

ON THE BRAUER GROUP OF TORIC VARIETIES

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ABSTRACT. We compute the cohomological Brauer group of a normal toric variety whose singular locus has codimension less than or equal to 2 everywhere.

Associated to each algebraic variety X is the cohomological Brauer group $B'(X) = \text{tors}(H^2(X, G_m))$ which is the torsion subgroup of the second étale cohomology group of X with coefficients in the sheaf of units. Except in the easiest cases, calculations of this group are scarce. Toric varieties over an algebraically closed field of characteristic 0 provide a nontrivial class of higher dimensional varieties for which calculations of $B'(X)$ can sometimes be made. These calculations are the purpose of this article.

Each toric variety X is determined by a combinatorical object Δ in real affine space called a fan. Tied into the structure of the fan are arithmetic properties of sublattices of free $\mathbb{Z}$ -lattices. Our arguments therefore ultimately reduce questions about $B'(X)$ to calculations with integer matrices.

In §1 we determine the Brauer group $B(X) = B'(X)$ of any nonsingular toric variety X (Theorem 1.1). This group is a direct sum of finitely many copies of finite cyclic groups and copies of $\mathbb{Q}/\mathbb{Z}$. The algebras generating this group are given explicitly as smash products of cyclic Galois extensions of X . In §2 we consider toric varieties whose singular locus has codimension at most 2 everywhere in X . Let T_N denote the torus identified with an open subset of X , $B'(T_N/X)$ the elements in $B'(X)$ split by T_N , and $\tilde{X}$ a T_N -invariant desingularization of X . In Theorem 2.2 we construct an exact sequence $0 \rightarrow B'(T_N/X) \rightarrow B'(X) \rightarrow B'(\tilde{X}) \rightarrow 0$ which reduces the calculation of $B'(X)$ to the calculation of $B'(T_N/X)$. The hypotheses on X in §2 imply we can assume the associated fan Δ contains cones of dimension at most 2. Corresponding to each cone τ_i of dimension 2 is an irreducible closed subvariety $V_i = \text{orb } \tau_i$ and an affine neighborhood U_{τ_i} of V_i which has a finite cyclic divisor class group $\text{Cl}(U_{\tau_i})$. If Δ has 2-dimensional cones $\tau_1, \dots, \tau_m$, we construct an exact sequence (Theorem 2.3)

$$0 \rightarrow \text{Pic}(X) \rightarrow \text{Cl}(X) \rightarrow \bigoplus_{i=1}^m \text{Cl}(U_{\tau_i}) \rightarrow B'(T_N/X) \rightarrow 0.$$

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For each prime number p we find a subset $\tau_1, \dots, \tau_e$ (after a suitable relabelling) of $\{\tau_1, \dots, \tau_m\}$ such that $[\bigoplus_{i=1}^e \text{Cl}(U_{\tau_i})]_p \cong [\text{B}'(T_N/X)]_p$. We calculate the Brauer group of any toric surface (Corollary 2.9). In this case $\text{B}'(T_N/X)$ is nontrivial when there is a cycle of divisors on X whose pairwise consecutive intersections are singular points on X whose local rings all have divisor class groups of order divisible by a common prime p . An analogous statement holds for X of higher dimension. We employ terminology and notation of [12] for toric varieties and [11] for étale cohomology.

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Following the notation terminology of [12] let $r > 0$ be an integer and $N = \mathbb{Z}^r$ a free abelian group of rank r . Let Δ be a finite fan on $N_{\mathbb{R}}$ and $X = T_N \text{emb}(\Delta)$ the associated toric variety containing the r -dimensional torus T_N as an open subset defined over the algebraically closed field k of characteristic 0. Let N' be the subgroup of N generated by $\bigcup_{\sigma \in \Delta} \sigma \cap N$. The basis theorem for finitely generated abelian groups gives a basis $n_1, \dots, n_r$ of N such that $N' = \mathbb{Z}a_1n_1 \oplus \mathbb{Z}a_2n_2 \oplus \dots \oplus \mathbb{Z}a_rn_r$ where the a_i are nonnegative integers and $a_i | a_{i+1}$ for $1 \leq i \leq r-1$. Call $\{a_1, \dots, a_r\}$ the set of invariant factors of Δ (or $X = T_N \text{emb}(\Delta)$). Let $\text{B}(X)$ denote the Brauer group of Azumaya algebras on X and $\text{B}'(X)$ the torsion subgroup of $\text{H}^2(X, \text{G}_m)$ the cohomological Brauer group of X . Our principal result of §1 is

Theorem 1.1. *If $X = T_N \text{emb}(\Delta)$ is nonsingular and $a_1, \dots, a_r$ is the set of invariant factors of X , then*

$$\text{B}(X) = \text{B}'(X) \cong \bigoplus_{i=1}^{r-1} \text{Hom}(\mathbb{Z}/a_i, \mathbb{Q}/\mathbb{Z})^{r-i}.$$

We list two special cases of Theorem 1.1.

Corollary 1.2. *If Δ contains a cone σ such that $\dim \sigma \geq r-1$, then $\text{B}(X) = (0)$.*

Proof. Since X is nonsingular, [12, Theorem 1.10] implies there is a basis $n_1, \dots, n_r$ of N such that $\mathbb{R}_{\geq 0}n_1 + \dots + \mathbb{R}_{\geq 0}n_{r-1} \subseteq \sigma$. Since each $n_i \in \sigma \cap N$, all of the invariant factors $a_i = 1$ for $1 \leq i \leq r-1$. So $\text{B}(X) = (0)$ by Theorem 1.1. $\square$

Corollary 1.3. *$\text{B}(X)$ is finite if and only if $\text{Rank}_{\mathbb{Z}}(N') \geq r-1$. In this case*

$$\text{B}(X) \cong \bigoplus_{i=1}^{r-1} (\mathbb{Z}/a_i)^{r-i}.$$

The rest of this section is devoted to a proof of Theorem 1.1. From now on we assume $X = T_N \text{emb}(\Delta)$ is nonsingular. Along the way we will obtain explicit information about the Azumaya algebras on X and show the Brauer group $\text{B}(X)$ of Azumaya algebras on X is equal to the cohomological Brauer group $\text{B}'(X) = \text{H}^2(X, \text{G}_m)$.

Let $\Gamma = \{0, \rho_1, \dots, \rho_n\}$ be the fan on $N_{\mathbb{R}}$ consisting of all cones in Δ of dimension ≤ 1 and let $U = T_N \text{emb}(\Gamma)$. The open immersion $U \hookrightarrow X$ induces the isomorphisms of the next lemma.

Lemma 1.4. *For each positive integer ν ,*

$$(a) \quad \text{H}^1(X, \mathbb{Z}/\nu) \cong \text{H}^1(U, \mathbb{Z}/\nu),$$

- (b) $H^2(X, \mathbb{Z}/\nu) \cong H^2(U, \mathbb{Z}/\nu)$,
 (c) $B'(X) \cong B'(U)$.

Proof. Let X, U be as above and let $Z = X - U$. Then part of the long exact sequence of cohomology with supports [11, Proposition 1.25] is

$$\begin{aligned} H_Z^1(X, \mathbb{Z}/\nu) \rightarrow H^1(X, \mathbb{Z}/\nu) \rightarrow H^1(U, \mathbb{Z}/\nu) \rightarrow H_Z^2(X, \mathbb{Z}/\nu) \\ \rightarrow H^2(X, \mathbb{Z}/\nu) \rightarrow H^2(U, \mathbb{Z}/\nu) \rightarrow H_Z^3(X, \mathbb{Z}/\nu) \end{aligned}$$

since the codimension of Z in X is ≥ 2 , [11, Lemma 9.1, p. 268] implies $H_Z^s(X, \mathbb{Z}/\nu) = (0)$ for $s < 4$, which proves (a) and (b) in our context. There is an exact sequence [3, Theorem 1.c]

$$0 \rightarrow H^2(X, G_m) \rightarrow H^2(U, G_m) \rightarrow H_Z^3(X, \mu)$$

and $H_Z^3(X, \mu) \cong \varinjlim H_Z^3(X, \mathbb{Z}/\nu) = (0)$, which proves (c). $\square$

Notice that $N' = \langle \bigcup_{\sigma \in \Delta} \sigma \cap N \rangle = \langle \bigcup_{i=1}^n \rho_i \cap N \rangle$ so as a consequence of Lemma 1.4(c) we can assume that $\Delta = \{0, \rho_1, \dots, \rho_n\}$ and $X = U$. We write $\rho_k = \mathbb{R}_{\geq 0} \eta_k$ where $\eta_k \in N$ and η_k is primitive (the GDC of the coordinates of η_k is 1). Let $n_1, \dots, n_r$ be a basis for N with $N' = \mathbb{Z}a_1n_1 \oplus \dots \oplus \mathbb{Z}a_rn_r$ and $a_i | a_{i+1}$ for $1 \leq i \leq r$. ($\{a_1, \dots, a_r\}$ is the set of invariant factors of Δ .) Let $m_1, \dots, m_r$ be a dual basis for $M = \text{Hom}(N, \mathbb{Z})$. Then $T_N = \text{Spec } k[M]$. An element $\sum a_i m_i$ in M is usually identified with the Laurent monomial $x_1^{a_1} x_2^{a_2} \dots x_r^{a_r}$ and $k[M]$ with $k[x_1, x_1^{-1}, \dots, x_r, x_r^{-1}]$. Let ν be a positive integer and fix a primitive ν th root of unity ζ . Given units α, β in $k[x_1, x_1^{-1}, \dots, x_r, x_r^{-1}]$, the symbol algebra $(\alpha, \beta)_\nu$ is the associative k -algebra generated by elements u, v subject to the relations $u^\nu = \alpha, v^\nu = \beta$, and $uv = \zeta vu$. In what follows, we choose to identify $(x_i, x_j)_\nu$ as $(m_i, m_j)_\nu$ and work in $k(M)$. By [10, Theorem 6], ${}_\nu B(T_N)$ is a free $\mathbb{Z}/\nu$ -module with basis given by the set of symbol algebras $\{(m_i, m_j)_\nu\}_{1 \leq i < j \leq r}$ for each $\nu \geq 2$. Since T_N is an open subset of X and X is nonsingular, $B(X)$ is a subgroup of $B(T_N)$ by restriction and our object is to identify this subgroup explicitly.

From [4, Corollary 1.4] there is an exact sequence

$$(1) \quad 0 \rightarrow B'(X) \rightarrow B(T_N) \xrightarrow{a} \bigoplus_{i=1}^n H^1(\text{orb } \rho_i, \mathbb{Q}/\mathbb{Z})$$

where $\text{orb } \rho_i$ is the T_N -invariant divisor on $X = T_N \text{emb}(\Delta)$ corresponding to the face ρ_i of Δ . Given a symbol algebra $(\alpha, \beta)_\nu$ representing a class in $B(T_N)$, the ramification map a agrees with the tame symbol (see the discussion following [4, Remark 1.7] and [14, Theorem 8, p. 155]). This means the k th coordinate of $a((\alpha, \beta)_\nu)$, the ramification of $(\alpha, \beta)_\nu$ along $\text{orb } \rho_k$, is identified with a cyclic Galois extension of $\text{orb } \rho_k$ of degree ν . Over the function field $K(\text{orb } \rho_k)$ this extension is given by adjoining the ν th root of $\alpha^{v_k(\beta)} / \beta^{v_k(\alpha)}$ where v_k is the valuation on $K(X)$ determined by the prime divisor $\text{orb } \rho_k$.

From the remarks above, to determine the ramification of an arbitrary algebra Λ representing an element in $B(T_N)$ along $\text{orb } \rho_k$ it suffices to determine $K(\text{orb } \rho_k)$ and $v_k(m_j)$ for each k, j . The following lemma is well known. We include its short proof for completeness and to fix notation.

Lemma 1.5. *Let η_k be the primitive vector in $N \cap \rho_k$ and $\langle \cdot, \cdot \rangle$ the natural inner product from $M \times N \rightarrow \mathbb{Z}$.*

- (a) $K(\text{orb } \rho_k)$ is the quotient field of $k[\eta_k^\perp]$.
- (b) $v_k(m) = \langle m, \eta_k \rangle$.

Proof. Since η_k is primitive there is a primitive $\mu_k \in M$ with $\langle \mu_k, \eta_k \rangle = 1$. Let $\eta_k^\perp = \{m \in M \mid \langle m, \eta_k \rangle = 0\}$. Then $M = \eta_k^\perp \oplus \mathbb{Z}\mu_k$ since

$$0 \rightarrow \eta_k^\perp \rightarrow M \xrightarrow{\varphi} \mathbb{Z} \rightarrow 0$$

splits, where $\varphi(m) = \langle m, \eta_k \rangle$. The affine coordinate ring of

$$U_{\rho_k} = T_N \text{ emb}\{0, \rho_k\}$$

is $k[\eta_k^\perp, \mu_k]$. Localizing X along $\text{orb } \rho_k$ is equivalent to localizing U_{ρ_k} along $\text{orb } \rho_k$. The prime ideal corresponding to $\text{orb } \rho_k$ is the principal ideal in $k[\eta_k^\perp, \mu_k]$ generated by μ_k . Hence μ_k is a local parameter along $\text{orb } \rho_k$. $K(\text{orb } \rho_k)$ is the quotient field of $k[\eta_k^\perp, \mu_k]/(\mu_k)$ giving (a). The valuation v_k of any $m \in M$ is the μ_k -coordinate when m is written in terms of the decomposition $M = \eta_k^\perp \oplus \mathbb{Z}\mu_k$. Thus $v_k(m) = \langle m, \eta_k \rangle$. $\square$

Keeping the notation above, define a homomorphism $\text{ram}_{\text{orb } \rho_k}: {}_\nu B(T_N) \rightarrow M/\nu M$ by letting $\text{ram}_{\text{orb } \rho_k}(m_i, m_j)_\nu = \langle m_j, \eta_k \rangle m_i - \langle m_i, \eta_k \rangle m_j + \nu M$ be the assignment on the basis for ${}_nu B(T_N)$, and extending by $\mathbb{Z}/\nu$ -linearity.

Lemma 1.6. $(m_i, m_j)_\nu$ is unramified along $\text{orb } \rho_k$ if and only if

$$\text{ram}_{\text{orb } \rho_k}(m_i, m_j)_\nu = 0.$$

Proof. The ramification of $(m_i, m_j)_\nu$ along $\text{orb } \rho_k$ corresponds to the cyclic extension of the affine coordinate ring $k[\eta_k^\perp]$ of $\text{orb } \rho_k$ obtained by adjoining the ν th root of $v_k(m_j)m_i - v_k(m_i)m_j = \langle m_j, \eta_k \rangle m_i - \langle m_i, \eta_k \rangle m_j$. (Note $\langle \langle m_j, \eta_k \rangle m_i - \langle m_i, \eta_k \rangle m_j, \eta_k \rangle = 0$ so $\langle m_j, \eta_k \rangle m_i - \langle m_i, \eta_k \rangle m_j \in \eta_k^\perp$.) Thus, $(m_i, m_j)_\nu$ is unramified along $\text{orb } \rho_k$ if and only if $\langle m_j, \eta_k \rangle m_i - \langle m_i, \eta_k \rangle m_j$ is a ν th power in $k[\eta_k^\perp]$ if and only if $\langle m_j, \eta_k \rangle m_i - \langle m_i, \eta_k \rangle m_j \in \nu M$ if and only if $\text{ram}_{\text{orb } \rho_k}(m_i, m_j)_\nu = 0$. $\square$

Let Λ be any Azumaya algebra representing a class in ${}_nu B(T_N)$. We have seen Λ is equivalent to $\prod_{i < j} (m_i, m_j)_\nu^{e_{ij}}$ where $0 \leq e_{ij} < \nu$. Moreover the class of Λ determines and is determined by the integers e_{ij} . Associate to the class represented by Λ in ${}_nu B(T_N)$ the matrix

$$M_\Lambda = \begin{bmatrix} 0 & e_{12} & e_{13} & \cdots & e_{1r} \\ -e_{12} & 0 & e_{23} & \cdots & e_{2r} \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ -e_{1r} & -e_{2r} & \cdots & -e_{r-1,r} & 0 \end{bmatrix}.$$

Lemma 1.7. (a) *The assignment $\Lambda \rightarrow M_\Lambda$ induces a monomorphism*

$$\phi: {}_\nu B(T_N) \rightarrow \text{Hom}_{\mathbb{Z}}(N, M/\nu M).$$

- (b) Λ is unramified along $\text{orb } \rho_k$ if and only if $M_\Lambda \cdot \eta_k = 0$.

Proof. (a) The matrix M_Λ defines the indicated homomorphism $\phi(\Lambda)$ by representing elements in N as column vectors with respect to the basis $n_1, \dots, n_r$; the elements in M as column vectors with respect to the dual basis $m_1, \dots, m_r$ and following left multiplication by M_Λ by reduction modulo νM . Since multiplication of symbols corresponds to addition of exponents modulo ν , it is clear that ϕ is a homomorphism. If $\phi(\Lambda) = M_\Lambda = 0$, then each $e_{ij} = 0$, so $\Lambda = 0$ in $B(T_N)$. Thus ϕ is a monomorphism.

(b) Write $\eta_k = \sum_{i=1}^r \eta_{ki} n_i$ and let $\Lambda = \prod_{i < j} (m_i, m_j)_{\nu}^{e_{ij}}$. Then

$$\text{ram}_{\text{orb } \rho_k}(m_i, m_j)_{\nu} = \langle m_j, \eta_k \rangle m_i - \langle m_i, \eta_k \rangle m_j = \eta_{kj} m_i - \eta_{ki} m_j.$$

Hence

$$\begin{aligned} \text{ram}_{\text{orb } \rho_k}(\Lambda) &= \text{ram}_{\text{orb } \rho_k} \left(\prod_{i < j} (m_i, m_j)_{\nu}^{e_{ij}} \right) \\ &= \sum_{i < j} e_{ij} \eta_{kj} m_i - e_{ij} m_j + \nu M \\ &= \sum_{i=1}^r \sum_{j=i+1}^r e_{ij} \eta_{kj} m_i - \sum_{i=1}^r \sum_{j=i+1}^r e_{ij} \eta_{ki} m_j + \nu M \\ &= \sum_{i=1}^r \sum_{j=i+1}^r e_{ij} \eta_{kj} m_i - \sum_{j=1}^r \sum_{i=j+1}^r e_{ji} \eta_{kj} m_i + \nu M \\ &= M_\Lambda \cdot \begin{bmatrix} \eta_{k1} \\ \vdots \\ \eta_{kr} \end{bmatrix}. \end{aligned}$$

As we observed in the proof of Lemma 1.6, $\text{ram}_{\text{orb } \rho_k}(\Lambda) = m + \nu M$ for some $m \in \eta_k^\perp$. The ramification of Λ along $\text{orb } \rho_k$ is the cyclic extension of $k[\eta_k^\perp]$ obtained by adjoining the ν th root of m and this extension is split (Λ is unramified along $\text{orb } \rho_k$) if and only if $m \in \nu \eta_k^\perp$. Since $\eta_k^\perp$ is a direct summand of M , Λ is unramified along $\text{orb } \rho_k$ if and only if $m \in \nu M$ if and only if

$$M_\Lambda \cdot \begin{bmatrix} \eta_{k1} \\ \vdots \\ \eta_{kr} \end{bmatrix} = 0. \quad \square$$

Theorem 1.8. *Let $X = T_N \text{emb}(\Delta)$ be a nonsingular toric variety and $a_1, \dots, a_r$ the set of invariant factors of X . Then $B'(X)$ is the subgroup of $B(T_N)$ represented by algebra classes $\prod_{i < j} (m_i, m_j)_{\nu_i}^{e_{ij}}$ where $\nu_i | a_i$, $1 \leq i \leq r$.*

Proof. The exact sequence (1) and Lemma 1.7 imply ${}_\nu B'(X)$ consists of those algebra classes Λ in ${}_\nu B(T_N)$ such that $M_\Lambda \cdot \eta_k = 0$ for the primitive vector η_k on each 1-dimensional cone ρ_k in Δ ($1 \leq k \leq n$). If N' is the subgroup of N generated by $\bigcup_{\sigma \in \Delta} \sigma \cap N$ then N' is generated by $\{\eta_k\}_{k=1}^n$ so Λ represents a class in ${}_\nu B'(X)$ if and only if M_Λ vanishes on N' . For each $\nu > 0$ we have

the commutative diagram with exact rows and columns

$$\begin{array}{ccccc}
 0 & \longrightarrow & {}_{\nu}\mathbf{B}(T_N) & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}}(N, M/\nu M) \\
 & & \uparrow & & \uparrow \\
 0 & \longrightarrow & {}_{\nu}\mathbf{B}'(X) & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}}(N/N', M/\nu M) \\
 & & \uparrow & & \uparrow \\
 & & 0 & & 0
 \end{array}$$

Taking the direct limit over all $\nu \geq 2$ gives a monomorphism

$$\mathbf{B}'(X) \rightarrow \mathrm{Hom}_{\mathbb{Z}}(N/N', M \otimes \mathbb{Q}/\mathbb{Z}).$$

Let $n_1, \dots, n_r$ be a basis for N such that $N' = \mathbb{Z}a_1n_1 \oplus \dots \oplus \mathbb{Z}a_rn_r$ and $a_i|a_{i+1}$ for $1 \leq i \leq r-1$. That is $a_1, \dots, a_r$ is the set of invariant factors for X . Then $\mathrm{Hom}_{\mathbb{Z}}(N/N', M \otimes \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(\bigoplus (\mathbb{Z}n_i/\mathbb{Z}a_in_i), M \otimes \mathbb{Q}/\mathbb{Z})$. This means $\mathbf{B}'(X)$ is contained in the subgroup of $\mathbf{B}(T_N)$ of algebra classes $\prod_{i < j} (m_i, m_j)_{\nu_i}^{e_{ij}}$ where $0 < \nu_i$ and $\nu_i|a_i$, $1 \leq i \leq r$. Conversely, if $\nu_i|a_i$ and $\nu_i \geq 1$ then the matrix M_{Λ} for $(m_i, m_j)_{\nu_i}$ has a $+1$ in the ij th entry and a -1 in the ji th entry. A typical element in N' is $x = \lambda_1 a_1 n_1 + \dots + \lambda_r a_r n_r$ and $M_{\Lambda} \cdot x = \lambda_j a_j m_i - \lambda_i a_i m_j \in a_i M$. Thus $(m_i, m_j)_{\nu_i}$ represents an element in $\mathbf{B}'(X)$. So $\mathbf{B}'(X) = \{\prod_{i < j} (m_i, m_j)_{\nu_i}^{e_{ij}} | 0 < \nu_i \text{ and } \nu_i|a_i\}$. $\square$

Now it follows that $\mathbf{B}'(X) \cong \bigoplus_{i=1}^{r-1} \mathrm{Hom}(\mathbb{Z}/a_i, \mathbb{Q}/\mathbb{Z})^{r-i}$. To complete the proof of Theorem 1.1 it suffices to show $\mathbf{B}(X) = \mathbf{B}'(X)$. It suffices to find an Azumaya algebra Λ on X such that $K(X) \otimes \Lambda$ is equivalent to $(m_i, m_j)_{\nu_i}$ for each $\nu_i|a_i$.

Lemma 1.9. *Let X be as in Theorem 1.8 and let $N' = \langle \bigcup_{\sigma \in \Delta} \sigma \cap N \rangle$. Let $\nu \geq 2$ and let $M_{\nu} = \{m \in M \mid \langle m, n' \rangle \equiv 0 \pmod{\nu} \text{ for all } n' \in N'\}$. If (ν, a_i) is the greatest common divisor of ν and a_i , then*

$$\mathrm{H}^1(X, \mathbb{Z}/\nu) \cong M_{\nu}/\nu M \cong \bigoplus_{i=1}^r \mathbb{Z}/(\nu, a_i).$$

Proof. Restriction induces an embedding $\mathrm{H}^1(X, \mathbb{Z}/\nu) \rightarrow \mathrm{H}^1(T_N, \mathbb{Z}/\nu)$. The correspondence which assigns to each element $m \in M$ the cyclic extension of T_N obtained by adjoining the ν th root of m induces an isomorphism $M/\nu M \cong \mathrm{H}^1(T_N, \mathbb{Z}/\nu)$. An element $m + \nu M$ corresponds to an element of $\mathrm{H}^1(X, \mathbb{Z}/\nu)$ if and only if $K(X)(m^{1/\nu})$ is unramified along $\mathrm{orb} \rho_k$ for $1 \leq k \leq n$ if and only if the restriction of m to $\mathrm{orb} \rho_k$ is a unit in the coordinate ring $k[\eta_k^{\perp}, \mu_k]$ of $\mathrm{orb} \rho_k$ if and only if $v_k(m) \equiv 0 \pmod{\nu}$ if and only if $\langle m, \eta_k \rangle \equiv 0 \pmod{\nu}$ where unexplained notation is as in Lemma 1.5. Thus

$$\begin{aligned}
 \mathrm{H}^1(X, \mathbb{Z}/\nu) &= \{m \in M \mid \langle m, \eta_k \rangle \equiv 0 \pmod{\nu} \ (1 \leq k \leq n)\} + \nu M \\
 &= \{m \in M \mid \langle m, n' \rangle \equiv 0 \pmod{\nu} \text{ for all } n' \in N'\} + \nu M \\
 &= M_{\nu}/\nu M.
 \end{aligned}$$

But $N' = \mathbb{Z}a_1n_1 \oplus \dots \oplus \mathbb{Z}a_rn_r$ so it is easy to check that

$$M_{\nu}/\nu M \cong \bigoplus_{i=1}^r \mathbb{Z} \left(\frac{\nu}{(a_i, \nu)} m_i \right) / \mathbb{Z}(\nu m_i) \cong \bigoplus_{i=1}^r \mathbb{Z}/(a_i, \nu). \quad \square$$

Lemma 1.10. *If X is as in Theorem 1.8, then $B(X) = B'(X)$.*

Proof. It suffices to show each $(m_i, m_j)_{\nu_i}$ is in the image of the cup product map $H^1(X, \mathbb{Z}/\nu_i) \times H^1(X, \mu_{\nu_i}) \rightarrow B(X)$ when $\nu_i | a_i$, since cup products correspond to taking smash products of cyclic Galois extensions and thus are Azumaya algebras (e.g. [6]).

If $a_j = 0$, both $m_i, m_j \in M_{\nu_i}$ since $\nu_i/(a_i, \nu_i) = \nu_i/\nu_i = 1 = \nu_i/(0, \nu_i)$. If $a_j \neq 0$, then $(m_i, m_j)_{\nu_i} \sim (m_i, m_j)_{a_j/\nu_i}^{a_j/\nu_i} \sim ((a_j/\nu_i)m_i, m_j)_{a_j}$. But $a_j/(a_i, a_j) = a_j/a_i$ which divides a_j/ν_i since $\nu_i | a_i$. Thus $(a_j/\nu_i)m_i$ and m_j are both in M_{a_j} and $(m_i, m_j)_{\nu_i}$ is equivalent to an algebra in the image of the cup product map $H^1(X, \mathbb{Z}/a_j) \times H^1(X, \mu_{a_j}) \rightarrow_{a_j} B(X)$. So $B(X) = B'(X)$. $\square$

As a result of observations made so far, we can show the following proposition.

Proposition 1.11. *Let $\prod_{i < j} (m_i, m_j)_{\nu}^{e_{ij}}$ represent a class in $B(T_N)$ of order ν . Let M_{Λ} be the matrix transformation in $\text{Hom}(N, M/\nu M)$ defined in Lemma 1.7 and let t be the rank of $\ker(M_{\Lambda})$. Then there exists a direct summand P of M with $\text{rank}(P) = r - t$ and an Azumaya algebra L over $k[P]$ with $\Lambda \cong k[M] \otimes_{k[P]} L$. No direct summand of M of smaller rank has this property.*

Proof. Find a basis $n_1, \dots, n_r$ of N such that $\ker(M_{\Lambda}) = \mathbb{Z}b_1n_1 \oplus \mathbb{Z}b_2n_2 \oplus \dots \oplus \mathbb{Z}b_rn_r$ and $b_i | b_{i+1}$ for $1 \leq i \leq r-1$. Since $\ker(M_{\Lambda})$ has rank t , $b_i \neq 0$ and $b_{t+i} = 0$ for $i \geq 1$.

Let P be a direct summand of M and assume Λ is obtained by extending an algebra over $k[P]$. Let $m'_1, \dots, m'_s$ be a basis for P and extend this basis to a basis for M . We can assume $\Lambda = \prod_{i < j \leq s} (m'_i, m'_j)_{\nu}^{e'_{ij}}$. If $n'_1, \dots, n'_r$ is the dual basis to $m'_1, \dots, m'_r$, then the matrix of the transformation M_{Λ} with respect to this new basis pair has a kernel which contains a direct summand of N of rank $r - s$. Therefore $t \geq r - s$ so $s \geq r - t$. Now let $m_1, \dots, m_r$ be a dual basis for M with respect to $n_1, \dots, n_r$. The matrix M_{Λ} with respect to this new basis is

$$M_{\Lambda} = \left[\begin{array}{c|cccc} 0 & & & & 0 \\ \hline & 0 & e_{t+1,t} & \cdots & e_{t+1,r} \\ 0 & -e_{t+1,t} & 0 & \cdots & \\ & \vdots & & & \vdots \\ & -e_{t+1,r} & & \cdots & 0 \end{array} \right].$$

So Λ is defined on the torus $k[m_{t+1}, \dots, m_r, -m_{t+1}, \dots, -m_r]$ and we can take $P = \langle m_{t+1}, \dots, m_r \rangle$. The rank of P is $r - t$. $\square$

2

In this section we continue to let Δ be a finite fan on $N_{\mathbb{R}}$ and $X = T_N \text{emb}(\Delta)$ the associated toric variety containing the r -dimensional torus T_N as an open subset. Assume Δ consists of cones of dimension ≤ 2 and let $\Delta(2) = \{\tau_1, \dots, \tau_m\}$. Let $U_i = U_{\tau_i}$, $V_i = V(\tau_i) = \text{orb}(\tau_i)$ and let $V = V_1 \cup \dots \cup V_m$. Then $X - V = T_N \text{emb}(\Delta - \Delta(2))$ is nonsingular. In this situation our first lemma gives information about the étale cohomology groups of the affine open subsets U_i of X .

Lemma 2.1. (a) For each i and each $p \geq 0$, we have a short exact sequence

$$0 \rightarrow H^p(U_i, G_m) \rightarrow H^p(U_i - V_i, G_m) \rightarrow H_{V_i}^{p+1}(U_i, G_m) \rightarrow 0.$$

(b) $H^p(U_i, G_m) \cong H^p(T_{r-2}, G_m)$ where T_{r-2} is a torus of dimension $r - 2$.

Proof. First we check that $H^p(U_i, G_m)$ is torsion for $p \geq 2$. For notational simplicity we suppress the subscript i from τ_i , U_i , and V_i . Now τ is a two-dimensional cone in $N_{\mathbb{R}}$. Let $\bar{\tau}$ be τ viewed as a two-dimensional cone in $\mathbb{R}\tau$. Then $U = U_{\bar{\tau}} \times T_{r-2}$ where T_{r-2} is an $(r - 2)$ -dimensional torus. Let R be the affine coordinate ring of $U_{\bar{\tau}}$, and R^h the henselization of R at the maximal ideal m corresponding to the closed point $\text{orb } \bar{\tau}$. Let $R[X, X^{-1}]$ denote the affine coordinate ring of U and let $U^h = \text{Spec } R^h[X, X^{-1}]$. Let $V^h = V \times U^h$. Then V^h is the closed set corresponding to $I = mR^h[X, X^{-1}]$. The completion of $R^h[X, X^{-1}]$ in the I -adic topology is $\hat{R}[X, X^{-1}]$ where $\hat{R}$ is the m -adic completion of R . By [13, p. 127], we see that $(R^h[X, X^{-1}], I)$ is a Hensel pair. By [5, p. 35] $\text{Cl}(R^h[X, X^{-1}])$ embeds into $\text{Cl}(\hat{R}[X, X^{-1}])$. Since the singularity of U is given by a finite cyclic group action [12, p. 30], it is well known that $\text{Cl}(\hat{R}[X, X^{-1}]) = \text{Cl}(\hat{R})$ is also finite cyclic [2, Satz 2.11]. Thus $\text{Cl}(U^h)$ is finite. The long exact sequences for the pairs $V \subseteq U$ and $V^h \subseteq U^h$ give the commutative diagram

$$\begin{array}{ccccccc} \rightarrow H^{p-1}(U - V, G_m) & \rightarrow & H_V^p(U, G_m) & \rightarrow & H^p(U, G_m) & \rightarrow & H^p(U - V, G_m) \\ & & \downarrow & & \downarrow \cong & & \downarrow \\ \rightarrow H^{p-1}(U^h - V^h, G_m) & \rightarrow & H_{V^h}^p(U^h, G_m) & \rightarrow & H^p(U^h, G_m) & \rightarrow & H^p(U^h - V^h, G_m) \end{array}$$

with exact rows. By excision $H_V^p(U, G_m) \cong H_{V^h}^p(U^h, G_m)$ [11, p. 92]. By [15] $H^p(U^h, G_m) \cong H^p(V^h, G_m) = H^p((\text{orb } \bar{\tau}) \times T_{r-2}, G_m)$ which is torsion for $p \geq 2$ since T_{r-2} is smooth [7, p. 71]. Again by [7, p. 71] $H^p(U^h - V^h, G_m)$ and $H^p(U - V, G_m)$ are torsion for $p \geq 2$. But $H^1(U^h - V^h, G_m) = \text{Pic}(U^h - V^h) = \text{Cl}(U^h - V^h) = \text{Cl}(U^h)$ is torsion. It now follows that $H^p(U, G_m)$ is torsion for $p \geq 2$.

The natural map $U \times A^1 \rightarrow U$ and Kummer theory induce the commutative diagram

$$\begin{array}{ccccccc} 0 \rightarrow H^{p-1}(U \times A^1, G_m) \otimes \mathbb{Z}/n & \rightarrow & H^p(U \times A^1, \mu_n) & \rightarrow & {}_n H^p(U \times A^1, G_m) & \rightarrow & 0 \\ & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma \\ 0 \rightarrow H^{p-1}(U, G_m) \otimes \mathbb{Z}/n & \rightarrow & H^p(U, \mu_n) & \rightarrow & {}_n H^p(U, G_m) & \rightarrow & 0 \end{array}$$

for all $p \geq 2$ and $n \geq 2$. By [11, p. 240] β is an isomorphism for $p \geq 2$. Since $\text{Pic } U = 0 = \text{Pic}(U \times A^1)$, α is an isomorphism for $p = 2$. Therefore, γ is an isomorphism for $p = 2$ and all $n \geq 2$. Taking the inductive limit over all n , we have $H^2(U \times A^1, G_m) \cong H^2(U, G_m)$. By induction on p we see that $H^p(U \times A^1, G_m) \cong H^p(U, G_m)$ for all $p \geq 2$.

We can give the coordinate ring $k[\mathcal{S}_\tau]$ of U a grading by the nonnegative integers such that the degree $= 0$ subring is the coordinate ring of T_{r-2} . Since $H^p(U \times A^1, G_m) = H^p(U, G_m)$, [8, Theorem 1.1] implies $H^p(U, G_m) \cong$

$H^p(T_{r-2}, G_m)$, which proves (b). We have a commutative diagram

$$\begin{array}{ccccc} H^p(T_{r-2}, G_m) & \cong & H^p(U, G_m) & \xrightarrow{\alpha} & H^p(U - V, G_m) \\ & \searrow \beta & \downarrow \gamma & \swarrow \delta & \\ & & H^p(T_r, G_m) & & \end{array}$$

where the maps β, γ, δ are induced from restriction. Since β is injective, α is injective and Lemma 2.1 now follows. $\square$

Theorem 2.2. *Let Δ be a fan which consists of cones of dimension ≤ 2 . Let Δ' be a nonsingular fan obtained from Δ by subdividing the two-dimensional faces of Δ and let $\tilde{X} = T_N \text{emb}(\Delta')$. Then the sequence $0 \rightarrow B'(T_N/X) \rightarrow B'(X) \rightarrow B'(\tilde{X}) \rightarrow 0$ (with natural maps) is exact.*

Proof. Let $\pi: \tilde{X} \rightarrow X$ be the desingularization resulting from the subdivision Δ' of Δ [12, Corollary 1.18] and let $\tilde{U}_i = \pi^{-1}(U_i)$. From the long exact sequence of cohomology with supports, and the observation that V is a disjoint union of closed sets V_i (see pp. 92–93 of [11]) we have a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 \rightarrow B'(T_N/X) \rightarrow B'(X) \rightarrow B'(X - V) & \longrightarrow & \bigoplus_{i=1}^m H_{V_i}^3(U_i, G_m) \\ \downarrow \alpha & & \downarrow = & & \downarrow \beta \\ 0 \rightarrow B'(\tilde{X}) \rightarrow B'(\tilde{X} - \pi^{-1}(V)) & \rightarrow & \bigoplus_{i=1}^m H_{\pi^{-1}(V_i)}^3(\tilde{U}_i, G_m) \end{array}$$

The second row is exact since $\tilde{X}$ is nonsingular. First check β is injective. For each i , Lemma 2.1 yields the commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 \longrightarrow B'(U_i) \longrightarrow B'(U_i - V_i) & \longrightarrow & H_{V_i}^3(U_i, G_m) \longrightarrow 0 \\ \downarrow \alpha_i & & \downarrow = & & \downarrow \beta_i \\ 0 \longrightarrow B'(\tilde{U}_i) \longrightarrow B'(\tilde{U}_i - \pi^{-1}(V_i)) & \longrightarrow & H_{\pi^{-1}(V_i)}^3(\tilde{U}_i, G_m) \end{array}$$

Here $B'(U_i) = B'(\tilde{U}_i \times T_{r-2}) = B(T_{r-2})$ by Lemma 2.1 and $B'(U_i - V_i) = H^2(U_i - V_i, G_m)$ since $U_i - V_i$ is nonsingular. If $\Delta'(\tau_i)$ is the fan whose cones are the cones of Δ' contained in τ_i , then $\Delta'(\tau_i)$ is a nonsingular fan whose one dimensional faces lie in a plane. The invariants for $\tilde{U}_i = T_N \text{emb}(\Delta'(\tau_i))$ are $\{1, 1, 0, \dots, 0\}$ and Theorem 1.1 implies $B'(\tilde{U}_i) = B'(\tilde{U}_i \times T_{r-2}) = B(T_{r-2})$ so α_i is an isomorphism. Since $\ker \beta_i = \text{coker } \alpha_i$, β_i is injective so β is injective. But $\ker \beta = \text{coker } \alpha$, so α is an epimorphism and the theorem follows. $\square$

As a result of Theorem 2.2 and our analysis of the Brauer groups of nonsingular toric varieties in §1, we are left with the study of $B'(T_N/X)$.

Theorem 2.3. *Let Δ be a fan which consists of cones of dimension ≤ 2 . Let $\Delta(2) = \{\tau_1, \dots, \tau_m\}$. Let $X = T_N \text{emb}(\Delta)$ and let $U_i = U_{\tau_i}$ be the open subsets*

of X associated to the τ_i . Then there is an exact sequence

$$0 \rightarrow \text{Pic}(X) \rightarrow \text{Cl}(X) \rightarrow \bigoplus_{i=1}^m \text{Cl}(U_i) \rightarrow \text{B}'(T_N/X) \rightarrow 0.$$

Proof. Let $V_i = V(\tau_i) = \text{orb}(\tau_i)$ and let $V = V_1 \cup \cdots \cup V_m$. From the long exact sequence of cohomology with supports in the closed set V we have (since V is the disjoint union of the closed sets V_i)

$$(1) \quad \begin{aligned} \cdots \rightarrow H^1(X, G_m) \rightarrow H^1(X - V, G_m) \rightarrow \bigoplus_{i=1}^m H_{V_i}^2(X, G_m) \\ \rightarrow H^2(X, G_m) \rightarrow H^2(X - V, G_m) \rightarrow \bigoplus_{i=1}^m H_{V_i}^3(X, G_m) \rightarrow \cdots \end{aligned}$$

Since V has codimension 2 in X , and $X - V$ is nonsingular, $H^1(X - V, G_m) = \text{Pic}(X - V) = \text{Cl}(X - V) = \text{Cl}(X)$. Since U_i is an open neighborhood of V_i , $H_{V_i}^p(X, G_m) = H_{V_i}^p(U_i, G_m)$ for all $p \geq 0$ [11, p. 93]. From Lemma 2.1 with $p = 1$ we get the exact sequences

$$0 \rightarrow \text{Pic } U_i \rightarrow \text{Cl } U_i \rightarrow H_{V_i}^2(U_i, G_m) \rightarrow 0 \quad (1 \leq i \leq m).$$

Lemma 2.1(b) gives $\text{Pic } U_i = \text{Pic } T_{r-2} = 0$ so $\text{Cl}(U_i) = H_{V_i}^2(U_i, G_m)$. Since τ_i is simplicial, $\text{Cl}(U_i) = \text{Pic}(U_i - V_i)$ is torsion [12, Proposition 2.1]. Since $X - V$ is nonsingular, $\text{B}'(X - V) = H^2(X - V, G_m)$ and $\text{B}'(X - V) \rightarrow \text{B}(T_N)$ is injective. But $\text{Pic } X \rightarrow \text{Cl}(X)$ is injective [5]. With these identifications (1) reduces to the sequence of the theorem. $\square$

Corollary 2.4. *In the context of Theorem 2.3, if $\text{rank}_{\mathbb{Z}}(N) = r \leq 3$ and $m \geq 1$, then*

$$0 \rightarrow \text{Pic}(X) \rightarrow \text{Cl}(X) \rightarrow \bigoplus_{i=1}^m \text{Cl}(U_i) \rightarrow \text{B}'(X) \rightarrow 0$$

is exact.

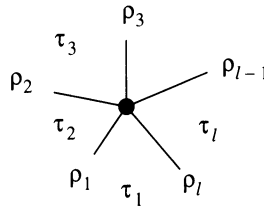
Proof. We need to check $\text{B}'(X) = \text{B}'(T_N/X)$. $H^2(U_i, G_m) = H^2(T_{r-2}, G_m) = 0$ for $r \leq 3$ (Lemma 2.1 and [10]). From Lemma 2.1(a) we have (since $U_i - V_i$ is nonsingular) $\text{B}'(U_i - V_i) = H^2(U_i - V_i, G_m) = H_{V_i}^3(U_i, G_m) = H_{V_i}^3(X, G_m)$ so (1) becomes

$$0 \rightarrow \text{Pic } X \rightarrow \text{Cl}(X) \rightarrow \bigoplus_{i=1}^m \text{Cl}(U_i) \rightarrow \text{B}'(X) \rightarrow \text{B}'(X - V) \rightarrow \bigoplus_{i=1}^m \text{B}'(U_i - V_i).$$

Since $X - V$ is nonsingular, restriction induces a monomorphism $\text{B}'(X - V) \rightarrow \text{B}'(U_i - V_i)$ for each i and the corollary follows. $\square$

The object of the rest of this section is to give an algorithm for finding for each prime p a subset $\tau_1, \dots, \tau_e$ of the two-dimensional faces of Δ such that $[\bigoplus_{i=1}^e \text{Cl}(U_{\tau_i})]_p \cong [\text{B}'(T_N/X)]_p$. (For G a finite abelian group, G_p is the Sylow p -subgroup.) In particular, the exact sequence of Theorem 2.3 is split-exact. To the fan Δ we associated a bipartite graph Γ . The vertex set of Γ is $\Delta(1) \cup \Delta(2) = \{\rho_1, \dots, \rho_n\} \cup \{\tau_1, \dots, \tau_m\}$ and there is an edge in Γ connecting ρ_j and τ_i if and only if ρ_j is a face of τ_i . If Y is the T_N -invariant divisor $X - T_N$ on X , then Γ is the graph associated to Y in the sense of [4]. A

cycle Z in Γ (i.e., Z is homeomorphic to the unit circle) determines a finite set $\tau_1, \dots, \tau_l$ of two-dimensional cones and $\rho_1, \dots, \rho_l$ of one-dimensional faces of Δ configured as follows:



If Δ_Z is the subfan of Δ consisting of the cones $\{0, \rho_1, \dots, \rho_l, \tau_1, \dots, \tau_l\}$, we will show that the cohomological Brauer group of $T_N \text{emb}(\Delta_Z)$ is cyclic of order the greatest common divisor of $\{|\text{Cl } U_{\tau_i}|\}_{i=1}^l$. Of course, there may be many such cycles in Γ and the last step in the analysis is to choose for each prime p a list of cycles $\{Z_i\}_{i=1}^e$ and for each Z_i a face τ_i such that $[\bigoplus_{i=1}^e \text{Cl}(U_{\tau_i})]_p \cong [\mathbf{B}'(T_N/X)]_p$.

We adopt the following notation: for each two-dimensional cone τ_i in Δ ($1 \leq i \leq m$) let ρ_{i1} and ρ_{i2} be the one-dimensional faces of τ_i so $\tau_i = \rho_{i1} + \rho_{i2}$. We have observed $\text{Cl}(X) = \text{Cl}(X - V) = \text{Pic}(X - V)$ and $\text{Cl}(U_i) = \text{Cl}(U_i - V_i) = \text{Pic}(U_i - V_i)$. Now we want to present $\text{Pic}(X - V)$ and $\text{Pic}(U_i - V_i)$ in terms of support functions on the fan $\Delta - \{\tau_1, \dots, \tau_m\}$. If we let $\rho_1, \dots, \rho_n$ be the one-dimensional cones in Δ , then we can identify the support functions on $\Delta - \{\tau_1, \dots, \tau_m\}$ with the direct sum of copies of $\mathbb{Z}$ indexed by the ρ_i . If $\Delta_i = \{0, \rho_{i1}, \rho_{i2}\}$, then $U_i - V_i = T_N \text{emb}(\Delta_i)$. It follows from [12, Corollary 2.5] that the sequences

$$(2) \quad \begin{aligned} M &\rightarrow \bigoplus_{i=1}^n \mathbb{Z}\rho_i \rightarrow \text{Cl}(X) \rightarrow 0, \\ M &\xrightarrow{\beta} \mathbb{Z}\rho_{i1} \oplus \mathbb{Z}\rho_{i2} \rightarrow \text{Cl}(U_i) \rightarrow 0 \end{aligned}$$

are exact. Combining these sequences with the exact sequence of Theorem 2.3 gives a commutative diagram with exact rows and columns

$$\begin{array}{ccccccc} M & \longrightarrow & \bigoplus_{j=1}^n \mathbb{Z}\rho_j & \xrightarrow{\phi} & \text{Cl}(X) & \longrightarrow & 0 \\ & & \downarrow \alpha & & \downarrow \varepsilon & & \\ \bigoplus_{i=1}^m M & \xrightarrow{\beta} & \bigoplus_{i=1}^m (\mathbb{Z}\rho_{i1} \oplus \mathbb{Z}\rho_{i2}) & \xrightarrow{\gamma} & \bigoplus_{i=1}^m \text{Cl}(U_i) & \longrightarrow & 0 \\ & & & & \downarrow \delta & & \\ & & & & \mathbf{B}'(T_N/X) & & \\ & & & & \downarrow & & \\ & & & & 0 & & \end{array}$$

It is routine to check that $\text{im } \alpha + \text{im } \beta = \ker \delta\gamma$. As a result we have a fundamental exact sequence which we exploit for the remainder of this section: (in this sequence $\psi = \delta\gamma$),

$$(3) \quad \left(\bigoplus_{j=1}^n \mathbb{Z}\rho_j \right) \oplus \left(\bigoplus_{i=1}^m M \right) \xrightarrow{\alpha+\beta} \bigoplus_{i=1}^m (\mathbb{Z}\rho_{i1} \oplus \mathbb{Z}\rho_{i2}) \xrightarrow{\psi} B'(T_N/X) \rightarrow 0.$$

Let Γ be the graph associated to Δ . Observe that Γ has $2m$ edges since each τ_i has exactly two one-dimensional faces ρ_{i1} and ρ_{i2} . The free abelian group $\bigoplus_{i=1}^m (\mathbb{Z}\rho_{i1} \oplus \mathbb{Z}\rho_{i2})$ is called the edge space of Γ . If we write Γ as a union of its connected components Γ_i we get a corresponding decomposition of Δ into subfans Δ_i with $\Delta_i \cap \Delta_j = \{0\}$ whenever $i \neq j$. The decomposition of Δ gives an open cover of X where the elements in the open cover are $T_N \text{emb}(\Delta_i) = X_i$ and $X_i \cap X_j = T_N$ whenever $i \neq j$. With this notation we can prove

Proposition 2.5. *The natural map $B'(X) \rightarrow \bigoplus_i B'(X_i)$ induces an isomorphism $B'(T_N/X) \cong \bigoplus_i B'(T_N/X_i)$.*

Proof. Assume $\Delta = \Delta_1 \cup \Delta_2$ where Δ_1 and Δ_2 are fans with $\Delta_1 \cap \Delta_2 = \{0\}$. It is sufficient to prove $B'(T_N/X) \cong B'(T_N/X_1) \oplus B'(T_N/X_2)$ where $X_i = T_N \text{emb}(\Delta_i)$ ($i = 1, 2$). Let $\Delta_1(1) = \{\rho_1, \dots, \rho_{n_1}\}$ and $\Delta_1(2) = \{\tau_1, \dots, \tau_{m_1}\}$ and $\Delta_2(1) = \{r_1, \dots, r_{n_2}\}$ and $\Delta_2(2) = \{t_1, \dots, t_{m_2}\}$. Also let $\tau_i = \rho_{i1} + \rho_{i2}$ and $t_j = r_{j1} + r_{j2}$ where $\rho_{ik} \in \Delta_1(1)$ and $r_{jl} \in \Delta_2(1)$. With respect to this decomposition the exact sequence (3) decomposes as

$$\left[\bigoplus_{j=1}^{n_1} \mathbb{Z}\rho_j \oplus \bigoplus_{i=1}^{m_1} M \right] \oplus \left[\bigoplus_{j=1}^{n_2} \mathbb{Z}r_j \oplus \bigoplus_{i=1}^{m_2} M \right] \xrightarrow{(\alpha_1+\beta_1) \oplus (\alpha_2+\beta_2)} \left[\bigoplus_{i=1}^{m_1} (\mathbb{Z}\rho_{i1} \oplus \mathbb{Z}\rho_{i2}) \right] \oplus \left[\bigoplus_{i=1}^{m_2} (\mathbb{Z}r_{i1} \oplus \mathbb{Z}r_{i2}) \right] \rightarrow B'(T_N/X) \rightarrow 0.$$

But $\text{coker}(\alpha_1 + \beta_1) \oplus (\alpha_2 + \beta_2) = B'(T_N/X_1) \oplus B'(T_N/X_2)$ by (3) so $B'(T_N/X) = B'(T_N/X_1) \oplus B'(T_N/X_2)$. $\square$

Notice in Proposition 2.5 that if X_i corresponds to a connected component of Γ containing no two-dimensional faces τ_i as vertices, then $X_i = T_N \text{emb}\{0, \rho\}$ for some one-dimensional cone ρ in Δ . In this case X_i is nonsingular and $B'(T_N/X_i) = 0$. Thus, as a result of Proposition 2.5 we can assume Γ is connected and at least one vertex of Γ is a two-dimensional cone in Δ .

We now determine a matrix representation for the map $\alpha + \beta$ in (3). Let $\tau = \rho_1 + \rho_2 \in \Delta(2)$ and consider the map

$$M \xrightarrow{\beta} \mathbb{Z}\rho_1 \oplus \mathbb{Z}\rho_2 \rightarrow \text{Cl}(U_\tau) \rightarrow 0$$

as in (2). Pick a basis $n_1, \dots, n_r$ for N and a dual basis $m_1, \dots, m_r$ for M . Let η_i be a primitive element in N with $\rho_i = \mathbb{R}_{\geq 0}\eta_i$. The matrix of β with respect to the basis pair $\{m_1, \dots, m_r\}, \{\rho_1, \rho_2\}$ is the $2 \times r$ matrix whose i, j th entry is $\langle m_j, \eta_i \rangle$. But $\langle m_j, \eta_i \rangle$ is the j th coordinate of η_i so we can write this matrix as $\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$ where we think of η_i as a row vector. Therefore the map β in (3)

$$\bigoplus_{i=1}^m M \xrightarrow{\beta} \bigoplus_{i=1}^m (\mathbb{Z}\rho_{i1} \oplus \mathbb{Z}\rho_{i2})$$

has a matrix representation which is a direct sum of $2 \times r$ matrices $\begin{pmatrix} \eta_{i1} \\ \eta_{i2} \end{pmatrix}$ where η_{i1} and η_{i2} are the primitive generators of ρ_{i1} and ρ_{i2} expressed with respect to the basis $\{n_1, \dots, n_r\}$. To determine the matrix for the homomorphism

$$\bigoplus_{j=1}^n \mathbb{Z}\rho_j \xrightarrow{\alpha} \bigoplus_{i=1}^m (\mathbb{Z}\rho_{i1} \oplus \mathbb{Z}\rho_{i2})$$

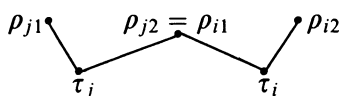
given in (3) we observe the j th column of this matrix is $\alpha(\rho_j)$. Thus the j th column has a 1 in the row determined by ρ_{ik} if $\rho_j = \rho_{ik}$. Otherwise this entry is 0. The matrix of the homomorphism $\alpha + \beta$ of (3) is then

$$Q = \left[\begin{array}{c|ccc} \alpha(\rho_1) \cdots \alpha(\rho_n) & \begin{pmatrix} \eta_{11} \\ \eta_{12} \end{pmatrix} & & \\ & & \ddots & \\ & & & \begin{pmatrix} \eta_{m1} \\ \eta_{m2} \end{pmatrix} \end{array} \right].$$

Note Q is an integral matrix with $2m$ rows and $n + rm$ columns, and we can identify $\text{im}(\alpha + \beta)$ with the column space of Q . Since $B'(T_N/X) \cong \text{coker}(\alpha + \beta)$ from (3), calculating $B'(T_N/X)$ is reduced to determining the column space of Q . Our first observation is a straightforward calculation:

$$(4) \quad Q \cdot \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_n \\ -I_r \\ \vdots \\ -I_r \end{bmatrix} = (0)$$

where I_r is the $r \times r$ identity matrix. Thus the last r columns of Q containing $\begin{pmatrix} \eta_{m1} \\ \eta_{m2} \end{pmatrix}$ are linear combinations of the preceding columns. We now assume Γ is connected, and let T be a spanning tree for Γ . We observe that in Γ each vertex τ_i is joined by edges $\tau_i - \rho_{i1}$, $\tau_i - \rho_{i2}$ to vertices ρ_{i1} , ρ_{i2} so there are $2m$ edges in Γ . Since Γ is connected, there are $n + m - 1$ edges in T [1].



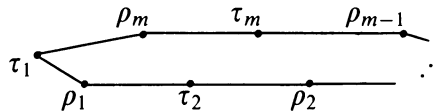
Thus, if $c_1, \dots, c_e$ denote the edges of Γ that are not in T , then $e = m - n + 1$ and for each i at least one of $\tau_i - \rho_{i1}$, $\tau_i - \rho_{i2}$ is in T . By reindexing we can assume $c_1 = \tau_1 - \rho_{12}$, $\dots$, $c_e = \tau_e - \rho_{e2}$. We identify c_j with the basis vector ρ_{j2} in the edge space $\bigoplus_{i=1}^m (\mathbb{Z}\rho_{i1} \oplus \mathbb{Z}\rho_{i2})$. For $1 \leq i \leq m$ let n_{i1} be a primitive vector in N with $\mathbb{R}_{\geq 0}n_{i1} = \rho_{i1}$ and choose n_{i2} in N with $\{n_{i1}, n_{i2}\}$ a basis for $\mathbb{R}\tau_i \cap N$. We can extend $\{n_{i1}, n_{i2}\}$ to a basis $\{n_{i1}, n_{i2}, \dots, n_{ir}\}$ for N . With respect to this basis we can write $\eta_{i1} = n_{i1}$, $\eta_{i2} = a_in_{i1} + b_in_{i2}$ where the η_{ij} are as in Q . With respect to these basis choices for N and corresponding dual basis choices for M , and after deleting columns consisting of zeros, the matrix Q for $\alpha + \beta$ becomes

$$Q = \begin{matrix} \tau_1 - \rho_{11} \\ \tau_1 - \rho_{12} \\ \tau_i - \rho_{i1} \\ \tau_i - \rho_{i2} \\ \tau_m - \rho_{m1} \\ \tau_m - \rho_{m2} \end{matrix} \left(\begin{array}{cccc|cc} \rho_1 & \rho_2 & \cdots & \rho_n & \alpha_1 \beta_1 & \alpha_i \beta_i & \alpha_m \beta_m \\ & & & & \begin{bmatrix} 1 & 0 \\ a_1 & b_1 \end{bmatrix} & & \\ & & & \alpha(\rho_1) & \alpha(\rho_2) \cdots \alpha(\rho_n) & \begin{bmatrix} 1 & 0 \\ a_i & b_i \end{bmatrix} & \\ & & & & & & \begin{bmatrix} 1 & 0 \\ a_m & b_m \end{bmatrix} \end{array} \right)$$

where the first n columns span $\text{im } \alpha$ and the last $2m$ columns span $\text{im } \beta$. We checked in (4) above that the last 2 columns labeled α_m and β_m are linear combinations of the preceding $n + 2(m - 1)$ columns. It follows from (2) that $|b_i| = |\text{Cl}(U_{\tau_i})| = |\text{Cl } U_i|$. The columns $\beta_1, \dots, \beta_e$ are $b_1 c_1, \dots, b_e c_e$.

Theorem 2.6. *Let Δ be a fan on $N_{\mathbb{R}}$ and let $X = T_N \text{emb}(\Delta)$. Assume all the cones in Δ have dimension ≤ 2 . Assume the two-dimensional faces $\tau_1, \dots, \tau_m$ and one-dimensional faces $\rho_1, \dots, \rho_n$ of Δ can be ordered so that $\tau_i \cap \tau_{i+1} = \rho_i$ ($1 \leq i \leq m - 1$) and $\tau_m \cap \tau_1 = \rho_m$. Let b_i be the order of $\text{Cl}(U_{\tau_i})$. Then $B'(T_N/X)$ is cyclic of order $\gcd\{b_1, \dots, b_m\}$.*

Proof. Using Proposition 2.5 and the hypotheses, we can assume that the graph Γ is connected and consists of one cycle as shown:



We take the spanning tree T for Γ to be the graph obtained from Γ by deleting the edge $c_1 = \tau_1 - \rho_m$. Let C be the matrix whose only column is c_1 and form the augmented matrix $[Q|C]$:

$$\begin{matrix} \tau_1 - \rho_1 \\ \tau_1 - \rho_m \\ \tau_2 - \rho_2 \\ \tau_2 - \rho_1 \\ \tau_3 - \rho_3 \\ \tau_3 - \rho_2 \\ \vdots \\ \tau_{m-1} - \rho_{m-1} \\ \tau_{m-1} - \rho_{m-2} \\ \tau_m - \rho_m \\ \tau_m - \rho_{m-1} \end{matrix} \left[\begin{array}{cccccccccccccccc|c} \rho_1 & \rho_2 & \rho_3 & \cdots & \rho_{m-2} & \rho_{m-1} & \rho_m & \alpha_1 & \beta_1 & \alpha_2 & \beta_2 & \cdots & \alpha_{m-1} & \beta_{m-1} & \alpha_m & \beta_m & c_1 \\ \hline 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 0 & & & & & & & & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 & a_1 & b_1 & & & & & & & & 1 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 & & & 1 & 0 & & & & & & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 & & & a_2 & b_2 & & & & & & 0 \\ 0 & 0 & 1 & & \cdots & & & 1 & & & & & & & & & \\ 0 & 1 & 0 & & \cdots & & & 1 & & & & & & & & & \\ \vdots & & & & & \vdots & & & & & & & \vdots & & & & \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 & & & & & 1 & 0 & & & & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 & & & & & a_{m-1} & b_{m-1} & & & & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 & & & & & & & 1 & 0 & & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 & & & & & & & a_m & b_m & & 0 \end{array} \right]$$

Here c_1 corresponds to the edge $\tau_1 - \rho_m$ and $\bigoplus_{i=1}^m (\mathbb{Z}\rho_i \oplus \mathbb{Z}\rho_{i-1})$ is the edge space of Γ where $\rho_0 = \rho_m$. We observed that $B'(T_N/X)$ is the quotient of the edge space by the column space of Q . Let $[B|C]$ be the matrix whose columns are the columns labeled $\rho_1, \dots, \rho_m, \alpha_1, \alpha_2, \dots, \alpha_{m-1}, c_1$. We check that the columns of $[B|C]$ form a basis for the edge space by using column operations to reduce to a permutation matrix. Use column c_1 to eliminate a_1 from column α_1 and the 1 in the entry with row index $\tau_1 - \rho_m$ and column index ρ_m . Then use the 1 in the new column α_1 to eliminate the 1 in the entry with row index $\tau_1 - \rho_1$ and column index ρ_1 . Use the new column ρ_1 to eliminate the a_2 in column α_2 . Continue inductively, eliminating $a_3, \dots, a_{m-2}$ from columns indexed $\alpha_3, \dots, \alpha_{m-2}$ and the ones in the entries with row index $\tau_i - \rho_i$ and column index ρ_i , for $2 \leq i \leq m-2$. At the last step use the remaining 1 in column ρ_{m-2} to eliminate a_{m-1} in column α_{m-1} . Use 1 in the new column α_{m-1} to eliminate the 1 in the entry in row $\tau_{m-1} - \rho_{m-1}$, column ρ_{m-1} . The result is a matrix whose $\mathbb{Z}$ -rank is $2m$, which shows that c_1 generates the quotient of the edge space $\bigoplus_{i=1}^m (\mathbb{Z}\rho_i \oplus \mathbb{Z}\rho_{i-1})$ ($\rho_0 = \rho_m$) by the column space of Q . Recall that the last two columns of Q are a linear combination of the preceding ones. Thus to calculate this quotient we simply project each of the columns $\beta_1, \dots, \beta_m$ on $\mathbb{Z}c_1$. These projections follow the recursive pattern:

<u>Column vector</u>	<u>Projection on $\mathbb{Z}c_1$</u>
$\beta_1 = b_1 c_1$	$b_1 c_1$
$\beta_2 = b_2(\rho_1 - (\alpha_1 - a_1 c_1))$	$b_2 a_1 c_1$
$\beta_3 = b_3(\rho_2 - (\alpha_2 - a_2(\rho_1 - (\alpha_1 - a_1 c_1))))$	$b_3 a_2 a_1 c_1$
$\vdots$	$\vdots$
β_m	$b_m a_{m-1} \cdots a_2 a_1 c_1$

The subgroup generated by the projections of the columns β_i on $\mathbb{Z}c_1$ is the subgroup generated by dc_1 where $d = \gcd\{b_1, b_2 a_1, \dots, b_m a_{m-1} \cdots a_2 a_1\}$. Since $\gcd(a_i, b_i) = 1$ for $1 \leq i \leq m$, we see $d = \gcd\{b_1, \dots, b_m\}$. But $|b_i|$ is the order of $\text{Cl}(U_{\tau_i})$, so the theorem follows. $\square$

To extend Theorem 2.6 it is necessary to introduce some additional notation. Suppose the graph Γ we have associated to the fan Δ is connected and let T be a spanning tree for Γ . Since each vertex labeled by a two-dimensional face τ_i is connected by exactly two edges to vertices ρ_{i1} and ρ_{i2} corresponding to the one-dimensional faces of τ_i in Δ , each τ_i is a vertex in T . If $\Delta(2) = \{\tau_1, \dots, \tau_m\}$, designate τ_m as the root node for T . Let C be the matrix whose columns are $c_1, \dots, c_e$ and let $[Q|C]$ be the augmented matrix similar to that used in the proof of Theorem 2.6. Then $\tau_1, \dots, \tau_e$ are leaf nodes of T and $c_i = \tau_i - \rho_{i2}$ for $1 \leq i \leq e$. For $e < i < m$ relabel ρ_{i1} and ρ_{i2} if necessary so the edge $\tau_i - \rho_{i1}$ is closer to the root node τ_m than the edge $\tau_i - \rho_{i2}$. In our previous analysis this amounts to permuting the basis of the edge space $\bigoplus_{i=1}^m (\mathbb{Z}\rho_{i1} \oplus \mathbb{Z}\rho_{i2})$. This does not affect the columns labeled $\alpha_1, \beta_1, \dots, \alpha_e, \beta_e$ in Q . Let $[B|C]$ be the matrix obtained from $[Q|C]$ by deleting from Q the columns labeled $\beta_1, \dots, \beta_{m-1}, \beta_m, \alpha_m$. We note that the column space of B depends on the choice of τ_m .

Lemma 2.7. *The columns of $[B|C]$ form a basis for $\bigoplus_{i=1}^m (\mathbb{Z}\rho_{i1} \oplus \mathbb{Z}\rho_{i2})$, the edge space, for any choice of root node τ_m .*

Proof. If Γ is a tree, then $T = \Gamma$ and $e = 0$. In this case we need to show that the columns of B span the edge space. Each leaf node of Γ must be ρ_{i2} for some i since each τ_i is incident to two edges in $\Gamma = T$ and ρ_{i1} is closer to the root node τ_m than ρ_{i2} . We call the pair (τ_i, ρ_{i2}) a leaf node pair. If $i = m$, then it is possible for ρ_{i1} to be a leaf node. This is the only exception and will be treated in the basis step for our induction below. Assume (τ_i, ρ_{i2}) is a leaf node pair and $i \neq m$. In Δ , ρ_{i2} is a face of exactly one two-dimensional cone τ_i . Thus the column indexed by ρ_{i2} in B has exactly one nonzero entry which is a 1 in the row indexed $\tau_i - \rho_{i2}$ as indicated below.

$$\tau_i - \rho_{i1} \begin{bmatrix} \rho_{i1} & \rho_{i2} & \alpha_i \\ 1 & 0 & 1 \\ 0 & 1 & a_i \end{bmatrix}$$

Use the column indexed ρ_{i2} to eliminate a_i in the column α_i by an elementary column operation, then use the new column α_i to eliminate the 1 in the $\tau_i - \rho_{i1}$ entry of column ρ_{i1} . After these two steps are performed, we say we have pruned the leaf node pair from the tree $\Gamma = T$. The two columns indexed ρ_{i2} and α_i are now elementary basis vectors in our basis for the edge space and appear in no further column operations. After the columns indexed ρ_{i2} and α_i are deleted, the remaining matrix is the matrix we would associate to the fan Δ' obtained from Δ by deleting the cones ρ_{i2} and τ_i . Apply this leaf pruning algorithm iteratively to reduce to the case where Γ is the tree $\rho_{m1} - \tau_m - \rho_{m2}$. The matrix B for this tree is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Thus our algorithm reduces the original matrix B , using elementary column operations, to a permutation matrix.

If Γ is not a tree, let $\tau_1 - \rho_{12}, \dots, \tau_e - \rho_{e2}$ be the edges of Γ which are not in T . Fix i , $1 \leq i \leq e$. Since $\tau_i - \rho_{i1}$ is in T , it follows that τ_i is a leaf node of T . We can use the column indexed c_i and elementary column operations to eliminate the entry a_i from the column indexed α_i and the entry 1 in the column indexed ρ_{i2} and row indexed $\tau_i - \rho_{i2}$ in the matrix $[B|C]$. Use the new column α_i to eliminate the 1 in the $\tau_i - \rho_{i1}$ entry of column ρ_{i1} . Repeat this step for $i = 1, \dots, e$. Observe that the $2e \times 2e$ submatrix of $[B|C]$ whose columns are indexed α_i ($1 \leq i \leq e$), c_i ($1 \leq i \leq e$) and whose rows are indexed $\tau_i - \rho_{ij}$ ($1 \leq i \leq e$ and $1 \leq j \leq 2$) has rank $2e$. If we delete this submatrix from $[B|C]$ (which corresponds to deleting $\tau_1, \dots, \tau_e$ from the fan Δ), the resulting matrix is the one we would associate to the fan $\Delta' = \Delta - \{\tau_1, \dots, \tau_e\}$. The graph of Δ' is a tree so the result follows from the first part of the proof. $\square$

Corollary 2.8. *Let Δ be a fan on $\mathbb{R}^r$ whose cones all have dimension ≤ 2 . If the graph Γ associated to Δ is a disjoint union of trees and $X = T_N \text{emb}(\Delta)$, then $B'(T_N/X) = 0$.*

Proof. By Proposition 2.5 we can assume Γ is connected, so the hypotheses imply Γ is a tree. By Lemma 2.7 the columns of B span the edge space of Γ . But the column space of B is contained in $\text{im}(\alpha + \beta)$ in (3) $B'(T_N/X) = 0$. $\square$

Corollary 2.9. *Let Δ be a fan on $\mathbb{R}^2$ and $X = T_N \text{emb}(\Delta)$ the associated toric surface.*

- (a) *If $\Delta = \{0\}$, then $B(X) \cong \mathbb{Q}/\mathbb{Z}$.*
- (b) *If $\Delta \neq \{0\}$ and $|\Delta| \neq \mathbb{R}^2$ (i.e., X is not complete), then $B(X) = 0$.*
- (c) *If $|\Delta| = \mathbb{R}^2$ (i.e., X is complete), $\Delta(1) = \{\rho_1, \dots, \rho_n\}$ and $N' = \langle \rho_1 \cap N, \dots, \rho_n \cap N \rangle$, then $B(X) \cong N/N'$.*

Proof. Every toric surface is a normal projective surface [12] so by [9, Corollary 9] $B(X) = B'(X)$. If $\Delta = \{0\}$, then $X = T_N$ is nonsingular and since $r = 2$, $B(X) = \mathbb{Q}/\mathbb{Z}$ [10]. If $\Delta \neq \{0\}$ and $|\Delta| \neq \mathbb{R}^2$, then the graph Γ associated to Δ is a disjoint union of trees. By Proposition 2.5 we assume Γ is a tree and contains at least one two-dimensional cone τ . By Corollary 2.8 and Corollary 2.4, $B(X) = 0$. If $|\Delta| = \mathbb{R}^2$, then Γ is a cycle. Corollary 2.4 implies $B(X) = B(T_N/X)$. If $\Delta(2) = \{\tau_1, \dots, \tau_m\}$, then Theorem 2.6 implies $B(X)$ is cyclic of order $\gcd\{|\text{Cl}(U_{\tau_i})|\}_{i=1}^m$. Let $\rho_1, \dots, \rho_m$ be the one-dimensional cones in Δ with $\tau_i = \rho_i + \rho_{i+1}$ ($1 \leq i \leq m$) where $\rho_{m+1} = \rho_1$ and let $\rho_i = \mathbb{R}_{\geq 0}\eta_i$ for primitive vectors $\eta_1, \dots, \eta_m$ in N . Choose a basis n_1, n_2 for N with $n_1 = \eta_1$ and write $\eta_1 = a_1 n_1 + b_1 n_2$. Then $N/N' = N/\langle \eta_1, \dots, \eta_m \rangle$ is cyclic of order $\gcd\{b_1, \dots, b_m\}$. On the other hand for ($1 \leq i \leq m-1$), $\text{Cl}(U_{\tau_i})$ is cyclic of order $|\det \begin{bmatrix} a_i & a_{i+1} \\ b_i & b_{i+1} \end{bmatrix}|$ and $|\text{Cl}(U_{\tau_m})| = |\det \begin{bmatrix} a_m & 1 \\ b_m & 0 \end{bmatrix}|$ by (2). Since $\gcd(a_i, b_i) = 1$ for each i , an easy calculation shows $\gcd\{|\text{Cl}(U_{\tau_i})|\}_{i=1}^m = \gcd\{b_1, \dots, b_m\}$. $\square$

To determine the p -subgroups of $B'(T_N/X)$ for each prime number p , we introduce some additional notation and terminology. Let Γ be a finite edge-weighted graph such that to each edge E is associated the positive integer $\text{weight}(E)$. Let v_p be the p -adic valuation on $\mathbb{Z}$ and set the p -weight of $E = \text{weight}_p(E) = v_p(\text{weight}(E))$. If Γ_1 is a subgraph of Γ , let $\text{weight}_p(\Gamma_1) = \sum \text{weight}_p(E)$ where the summation is over all edges E in Γ_1 . A p -maximal spanning tree for Γ is a spanning tree T for Γ such that $\text{weight}_p(T)$ is maximal among the p -weights of all spanning trees. It is clear that every connected graph has a p -maximal spanning tree. Let T be a p -maximal spanning tree for Γ and let c denote an edge of Γ which is not an edge of T . Since T is a spanning tree, the subgraph Γ_1 of Γ obtained by adding the edge c to T contains a cycle Z which is unique since there is a unique path between any two vertices of the tree T . Suppose there is some edge E in Z with $\text{weight}_p(E) < \text{weight}_p(c)$. Then we could obtain a spanning tree of larger p -weight by deleting the edge E from Γ_1 . This means that if T is a p -maximal spanning tree for Γ , c is an edge of Γ not in T and Z is the unique cycle in the graph $T \cup \{c\}$, then c is an edge of minimal p -weight in Z .

Let Γ be the (connected) graph we have associated to the fan Δ whose cones all have dimension ≤ 2 . Assign weights to the edges $\tau_i - \rho_{ij}$ of Γ by setting $\text{weight}(\tau_i - \rho_{ij}) = b_i = |\text{Cl}(U_{\tau_i})|$ (recall $\text{Cl}(U_{\tau_i}) = \mathbb{Z}/b_i$ is cyclic from (2)). Let T be a p -maximal spanning tree for Γ . We have labeled the edges of Γ not in T as $\tau_1 - \rho_{12}, \tau_2 - \rho_{22}, \dots, \tau_e - \rho_{e2}$. We call the set of 2-dimensional cones $\{\tau_1, \dots, \tau_e\}$ in Δ a p -minimal set of cones in Δ . If Γ is not connected, then we can decompose Δ as a union of fans Δ_i with $\Delta_i \cap \Delta_j = \{0\}$ when $i \neq j$ and the graphs Γ_i associated to Δ_i are connected. We define a p -minimal set of cones in Δ to be the union of p -minimal sets of cones in each Δ_i .

Theorem 2.10. *Let Δ be a fan on $N_{\mathbb{R}}$ and assume every cone in Δ has dimension ≤ 2 . Let $\{\tau_1, \dots, \tau_e\}$ be a p -minimal set of cones in Δ and let $|\text{Cl}(U_{\tau_i})| = b_i$. If $X = T_N \text{ emb}(\Delta)$, then $B'(T_N/X)_p \cong \bigoplus_{i=1}^e \mathbb{Z}/b_i$. This isomorphism is induced by the epimorphism ψ of (3).*

Proof. By Proposition 2.5 and the discussion preceding the theorem, we can assume the graph Γ associated to the fan Δ is connected. Let T be a p -maximal spanning tree for Γ . Continuing the analysis that was begun in

the proof of Theorem 2.6 we consider the matrix $[Q|C]$ defined there. If $\psi: \bigoplus_{i=1}^m (\mathbb{Z}\rho_{i1} \oplus \mathbb{Z}\rho_{i2}) \rightarrow B'(T_N/X)$ is the epimorphism given in (3) and $\widehat{\mathbb{Z}}_p$ is the p -adic integers, we have an epimorphism

$$\psi_p: \bigoplus_{i=1}^m (\widehat{\mathbb{Z}}_p \rho_{i1} \oplus \widehat{\mathbb{Z}}_p \rho_{i2}) \rightarrow B'(T_N/X)_p.$$

It follows from Lemma 2.7 that $\{c_1, \dots, c_e\}$ generates $\text{coker}(\alpha + \beta)$ so $\{\psi_p(c_1), \dots, \psi_p(c_e)\}$ generates $B'(T_N/X)_p$. We check

$$0 = \langle \psi_p(c_j) \rangle \cap \langle \psi_p(c_1), \dots, \psi_p(c_{j-1}), \psi_p(c_{j+1}), \dots, \psi_p(c_e) \rangle$$

and $\psi_p(c_j)$ has order $p^{v_p(b_j)}$ for $1 \leq j \leq e$ by identifying these elements in $B'(T_N/X)$ with their corresponding preimages $c_j + \text{image}(\alpha + \beta) \in \text{coker}(\alpha + \beta)$ in (3) and then checking the corresponding statements in $\text{coker}(\alpha + \beta)$.

Fix j and let π be a permutation of $\{1, \dots, m\}$ with π chosen so $\pi(1) = j$ where the edge $c_j = \tau_{j-1}\rho_s$ and the cycle in $T \cup \{c_j\}$ is

$$\tau_j = \tau_{\pi(1)} - \rho_1 - \tau_{\pi(2)} - \rho_2 - \dots - \tau_{\pi(s)} - \rho_s - \tau_{\pi(1)}.$$

Choose the vertex $\tau_{\pi(s)}$ as the root node for T . By Lemma 2.7 we know the columns of $[B|C]$ form a basis for $\bigoplus_{i=1}^m (\mathbb{Z}\rho_{i1} \oplus \mathbb{Z}\rho_{i2})$. The column space of B is a submodule of $\text{image}(\alpha + \beta)$. Project the submodule $\widehat{\mathbb{Z}}_p \beta_1 + \dots + \widehat{\mathbb{Z}}_p \beta_m$ of $\text{image}(\alpha + \beta)$ onto a $\widehat{\mathbb{Z}}_p$ -submodule of the column space of C over $\widehat{\mathbb{Z}}_p$. Then $\text{coker}(\alpha + \beta)_p$ is the quotient module.

If $s+1 \leq i \leq m$ we check the projection of $\beta_{\pi(i)}$ on $\widehat{\mathbb{Z}}_p c_j$ is 0. The selection of $\tau_{\pi(s)}$ as the root node for T gives a partial order on the vertices of T . Let T_i be the subtree of T with root node $\rho_{\pi(i)2}$. This means the vertices v in T_i are those for which the unique path from v to $\tau_{\pi(s)}$ contains $\rho_{\pi(i)2}$. In the expression for $\beta_{\pi(i)}$ as a linear combination of the columns of $[B|C]$, the columns of C that appear are those c_k which when considered as edges of Γ are incident to some vertex in T_i (see the proof of Theorem 2.6). But neither τ_j nor ρ_s are in T_i since $i \geq s+1$ so the projection of $\beta_{\pi(i)}$ on c_j has coefficient = 0.

If $1 \leq i \leq s$ let the projection of $\beta_{\pi(i)}$ on $\bigoplus_{k=1}^e \widehat{\mathbb{Z}}_p c_k$ be $\sum_{k=1}^e d_{ki} c_k$. We say in the proof of Theorem 2.6 that $b_{\pi(i)} | d_{ji}$ (where $b_{\pi(i)} = |\text{Cl}(U_{\pi(i)})|$). The projections of $\beta_{\pi(1)}, \dots, \beta_{\pi(m)}$ on $\bigoplus_{k=1}^e \widehat{\mathbb{Z}}_p c_k$ are the columns of the $e \times m$ matrix (d_{ki}) . We have observed above that $d_{j(s+1)} = \dots = d_{jm} = 0$. The definitions of β_j and c_j imply $\beta_j = b_j c_j$ so $d_{k1} = 0$ if $k \neq j$. Also $b_{\pi(i)} | d_{ji}$ for $2 \leq i \leq s$ and we chose j with $v_p(b_j) = \min\{v_p(b_{\pi(1)}), \dots, v_p(b_{\pi(s)})\}$. Thus after elementary column operations over $\widehat{\mathbb{Z}}_p$ the column space of (d_{ki}) is equal to the column space of

$$\begin{bmatrix} 0 & d_{12} & \cdots & d_{1m} \\ \vdots & \vdots & & \vdots \\ 0 & d_{(j-1)2} & \cdots & 0 \\ b_j & 0 & \cdots & 0 \\ 0 & d_{(j+1)2} & \cdots & \vdots \\ \vdots & \vdots & & \vdots \\ 0 & d_{e2} & \cdots & d_{em} \end{bmatrix}$$

Therefore $c_1, \dots, c_e$ represents a basis for $\text{coker}(\alpha + \beta)_p$, $B'(T_N/X)_p = \langle \psi_p(c_1) \rangle \oplus \dots \oplus \langle \psi_p(c_e) \rangle$, and $\langle \psi_p(c_j) \rangle$ is cyclic of order $p^{v_p(b_j)}$. $\square$

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REFERENCES

1. N. Biggs, *Algebraic graph theory*, Cambridge Univ. Press, London, 1974.
2. E. Brieskorn, *Rational singularitäten komplexer Flächen*, Invent. Math. **4** (1968), 336–358.
3. T. Ford, *On the Brauer group of $k[x_1, \dots, x_n, 1/f]$* , J. Algebra **122** (1989), 410–424.
4. —, *On the Brauer group of a localization*, J. Algebra **147** (1992), 365–378.
5. R. Fossum, *The divisor class group of a Krull domain*, Springer-Verlag, New York, 1973.
6. J. Gamst and K. Hoechsmann, *Quaternions generalises*, C. R. Acad. Sci. Paris **269A** (1969), 560–562.
7. A. Grothendieck, *Le groupe de Brauer. II*, Dix Exposés sur la Cohomologie des Schémas, North-Holland, Amsterdam, 1968.
8. R. Hoobler, *Functors of graded rings*, Methods in Ring Theory (F. van Oystaeyen, ed.), NATO ASI Series, Reidel, Dordrecht, 1984, pp. 161–170.
9. —, *When is $\text{Br}(X) = \text{Br}'(X)$?*, Brauer Groups in Ring Theory and Algebraic Geometry, Lecture Notes in Math., Vol. 917, Springer-Verlag, New York/Berlin, 1982, pp. 231–245.
10. A. Magid, *Brauer groups of linear algebraic groups with characters*, Proc. Amer. Math. Soc. **71** (1978), 164–168.
11. J. Milne, *Etale cohomology*, Princeton Univ. Press, Princeton, N. J., 1980.
12. T. Oda, *Convex bodies and algebraic geometry*, Springer-Verlag, Berlin and Heidelberg, 1988.
13. M. Raynaud, *Anneaux locaux Henséliens*, Lecture Notes in Math., Vol. 169, Springer-Verlag, Berlin, 1970.
14. O. F. G. Schilling, *The theory of valuations*, Math. Surveys, No. 4, Amer. Math. Soc., Providence, R. I., 1950.
15. R. Strano, *On the étale cohomology of Hensel rings*, Comm. Algebra **12** (1984), 2195–2211.

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