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RESIDUAL FINITENESS OF COLOR LIE SUPERALGEBRAS

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ABSTRACT. A (color) Lie superalgebra L over a field K of characteristic $\neq 2, 3$ is called residually finite if any of its nonzero elements remains nonzero in a finite-dimensional homomorphic image of L. In what follows we are looking for necessary and sufficient conditions under which all finitely generated Lie superalgebras satisfying a fixed system of identical relations are residually finite. In the case $\operatorname{char} K = 0$ we show that a variety V satisfies this property if and only if V does not contain all center-by-metabelian algebras and every finitely generated algebra of V has nilpotent commutator subalgebra.

1. INTRODUCTION

Many authors have considered a natural generalization of the class of Lie algebras called color Lie superalgebras [1, 2, 3]. We recall that, given a field F of characteristic different from 2, an abelian group G, and an alternating bilinear form $\varepsilon : G \times G \to F^*$, i.e.,

$$\varepsilon(g+h, k) = \varepsilon(g, k)\varepsilon(h, k), \qquad \varepsilon(g, h) = \varepsilon(h, g)^{-1},$$

we call a G-graded algebra $L = \sum_{g\in G} L_g$ an (ε-)color Lie superalgebra if for any $x \in L_g$, $y \in L_h$, and $z \in L$ we have

$$(1) \qquad [x, y] = -\varepsilon(g, h)[y, x],$$

$$(2) \qquad [[x, y], z] = [x, [y, z]] - \varepsilon(g, h)[y, [x, z]].$$

If ε is trivial, i.e., $\varepsilon(g, h) = 1$ for all $g, h \in G$, then L becomes an ordinary G-graded Lie algebra. If $G = \mathbf{Z}_2$ and $\varepsilon(1, 1) = -1$, then we arrive at ordinary Lie superalgebras.

The aim of the present paper is to discuss the question about residual finiteness of color Lie superalgebras. Here we say that a color Lie superalgebra $L = \sum_{g\in G} L_g$ is residually finite if for any $x \neq 0$, $x \in L$, there exists a homomorphism $\phi : L \to M = \sum_{g\in G} M_g$ onto a finite-dimensional color superalgebra such that $\phi(x) \neq 0$. It is assumed here that ϕ is homogeneous, that is, we have $\phi(L_g) \subset M_g$ for all $g \in G$. An example of a color Lie superalgebra which is not residually finite is given by the Heisenberg superalgebra $\Gamma = \Gamma(g, h)$, where $g, h \in G$. The basis of this superalgebra is formed by the set $\{a_i, b_i, z | i \in \mathbf{Z}\}$ and $\Gamma_\alpha = 0$ for $\alpha \neq g, h, g+h$, $a_i \in \Gamma_g$, $b_i \in \Gamma_h$ $(i \in \mathbf{Z})$, $z \in \Gamma_{g+h}$. The commutator is given by the formula

$$[a_i, b_i] = -\varepsilon(g, h)[b_i, a_i] = z, \qquad i \in \mathbf{Z},$$

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with all other commutators being zero. It is easy to see that Γ is not residually finite since the image of z under any homomorphism onto a finite-dimensional superalgebra is trivial.

To obtain an example of a finitely generated color Lie superalgebra it is necessary to adjoin to $\Gamma(g, h)$ a binding derivation $d\colon \Gamma(g, h) \to \Gamma(g, h)$ and then to place it into the zero component of the newly born algebra. As a result, we obtain a superalgebra $B = B(g, h)$ all of whose components, except zero, are the same as in $\Gamma(g, h)$ and $B_0 = \Gamma_0 \oplus \langle d\rangle$, with commutator given by:

$$\begin{aligned}[d, a_i] &= \varepsilon(g, h)a_{i+1} - \varepsilon(h, g)a_{i-1}, \\ [d, b_j] &= \varepsilon(h, g)b_{j+1} - \varepsilon(g, h)b_{j-1}.\end{aligned} \tag{3}$$

We ask the reader to verify that each of the algebras $B(g, h)$ is indeed an (ε-) color Lie superalgebra, that it is finitely generated, and that it is not residually finite (since it contains $\Gamma(g, h)$).

Any $B(g, h)$ is center-by-metabelian in the sense that it satisfies an identical relation of the form

$$[x, [y, z], [u, v]] = 0, \tag{4}$$

where x, y, z, u, and v are arbitrary variables and we are using right-normed notation. Our first result is

Theorem 1. *Any finitely generated metabelian color Lie superalgebra with finite grading group is residually finite.*

The following example shows that we cannot extend Theorem 1 to infinite grading groups. For this we consider a metabelian Lie algebra $L = P \oplus M$, where P is abelian with basis $\{x, y\}$ and M is abelian with basis $\{z_i | i \in \mathbf{Z}\}$. We set $[x, z_i] = z_{i-1}$ and $[y, z_i] = z_{i+1}$, $i \in \mathbf{Z}$. Now L is $\mathbf{Z}$-graded if $L_i = \langle z_i\rangle$ for all $i \neq +1, -1$, $L_{-1} = \langle x, z_{-1}\rangle$, and $L_1 = \langle y, z_1\rangle$. If we set $\varepsilon(m, n) = 1$ for all $m, n \in \mathbf{Z}$, then L becomes a color Lie superalgebra, and M becomes the minimal nonzero graded ideal of L, i.e., its monolith. Thus, L is a 3-generator monolithic metabelian algebra of infinite dimension, hence not residually finite.

In fact Theorem 1 is a particular case of a theorem which is formulated by using the following notation. We denote by G_+ the set of elements g in the grading group G such that $\varepsilon(g, g) = 1$. Similarly, $G_- = \{g \in G | \varepsilon(g, g) = -1\}$. Then $G = G_+ \cup G_-$ and G_+ is a subgroup of G. We also write

$$L_+ = \sum_{g\in G_+} L_g, \qquad L_- = \sum_{g\in G_-} L_g.$$

If we use this notation then the following holds.

Theorem 2. *Let F be a field,* $\operatorname{char} F \neq 2$, *$G$ be a finite group, and L be a finitely generated color Lie superalgebra with an abelian ideal A such that L/A is finite-dimensional and, in addition, if* $\operatorname{char} F = 0$, *we assume that $L_+^2 \subset A$. Then L is residually finite.*

The largest portion of this paper is devoted to varieties of color Lie superalgebras. As usual, by a variety we mean a class of superalgebras (with G and ε

fixed) satisfying a fixed system of identities. Now, given a G-graded alphabet $X = \bigcup_{g\in G} X_g$, an equation of the form

$$f(x_1, \dots, x_n) = 0 \qquad (x_i \in X_{g_i},\ g_i \in G,\ i = 1, \dots, n),$$

where $f(x_1, \dots, x_n)$ is a (nonassociative, commutator) polynomial with coefficients in F, is called an identity in a color Lie superalgebra L if we have $f(a_1, \dots, a_n) = 0$ for any choice of $a_1, \dots, a_n \in A$ such that $a_i \in A_{g_i}$, $i = 1, \dots, n$.

We discuss locally residually finite varieties, i.e., in which all finitely generated superalgebras are residually finite. It is obvious that no such variety $\mathscr{V}$ contains an algebra of the form $B(g, h)$. We will show (Theorem 3) that any locally residually finite variety $\mathscr{V}$ over an infinite field satisfies a system of identities of the form

$$(5)\quad [x, y^{(n)}, z] = \sum_{j=1}^{n} \alpha_j [y^{(j)}, x, y^{(n-j)}, z], \qquad x \in X_g,\ y \in X_0,\ z \in X_h.$$

Here $[y^{(k)}, x]$ stands for the right-normed commutator $[y, \dots, y, x]$ with k entries of y. A surprising fact is that if such a system, for all g, h, is satisfied in a variety $\mathscr{V}$ of a locally soluble Lie superalgebras over an infinite field F of positive characteristic, then $\mathscr{V}$ is residually finite (Theorem 4). In the case when F is of characteristic zero, for a variety $\mathscr{V}$ of Lie superalgebras to be residually finite it is necessary and sufficient to require in addition to the identities (5) that the commutator L_0^2 of the even component L_0 of any finitely generated Lie superalgebra $L = L_0 \oplus L_1$ in $\mathscr{V}$ acts on the whole of L as a nilpotent space of transformations (Theorem 5).

We remark that all of the results in this paper are a natural generalization of certain results about ordinary Lie algebras in [4, 5, 6].

2. Sufficient conditions for residual finiteness

In this section we prove Theorem 2 stated in the Introduction. We recall that the enveloping algebra $U(L)$ is a G-graded associative algebra A with imbedding $\imath : L \to A$ such that

$$(6)\qquad \imath([x, y]) = \imath(x)\imath(y) - \varepsilon(g, h)\imath(y)\imath(x),$$

where $x \in L_g$ and $y \in L_h$, and such that if $\tau : L \to B$ is a similar homomorphism into a G-graded associative algebra B, then there exists a unique homogeneous homomorphism $f : A \to B$ such that $f(\imath(x)) = \tau(x)$ for all $x \in L$.

It is known (in the case of ordinary Lie superalgebras of characteristic zero see, for example, [7, p. 26]) that if F is a field, then $\imath$ is a monomorphism which enables us to identify L with a subalgebra in A (under the operation

$$(7)\qquad [x, y] = xy - \varepsilon(g, h)yx,$$

$x \in A_g$, $y \in A_h$). Moreover, if $E = \bigcup_{g\in G} E_g$ is a totally ordered basis of L such that E_g is a basic of L_g, then $U(L)$ is formed by 1 together with all "ordered" monomials of the form

$$(8)\qquad e_1 e_2 \cdots e_n \qquad (e_i \in E,\ e_1 \le e_2 \le \cdots \le e_n),$$

where we cannot have $e_i = e_{i+1}$ for $e_i \in E_g$ with $g \in G_-$.

In particular, if $L = L_+ \oplus L_-$ is a finite-dimensional color Lie superalgebra, then $U(L)$ is a free left and right module of finite type 2^s over its subalgebra $U(L_+)$, where $s = \dim L_-$. As a vector space, $U(L)$ takes the form

$$U(L) = S(L_+) \otimes \Lambda(L_-), \tag{9}$$

where $S(V)$ is the symmetric algebra of the vector space V and $\Lambda(V)$ is the Grassmann algebra for V. It is clear that, in general, the equality in (9) is not a homomorphism of algebras. However, following the proofs of well-known results, we can prove

Proposition 1. (1) *$U(L_+)$ has no zero-divisors.*

(2) *If L is finite dimensional, then $U(L)$ is noetherian.*

(3) *If L is finite dimensional and abelian and G is finite, then any irreducible L-module is finite dimensional.*

(4) *If L is finite dimensional, G is finite, and* $\operatorname{char} F = p > 0$, *then any irreducible L-module is finite dimensional.*

Proof. We have a filtration in $U(L)$ of the form

$$\{0\} = U_{-1} \subset U_0 \subset U_1 \subset \cdots \subset U_n \subset \cdots,$$

where U_n is the linear span of all products of elements in L with at most n factors. The associated graded algebra is the enveloping algebra for the abelian algebra $\overline{L}$ with basis $E = \bigcup_{g \in G} E_g$. Therefore, $U(\overline{L})$ is generated by E with respect to defining relations of the form

$$ee' = \varepsilon(g, h)e'e, \quad \text{where } e \in E_g, \ e' \in E_h. \tag{10}$$

It is obvious then that, in $U(L)$, the degree of elements is defined with the usual properties. If we restrict to the subalgebra generated by $E_+ = \bigcup_{g \in G_+} E_g$, then the degree of the product is equal to the sum of the degrees of the factors (more exactly, that the lexicographically leading term of the product is the product of the lexicographically leading terms of the factors). Now (1) follows since E_+ generates $U(\overline{L}_+) = \operatorname{gr} U(L_+)$.

It is sufficient to verify the noetherian property for $U(L_+)$ only, since $U(L)$ is of finite type over $U(L_+)$. Again the key point is passing to $U(\overline{L})$. It is easy to see that, under our hypotheses, we can apply for our proof the usual procedure as found in the proof of Hilbert's Basis Theorem with induction over the number of variables $\dim \overline{L}_+$.

Let V be an irreducible (graded) L-module, L being a Lie superalgebra with finite grading group G and with form $\varepsilon\colon G \times G \to F^*$. Then V is a left module over $U(L) = U$. Every nonzero component V_h is obviously an irreducible U_0-module. Since G is finite it is sufficient to establish that any irreducible (nongraded) U_0-module is finite dimensional. If $n = |G|$ then, clearly, any $\varepsilon(g, h)$ is an nth root of unity. In this case, given a basis $E = \{e_1, \ldots, e_m\}$ of L, the elements $e_1^{2n}, \ldots, e_m^{2n}$ generate a central polynomial subalgebra Z such that U_0 is a free (left, right) Z-module of finite type. For, if $e_i \in L_g$ and $e_j \in L_h$, we have

$$\begin{aligned} e_i^{2n} e_j &= \varepsilon(g, h) e_i^{2n-1} e_j e_i = \varepsilon^2(g, h) e_i^{2n-2} e_j e_i^2 \\ &= \cdots = \varepsilon^{2n}(g, h) e_j e_i^{2n} = \varepsilon(2ng, h) e_j e_i^{2n} = e_j e_i^{2n}. \end{aligned}$$

By Curtis' theorem [8] all irreducible representations of U_0 are finite dimensional, proving (3).

Finally, let L be a finite-dimensional Lie superalgebra over a field F of characteristic p, $p>0$, $x \in L_g$, $g \in G_+$, and f a p-polynomial annihilating $\operatorname{ad} x$. Then $f(\operatorname{ad} x) = \operatorname{ad} f(x)$. For this, it is useful to notice that, given $u \in U(L)$, $u = u_{g_1} + \cdots + u_{g_k}$, $v \in U(L)_h$, we have

$$(\operatorname{ad} u)(v) = (u_{g_1}v - \varepsilon(g_1, h)vu_{g_1}) + \cdots + (u_{g_k}v - \varepsilon(g_k, h)vu_{g_k}).$$

Now if $x \in L_g$, $g \in G_+$, and $y \in L_h$ then

$$(\operatorname{ad} x)^p(y) = (L_x - \varepsilon(g, h)R_x)^p(y) = (L_{x^p} - \varepsilon(pg, h)R_{x^p})(y) = (\operatorname{ad} x^p)(y).$$

Here, under the consecutive actions of $\operatorname{ad} x$ the value of the form ε does not change since on the $(k+1)$th step $\operatorname{ad} x$ transforms $(\operatorname{ad} x)^k(y) \in L_{kg+h}$ and

$$\varepsilon(g, kg+h) = \varepsilon(g, g)^k \varepsilon(g, h) = \varepsilon(g, h).$$

It is easy to observe that if $u = f(x) = u_{g_1} + \cdots + u_{g_k}$ is an element with the property $\operatorname{ad} f(x) = 0$ and $g_i \neq g_j$ for $i \neq j$, then $\operatorname{ad} u_{g_1}, \ldots, \operatorname{ad} u_{g_k}$ have the same property. Now if we take $u_{g_1}^n, \ldots, u_{g_k}^n$ with $n = |G|$, then these elements belong to the usual center of $U(L)$. The subalgebra Z generated by all $u_{g_1}^n, \ldots, u_{g_k}^n$ with x running through the basis $\{e_1, \ldots, e_r\}$ of L_+ is in the center of $U(L)$, which is a module of finite type over Z. Now the same argument as before proves the finiteness of the dimension of irreducible L-modules. The proof of the proposition is complete.

Now we are able to pass to the proof of Theorem 2 stated in §1. This is quite similar to the proof in [4].

Proof of Theorem 2. Let $g_1, \ldots, g_t$ be a generating system of L such that $g_1 + A, \ldots, g_t + A$ contains a basis of $H = L/A$. By a linear change of variables we can assume that some $\{g_1 + A, \ldots, g_s + A\}$ is a basis in L/A and $g_{s+1}, \ldots, g_t \in A$. Let $\{c_{ij}^k\}$ be the set of structure constants with respect to the given basis. For any i, j $(1 \leq i, j \leq s)$ we set

$$r_{ij} = [g_i, g_j] - \sum_k c_{ij}^k g_k.$$

It is obvious that $r_{ij} \in A$. If $J = \operatorname{id}_L\{r_{ij}, g_{s+1}, \ldots, g_t\}$, then $A \supset J$. On the other hand, $\dim L/J \leq s = \dim L/A$. Then $A = J$, i.e., A is the ideal generated by $\{r_{ij}, g_{s+1}, \ldots, g_t | 1 \leq i, j \leq k\}$.

Now we set $H = L/A$. Then A is a finitely generated H-module with respect to the adjoint action. According to the above proposition, $U(H)$ is a noetherian algebra, hence, A is a noetherian module. For the proof of the residual finiteness of L it is sufficient to verify the finiteness of the dimension of its monolithic homomorphic images, i.e., those which have the least nonzero ideal, the monolith. Without any loss of generality we can assume that already L itself is monolithic, and M is its monolith. According to claims (3) and (4) in the above proposition, its monolith M is a finite-dimensional space. Hence the proof of the theorem will be complete if we manage to prove the following.

Lemma 1. *Let L be a finitely generated color Lie superalgebra with an abelian ideal A satisfying the hypotheses of Theorem 2, and let M be a* *finite-dimensional (homogeneous) ideal in L having nonzero intersection with any other ideal K in L, $K \neq 0$. Then L is finite dimensional.*

Proof. Let $H = L/A$. If we consider the proof of Proposition 1 then we see that $U(H)$ has a central noetherian G-homogeneous subalgebra Z such that $U(H)$ is a Z-module of finite type. Let z be an arbitrary G-homogeneous element in $Z \cap \operatorname{Ann} M$. Then the chain of subspaces of the form

$$zA \supset z^2 A \supset \cdots \supset z^t A \supset \cdots \tag{11}$$

is, in fact, a chain of $U(H)$-submodules. Let us assume that none of the subspaces in the chain (11) is zero; in this case for any t we also have $z^t A \cap M \neq 0$. We consider also the chain

$$M \cap A \subset \operatorname{Ann}_A z \subset \operatorname{Ann}_A z^2 \subset \cdots \subset \operatorname{Ann}_A z^u \subset \cdots . \tag{12}$$

Since A is a noetherian $U(H)$-module, we have, for a suitable u, the following:

$$\operatorname{Ann}_A z^u = \operatorname{Ann}_A z^{u+1} = \cdots . \tag{13}$$

We consider $m \in M \cap z^u A$, $m \neq 0$. Then, for some $a \in A$, we have $m = z^u a$. Furthermore, $0 = zm = z(z^u a) = z^{u+1} a$, i.e., $a \in \operatorname{Ann}_A z^{u+1}$. It follows from (13) that then $z^u a = 0$, hence also $m = 0$, a contradiction. Hence in the chain (11) not all the spaces are nonzero, that is, the action of z on A is nilpotent. Now if $b_1, \ldots, b_k$ are the Z-generators of $U(H)$, and $z_1, \ldots, z_l$ the generators of Z, then there exist polynomials f_i so that $u_1 = f_1(z_1), \ldots, u_l = f_l(z_l)$ are elements in $Z \cap \operatorname{Ann} M$. Hence any element in $U(H)$ can be represented as a linear combination of monomials of the form

$$b_i z_1^{k_l} \cdots z_l^{k_l} u_1^{t_1} \cdots u_l^{t_l}, \quad 0 \leq k_i < d_i, \; d_i = \deg f_i, \; i = 1, \ldots, l, \; t_i \geq 0. \tag{14}$$

If n_i is the nilpotent index for the action of u_i on A then only a finite number of elements of the form (14), with $t_i < n_i$ for all $i = 1, \ldots, l$, can act on A in a nonzero way. Since A is finitely generated, we deduce that it is finite dimensional. It follows by $\dim L/A < \infty$ that also $\dim L < \infty$. Now the proof of Lemma 1 and, with it, the proof of Theorem 2 is complete.

3. Identical relations in locally residually finite varieties

Theorem 3. *Let $\mathscr{V}$ be a locally residually finite variety of color Lie superalgebras over an infinite field (with respect to a form $\varepsilon \colon G \times G \to F^*$). Then for any $g, h \in G$ an identity of the following form holds in $\mathscr{V}$:*

$$[x, y^{(n)}, z] = \sum_{j=1}^{n} \alpha_j [y^{(j)}, x, y^{(n-j)}, z], \tag{15}$$

where $\alpha_j \in F$, $j = 1, \ldots, n$, $x \in X_g$, $y \in X_0$, and $z \in X_h$.

Proof. Since F is infinite, it is sufficient to restrict to multihomogeneous identities only. It is known [9] that a variety of ordinary Lie algebras which does not contain $B(0, 0)$ consists of algebras satisfying (15). It follows that $\mathscr{V}$ satisfies (15) with $x, y, z \in X_0$. To prove (15) in the case $g = 0$ we consider $B = B(0, h)$, where $h \neq 0$. We have

$$B_0 = \langle d, a_i \; (i = \cdots -1, 0, 1, \ldots) \rangle, \qquad B_h = \langle z, b_i \; (i = \cdots -1, 0, 1, \ldots) \rangle .$$

Since B is not locally residually finite, $\mathscr{V} \not\ni B$. Any multihomogeneous identity with variable $x \in X \backslash (X_0 \cup X_h)$ is satisfied in B. Hence $\mathscr{V}$ must satisfy a multihomogeneous identity in the variables of $X_0 \cup X_h$ which is not satisfied by B. Since B_h is an abelian ideal, B satisfies any multihomogeneous identity depending on at least two variables in X_h. Assume first that there is no identity with one variable in X_h which is satisfied in $\mathscr{V}$ but does not hold in B. Now since B_0 is a metabelian algebra, any multihomogeneous identity in the variables of X_0 satisfied by $\mathscr{V}$ but not B can be brought to the form

$$\sum_{k=1}^{n} \beta_k [x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n, x_k, x_0] \equiv 0$$

modulo the metabelian identity, where like terms are reduced. (Note that this identity as well as some others to follow are not assumed multilinear.) If, say, $\beta_1 \neq 0$, then setting $x_1 = y + z$, $x_i = y$ if $x_i \neq x_1$, and taking the sum of monomials of degree 1 in z we get $[y^{(n)}, z] \equiv 0$. It is obvious that the same substitution in each of the consequences of $[[x, y], [z, u]] \equiv 0$ gives zero. Now, clearly, $[y^{(n)}, z] \equiv 0$ implies (15) with all $\alpha_j = 0 : [x, y^{(n)}, z] \equiv 0$ with $x \in X_h$. If we want to have (15) with $z \in X_h$ then we know already that $[z, y^{(n)}, x] \equiv 0$, hence $[[y^{(n)}, x], z] \equiv 0$. Since we have

$$[(\operatorname{ad} y)^n, \operatorname{ad} x] = (L_{\operatorname{ad} y} - R_{\operatorname{ad} y})^n (\operatorname{ad} x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (\operatorname{ad} y)^{n-k} (\operatorname{ad} x)(\operatorname{ad} y)^k,$$

it follows that

$$[[y^{(n)}, x], z] = \sum_{k=0}^{n} (-1)^k \binom{n}{k} [y^{(n-k)}, x, y^{(k)}, z],$$

which gives (15) with $x, y \in X_0$ and $z \in X_h$.

Now suppose $\mathscr{V}$ satisfies an identity with one variable in X_h which is not satisfied in $B(0, h)$. Since $B(0, h)$ satisfies

$$(16) \qquad [x_1, [x_2, x_3], z] = 0, \qquad x_i \in X_0, \; z \in X_h,$$

our additional identity is not a consequence of (16). We reduce that identity modulo (16). Then it takes the form

$$\sum_{k=1}^{n+1} \gamma_k [x_k, x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n, z] \equiv 0.$$

Let $\gamma_1 \neq 0$. Then, setting $x_1 = x + y$, $x_i = y$ if $x_i \neq x_1$, and taking the sum of monomials of degree 1 in x yields

$$\gamma_1 [x, y^{(n)}, z] + \left(\sum_{k=2}^{n+1} \gamma_k \right) [y, x, y^{(n-1)}, z] \equiv 0.$$

On the other hand, if we apply the same substitution in the consequences of (16), we get terms on the right-hand side of (15), proving that, indeed, $\mathscr{V}$ satisfies the identities of the form as claimed.

To get (15) of arbitrary form we consider $B = B(g, h)$ with $g, h \neq 0$. Since B is not residually finite we have $B \notin \mathscr{V}$. As before, $\mathscr{V}$ must satisfy an identity in the variables of $X_0 \cup X_g \cup X_h \cup X_{g+h}$. If $g+h \neq 0$ then $B_{g+h} = \langle z \rangle$ and the only identity including a variable $u \in X_{g+h}$ which does not hold in B is $u \equiv 0$. However, it follows from this identity that $[x, z] = 0$ with $x \in X_g$ and $z \in X_h$, and this implies $[x, y, z] = 0$ with $x \in X_g$, $y \in X_0$, and $z \in X_h$, i.e., the identity as required. Hence, we may assume that the identities of $\mathscr{V}$ which do not hold in B include only the variables in $X_0 \cup X_g \cup X_h$. It is very clear that an "additional" identity cannot contain more than two variables in $X_g \cup X_h$ since $B_g + B_h + \langle z \rangle$ is a 2-step nilpotent ideal in B. If this identity depends only on variables in X_0 then it must be $y \equiv 0$, with $y \in X_0$, which, obviously implies (15). If the additional identity contains only one variable which is not in X_0, say, $z \in X_h$, then, since we have $[y_1, y_2] = 0$ in B for $y_1, y_2 \in X_0$, this identity is equivalent to $[y_1, \dots, y_n, z] = 0$. If we identify all $y_1, \dots, y_n$ with some y we have $[y^{(n)}, z] = 0$, hence $[x, y^{(n)}, z] \equiv 0$, hence (15). Now let the additional identity contain two variables $x \in X_g$ and $z \in X_h$ with all the rest in X_0. Since $B = B(g, h)$ satisfies

$$(17) \qquad \begin{aligned} &[y, x, z] = 0, \qquad y \in X_0,\ x \in X_g,\ z \in X_h, \\ &[y_1, y_2] = 0, \qquad y_1, y_2 \in X_0, \end{aligned}$$

our additional identity reduces, modulo (17), to the form

$$[x, y_1, \dots, y_n, z] \equiv 0.$$

If we identify $y_1, \dots, y_n$ then we get $[x, y^{(n)}, z] \equiv 0$. As for the consequences of the first identity in (17), either these are zero or else they have the form of a linear combination of monomials of the form $[y, [y^{(m)}, x], [y^{(k)}, z]]$, $m+k = n-1$; if $g+h=0$, the consequences of the second identity in (17) have the same form, and if $g+h \neq 0$ then all the consequences of the second identity are zero. It is clear the rewriting these monomials in the right-normed form gives only the monomials in the right-hand side of (15), proving the theorem.

4. Two lemmas on representations of soluble Lie algebras

This section contains some auxiliary results. The motivation for the lemmas that follow lies in the fact that, given an arbitrary color Lie superalgebra $L = \sum_{g \in G} L_g$, any of its components L_g becomes an L_0-module, where L_0 is an ordinary Lie algebra.

Lemma 2. *Let L be a finitely generated soluble Lie algebra over an infinite field F, and let V be an L-module. If a representation of L by linear operators of a vector space V satisfy identities of the form*

$$(18) \qquad [y^{(n)}, x] = \sum_{j=1}^{n} \lambda_j y^j [y^{(n-j)}, x],$$

$$(19) \qquad [z, y^{(n)}, x] = \sum_{j=1}^{n} \alpha_j [y^{(j)}, z, y^{(n-j)}, x],$$

then, for some m, L^m acts on V as a nilpotent space of transformations, where $L^m = [L, L, \dots, L]$ with m factors.

Proof. We apply to (18) the method of divided variables due to Mishchenko [10]. If $\lambda_j = 0$ for all $j = 1, \dots, n$ in (18), then $g(x, y) = [y^{(n)}, x]$ is equal to zero on V identically. Since F is an infinite field, the linear span of values of g is an ideal. This ideal contains L^m for some m because any finitely generated soluble Lie algebra with Engel condition is nilpotent (see, for example, [11, Theorem 4.7.2]).

Let s be the greatest of the indices in (18) such that $\lambda_s \neq 0$ and $z, y \in L$. We consider the operators $A, B, C, D : L \to \operatorname{End} V$ such that for all $x \in L$ we have

$$A(x) = yx, \quad B(x) = [y, x], \quad C(x) = zx, \quad D(x) = [z, x].$$

Then (18) can be written in the form $fB^{n-s} = 0$ on L, where

$$f = f(A, B) = B^s - \sum_{j=1}^{s} \lambda_j A^j B^{s-j}.$$

Now let f_i be the sum of monomials of total degree i in C, D in the polynomial $f(A+C, B+D)$. Then $f_0 = f(A, B)$ and $f_s = f(C, D)$. We will show that for each $i = 0, \dots, s$ there exists a number k_i such that

$$f_i B^{k_i} = 0 \tag{20}$$

on L.

If $i = 0$ then such a relation follows from (18) if we set $k_0 = n - 1$. Now we assume that the numbers $k_0, k_1, \dots, k_{i-1}$ have been found. For any nonnegative t we have by (19) an equation of the form

$$f(A+C, B+D)[(y+z)^{(n-s)}, y^{(t)}, x] = 0. \tag{21}$$

Since F is infinite, the sum of monomials of degree i in z in (21) is also trivial. It takes the form

$$f_i[y^{(n-s+t)}, x] + \sum_j f_j g_j(\operatorname{ad} y, \operatorname{ad} z)(x). \tag{22}$$

Here the summation over j goes from 0 to $i-1$ if $i \leq n-s$ and from $i-n+s$ to $i-1$ if $i > n-s$, and g_j is a homogeneous polynomial of degree $i-j$ in $\operatorname{ad} z$ and of degree $d = n-s+t-i+j$ in $\operatorname{ad} y$. It follows from (19) that $g_j(\operatorname{ad} y, \operatorname{ad} z)$ has the form $(\operatorname{ad} y)^{k_j} h_j(\operatorname{ad} y, \operatorname{ad} z)$ if $d \geq k_j + n(i-j)$, i.e., $t \geq r_{ij} = k_j + n(i-j-1) + i - j + s$. Now it is sufficient to define k_i as the greatest of the numbers r_{ij}. Since the homogeneous component of degree i in z is equal to zero in (21), we have (20). If $i = s$, then (20) takes the form

$$[z^{(s)}, T] = \sum_{j=1}^{s} \lambda_j z^j [z^{(s-j)}, T], \tag{23}$$

where $T = [y^{(k)}, x]$ for suitable k and where the last summand in the right-hand side takes the form $\lambda_s z^s T$ with $\lambda_s \neq 0$. Since F is an infinite field the linear span of values of T is an ideal, while it follows from the local nilpotence of soluble Engel Lie algebras that this ideal contains L^m for some m. Having substituted $z = T = b \in L^m$ in (23), we get the equation of the form $b^{s+1} = 0$.

Now we want to prove that L^m acts on V as a nilpotent space of transfomations. It was proved in [6] that any finitely generated soluble Lie algebra with an identity of the form (19) is in the product of two nilpotent varieties (Lemma

6). Hence, without loss of generality, we may assume that L^m is a nilpotent Lie algebra.

Since L is finitely generated, L/L^m is finite dimensional. Now let $\{e_1, \dots, e_q\}$ be a basis of L modulo L^m such that $[e_i, e_j]$ can be written as a linear combination (modulo L^m) of elements e_k with $k < i, j$. Let $a_1, \dots, a_r$ be generators of the ideal L^m including the elements of the form $[e_i, e_j] - \sum_{k=1}^q c_{ij}^k e_k$, where the c_{ij}^k are the structure constants of L/L^m. Using the nilpotence of L^m it is easy to verify that, as a Lie algebra, L^m can be generated by all commutators of the form

$$[e_1^{(t_1)}, \dots, e_q^{(t_q)}, a_i], \qquad t_1, \dots, t_q \geq 0, \ i = 1, \dots, r. \tag{24}$$

We denote by H the subalgebra in L^m generated by elements of the form (24) with $t_1, \dots, t_q \leq n-1$. Then H is finite dimensional and acts on V by nilpotent operators. Let $w_1, \dots, w_q$ be a basis of H. If $v \in V$ then the H-submodule W generated by v is a linear span of elements $w_1^{r_1} \cdots w_q^{r_q} v$. For any $h \in L^m$ and $w \in W$, the element $h^{s+1}w$ is equal to zero. Hence, $\dim W \leq q(s+1) = N$. It is known that H can be represented as a set of upper triangular $(N \times N)$-matrices (see, for example, [11, Chapter 1]). Therefore, $h_1 \cdots h_N v = 0$ for any $h_1, \dots, h_N \in H$.

For the proof it is sufficient to verify that $b_1 \cdots b_N V = 0$ for all $b_1, \dots, b_N$ of the form (24). We introduce a partial ordering on the set of such elements by setting

$$[e_1^{(t_1)}, \dots, e_q^{(t_q)}, a_i] < [e_1^{(u_1)}, \dots, e_q^{(u_q)}, a_j]$$

if $(t_1, \dots, t_q)$ is lexicographically less than $(u_1, \dots, u_q)$ (we compare the components from the right to the left). Similarly we can define the ordering of the sets of the form $\{b_1, \dots b_N\}$ by comparing their terms from the right to the left considering the ordering introduced just above.

If all b_i in the set $\{b_1, \dots, b_N\}$ are generators of H, then $b_1 \cdots b_N v = 0$ for any $v \in V$. Otherwise, let b_i be the first from right to left generator of L^m of the form

$$b_i = [e_1^{(t_1)}, \dots, e_q^{(t_q)}, a_j],$$

where at least one exponent, say t_q, is not less than n.

If $i_1 > i_2$ and $w \in L$ then by the Jacobi identity the commutator $[e_{i_1}, e_{i_2}, w]$ is a linear span of elements $[e_{i_3}, w]$, where $i_3 < i_2$, and $[e_{i_2}, e_{i_1}, w]$. It follows that if we replace one of the entries of e_p in b_i by e_l with $l < p$ then we get a linear span of some elements b_i' of the type (24) and $b_i' < b_i$.

If we replace one of the entries of e_p in b_i by a_k then we get a linear span of commutators $[d_1, d_2]$, where d_1, d_2 are of the form (24) and $d_1, d_2 < b_i$. We set

$$c = [e_q^{(t_q)}, e_1^{(t_1)}, \dots, e_{q-1}^{(t_{q-1})}, a_j].$$

The commutator $[e_q, e_p]$ is equal to $\mu_1 e_1 + \cdots + \mu_{p-1} e_{p-1} + a_k$ for some k, where the $\mu_1, \dots, \mu_{p-1}$ are in F. It follows that $b_i - c$ is a linear span of some commutators d and $[d_1, d_2]$, where d, d_1, and d_2 are of the type (24) and less than b_i.

Now the element $b_1 \cdots b_{i-1}(b_i - c)b_{i+1} \cdots b_N v$ takes the form of a linear combination of products of the form

$$b_1 \cdots b_{i-1} d_{j_1} d_{j_2} b_{i+1} \cdots b_N v; \qquad b_1 \cdots b_{i-1} d_j b_{i+1} \cdots b_N v$$

in each of which the "tail" on N rightmost operators, acting on v, is strictly less than $\{b_1, \dots, b_N\}$, hence we may apply our induction. So, our element $b_1 \cdots b_N v$ is equal to an element of the form $b_1 \cdots b_{i-1} c b_{i+1} \cdots b_N v$. We set $w = b_{i+1} \cdots b_N v$. According to (18), the element cw is equal to a linear combination of the form $e_q^k c_k w$, where

$$c_k = [e_q^{(t_q-k)}, e_1^{(t_1)}, \dots, e_{q-1}^{(t_{q-1})}, a_j], \qquad k > 0.$$

The vector $b_1 \cdots b_{i-1} e_q^k c_k w$ is in the L-module generated by some elements of the form $d_1 \cdots d_{i-1} c_k w$ with $d_1, \dots, d_{i-1} \in L^m$. As all elements (24) generate L^m it follows that $b_1 \cdots b_{i-1} e_q^k c_k w$ is in the L-module generated by elements $g_1 \cdots g_m c_k w$, where $g_1, \dots, g_m$ are of the form (24) and $m \geq i-1$.

Element c_k is not of the type (24). If we move e_q in c_k from left to right using the Jacobi identity and the relation $[e_q, e_p] = \mu_1 e_1 + \cdots + \mu_{p-1} e_{p-1} + a_k$, then we express c_k as a linear span of some f_k of the form (24) with $f_k < b_i$. Hence, $b_1 \cdots b_{i-1} e_q^k c_k w$ is in the L-module generated by

$$g_1 \cdots g_m f_k w = g_1 \cdots g_m f_k b_{i+1} \cdots b_N v.$$

Since $f_k < b_i$ and $m \geq i-1$, we may apply induction over the partial ordering on the sets of the form $\{b_1, \dots, b_N\}$. By the inductive hypothesis,

$$g_1 \cdots g_m f_k b_{i+1} \cdots b_N v = 0$$

and $b_1 \cdots b_N v = 0$.

If $t_k > n-1$ for b_i with $k \leq q-1$ then the proof is quite similar to the case $k = q$. The proof of Lemma 2 is complete.

Lemma 3. *Let L be a finitely generated soluble Lie algebra and V a finitely generated L-module with identities* (18) *and* (19). *Then V is a noetherian L-module.*

Proof. By Lemma 2 for some m the ideal L^m acts on V by nilpotent transformations. Now V possesses a finite series of submodules of the form

$$V = V_0 \supset V_1 = L^m V_0 \supset \cdots \supset V_{k+1} = L^m V_k \supset \cdots \supset L^m V_N = \{0\}.$$

To prove that V is a noetherian L-module it is sufficient to verify the noetherian property of each factor V_k/V_{k+1}, $k = 1, \dots, N$, where, in fact, each V_k/V_{k+1} is an L/L^m-module. Since L/L^m is finite dimensional, $U(L/L^m)$ is noetherian and it is sufficient to verify that each module V_k/V_{k+1} is finitely generated.

Let $\{e_1, \dots, e_q\}$ be a basis of L modulo L^m and $\{a_1, \dots, a_r\}$ a generating set for L^m as an ideal of L. The number of commutators of the form

$$(25) \qquad [e_1^{(t_1)}, \dots, e_q^{(t_q)}, a_i], \qquad 0 \leq t_1, \dots, t_q \leq n-1,$$

is finite. Using induction over k we want to show that $V_k = V_{k+1} + P_k$, where P_k is the L-submodule generated by all $b_1 \cdots b_k v$ with each b_i of the form (25) and v is one of the generators of V as an L -module, the number of such generators being finite. If $k = 0$, then it is obvious.

Now let $k \geq 1$ and suppose that $V_{k-1} = V_k + P_{k-1}$ is already true. Then $V_k = V_{k+1} + H$, where H is the linear span of the set of the elements of the form

$$d e_1^{t_1} \cdots e_q^{t_q} b_2 \cdots b_k v,$$

where $b_2, \dots, b_k$ is of the form (25) and $d \in L^m$. It follows from this that V_k, as an L-module, is generated by V_{k+1} together with the set of elements of the form $db_2 \cdots b_k v$, where $d = [e_1^{(t_1)}, \dots, e_q^{(t_q)}, a_i]$. If $t_1, \dots, t_q \le n-1$ then d is an element of the form (25) as required. Otherwise, let us assume that one of the t_j, say t_q, is greater than $n-1$. We set $B = b_2 \cdots b_k v$ and $c = [e_q^{(t_q)}, e_1^{(t_1)}, \dots, e_{q-1}^{(t_{q-1})}, a_i]$. It follows from (18) that $cB = \sum_j \lambda_j e_q^j c_j B$, where each c_j, as well as $d - c$, can be expressed modulo $(L^m)^2$ as a linear combination of commutators $[e_1^{(s_1)}, \dots, e_q^{(s_q)}, a_i]$ with $s_1 + \dots + s_q < t_1 + \dots + t_q$. By lowering the total degree consecutively we get the inclusion $dB \in V_{k+1} + P_k$. The proof is complete.

5. Sufficient conditions for finite generation of color Lie superalgebras and their modules

We recall that for any color Lie superalgebra L we have a decomposition of the form $L = L_+ \oplus L_-$.

Lemma 4. *Let $L = \sum_{g \in G} L_g$ be a finitely generated soluble color Lie superalgebra over a field F, $\operatorname{char} F \neq 2$, such that for all $g, h \in G$ and $r \in G_+$ we have an identity of the following form in L:*

$$[x, y^{(n)}, z] = \sum_{j=1}^{n} \alpha_j [y^{(j)}, x, y^{(n-j)}, z], \tag{26}$$

where $\alpha_1, \dots, \alpha_n \in F$, $x \in L_g$, $z \in L_h$, and $y \in L_r$. Then L_+ is a finitely generated color Lie superalgebra and L_- is a finitely generated. L_+-module.

Proof. Let L be generated by homogeneous elements $x_1, \dots, x_m, y_1, \dots, y_m$ with $x_i \in L_-$ and $y_i \in L_+$, $i = 1, \dots, m$. We denote by H the subalgebra in L generated by y_i, x_i^2, $i = 1, \dots, m$. Then $H \subset L_+$. We denote by M the H-submodule in L generated by all commutators of the form $[a_1, \dots, a_r]$, where all the $a_1, \dots, a_r$ are commutators of the form

$$[y_1^{(t_1)}, \dots, y_m^{(t_m)}, x_1^{(q_1)}, \dots, x_m^{(q_m)}, u] \tag{27}$$

such that $u \in \{x_1, \dots, x_m, y_1, \dots, y_m\}$ and the following restrictions take place: $0 \le t_1, \dots, t_m \le n$, $0 \le q_1, \dots, q_m \le 2n$.

Our next goal is to prove the equation $L = M$. For the elements a_λ and a'_λ of the form (27) we set $a_\lambda < a'_\lambda$ if and only if $(t_1, \dots, t_m, q_1, \dots q_m)$ is lexicographically less than $(t'_1, \dots, t'_m, q'_1, \dots, q'_m)$ if we compare the components from the right to the left. We extend this partial ordering to all ordered sets of the form $\{a_1, \dots, a_r\}$ by comparing the components from the right to the left.

We want to prove that any commutator $[a_1, \dots, a_r]$ of the elements of the form (27) is in M. This will prove $L = M$, because all elements (27) with the restriction $t_1 + \dots + t_m + q_1 + \cdots q_m \ge 1$ generate L^2 as a superalgebra. We proceed by induction over the partial ordering.

If the exponents t_i and q_j of each of the a_i do not exceed n and $2n$ respectively, then $[a_1, \dots, a_r] \in M$ by the construction. Suppose that the inductive hypothesis is satisfied and a_i is a commutator of the form (27) with $t_m > n$. Then the difference $a_i - [y_m^{(t_m)}, c_i]$, where $c_i = [y_1^{(t_1)}, \dots, y_{m-1}^{(t_{m-1})}, x_1^{(q_1)}, \dots,$

$x_m^{(q_m)}, u]$, can be written in the form of a linear combination of commutators $[d_1, d_2]$, where each factor d_j has the form (27) and it is strictly less than a_i. Using identities of the set (26) one can write the commutator $[[y_m^{(t_m)}, c_i], d]$, d being arbitrary homogeneous, as a linear combination of commutators of the form $[y_m^{(j)}, [y_m^{(t_m-j)}, c_i], d]$, $j \geq 1$. Any $[y_m^{(t_m-j)}, c_i]$ is equal to

$$[y_1^{(t_1)}, \dots, y_m^{(t_m-j)}, x_1^{(q_1)}, \dots, x_m^{(q_m)}, u]$$

plus a linear combination of commutators $[d_1, d_2]$, where d_1, d_2 are of the form (27) and $d_1, d_2 < a_i$.

Since $y_m \in H$, using the Jacobi identity for color Lie superalgebras proves that $[a_1, \dots, a_r]$ lies in the H-module generated by commutators of the form $[b_1, \dots, b_s, a_{i+1}, \dots, a_r]$, with $s \geq i$, where all factors have the form (27) with $b_s < a_i$. Now $\{b_1, \dots, b_s, a_{i+1}, \dots, a_r\}$ is strictly less than $\{a_1, \dots, a_r\}$, hence by the inductive hypothesis, $[a_1, \dots, a_r]$ is in M.

Arguing similarly in the case where one of the exponents $t_1, \dots, t_m$ in the expression for a_i is greater than n or one of the $q_1, \dots, q_m$ is greater than $2n$ and moving x_j^2 to the left, we get $[a_1, \dots, a_r] \in M$ in the general case.

To complete the proof of Lemma 4 it is sufficient to construct a finitely generated subalgebra B of L_+ such that L_+ and L_- are finitely generated B-modules. We use induction over the solubility length of L. If $L^2 = 0$ then $B = L_+ = \langle y_1, \dots, y_m \rangle$. Now suppose that $L^2 \neq 0$.

Let R denote the subalgebra in L generated by the elements of the form (27) with $0 \leq t_1, \dots, t_m \leq n$, $0 \leq q_1, \dots, q_m \leq 2n$, and $t_1 + \dots + t_m + q_1 + \dots + q_m \geq 1$. Since the solubility length of R is strictly less than that of L, it is possible to assume that R_+ contains a finitely generated subalgebra C such that R_+ and R_-, as C-modules, are generated by the set of elements of the form $\{Y_1, \dots, Y_N\}$, $\{X_1, \dots, X_N\}$ respectively.

It follows from $L = M$, proven above, that if

$$B = \operatorname{alg}\{C, y_1, \dots, y_m, x_1^2, \dots, x_m^2\}$$

then, as a B-module, L_+ is generated by Y_i and y_i, and L_- by all X_i and x_i, proving Lemma 4.

In the case of ordinary superalgebras we derive the following result from Lemmas 2, 3, and 4.

Proposition 2. *Let $\mathscr{V}$ be a locally soluble variety of Lie superalgebras over an infinite field F, $\operatorname{char} F \neq 2$. Then the following conditions are equivalent.*

(a) *Any finitely generated Lie superalgebra in $\mathscr{V}$ satisfies the maximality condition for ideals.*

(b) *Any finitely generated Lie superalgebra $L = L_0 \oplus L_1$ in $\mathscr{V}$ is noetherian as an L_0-module.*

(c) *The following identities hold in $\mathscr{V}$:*

$$[x, y^{(n)}, z] = \sum_{j=1}^{n} \alpha_j^i [y^{(j)}, x, y^{(n-j)}, z],$$

where $i = 1, 2, 3$. Here if $i = 1$ then all three variables x, y, z are in L_0, if $i = 2$ then $x, y \in L_0$ and $z \in L_1$, and if $i = 3$ then $y \in L_0$ and $x, z \in L_1$.

Proof. If the third condition holds then $\mathscr{V}$ has an identity

$$[z, y^{(n)}, x] = \sum_{j=1}^{n} \beta_j [y^{(j)}, z, y^{(n-j)}, x]$$

for $z \in L_1$, $x, y \in L_0$, and $\beta_1, \dots, \beta_n \in F$. Indeed this is another expression of the identity (c) for $i = 2$, because $[x, y^{(k)}, z]$ with an arbitrary k is equal to $(-1)^k [z, y^{(k)}, x]$ modulo the linear span of the commutators $[y^{(i)}, z, y^{(n-i)}, x]$, $i \geq 1$. It follows from Lemma 4 that both L_0 and L_1 are finitely generated L_0-modules. Applying Lemma 3 we get (b) and this, obviously, implies (a). Now we want to prove that the first condition of the proposition implies the third one. We consider the ideal I_k in the $\mathscr{V}$-free algebra L generated by $[x, z], [x, y, z], \dots, [x, y^{(k)}, z]$, where x, y, and z are free generators of L. Then $I_1 \subset I_2 \subset \cdots$ and, by the maximality condition, $I_n = I_{n-1}$ for some n. Hence, $[x, y^{(n)}, z] \in I_{n-1}$. It follows that

$$[x, y^{(n)}, z] = \sum_{j=1}^{n} g_j [x, y^{(n-j)}, z],$$

where g_j is a polynomial depending on $\operatorname{ad} x$, $\operatorname{ad} y$, and $\operatorname{ad} z$. Since F is an infinite field and x, y, z are free generators, it follows that $g_j = \beta_j (\operatorname{ad} y)^j$. Varying the parities of x, y, z, without any difficulty we get all three identities as claimed. The proof is complete.

Lemma 5. *Let $L = L_0 \oplus L_1$ be a finitely generated soluble Lie superalgebra with identities of the form*

$$(28) \qquad [x, y^{(n)}, z] = \sum_{j=1}^{n} \alpha_j [y^{(j)}, x, y^{(n-j)}, z], \qquad x, y, z \in L_0,$$

$$(29) \qquad [x, y^{(n)}, z] = \sum_{j=1}^{n} \beta_j [y^{(j)}, x, y^{(n-j)}, z], \qquad x, y \in L_0, \ z \in L_1,$$

$$(30) \qquad [x, y^{(n)}, z] = \sum_{j=1}^{n} \gamma_j [y^{(j)}, x, y^{(n-j)}, z], \qquad y \in L_0, \ x, z \in L_1.$$

Suppose also that $M = M_0 \oplus M_1$ is a finitely generated L-module, and the following identities hold for the representation of L on M:

$$(31) \qquad [y^{(n)}, z]v = \sum_{j=1}^{n} \lambda_{ij} y^j [y^{(n-j)}, z]v,$$

with $y \in L_0$, $z \in L_1$, $\lambda_{ij} \in F$, and $i = 0$ if $v \in M_0$, $i = 1$ if $v \in M_1$. Then M is a finitely generated module over the superalgebra $H = L_0 \oplus [L_0^c, L_1]$ for any $c \geq 2$.

Proof. By Lemma 4 we can say that L_0 is finitely generated and L_1, as an L_0-module, can be generated by a finite set of elements $x_1, \dots, x_q \in L_1$. Let $\{e_1, \dots, e_m\}$ be a basis of L_0 modulo L_0^c chosen in such a way that, modulo L_0^c, $[e_i, e_j]$ is equal to a linear combination of elements of the form e_k, $k <$

i, j. Then, modulo $[L_0^c, L_1]$, L_1 becomes the linear span of commutators of the form

$$[e_1^{(t_1)}, \dots, e_m^{(t_m)}, x_i]. \tag{32}$$

We order the commutators of the form (32) by comparing vectors $(t_1, \dots, t_m, i)$ lexicographically from the right to the left. Now if $\{u_j\}$ is a finite set of homogeneous generators of M as an L-module then, by Poincarè-Birkhoff-Witt's Theorem, M as an H-module can be generated by the elements of the form

$$b_1 \cdots b_r u_j, \qquad b_1 > \cdots > b_r, \ r \geq 0, \tag{33}$$

where $b_1, \dots, b_r$ are commutators of the form (32). Let T denote the H-submodule of M generated by all elements of the form (32) such that each b_j satisfies the conditions $t_1, \dots, t_m < n$. To prove our lemma it is sufficient to verify that each element of the form (33) is in T. For $r = 0$ the above inclusion is obvious. Now suppose that it holds for all numbers less than r. Then, by the Jacobi identity, it is sufficient to verify that T contains all elements of the form $b_1 b_2 \cdots b_r u_j$, where the restrictions $t_1, \dots, t_m < n$ hold for $b_2, \dots, b_r$ and $b_2 > \cdots > b_r$.

Now let $b_1 = [e_1^{(q_1)}, \dots, e_m^{(q_m)}, x_i]$. We consider first the case where $q_1, \dots, q_m < n$. If $b_1 > b_2$ then $b_1 \cdots b_r u_j$ is in T by construction. If $b_i > b_1 \geq b_{i+1}$, then applying induction over r shows that $b_1 \cdots b_r u_j$ is congruent, modulo T, to $B = b_2 \cdots b_i b_1 b_{i+1} \cdots b_r u_j$. In the case where $b_1 \neq b_{i+1}$ we find that B is in T, otherwise, if $b_1 = b_{i+1}$, we have $B \in T$ by the induction hypothesis, since $b_1^2 \in L_0 \subset H$.

Now let the exponents $q_1, \dots, q_m$, in the expression of b_1, be arbitrary. We set $b_2 \cdots b_r u_j = v$. If $q_1 \geq n$ then, using identity (31), we can write $b_1 v$ as a linear combination of elements of the form $e_1^j c_j v$, where c_j is a commutator of the form (32) with $t_1 \leq n-1$. If $q_j \geq n$ and $d = [e_j^{(q_j)}, e_1^{(q_1)}, \dots, e_m^{(q_m)}, x_i]$ then the difference $b_1 - d$ can be expressed, modulo H, as a linear combination of elements of the form (32) which are strictly less than b_1. This enables us to lower the degree of b_1 with respect to any of the variables e_j using (31). So, all reduces to the situation considered above, and the proof is complete.

6. Finite dimension of certain modules over Lie superalgebras

All results that follow from now on refer to Lie superalgebras over a field F of characteristic different from 2.

Lemma 6. *Let L, M, and H be as in Lemma 5. If Q is an infinite-dimensional H-submodule in M and $\dim M/Q < \infty$, then Q contains a nonzero L-submodule.*

Proof. Let $\{e_1, \dots, e_m\}$ be a basis of L_0 modulo L_0^c and $\{x_i\}$ a finite set of elements, generating L_1 as an L_0-module. Let $\{z_1, \dots, z_N\}$ be the set of all commutators of the form

$$[e_1^{(q_1)}, \dots, e_m^{(q_m)}, x_i], \qquad 0 \leq q_1, \dots, q_m \leq n-1. \tag{34}$$

We denote by $\{A_1, \dots, A_T\}$ the set of all linear operators on M of the form $z_{i_1} \cdots z_{i_k}$, $N \geq i_1 > \cdots > i_k \geq 1$. By our hypothesis we can choose in Q an infinite-dimensional subspace W_1 such that $A_1 W_1 \subset Q$. Now we can choose in

W_1 an infinite-dimensional subspace W_2 such that $A_2W_2 \subset Q$. By repeating this procedure we will arrive at a nonzero vector v in Q such that $A_iv \in Q$ for all $i = 1, \dots, T$.

Now let $U(L)$ be the enveloping algebra for L and let P be a subspace in $U(L)$ spanned by the elements of the form fA_i, $i = 0, \dots, T$, with $A_0 = 1$ and $f \in U(H)$. Then $Pv \subset Q$ by the choice of v. The proof of our lemma will be complete if we verify that Pv is an L-module. Since $H = L_0 \oplus [L_0^c, L_1]$, it is sufficient to verify the inclusion $aPv \subset Pv$, where a is a commutator of the form (32). As in the proof of Lemma 5 one can show that identity (31) implies the following relation:

$$aw \equiv \sum_j h_j z_j w \qquad (\mathrm{mod}\ U(H)w), \tag{35}$$

where a and w are arbitrary in L and M respectively, each z_j is a commutator of the form (34), and $h_j \in U(H)$. It follows from (35) that, to finish the proof, it remains to show that Pv contains all elements of the form z_jA_iv. Let $A_i = z_{j_1} \cdots z_{j_k}$. If $k = 1$ then $[z_j, A_i] \in L_0$, hence it is obvious that $z_jA_iv \in Pv$. Now suppose $k > 1$. If $j > j_1$ then the product z_jA_i is one of the operators $A_1, \dots, A_T$, hence $z_jA_iv \in Pv$. Otherwise, if $j_{s-1} > j \geq j_s$ for some s then moving z_j to the right in the expression of z_jA_iv we can write this latter, modulo Pv, as the sum of some of the products of the form

$$z_{j_1} \cdots z_{j_r} g_r z_{j_{r+2}} \cdots z_{j_k} v \tag{36}$$

in which $g_r \in L_0$, $r \leq s-1$. Moving g_r to the left we get an expression for any of (36) as the sum of some elements of the form $b_t = f_t z_{i_t} \cdots z_{i_k}$ with $i_t > i_{t+1} > \cdots > i_k$, $2 \leq t \leq k$, $f_t \in L$. Hence, considering (35), we express z_jA_iv modulo Pv as the sum of elements of the form hz_lA_lv with $h \in U(H)$, $A_l = z_{i_1} \cdots z_{i_s}$, and $s \leq k-1$. Applying induction over k we derive that all these elements are in Pv, proving the lemma.

Lemma 7. *Let L and M be as in Lemma 5 and the action of L_0^c on M and L_1 is nilpotent. Also let $c = 2$ if the characteristic of the ground field is zero. If M is an irreducible L-module then* $\dim M < \infty$.

Proof. We denote L_1 by S_0 and $[L_0^c, S_{i-1}]$ by S_i, if $i > 0$. Then $S_d = 0$ for some d. We proceed by induction over d. If $d = 1$ then, using the notation in Lemmas 5 and 6, $H = L_0$. If M is an irreducible H-module then $H^cM = 0$, hence M is an irreducible module over the finite-dimensional Lie algebra H/H^c, and $c = 2$ if $\operatorname{char} F = 0$. By Proposition 1, $\dim M < \infty$.

If M has a nonzero proper H-submodule then it has a maximal such submodule. We denote this latter by Q. Then $\dim M/Q < \infty$ and using Lemma 6 guarantees the finiteness of the dimension of M.

Now suppose $d > 1$. Then $H = L_0 \oplus [L_0^c, L_1]$ is a finitely generated Lie superalgebra by Proposition 2. By Lemma 5, M is a finitely generated H-module. Hence, by the induction hypothesis, all irreducible submodules and quotient-modules of M as a H-module are finite dimensional.

Let Q be a maximal H-submodule of M, $Q \neq M$. Then $\dim M/Q < \infty$ and Lemma 6 complete the proof of Lemma 7.

Lemma 8. *Let $L = L_0 \oplus L_1$ and $M = M_0 \oplus M_1$ be as in Lemma 7 and the action of L_0 on M_0 as well on M_1 satisfies the identities of the form* (18). *Also let*

M have a finite-dimensional L-submodule K which has a nonzero intersection with any nonzero L-submodule. Then $\dim M < \infty$.

Proof. First we consider the case where $L_1 = 0$. If $L_0^c M = 0$ then M is a finitely generated module over a finite-dimensional Lie algebra $L/\operatorname{Ann}_L(M)$. It follows by Lemma 1 that the semidirect product $(L/\operatorname{Ann}_L(M)) \oplus M$ (where $L/\operatorname{Ann}_L(M)$ is a subalgebra and M is an ideal) is finite dimensional, hence we will assume in what follows that $L_0^c M = T \neq 0$. Now Lemma 3 applies to both (L_0, M_0) and (L_0, M_1), hence T is a finitely generated L-module. In this case L, T, and $T \cap K$ satisfy the hypotheses of the lemma, while the nilpotent index of the action of L_0^c on T is by one less than in the case of (L, M). Therefore, the induction argument enables us to conclude that T is finite dimensional.

Now let D_j denote the L-submodule in M consisting of those v for which one has $b_1 \cdots b_j v = 0$ for any $b_1, \dots, b_j \in L^c$. We want to verify that all D_j have finite codimension in M. By our hypothesis, $D_q = M$ for some q. Suppose the finiteness of the dimension of M/D_{j+1} has been proven. Let $A_1, \dots, A_N$ be all the operators of End M of the form $z_1 \cdots z_j$, where each factor is of the form

$$[e_1^{(t_1)}, \dots, e_m^{(t_m)}, y_i] \tag{37}$$

with $0 \leq t_1, \dots, t_m \leq n-1$, $\{e_1, \dots, e_m\}$ being a basis of L modulo L^c, and $\{y_i\}$ a finite set of elements generating L^c as an ideal of L.

Set $B_j = \ker A_j$. Since $A_j M \subset T$ and $\dim T < \infty$, the codimension of B_j in M is finite. Then also for $Q = B_1 \cap \cdots \cap B_N \cap D_{j+1}$ we have $\dim M/Q < \infty$. Now we want to show that $D_j \supset Q$. If $w \in Q$ then it is sufficient to verify $a_1 \cdots a_j w = 0$ with $a_1, \dots, a_j$ of the form (37) without any restrictions on the exponents $t_1, \dots, t_m$.

For instance, let $a_r = [e_1^{(q_1)}, \dots, e_m^{(q_m)}, y_i]$ and $q_s \geq n$. Since $w \in D_{j+1}$ the factor a_r in $a_1 \cdots a_j w$ can be replaced by b_r and by a linear combination of a_r' of the same form but also with $q_1' + \cdots + q_m' < q_1 + \cdots + q_m$. Here b_r is the commutator of the form $[e_s^{(q_s)}, e_1^{(q_1)}, \dots, e_{s-1}^{(q_{s-1})}, e_{s+1}^{(q_{s+1})}, \dots, e_m^{(q_m)}, y_i]$. Using (18) we can replace this latter by a linear combination of the products of the form $e_s^k c_r$ in which any c_r is a commutator of the form (37) with $t_1 + \cdots + t_m < q_1 + \cdots + q_m$. This means that $a_1 \cdots a_j w$ belongs to the L-submodule generated by $b_1 \cdots b_j w$ with $b_1, \dots, b_j$ of the form (37) with bounded exponents $e_1, \dots, e_m$. Hence $a_1 \cdots a_j w = 0$ and the proof of $Q \subset D_j$ is complete.

Now from the finiteness of the dimension of M/D_j it follows that, for $j = 1$, M has a submodule D_1 of finite codimension such that $L^c D_1 = 0$. If $D_1 \neq 0$ then L, D_1, and $D_1 \cap K$ satisfy the hypothesis of the lemma, hence, as shown above, $\dim D_1 < \infty$. Thus, in the case $L = L_0$, M is a finite-dimensional L-module.

Now suppose $L_1 \neq 0$. We make use of the nilpotence of the action of L_0^c on L_1 and proceed by induction over the nilpotent index. We set $H = L_0 \oplus [L_0^c, L_1]$. By Lemma 5, M is finitely generated as an H-module and it follows by Proposition 2 that H is a finitely generated Lie superalgebra. By Zorn's Lemma there is a maximal H-submodule Q in M whose intersection with K is trivial. Then H, M/Q, and $K + Q/Q$ satisfy the hypotheses of the lemma. The induction hypothesis (the induction starts with the above case $L_1 = 0$) enables us to assume that M/Q is finite dimensional. If Q is infinite dimensional then, by Lemma 6, there exist a nonzero L-submodule W in Q. Then $W \cap K = 0$ by the choice of Q which contradicts the hypotheses of the lemma. Consequently, $\dim Q < \infty$ and the proof of Lemma 8 is complete.

7. Locally residually finite varieties of Lie superalgebras

Theorem 4. *Le $\mathscr{V}$ be a locally soluble variety of Lie superalgebras over an infinite field F of characteristic different from 2. Then the property of being locally residually finite for $\mathscr{V}$ is equivalent to the following two conditions:*

(a) *$\mathscr{V}$ satisfies the identities of the form* (28), (29), (30).

(b) *If* $\operatorname{char} F = 0$, *then any finitely generated algebra $L = L_0 \oplus L_1$ in $\mathscr{V}$ satisfies an identity of the form*

$$[[x_1, y_1], \dots, [x_m, y_m], z] = 0, \qquad x_i, y_i \in L_0, \ z \in L. \tag{38}$$

Proof. We prove first that the conditions above are necessary. We have obtained (a) in Theorem 3 for arbitrary locally residually finite varieties.

Now suppose $\operatorname{char} F = 0$ and let $L = L_0 \oplus L_1$ be a free algebra in $\mathscr{V}$ with free generators of the form $x_1, \dots, x_t, y_1, \dots, y_t, z_1, z_2$, where $z_2 \in L_1$ and z_1 and all the x_i, y_i are in L_0. Following the argument in Lemma 5 in [5] virtually verbatim we may assume that $f^N = 0$ for some N, where $f = \operatorname{ad}([y_1, x_1] + \dots + [y_t, x_t])$. If b is an arbitrary element in L_0^2, depending only on the variables in $y_1, \dots, y_t$, then b can be written in the form $b = [y_1, b_1] + \dots + [y_t, b_t]$, where $b_1, \dots, b_t \in L_0$. Then the mapping $x_i \mapsto b_i$ takes $f^N z_j$, which is zero, into the commutator $[b^{(N)}, z_j]$, $j = 1, 2$. Hence, if we denote by H the intersection of L_0^2 with the subalgebra generated by $y_1, \dots, y_t$ we will have a *weak* identity $x^N = 0$ satisfied in the representation of H by linear operators of L_0 and L_1. (Let A be an associative *enveloping* algebra of the Lie algebra B. This means that $B \subset A$ and elements of B generate A as an associative algebra. If the associative polynomial $f(x_1, \dots, x_n)$ is equal to zero in A for any $x_1, \dots, x_n \in B$ then this polynomial is called a *weak* identity of the pair (A, B).)

We shall prove that if H is a Lie algebra of linear transformations on the vector space W over a field F of characteristic zero and $x^N = 0$ for any $x \in H$, then $x_1 \cdots x_k = 0$ as a linear map on W for some k, where $x_1, \dots, x_k$ are arbitrary elements of H.

By Theorem 4.1 in [12] H is a nilpotent Lie algebra, since $(\operatorname{ad} x)^{2N+1} = 0$. Let us prove our statement for an abelian algebra H.

Since $[H, H] = 0$, the unique monomial which depends on all $b_1, \dots, b_N$ on the left-hand side of equation $(b_1 + \dots + b_N)^N = 0$ is equal to $N! b_1 \cdots b_N$. Since $\operatorname{char} F = 0$, it follows that $b_1 \cdots b_N = 0$, and the proof of the statement is complete for an abelian algebra.

Now suppose $H^t = 0$, $t \geq 2$. We proceed by induction over t. Denote by Z the center of H. Then Z acts on W as a nilpotent space of transformations. We may construct a finite chain of H-submodules

$$W = W_0 \supset W_1 \supset \cdots \supset W_j \supset \cdots \supset W_N = 0,$$

where $W_j = Z W_{j-1}$ for $j = 1, \dots, N$. For any $j \geq 0$, $M_j = W_j / W_{j+1}$ is an H/Z-module and, by the inductive hypothesis, H acts on M_j in a nilpotent way, since $(H/Z)^{t-1} = 0$. Hence, H is a nilpotent space of transformations on W, and the proof of the statement is complete.

This implies the nilpotence of the action of H on L, hence the validity of (38) since, by Theorem 3 and Lemma 4, L_0 is finitely generated.

Now we want to prove that the conditions of our theorem are sufficient. Let L be a finitely generated soluble Lie superalgebra over a field F with (28), (29), (30) and, if $\operatorname{char} F = 0$, with (38). It follows from (28) and (29) that the representations of L_0 in L_0 and L_1 satisfy the identities of the form (18) and (19). Similarly, it follows from (29) and (30) that the adjoint representation of L satisfies (31). It was shown in the proof of Proposition 2 that L satisfies all the conditions of Lemma 4. Hence, L_0 is a finitely generated algebra and, by Lemma 2, there exist such c that L_0^c acts on L as a nilpotent space of operators. One may assume that $c = 2$ is $\operatorname{char} F = 0$. As in the proof of Theorem 2 it is sufficient to show that if L has the least nonzero ideal K, then it is finite dimensional. Since L as a module over itself and K as an L-module satisfy the hypotheses of Lemmas 5, 7, and 8, it follows by Lemma 7 that $\dim K < \infty$ and by Lemma 8 that $\dim L < \infty$. Now the proof is complete.

It is possible to abandon the solubility condition in Theorem 4 in the case where the ground field is of characteristic zero.

Lemma 9. *Let* $L = L_0 \oplus L_1$ *be a Lie superalgebra over a field of characteristic zero satisfying* (28), (29), *and* (30), *and, for each* p, *let there exist an* N *such that* $(\operatorname{ad}([a_1, b_1] + \cdots + [a_p, b_p]))^N = 0$ *for all* $b_1, \ldots, b_p, a_1, \ldots, a_p \in L_0$. *Then* L *is locally soluble.*

Proof. We construct a subalgebra in L of the form

$$R = R_0 \oplus R_1 = L_0 \oplus [L_0, \ldots, L_0, L_1]$$

such that for some k we have $(\operatorname{ad} b^2)^{k+1} = 0$ in R for any b in R_1. First consider the case where not all the β_j in (29) and the γ_j in (30) are zero.

We can divide variables in (29) and (30) using (29) in the same way as was done in Lemma 2. This gives the identities of the form

$$[x, y^{(k)}, Z] = \sum_{j=1}^{k} \lambda_j [y^{(j)}, x, y^{(k-j)}, Z], \qquad x, y \in L_0, \tag{39}$$

$$[x, y^{(k)}, Z] = \sum_{j=1}^{k} \mu_j [y^{(j)}, x, y^{(k-j)}, Z], \qquad y \in L_0,\ x \in L_1. \tag{40}$$

Here, $Z = [t^{(r)}, z]$, $z \in L_1$, $t \in L_0$ and some r. Besides, $\lambda_k \neq 0$ and $\mu_k \neq 0$.

Let M denote the L_0-submodule in L_1 generated by all elements of the form $[a^{(r)}, b]$, $a \in L_0$, $b \in L_1$. Then the representation of L_0 in L_1/M satisfies the weak identity $u^r = 0$, whence we derive the nilpotence of the action of L_0 on L_1/M (it was shown in the proof of Theorem 4). Consequently, for some s, we have

$$T_s \subset M, \tag{41}$$

where $T_0 = L_1$ and $T_j = [L_0, T_{j-1}]$ if $j > 0$. We want to show that M is the linear span of the elements of the form $[a^{(r)}, b]$, $a \in L_0$, $b \in L_1$. If $a, d \in L_0$

and $b \in L_1$, then

$$[d, a^{(r)}, b] - [a^{(r)}, d, b] = \sum [a^{(j)}, [d, a], a^{(r-j-1)}, b]. \tag{42}$$

The sum on the right-hand side of (42) is one of the values of a partial linearization of $[y^{(r)}, x]$, $y \in L_0$, $x \in L_1$. Hence, $[d, a^{(r)}, b]$ can be written as a linear combination of values of the monomial $[y^{(r)}, x]$ in L. It follows then that M is the linear span of elements of the form $[a^{(r)}, b]$, $a \in L_0$, $b \in L_1$.

Now let $R = R_0 \oplus R_1$ be the subalgebra $L_0 \oplus T_s$. Suppose $a \in R_0$, $b \in R_1$, and $[a, b] = 0$. Since $R_1 \subset M$, it follows from (39) and (40) with $x = c$, $y = a$, and $z = b$ that $[a^{(k)}, c, b] = 0$ for any $c \in R$. Replacing a by b^2 in this relation we get $(\operatorname{ad} b)^{2k+1} z = 0$. It follows from this that, for any b in R_1, we have $(\operatorname{ad} b^2)^{k+1} = 0$.

If all the β_j in (29) are zero the the weak identity $x^{n+1} = 0$ holds in the representation of L_0 on L_1, hence $T_s = 0$ for some s and $R_1 = 0$. If the coefficients β_j are not equal to zero simultaneously then we have (39). As was shown, $(\operatorname{ad} b^2)^{k+1}$ is the zero map on R_0 for any b in R_1. If, in addition, all the γ_j in (30) are zero then we have $[x, (x^2)^{(n)}, z] = 0$ for any x, z in L_1. Hence, $(\operatorname{ad} b^2)^{(n+1)} = 0$ on L_1 for arbitrary b in L_1.

Let $a_1, \dots, a_q$ be arbitrary elements in R_0 such that $(\operatorname{ad} a_j)^T = 0$ in R. We want to show that there exists such a $P = P(q, T)$ that $(\operatorname{ad}(a_1 + \dots + a_q))^P = 0$ in R. If H is the Lie algebra generated by $a_1, \dots, a_q$ then the hypothesis of the lemma there exists a N such that $[b^{(N)}, u] = 0$ for all $u \in R$ and $b \in H^2$. It follows from this that H^2 acts on R in a nilpotent way, that is, there exists m, depending only on N, such that $[d_1, \dots, d_m, u] = 0$ for $d_1, \dots, d_m \in H^2$ and $u \in R$. As a Lie algebra, H^2 is generated by the commutators of the form

$$[a_1^{(t_1)}, \dots, a_q^{(t_q)}, a_i], \qquad 0 \le t_1, \dots, t_q \le T - 1, \ \sum t_i > 0. \tag{43}$$

Now $(\operatorname{ad}(a_1 + \dots + a_q))^P(v)$ can be written as a linear combination of commutators of the form

$$[b_1, \dots, b_r, a_1^{(j_1)}, \dots, a_q^{(j_q)}, v], \qquad 0 \le j_1, \dots, j_q \le T - 1,$$

where $b_1, \dots, b_r$ have the form (43). Since the degree of each element of the form (43) over all variables a_i does not exceed qT, we have $r \ge P/qT - 1$. It follows then that $(\operatorname{ad}(a_1 + \dots + a_q))^P = 0$ on R if $P \ge (m+1)qT$.

It follows from what has been proved that there exists a t such that for any $b_1, b_2 \in R_1$ we have $(\operatorname{ad}[b_1, b_2])^t = 0$ since $2[b_1, b_2] = (b_1 + b_2)^2 - b_1^2 - b_2^2$. In its turn, this means that the action of any element in R_1^2 on R is nilpotent.

Now we can prove the local solubility of the Lie superalgebra $H = H_0 \oplus H_1 = R_1^2 \oplus [R_0, R_1]$. Let $S = S_0 \oplus S_1 = \operatorname{alg}\{y_1, \dots, y_q, x_1, \dots, x_q\}$ be a finitely generated subalgebra in H, where $y_i \in H_0$ and $x_i \in H_1$. We will first prove the nilpotence of the action of S_0 on S_1.

If $b \in S_0$, then $b = a_1 + \dots + a_q + b_1 + \dots + b_q + y$, where $b_i = [z_i, x_i]$, $z_i \in H_1$, $a_i = [t_i, y_i]$, $t_i \in H_0$, and y is a linear combination of generators $y_1, \dots, y_q$. It follows from the nilpotence of the action of R_1^2 on R that there exists such an N_0 that $(\operatorname{ad} y_i)^{N_0} = 0$, $i = 1, \dots, q$. Since $(\operatorname{ad} b_i)^t = 0$ there exists an N_1, which does not depend on $b_1, \dots, b_q$, such that $(\operatorname{ad}(b_1 + \dots + b_q))^{N_1} = 0$. By the hypothesis of the lemma there exists an N_2, depending only on q, such that $(\operatorname{ad}(a_1 + \cdots + a_q))^{N_2} = 0$. It follows that $(\operatorname{ad} b)^N = 0$ in S, and that N does not depend on b. Consequently S_0 acts on S_1 in a nilpotent way. Since $S^2 \subset S_0 \oplus [S_0, S_1]$, it follows from this that for some j we have $S^{(j)} \subset S_0$, where $S^{(j)}$ is the jth term of the derived series of S. By our hypothesis, S_0 satisfies (28), which implies solubility (see [6]). Hence S is a soluble Lie superalgebra.

It follows from the definition of R and from (41) that $L^{(s)} \subset R$. Now R has an ideal H such that the derived algebra of R/H is an ordinary Lie algebra since $R^2 \subset H + R_0$. As $(R/H)^2$ is a Lie algebra with (28), it is soluble. Hence, $R^{(N)} \subset H$ for some N. It follows then that $L^{(q)} \subset H$ for some q.

So far we have proved the local solubility of $L^{(q)}$, an ideal of the Lie superalgebra L. To derive the local solubility of L we need two auxiliary lemmas.

Lemma 10. *Let $L = L_0 \oplus L_1$ be a finitely generated Lie superalgebra satisfying the conditions of Lemma 9 with locally soluble derived algebra. Then L_0 is a finitely generated Lie algebra.*

Proof. If L is a Lie superalgebra generated by even elements $y_1, \dots, y_m$ and odd elements $x_1, \dots, x_m$ then this is the linear span (modulo $\operatorname{alg}\{y_1, \dots, y_m\}$) of the commutators of the form $[b_1, \dots, b_r]$ such that each factor is a commutator of the form $[y_{j_1}, \dots, y_{j_s}, x_i]$. Each commutator of this kind can be replaced by a linear combination of elements of the form

$$[y_1^{(\lambda_1)}, \dots, y_m^{(\lambda_m)}, h_1, \dots, h_t, x_i], \tag{44}$$

where $h_1, \dots, h_t$ are commutators of the form

$$[y_1^{(\mu_1)}, \dots, y_m^{(\mu_m)}, y_j], \qquad \mu_1 + \cdots + \mu_m > 0. \tag{45}$$

If $H_0 = \operatorname{alg}\{y_1, \dots, y_m\}$ then, by the hypothesis, H_0^2 acts on L in a nilpotent way, hence t in (44) does not exceed a certain number t_0.

Using (28) and (29), as was done in Lemma 5, it is easy to observe that $[b_1, \dots, b_r]$ is in the H_0-submodule generated by the commutators of the form $[c_1, \dots, c_r]$, where each c_i has the form (44), (45) with $\lambda_1, \dots, \lambda_m$, $\mu_1, \dots, \mu_m \leq n-1$. The number of these elements is finite. If we denote these elements by $z_1, \dots, z_p$ then L coincides with the H_0-submodule generated by $B + H_0$, where $B = B_0 \oplus B_1 = \operatorname{alg}\{z_1, \dots, z_p\}$. Since all the z_i are in B_1, the derived algebra B^2 of B is generated by a finite number of commutators of the form $[z_1^{(\gamma_1)}, \dots, z_p^{(\gamma_p)}, z_i]$ with $0 \leq \gamma_1, \dots, \gamma_p \leq 2$, $\sum \gamma_i > 0$.

Since L^2 is locally soluble and $B^2 \subset L^2$, B is a soluble Lie superalgebra satisfying the hypotheses of Lemma 4. Hence, B_0 is a finitely generated Lie algebra. Since $L_0 = \operatorname{alg}\{y_1, \dots, y_m, B_0\}$, it follows that the proof of Lemma 10 is complete.

Lemma 11. *Let L be a Lie superalgebra satisfying the conditions of Lemma 9. If L^2 is a locally soluble Lie superalgebra then L is locally soluble.*

Proof. Let L be finitely generated. Then, by Lemma 10, its even component L_0 is finitely generated, too. By the hypothesis of Lemma 9, L_0^2 acts on L in a nilpotent way.

As before, $L^{(q)} \subset H$, where $H = R_1^2 \oplus [R_0, R_1]$. If $d = \dim L_0/L_0^2$ then any element h in H_0 can be written as the sum of at most $d+1$ summands, one of them being in L_0^2, while the remaining have the form $[a_1, a_2]$, where

$a_1, a_2 \in R_1$. Since $(\mathrm{ad}[a_1, a_2])^t = 0$ there exists a p such that $(\mathrm{ad}\, h)^p = 0$, p not depending on $h \in H_0$. It follows then that the action of H_0 on H_1 is nilpotent and that H is soluble (as in the proof of Lemma 9). Since $L^{(q)} \subset H$, L is soluble, proving the lemma.

To finish the proof of Lemma 9 it is sufficient to recall that $L^{(q)}$ is locally soluble for some q. Now Lemma 11 enables us to derive the solubility of $L^{(q-1)}, L^{(q-2)}, \dots, L$, and the proof is complete.

Theorem 5. *A variety $\mathscr{V}$ of Lie superalgebras over a field of characteristic zero is locally residually finite if and only if it satisfies identities of the form* (28), (29), *and* (30), *and, in any finitely generated algebra of $\mathscr{V}$, we have an identity of the form* (38).

Proof. If a variety $\mathscr{V}$ satisfies all identities listed above then, by Lemma 9, it is locally soluble. By Theorem 4, $\mathscr{V}$ is a locally residually finite variety.

Now let $\mathscr{V}$ be a locally residually finite variety of Lie superalgebras. By Theorem 3 then (28), (29), and (30) hold. As in Theorem 4, for even free generators $x_1, \dots, x_t, y_1, \dots, y_t$ of a relatively free algebra in $\mathscr{V}$, there exists a number N such that $(\mathrm{ad}([x_1, y_1] + \cdots + [x_t, y_t]))^N = 0$. Consequently, any finitely generated algebra in $\mathscr{V}$ is soluble by Lemma 9 and, as was shown in Theorem 4, it satisfies an identity of the form (38). Now the proof of Theorem 5 is complete.

References

1. Yu. A. Bahturin and V. S. Drensky, *Identities of soluble colour Lie superalgebras*, Algebra i Logika **26** (1987), 403–418. (Russian)
2. A. A. Mikhalev, *Subalgebras of colour Lie superalgebras*, Mat. Zametki **37** (1985), 653–662. (Russian)
3. M. A. Vasiliev, *De Sitter supergravity with positive constant and generalized Lie superalgebras*, Classical Quantum Gravity **2** (1985), 645–662.
4. Yu. A. Bahturin, *On residual properties of Lie algebras*, Mat. Zametki **12** (1972), 713–716. (Russian)
5. M. V. Zaicev, *Locally residually finite varieties of Lie algebras*, Mat. Zametki **44** (1988), 352–361. (Russian)
6. ——, *Residual finiteness and Noetherian property of finitely generated Lie algebras*, Mat. Sbornik **136** (1988), 500–509. (Russian)
7. M. Scheunert, *The theory of Lie superalgebras. An introduction.* Lecture Notes in Math., vol. 716, Springer-Verlag, 1979.
8. C. W. Curtis, *Noncommutative extensions of Hilbert rings*, Proc. Amer. Math. Soc. **4** (1953), 945–955.
9. I. B. Volichenko, *On the varieties of the center-by-metabelian Lie algebras*, Inst. Math. Akad. Nauk Byelorussian SSR, Preprint No. 16(96), 1980.
10. S. P. Mishchenko, *Structure and identities of certain soluble Lie algebras*, Ph.D. Thesis, Moscow University, 1982. (Russian)
11. Yu. A. Bahturin, *Identical relations in Lie algebras*, VNU Science Press, Utrecht, 1987.
12. A. I. Kostrikin, *Around Burnside*, Springer-Verlag, 1990.

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