_i \circ \phi_n(x) = 1$ for all $i \in D_1 \cup \cdots \cup D_n$, for all $x \in K_n - K_{n-1}$ such that $h(x \wedge y) > q_i$ for every $y \in K_{n-1}$, and

$(n.4)$ $\phi_n$ is level-preserving.

We may choose the embedding $\phi_k : K_k \to D^*$ so that there exist disjoint sequences $N_k$ and $D_k$ in $N_{k-1}$ such that conditions $(k.1)$–$(k.4)$ corresponding to $(n.1)$–$(n.4)$ above are satisfied. Thus, embeddings $\phi_n : K_n \to D^*$ are defined for every $n \in \mathbf{Z}^+$ satisfying conditions $(n.1)$–$(n.4)$.

We must now show that $(*)$ holds for $\phi : S_{\aleph_0} = \bigcup_{n=1}^{\infty} K_n \to D^*$, the common extension of these $\phi_n$'s. Suppose $y_n = \phi_n(x_n)$, $x_n \in K_n - K_{n-1}$, and $y_n \to y = \phi(x)$. For some $m$, $x \in K_m - K_{m-1}$. Then $\pi_i \circ \phi(x) = \pi_i \circ \phi_m(x) = 0$ for all $i \in N_m$ by condition $(m.2)$. Suppose by way of contradiction that $x_n \wedge x \not\to x$. Then there exists a $j \in D_{m+1}$ such that $f(x) > q_j > f(x_n \wedge x)$ for infinitely many $n$. Moreover, by condition $(j.3)$, $\pi_j \circ \phi(x_n) = 1$ for infinitely many $n$. Since $\pi_j \circ \phi(x) = 0$, it follows that $\phi(x_j) \not\to \phi(x)$, a contradiction.

We have constructed a level-preserving embedding $\phi : S_{\aleph_0} \to D^*$. Note that $D^*$ is a complete topological space (since $\mathrm{End}(D)$ is closed). Moreover, $D^*$ admits a complete metric $\rho$, which generates the topology that $D^*$ inherits from $D$, such that $h(x) = \rho(d, x)$. To see this, recall that a point $x \in D^*$ may be represented by $x = (t, z_1, z_2, \ldots)$, where $t = h(x)$ and $z_i \in \{0, 1\}$. This representation is not unique because of the equivalence relations defining the factor spaces. So, for $x = (t, z_1, z_2, \ldots)$, $y = (s, w_1, w_2, \ldots) \in D^*$, let $m(z_i, w_i)$ denote the minimum of $|z_i - w_i|$ computed over all equivalent representatives of $x$ and $y$. Then the $\rho$-distance between points $x$ and $y$ may be defined by

$$\rho(x, y) = |t - s| + \sum_{i=1}^{\infty} m(z_i, w_i)/2^i.$$

It follows that $\phi$ admits an extension (also called $\phi$) to $T_{\aleph_0}$. (See the proof of Theorem 1.11.) An argument similar to that in the preceding paragraph shows that $\phi$ is an embedding. □

4.4. **Theorem.** *Let $X$ be a separable* **R***-tree and let $p \in X$. Then there is an embedding $\phi : X \to D^*$ with $\phi(p) = d$.*

*Proof.* Since $X$ embeds in $T_{\aleph_0}$, and $T_{\aleph_0} \subset D^*$, $X$ embeds in $D^*$ as stated. □

4.5. **Corollary.** *Each separable* **R***-tree $X$ has a compactification into a smooth dendroid.*

*Proof.* Take the closure in $D$ (not $D^*$) of the embedding $\phi(X)$ obtained in Theorem 4.4. □

4.6. **Arboroids and dendrons.** An *arboroid* is a Hausdorff arc-connected, hereditarily unicoherent, Hausdorff continuum. A *dendron* is a locally Hausdorff arc-connected arboroid. These are the generalizations of dendroid and dendrite to Hausdorff continua. They retain many of the properties of dendroids and dendrites; e.g., they have covering dimension one. The definition of smooth with respect to $p \in X$ is extended to an arboroid $X$ by replacing "arc" with "Hausdorff arc" in the definition of the weak cut-point order. A dendron is smooth with respect to every point. A *dendritic space* is a connected space with the property that for every $x \neq y \in X$, there exists a $z \in X$ such that $x$ and $y$ lie in different components of $X - \{z\}$. A dendron is thus a compact dendritic space. Conversely, a compact dendritic space is a dendron.

4.7. **Subbases for R-trees.** Let $(X, d)$ be an **R**-tree. For a fixed $p \in X$, define $m(x) = \{y \in X \mid x \in [p, y] \ \& \ x \neq y\}$. Let $g(x) = d(p, x)$ be the usual level function associated to $p$. Consider the family

$$\mathscr{S} = \{g^{-1}([0, t)) \mid t \in \mathbf{R}^+\} \cup \{m(x) \mid x \in X\}.$$

4.7.1. **Proposition.** *Let $(X, d)$ be an **R**-tree and $p \in X$. Then the family $\mathscr{S}$ is a subbasis for the topology on $X$.*

*Proof.* If $x \in X$, then $m(x)$ is the union of all arc-components of $X - \{x\}$ which do not contain $p$. Since $X$ is locally arcwise connected, $m(x)$ is open. Therefore, $\mathscr{S}$ consists of open sets.

Note $\{g^{-1}([0, t)) \mid t \in \mathbf{R}^+\}$ is a neighborhood basis at $p$. Let $x \in X$ with $x \neq p$ and let $B(x, \varepsilon)$ be the open $\varepsilon$-ball about $x$ for some $\varepsilon > 0$. Let $g$ denote the level function for $(X, d)$ with respect to $p$. We may assume that $\varepsilon < g(x)$. Let $z \in (p, x)$ be the unique point such that $g(z) = g(x) - \varepsilon/3$. Let $U = g^{-1}([0, g(x) + \varepsilon/3)) \cap m(z)$. Then $U$ is a basic open neighborhood of $X$ in the basis generated by $\mathscr{S}$. It remains to show that $U \subset B(x, \varepsilon)$. See Figure 4.1.

Let $y \in U$ and $z' = x \wedge y$. Then $g(x) - \varepsilon/3 < g(y) < g(x) + \varepsilon/3$, because $g(z) < g(y)$ since $y \in m(z)$. Moreover, $z' \in [z, x]$, whence $g(x) - \varepsilon/3 \leq g(z') \leq g(x)$. Therefore, it follows by the railroad track equation that

$$\begin{aligned} d(x, y) &= g(x) + g(y) - 2g(z') \\ &= (g(y) - g(x)) + 2(g(x) - g(z')) < \varepsilon/3 + 2(\varepsilon/3) = \varepsilon. \end{aligned}$$

Thus, $y \in B(x, \varepsilon)$. $\square$

4.8. **Hausdorff compactifications.** We wish to thank Jan Aarts for discussions which led to a simplification of the proof of Theorem 4.9, below. A *normal basis* for closed sets for a space $X$ is a collection $\mathscr{B}$ of closed sets satisfying the following conditions [W, p. 20]:

(1) $\mathscr{B}$ is a ring (closed under finite unions and finite intersections),
(2) for all $x \in X$ and for all $B \in \mathscr{B}$ such that $x \notin B$, there is a $C \in \mathscr{B}$ such that $x \in C \subset X - B$, and

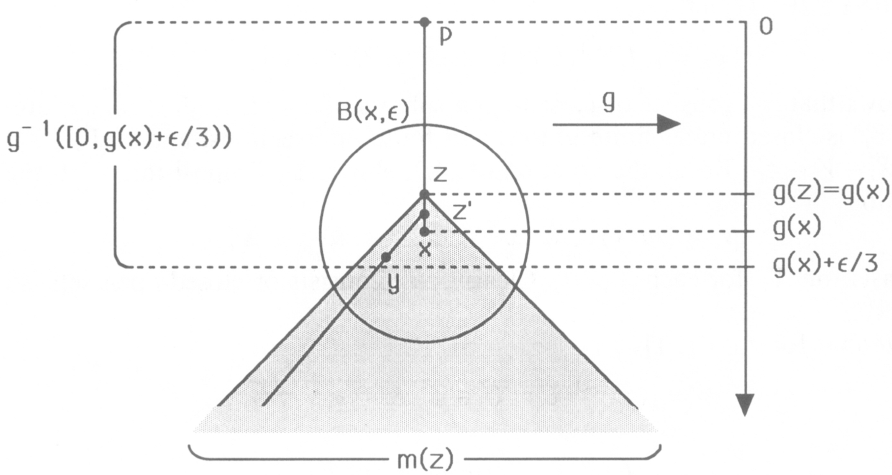


FIGURE 4.1

(3) for all $A, B \in \mathscr{B}$ with $A \cap B = \varnothing$, there are $C, D \in \mathscr{B}$ such that $X = C \cup D$, $A \cap D = \varnothing$, and $B \cap C = \varnothing$.

If, moreover, the following condition is satisfied:

(4) if $B \in \mathscr{B}$, then $X - B \in \mathscr{B}$,

then we say $\mathscr{B}$ is a *complemented normal basis.*

Note that (4) is stronger than (3). Obviously, if $\mathscr{B}$ is a complemented normal basis for closed sets of a space $X$, then $\mathscr{B}$ consists of closed-open sets, and it is a basis for open sets of $X$ as well. We will need the following version of the well-known Frink theorem.

4.8.1. **Theorem.** *If $\mathscr{B}$ is a normal basis for closed sets of a space $X$, then there is a unique Hausdorff compactification $\omega(X)$ such that $\omega(\mathscr{B}) = \{\mathrm{Cl}_{\omega(X)}(B) | B \in \mathscr{B}\}$ is a normal basis for closed sets of $\omega(X)$. If, moreover, $\mathscr{B}$ is a complemented normal basis for closed sets of $X$, then $\omega(\mathscr{B})$ is a complemented normal basis for closed sets of $\omega(X)$.*

*Proof.* See [W, p. 21]. The proof given there shows that if $B \in \mathscr{B}$, then

$$\mathrm{Cl}_{\omega(X)}(B) \cap \mathrm{Cl}_{\omega(X)}(X - B) = \varnothing$$

provided $X - B \in \mathscr{B}$. Clearly,

$$\mathrm{Cl}_{\omega(X)}(B) \cup \mathrm{Cl}_{\omega(X)}(X - B) = \mathrm{Cl}_{\omega(X)}(B \cup (X - B)) = \omega(X).$$

It follows that $\omega(\mathscr{B})$ is a complemented normal basis for closed sets of $\omega(X)$. □

4.9. **Theorem.** *Each* **R**-*tree $X$ admits a compactification $Y$ which is a smooth arboroid, each Hausdorff arc of which is homeomorphic to* $[0, 1]$.

*Proof.* Let $X$ be an **R**-tree, $p \in X$, and $d$ an **R**-tree metric on $X$ such that $\mathrm{diam}(X) \leq 1$. Let $g : X \to [0, 1]$ be defined by $g(x) = d(p, x)$. For each $s \in [0, 1]$, let $X_s = g^{-1}(s)$. For every $s, t \in [0, 1]$, $s \leq t$, let $f_s^t : X_t \to X_s$ be defined by setting $f_s^t(x)$ to be the only point of $[p, x] \cap X_s$. Then the sets $X_s$ are each closed, and the functions $f_s^t$ are continuous. Moreover, $f_r^t = f_r^s \circ f_s^t$ if $r \leq s \leq t$.

For any $s \in (0, 1]$, let $\mathscr{B}_s$ denote the family of all closed-open subsets of $X_s$. For $t \in (0, 1]$, let

$$\mathscr{C}_t = \{(f_s^t)^{-1}(U) \mid U \in \mathscr{B}_s \ \& \ 0 \leq s < t\}.$$

It follows that $\mathscr{C}_t$ consists of closed-open subsets of $X_t$, i.e., $\mathscr{C}_t \subset \mathscr{B}_t$. Moreover, $\mathscr{C}_t$ is closed under finite unions and finite intersections, and if $V \in \mathscr{C}_t$, then $X_t - V \in \mathscr{C}_t$. Recall the notation of §4.7, above. By Proposition 4.7.1, the collection

$$\mathscr{A}_s = \{m(x) \cap X_s \mid 0 \leq g(x) < s \ \& \ x \in X\}$$

is a basis for $X_s$ for each $s > 0$. Clearly, $\mathscr{A}_s$ consists of closed-open sets, so $\mathscr{A}_s \subset \mathscr{B}_s$.

Note that for $t \in (0, 1]$,

$$\mathscr{C}_t = \{(f_s^t)^{-1}(U) \mid U \in \mathscr{C}_s \ \& \ 0 < s < t\}$$

and

$$\mathscr{A}_t = \{(f_s^t)^{-1}(U) \mid U \in \mathscr{A}_s \ \& \ 0 < s < t\}.$$

This follows because if $U \in \mathscr{A}_s$ and $U = m(x) \cap X_s$, then $(f_s^t)^{-1}(U) = m(x) \cap X_t$. Therefore, $\mathscr{A}_t \subset \mathscr{C}_t$, and so $\mathscr{C}_t$ is a basis for $X_t$.

Observe that the family

$$\begin{aligned}\mathscr{T} &= \{g^{-1}([0, t)) \mid t \in (0, 1]\} \\ &\quad \cup \left\{\bigcup\{(f_s^t)^{-1}(U) \mid t \in (s, 1]\} \mid s \in (0, 1) \ \& \ U \in \mathscr{C}_s\right\}\end{aligned}$$

is a subbasis for the topology on $X$.

Note that $\mathscr{C}_t$ is a complemented normal basis for $X_t$. Let $\omega(X_t)$ denote the unique Hausdorff compactification of $X_t$ relative to the basis $\mathscr{C}_t$ as in Theorem 4.8.1. For $C \in \mathscr{C}_t$ let $\omega(C)$ denote $\mathrm{Cl}_{\omega(X_t)}(C)$. Then $\omega(\mathscr{C}_t) = \{\omega(C) \mid C \in \mathscr{C}_t\}$ is a complemented normal basis for $\omega(X_t)$. Consequently, it is a basis of closed-open sets for the open sets in $\omega(X_t)$.

Now we show that if $0 \leq s < t \leq 1$, then $f_s^t : X_t \to X_s$ has a (unique) continuous extension $F_s^t : \omega(X_t) \to \omega(X_s)$. Suppose that $y \in \omega(X_t)$. Let

$$F_s^t(y) = \bigcap\{\omega(C) \mid C \in \mathscr{C}_s \ \& \ y \in \omega((f_s^t)^{-1}(C))\}.$$

Then $F_s^t$ is a well-defined extension of $f_s^t$. Since $(F_s^t)^{-1}(\omega(C)) = \omega((f_s^t)^{-1}(C))$ for all $C \in \mathscr{C}_s$, the continuity of $F_s^t$ follows.

Having compactified each "level," we will now form the space we claim is the desired compactification of $X$. First, let $Y' = \bigcup\{\omega(X_t) | \ t \in [0, 1]\}$. Define $G' : Y' \to [0, 1]$ by $G'(y) = t$ if $y \in \omega(X_t)$, and define $J_y = \{F_s^t(y) \mid 0 \leq s \leq t\}$. Note that if $x \neq y \in \omega(X_t)$, then there exists $s < t$ such that $J_x \cap J_y \cap \omega(X_s) = \varnothing$. This follows because $x$ and $y$ are contained in disjoint basis elements in $\omega(X_t)$.

Let $\mathscr{K}$ be the collection of all maximal subsets $L$ of $Y'$ such that if $y, z \in L$, then either $J_y \subset J_z \subset L$ or $J_z \subset J_y \subset L$. Observe that if $L \in \mathscr{K}$, then either $G'(L) = [0, s]$ (so $L = J_x$ for some $x \in \omega(X_s)$) or $G'(L) = [0, s)$ for some $s \in (0, 1]$. Let

$$\mathscr{K}_0 = \{L \in \mathscr{K} \mid G'(L) = [0, s) \ \& \ s \in (0, 1]\}.$$

Second, let $Y''$ be a collection of points $\{y_L\}_{L \in \mathscr{K}_0}$. We assume that $y_L \notin Y'$ and $y_L \neq y_M$ for all $L \neq M$. Set $Y = Y' \cup Y''$. If $y_L \in Y''$, we define $G(y_L) = s$, where $G'(L) = [0, s)$, and if $y \in Y'$, we define $G(y) = G'(y)$. Then $G : Y \to [0, 1]$ is a well-defined extension of $g$ and $G'$. For each $t \in [0, 1]$, let $Y_t = G^{-1}(t)$. In particular, $Y_0 = X_0 = \{p\}$ and $\omega(X_t) \subset Y_t$. For $0 \leq s \leq t \leq 1$, we define $E_s^t : Y_t \to Y_s$ by the following formulas: $E_s^t(y) = F_s^t(y)$ if $y \in Y_t \cap Y'$, and $E_s^t(y) =$ the unique point of $Y_s \cap L$ if $y = y_L \in Y''$ for some $L \in \mathscr{K}_0$. Note that if $s < t$, then always $E_s^t(y) \in Y'$. Thus, $E_s^t$ is an extension of $f_s^t$ and $F_s^t$. Now if $y = y_L \in Y''$ for some $L \in \mathscr{K}_0$, we let $J_y = L \cup \{y\}$.

We are now going to define a topology on $Y$. For every $U \in \mathscr{C}_s$ and $s \in (0, 1)$ we let

$$U^* = \bigcup\{(E_s^t)^{-1}(\omega(U)) \mid t \in (s, 1]\}.$$

We set

$$\mathscr{T}^* = \{U^* \mid U \in \mathscr{C}_s \ \& \ s \in (0, 1)\} \cup \{G^{-1}([0, t)) \mid t \in (0, 1]\}.$$

We take $Y$ with the topology generated by $\mathscr{T}^*$ as a subbasis.

We now show that $Y$ is the desired compactification of $X$. Observe that if $0 < s < t \leq 1$ and $x \in Y'' \cap Y_s$, then $(E_s^t)^{-1}(x) = \varnothing$. It follows that if $t \in (0, 1]$, then the family

$$\{U^* \cap Y_t \mid U \in \mathscr{C}_s \ \& \ s \in (0, t)\} = \{(E_s^t)^{-1}(\omega(U)) \mid U \in \mathscr{C}_s \ \& \ s \in (0, t)\}$$

is a basis for the topology of $Y_t$ induced from $Y$. Hence, each $E_s^t : Y_t \to Y_s$ is continuous, and each $Y_t$ is 0-dimensional. The latter follows since our basis consists of closed-open sets: if $U \in \mathscr{C}_s$, then $X_s - U \in \mathscr{C}_s$, $U^* \cap (X_s - U)^* = \varnothing$, and $Y_t = (U^* \cup (X_s - U)^*) \cap Y_t$. Moreover, the topology on $\omega(X_t)$ induced from $Y$ coincides with the original topology of $\omega(X_t)$. Hence, for each $t$, $\omega(X_t)$ is a compact subspace of $Y$.

Since $X_t \in \mathscr{C}_t$ for $t \in (0, 1)$ and $X_t^* = G^{-1}((t, 1])$, it follows that $G : Y \to [0, 1]$ is continuous. Since $G|_{J_x} : J_x \to [0, G(x)]$ is one-to-one and open, it is a homeomorphism for each $x \in Y$. Thus, $J_x$ is homeomorphic to $[0, 1]$ provided $x \neq p$. Therefore, $Y$ is arcwise connected.

Since $\mathscr{T} = \{X \cap V^* \mid V^* \in \mathscr{T}^*\}$, the topology on $X$ induced from $Y$ coincides with the original topology of $X$. Since $X_t$ is dense in $\omega(X_t)$ for each $t \in (0, 1]$, $X$ is dense in $Y'$. Obviously, $Y'$ is dense in $Y$. Thus, $X$ is a dense subspace of $Y$.

4.9.1. *Claim.* $Y$ is a Hausdorff space.

*Proof of Claim.* Let $x \neq y \in Y$. If $G(x) \neq G(y)$, say $G(x) < G(y)$, then $G^{-1}([0, s))$ and $G^{-1}((s, 1])$ are disjoint neighborhoods of $x$ and $y$, where $s \in (G(x), G(y))$. So we may suppose that $G(x) = G(y) = t$. Clearly, $t \neq 0$. There are three cases to consider.

*Case* (1). Suppose $x \neq y \in \omega(X_t) \subset Y'$. Then $x \in \omega(U)$ and $y \in \omega(X - U)$ for some $U \in \mathscr{C}_t$. Since $U \in \mathscr{C}_t$, we have $U = (f_s^t)^{-1}(V)$ for some $s \in (0, t)$ and some closed-open $V \in \mathscr{C}_s$. Then $V^*$ and $(X_s - V)^*$ are disjoint neighborhoods of $x$ and $y$, respectively.

*Case* (2). Suppose that $x \in Y'$ and $y \in Y''$. Then there is an $L \in \mathscr{K}_0$ with $y = y_L$. Since $x \neq y$, there exists $s \in (0, t)$ such that $E_s^t(x) \neq E_s^t(y)$. Both $E_s^t(x)$ and $E_s^t(y)$ are in $\omega(X_s)$; hence, there exists $U \in \mathscr{C}_s$ such that $E_s^t(x) \in \omega(U)$ and $E_s^t(y) \in \omega(X_s - U)$. We have $U = (f_r^s)^{-1}(V)$ for some $r \in (0, s)$ and $V \in \mathscr{C}_r$. Then $V^*$ and $(X_r - V)^*$ are disjoint neighborhoods of $x$ and $y$, respectively.

*Case* (3). Suppose that $x \neq y \in Y''$. Then there are $L \neq M \in \mathscr{K}_0$ with $x = x_L$ and $y = y_M$. Therefore, there exist $s \in (0, t)$ such that $E_s^t(x) \neq E_s^t(y)$. We define $V^*$ and $(X_r - V)^*$ as in Case (2). □

4.9.2. *Claim.* $Y$ is compact.

*Proof of Claim.* By Alexander's Lemma [K, p. 139], in order to prove that $Y$ is compact, it suffices to show that if $\mathscr{R} \subset \mathscr{T}^*$ and $\mathscr{R}$ covers $Y$, then $\mathscr{R}$ has a finite subcover. Let $\mathscr{R}_0$ be the collection of all $V \in \mathscr{R}$ which are of the form $G^{-1}([0, t))$ for some $t \in (0, 1]$. Let $\mathscr{R}_d = \mathscr{R} - \mathscr{R}_0$. Since $\mathscr{R}$ is a cover,

$p \in \bigcup \mathscr{R}$, and so $\mathscr{R}_0 \neq \varnothing$. Let

$$t_0 = \sup\{t \mid G^{-1}([0, t)) \in \mathscr{R}_0\}.$$

Then $0 < t_0 \le 1$. We may assume that $\mathscr{R}_0$ does not contain a finite subcover of $Y$. Since $\omega(X_s)$ is compact, it suffices to prove that for some $s < t_0$, $\omega(X_s)$ is covered by $\mathscr{R}_d$.

Let $F_s = \omega(X_s) - \bigcup \mathscr{R}_d$. Note that $F_s$ is compact. Suppose $F_s$ is nonempty for every $s < t_0$. Since $E_r^s(F_s) \subset F_r$ if $r \le s \le t_0$, we have

$$F_{t_0} = \bigcap_{s<t_0} (E_s^{t_0})^{-1}(F_s).$$

Thus, $F_{t_0}$ is the intersection of a nested family of nonempty compacta. Hence, there is an $x \in F_{t_0} \subset Y_{t_0}$. Clearly, $x \notin \bigcup \mathscr{R}_0$. Therefore, there is an $R \in \mathscr{R}_d$ such that $x \in R = m(y)$ for some $y \in J_x - \{x\} = \{(E_s^{t_0})^{-1}(x) \mid 0 \le s < t_0\}$. Hence, for any $s$ with $G(y) < s < t_0$, $E_s^{t_0}(x) \in F_s \cap R$, a contradiction. □

It remains to show that $Y$ is hereditarily unicoherent, and that $Y$ is smooth with respect to the point $p$. Note that for each $x, y \in Y$, there is a largest $t \in [0, 1]$ such that $J_x \cap J_y \cap Y_t \neq \varnothing$. But then there is a $z \in Y_t$ such that $J_x \cap J_y \cap Y_t = \{z\}$. Clearly, $J_x \cap J_y = J_z$. We will write $z = x \wedge y$ as usual. Moreover, we let

$$J_{xy} = [(J_x \cup J_y) - J_z] \cup \{z\}.$$

Obviously, $J_{xy}$ is an arc with endpoints $x$ and $y$ provided $x \neq y$ and $J_{xx} = \{x\}$.

4.9.3. *Claim.* If $Z$ is a subcontinuum of $Y$ and $x, y \in Z$, then $J_{xy} \subset Z$.

*Proof of Claim.* By way of contradiction, suppose there is a $z \in J_{xy} - Z$. Without loss of generality, we may assume that $z \in J_x - J_y$ and that $z \neq x \wedge y$ (in particular, $z \neq p$). There exist $s, t \in (0, 1]$ with $s < t$, and $U \in \mathscr{C}_s$ such that the basic open set $U' = U^* \cap G^{-1}([0, t))$ is a neighborhood of $z$ and $\mathrm{Cl}(U') \cap Z = \varnothing$. We take a number $r$ such that $s < r < G(z)$ and let $V = (f_s^r)^{-1}(U)$. Then $V \in \mathscr{C}_r$, $V^* = U^* \cap G^{-1}((r, 1])$, and $\mathrm{Cl}(V^*) = V^* \cup \omega(V)$. Therefore, $V' = V^* \cap G^{-1}([0, t))$ is a neighborhood of $z$, and $\mathrm{Cl}(V') \cap Z = \varnothing$. It follows that $A = \mathrm{Cl}(V^*) \cap Z$ and $B = Z - V^*$ are disjoint closed subsets of $Z$ such that $Z = A \cup B$, $x \in A$ (because $z \in J_x$ and $z \in V^*$), and $y \in B$. Hence $Z$ is not connected, a contradiction. □

Now if $Z, Z'$ are subcontinua of $Y$ and $v, w \in Z \cap Z'$, then $J_{vw} \subset Z \cap Z'$. Hence $Z \cap Z'$ is arcwise connected. It follows that $Y$ is hereditarily unicoherent.

Let $\le$ denote the weak cut-point order on $Y$ with respect to $p$: $x \le y \Leftrightarrow x \in J_y$. It is easy to see that $P = \{(x, y) \in Y \times Y \mid x \le y\}$ is a closed subset of $Y \times Y$. Therefore, $Y$ is smooth with respect to $p$. □

4.10. *Remark.* The smooth arboroid $Y$ obtained in Theorem 4.9 is contractible to its vertex $p$. In fact, let $H : Y \times [0, 1] \to Y$ be defined by

$$H(x, 1-t) = \begin{cases} x, & \text{if } x \in Y_s \text{ for some } s \le t, \\ E_t^s(x), & \text{if } x \in Y_s \text{ for some } s > t. \end{cases}$$

Then $H$ is the desired contraction. (The proof of continuity of $H$ is straightforward.)

## 5. EXAMPLES

We conclude with some examples. In [Mr] Morgan proves that if $(X, d)$ is an **R**-tree, then $X$ is locally arcwise connected, uniquely arcwise connected, contractible, and one-dimensional (Theorem 1.4 in our §1). Morgan then asks (1) if every first countable space $X$ satisfying the conclusion of Theorem 1.4 admits a metric making it an **R**-tree, and if not, (2) what additional properties must one assume?

**R**-trees have been characterized among metric spaces by the properties of local arcwise connectivity and unique arcwise connectivity (Theorem 1.5). Here we show by several examples that the answer to Morgan's question (1) is no, and that for (2) one would have to assume such powerful additional hypotheses, that one might as well assume that $X$ is metric to begin with.

5.1. **Example.** Let $X$ be the quotient space $X = (\mathbf{Z} \times [0, 1])/(\mathbf{Z} \times \{0\})$. Since the quotient map is closed, $X$ is normal. Moreover, $X$ is uniquely and locally arcwise connected, contractible, and one-dimensional. However, $X$ is not metrizable, because it fails to be first countable at the point $\mathbf{Z} \times \{0\}$. Obviously, $X$ is separable. However, with the "arc-length" metric topology, as opposed to the quotient topology, $X$ is an **R**-tree.

5.2. **Example** (The Cantor dendrite). Let $\mathbf{H}$ denote the closed upper half-plane in $\mathbf{R}^2$, and let $\mathrm{Int}(\mathbf{H})$ denote the open upper half-plane. Employ the standard construction of the Cantor dendrite $T$ as the closure of an infinite binary tree in $\mathrm{Int}(\mathbf{H})$ with the deleted middle third Cantor set $C$ on the $x$-axis as its set of accumulation points. (See Figure 5.1.)

Let $E$ denote that subset of $C$ consisting of endpoints of open intervals deleted in the construction of $C$. Given the standard (Euclidean) topology $\mathscr{G}_0$ on $\mathbf{H}$, $T$ with the subspace topology $\mathscr{T}_0$ is a dendrite. Moreover, $(T, \mathscr{T}_0)$ admits an equivalent "arc-length" metric which makes it an **R**-tree. ($T$ is sometimes called the Gehman dendrite.)

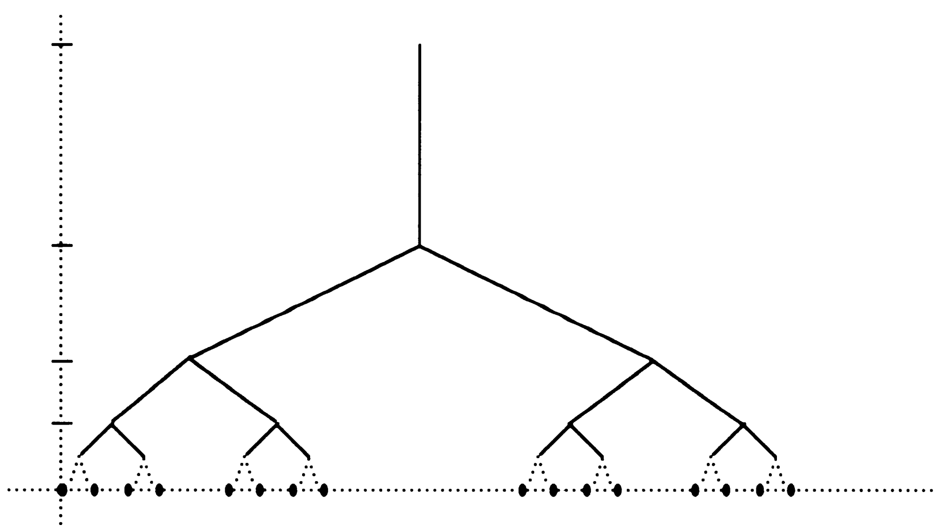

FIGURE 5.1

5.3. **Example** (A nonregular Cantor dendritic space). We can topologize $\mathbf{H}$ so that it is Hausdorff, but not regular. The topology $\mathscr{G}_1$ is the "half-disk" topology on the $x$-axis (i.e., a basic neighborhood of a point $p$ on the $x$-axis is $(B \cap \operatorname{Int}(\mathbf{H})) \cup \{p\}$, where $B$ is an open disk centered on $p$), and the standard topology on $\operatorname{Int}(\mathbf{H})$ (see [ST, p. 28]). Give $T$ the subspace topology $\mathscr{T}_1$ induced by $(\mathbf{H}, \mathscr{G}_1)$. Note that the subspace topology on the Cantor set $C$ is discrete. With the half-disk basis elements on $C$, one cannot find disjoint open sets about a point $p \in C$ and the closed set $C - \{p\}$. Consequently, $(T, \mathscr{T}_1)$ is a first countable, locally arcwise connected, uniquely arcwise connected, contractible, one-dimensional Hausdorff space, thus a dendritic space, which is not regular; hence, not metrizable.

Note that $(T, \mathscr{T}_1)$ has a second countable subspace $T'$ which consists of only those branches which terminate at points of $E$. Since $T'$ is still not regular, it is not metrizable.

5.4. **Example** (A regular, nonnormal Cantor dendritic space). We can topologize $\mathbf{H}$ so that it is regular, but not normal. The topology $\mathscr{G}_2$ is the "tangent disk" topology on the $x$-axis (i.e., a basic open neighborhood of a point $p$ on the $x$-axis is $B \cup \{p\}$, where $B$ is an open disk in $\operatorname{Int}(\mathbf{H})$ tangent to $p$), and the standard topology on $\operatorname{Int}(\mathbf{H})$ (see [ST, p. 46]). Give $T$ the subspace topology $\mathscr{T}_2$ induced by $(\mathbf{H}, \mathscr{G}_2)$. Note that in the subspace topology, the Cantor set $C$ is again discrete. With the tangent disk basis elements on $C$, one can find disjoint open sets about a point $p \in C$ and the closed set $C - \{p\}$; hence, $(T, \mathscr{T}_2)$ is regular. However, one cannot find disjoint open sets about the sets $E$ and $C - E$, so $(T, \mathscr{T}_2)$ is not normal. Consequently, $(T, \mathscr{T}_2)$ is a first countable, locally arcwise connected, uniquely arcwise connected, contractible, one-dimensional regular Hausdorff space, thus a dendritic space, which is not normal; hence, not metrizable.

In this case, however, the subspace $T'$ is second countable and regular, hence, metrizable by the Urysohn Metrization Theorem.

5.5. **Example** (A Souslin dendron). A dendron $X$ is said to be a *Souslin dendron* iff the following conditions hold (see [vMW]; other references may be found in [N]):

(1) $X$ is not metrizable (i.e., $X$ is not separable).
(2) Each countable subset of $X$ is contained in a metrizable subcontinuum of $X$. (Equivalently, each Hausdorff subarc of $X$ is homeomorphic to $[0, 1]$.)
(3) Each family of pairwise disjoint open subsets of $X$ is countable.

The existence of a Souslin dendron is equivalent to the existence of a Souslin line [vMW], hence, depends upon the axioms of set theory [DJ].

Suppose that $X$ is a Souslin dendron. Then density$(X) = \aleph_1$, and $X$ is first-countable. By [vMW, Theorem 3.1], $X$ is hereditarily perfectly normal and hereditarily Lindelöf; hence, hereditarily paracompact. Though $X$ is arcwise connected, $X$ is not contractible [N, 8.27(vi), 8.21(iii)]. However, $X$ is *long-arc contractible* [N, 8.22(ii)].

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