

A CAMERON-MARTIN TYPE QUASI-INVARIANCE THEOREM FOR PINNED BROWNIAN MOTION ON A COMPACT RIEMANNIAN MANIFOLD

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ABSTRACT. The results in Driver [13] for quasi-invariance of Wiener measure on the path space of a compact Riemannian manifold (M) are extended to the case of pinned Wiener measure. To be more explicit, let $h: [0, 1] \rightarrow T_oM$ be a C^1 function where M is a compact Riemannian manifold, $o \in M$ is a base point, and T_oM is the tangent space to M at $o \in M$. Let $W(M)$ be the space of continuous paths from $[0, 1]$ into M , ν be Wiener measure on $W(M)$ concentrated on paths starting at $o \in M$, and $H_s(\omega)$ denote the stochastic-parallel translation operator along a path $\omega \in W(M)$ up to "time" s . (Note: $H_s(\omega)$ is only well defined up to ν -equivalence.) For $\omega \in W(M)$ let $X^h(\omega)$ denote the vector field along ω given by $X_s^h(\omega) \equiv H_s(\omega)h(s)$ for each $s \in [0, 1]$. One should interpret X^h as a vector field on $W(M)$. The vector field X^h induces a flow $S^h(t, \cdot): W(M) \rightarrow W(M)$ which leaves Wiener measure (ν) quasi-invariant, see Driver [13]. It is shown in this paper that the same result is valid if $h(1) = 0$ and the Wiener measure (ν) is replaced by a pinned Wiener measure (ν_e). (The measure ν_e is proportional to the measure ν conditioned on the set of paths which start at $o \in M$ and end at a fixed end point $e \in M$.) Also as in [13], one gets an integration by parts formula for the vector-fields X^h defined above.

1. INTRODUCTION

Let μ denote standard Wiener measure on $W(\mathbb{R}^d) \equiv C([0, 1], \mathbb{R}^d)$, and \mathbf{H} denote the set of $h \in W(\mathbb{R}^d)$ such that h is C^1 and $h(0) = 0$. The classical Cameron-Martin theorem states (see Cameron and Martin [7]) that $\mu^h \equiv \mu(\cdot - h)$ is equivalent to μ and the Radon-Nikodym derivative of μ^h with respect to μ is

$$(1.1) \quad \frac{d\mu^h}{d\mu}(\omega) = \exp \left\{ \int_0^1 h(s) \cdot d\omega(s) - \frac{1}{2} \int_0^1 |h(s)|^2 ds \right\}.$$

There has since been many extensions to this theorem, see for example Cameron and Martin [8, 9], Girsanov [18], Gross [19], Kuo [27, 28], Kusuoka [30–33], and Ramer [45]. There is also the group of extensions of the Cameron-Martin theorem used in the study of loop groups; see Albeverio and Hoegh-Krohn [2], Frenkel [16], Gross [22], and Malliavin and Malliavin [35]. In Driver

Received by the editors March 12, 1992.

1991 *Mathematics Subject Classification.* Primary 60D05, 28D05; Secondary 58D20, 58D25.

This research was partially supported by NSF Grant No. DMS 91-01720.

[13] it is shown that the classical Cameron-Martin theorem extends to the case of compact Riemannian manifolds (see Theorem 3.1), which includes Wiener measure on the path space $W(G)$ of a compact Lie group G . The purpose of this paper is to extend the results in [13] to the case of “pinned Wiener measure” on a compact Riemannian manifold M ; see Proposition 3.3 and Theorem 3.4. We also derive an integration by parts formula for “ h -derivatives”; see Theorem 3.13.

2. NOTATION AND BASIC PROPERTIES OF PINNED BROWNIAN MOTION

In this section we setup some notation and review a number of prerequisites for this article. The reader may also wish to consult Driver [13] for a more detailed exposition of some of the material in this section.

2.1. Differential geometric preliminaries. In this paper the data $(M^d, g, \nabla, o, e, u_o)$ will be fixed, where M is a compact connected manifold (with boundary) of dimension d , g is a Riemannian metric on M , ∇ is a g -compatible covariant derivative, o and e are fixed points in M , and u_o is an orthonormal frame at $o \in M$. (We view u_o as an isometry from \mathbb{R}^d to T_oM , the tangent space to M at o .) The symbol ∇ will also be used to denote the gradient operator acting on functions on M . The orthogonal frame bundle over M will be denoted by $O(M)$ and $\pi: O(M) \rightarrow M$ will be the fiber projection map. Throughout this paper the covariant derivative (∇) is *always* assumed to be g -compatible and “torsion skew symmetric” or TSS for short. The connection (∇) is said to be torsion skew symmetric if the torsion tensor (T) satisfies $g(T(v, w), w) = 0$ for all $v, w \in T_mM$ and $m \in M$.

The Laplacian (Δ) with respect to (∇) is the second order differential operator acting on the smooth functions $f \in C^\infty(M)$ given by $\Delta f \equiv sp \nabla df \equiv \sum_{i=1}^n \{E_i E_i f - df \langle \nabla_{E_i} E_i \rangle\}$, where $\{E_i\}_{i=1}^n$ is a local orthonormal frame. We recall from Driver [13] that this Laplacian is the same as the Levi-Civita Laplacian due to TSS condition on (∇).

The Riemannian distance on M will be denoted by $d(\cdot, \cdot)$. That is for $x, y \in M$, $d(x, y) \equiv \inf_\sigma \int_0^1 |\sigma'(s)| ds$, where the infimum is taken over all C^1 -paths $\sigma: [0, 1] \rightarrow M$ such that $\sigma(0) = x$ and $\sigma(1) = y$. I have also written $|\sigma'(s)|$ for $(g(\sigma'(s), \sigma'(s)))^{1/2}$.

The standard horizontal vector fields on $O(M)$ will be denoted by $\{B\langle a \rangle\}_{a \in \mathbb{R}^d}$ —recall that $B\langle a \rangle(u)$ is by definition the horizontal lift of the tangent vector $ua \in T_{\pi(u)}M$ to $T_uO(M)$. The fundamental 1-form on $O(M)$ is the \mathbb{R}^d -valued 1-form (θ) on $O(M)$ given by $\theta\langle u' \rangle = u^{-1} \pi_* u'$, where $u(s)$ is a smooth path in $O(M)$. The connection 1-form is the $so(n)$ -valued 1-form on $O(M)$ defined by $\omega\langle u' \rangle \equiv u^{-1} \nabla u/ds$, where $\nabla u/ds$ denotes the covariant derivative of u along $\sigma \equiv \pi \circ u$. (Here $\nabla u(s)/ds: \mathbb{R}^d \rightarrow T_{\pi(u(s))}M$ and $\nabla u(s)a/ds \equiv \nabla(u(s)a)/ds$ for all $a \in \mathbb{R}^d$.)

It will also be convenient to define the “equivariant” form of the curvature and torsion tensors. These forms are given by $\Omega_u\langle a, b \rangle \equiv u^{-1} R\langle ua, ub \rangle u \in so(n)$, and $\Theta_u\langle a, b \rangle \equiv u^{-1} T\langle ua, ub \rangle \in \mathbb{R}^d$, where R is the curvature tensor of (∇), T is the torsion tensor, $u \in O(M)$, and a and b are in \mathbb{R}^d . For more details on this notation see Driver [13] and Kobayashi and Nomizu [26].

2.2. Heat kernel estimates. This section collects a number of estimates involving the heat kernel which will be needed in the sequel. Let $p_s(x, y)$ be the kernel (with respect to the Riemannian volume form on M) of the heat operator $e^{s\Delta/2}$. Alternatively, $p_s(x, y)$ is the value at $x \in M$ of the fundamental solution to the heat equation $\partial f/\partial s = \frac{1}{2}\Delta f$ with initial condition $f(0, \cdot) = \delta_y(\cdot)$. We summarize the estimates that will be used in this paper in the following theorem.

Theorem 2.1. *There are constants $\epsilon > 0$, $K > 0$, and $C_i > 0$ for $i = 1, \dots, 4$ such that the following estimates hold for all x, y , and $e \in M$, and $0 < s \leq 1$:*

$$(2.1) \quad \epsilon s^{-d/2} e^{-C_1 d(x, e)^2/s} \leq |p_s(x, e)| \leq K s^{-d/2} e^{-C_2 d(x, e)^2/s},$$

$$(2.2) \quad |\nabla_x \ln p_s(x, e)| \leq C_3 [d(x, e)/s + 1/\sqrt{s}],$$

$$(2.3) \quad 0 < p_s(x, e)/p_s(y, e) \leq e^{C_4 d(x, y)/s}.$$

Furthermore, the kernel $p_s(x, e)$ has the asymptotic expansion valid for x near e and s near zero:

$$(2.4) \quad p_s(x, e) = (2\pi s)^{-1} e^{-d(x, e)^2/2s} \{1 + O(s)\}.$$

Proof. The estimates in (2.1) are well known and can be found in a number of references, see for example, Li and Yau [34], Cheeger and Yau [10], Davies [11], Hamilton [23], Setti [49], and Varopoulos [54, 55]. Probabilistic expressions for the left member of (2.2) may be found for example in Bismut [4], Jones and Leandre [25], and Norris [43].

For the purposes of this paper the author has found the Hamilton paper most useful. Equation (2.2) is an easy consequence of the results in [23] and equation (2.1). Indeed, Corollary 1.3 in [23] asserts that there are constants A and B depending only on (M, g) such that

$$s|\nabla \log p_s(x, e)|^2 \leq A \log(B/s^{d/2} p_s(x, e)).$$

It is now trivial to conclude (2.2) from this estimate and the lower bound in (2.1).

To prove (2.3) let $\sigma: [0, 1] \rightarrow M$ be a smooth path such that $\sigma(0) = y$ and $\sigma(1) = x$. To simplify notation let $l(t) \equiv \log p_s(\sigma(t), e)$. Then it follows from (2.2) (with C_4 sufficiently large) that

$$|l(1) - l(0)| \leq \int_0^1 |\nabla \log p_s(\sigma(t), e) \cdot \dot{\sigma}(t)| dt \leq \frac{C_4}{s} \int_0^1 |\dot{\sigma}(t)| dt,$$

where we have used the fact that $d(\cdot, e)$ is bounded since M is compact. Now take the infimum of the right-hand member of this last inequality to find that

$$|\log(p_s(x, e)/p_s(y, e))| = |\log p_s(x, e) - \log p_s(y, e)| \leq \frac{C_4}{s} d(x, y).$$

Equation (2.3) now easily follows.

Finally the asymptotic expansion in (2.4) is well known and can be found in a number of references; see for example [18, 47]. \square

2.3. Brownian motions and pinned Brownian motions. We start this section with some measure theoretic notation and conventions. Suppose that $(\Omega, \{\mathcal{F}_s\}, \mathcal{F})$ is a filtered measurable space, P and Q are measures on (Ω, \mathcal{F}) , and $f : \Omega \rightarrow \mathbb{R}$ is a \mathcal{F} -measurable function. The measures P and Q are said to be equivalent if P and Q are absolutely continuous with respect to one another. If $\rho = dQ/dP$ is the Radon-Nikodym derivative of Q with respect to P we will write $Q = \rho \cdot P$. The integral $\int_{\Omega} f(\omega) dP(\omega)$ will often be denoted simply by $P(f)$. More generally if $\mathcal{H} \subset \mathcal{F}$ is a sub-sigma field of \mathcal{F} then $P(f|\mathcal{H})$ will denote the conditional expectation of f relative to \mathcal{H} and P .

Remark 2.2. We will find it convenient to work with filtered measure spaces $(\Omega, \{\mathcal{F}_s\}, \mathcal{F}, P)$ where P is a finite but not necessarily a probability measure. In these cases we say that a process $\{X_s\}$ is an $(\Omega, \{\mathcal{F}_s\}, \mathcal{F}, P)$ -martingale or Brownian motion if $\{X_s\}$ is an $(\Omega, \{\mathcal{F}_s\}, \mathcal{F}, P(\Omega)^{-1}P)$ -martingale or Brownian motion respectively.

Suppose that Q is any manifold and that $q_0 \in Q$ is a fixed base point. Let $W(Q) \equiv C([0, 1], Q)$ be the set of continuous paths from $[0, 1]$ to Q . Let $W_{q_0}(Q)$ be the subset of paths in $W(Q)$ which start at q_0 . Given a function $\phi : \Omega \rightarrow W(Q)$ and $s \in [0, 1]$, let $\phi_s : \Omega \rightarrow Q$ be defined by $\phi_s(\omega) = \phi(\omega)(s)$. Clearly to specify ϕ it suffices to specify ϕ_s for each $s \in [0, 1]$ in such a way that $s \rightarrow \phi_s(\omega)$ is continuous for each $\omega \in \Omega$. Finally, we denote by ϕ_*P the measure on $W(Q)$ such that $\phi_*P(A) = P(\phi^{-1}(A))$ for all $A \subset W(Q)$ such that $\phi^{-1}(A) \in \mathcal{F}$. That is to say ϕ_*P is the law of ϕ under the measure P . For the rest of this section we restrict our attention to $Q = M$ and $q_0 = o$.

For each $s \in [0, 1]$ let $\Sigma_s : W(M) \rightarrow M$ denote the coordinate functions given by $\Sigma_s(\sigma) = \sigma(s)$. (Note under the above convention, Σ can be identified with the identity map from $W(M)$ to $W(M)$.) Let \mathcal{G}_s^o be the σ -algebra on $W(M)$ generated by the coordinate maps $\{\Sigma_u : u \leq s\}$, and \mathcal{G}^o be the σ -algebra \mathcal{G}_1^o . We will refer to \mathcal{G}^o as the raw σ -algebra and $\{\mathcal{G}_s^o | s \in [0, 1]\}$ as the raw filtration on $W(M)$. The next theorem supplies the definition and some of the basic properties of Wiener and pinned Wiener measure on $(W(M), \mathcal{G}^o)$.

Theorem 2.3. *There exist two finite measures (ν) and (ν_e) on $(W(M), \mathcal{G}^o)$ which are uniquely determined by specifying their finite dimensional distributions as follows. For all $k \in \{1, 2, 3, \dots\}$, partitions $0 = s_0 < s_1 < s_2 < \dots < s_{k-1} < s_k = 1$ of $[0, 1]$, and for all bounded measurable functions $f : M^k \rightarrow \mathbb{R}$;*

$$\begin{aligned}
 &\nu(f(\Sigma_{s_1}, \dots, \Sigma_{s_k})) \\
 (2.5) \quad &= \int_{M^k} f(x_1, \dots, x_k) \prod_{i=1}^k p_{\Delta s_i}(x_{i-1}, x_i) dx_1 \cdots dx_k,
 \end{aligned}$$

and

$$\begin{aligned}
 &\nu_e(f(\Sigma_{s_1}, \dots, \Sigma_{s_k})) \\
 (2.6) \quad &= \int_{M^{k-1}} f(x_1, \dots, x_{k-1}, e) \prod_{i=1}^k p_{\Delta s_i}(x_{i-1}, x_i) dx_1 \cdots dx_{k-1},
 \end{aligned}$$

where $x_0 \equiv o$, $\Delta s_i \equiv s_i - s_{i-1}$, $p_t(x, y)$ is the heat kernel, and $x_k \equiv e$ in (2.6). (Informally $\nu_e(\cdot) = \nu(\cdot | \Sigma_1 = e) \cdot p_1(o, e)$.) Furthermore the two measures (ν) and (ν_e) enjoy the following properties:

- (1) Both measures ν and ν_e are concentrated on the set of Hölder continuous paths for any Hölder coefficient $\alpha < 1/2$.
- (2) The two measures ν and ν_e are equivalent on \mathcal{G}_s^o for each $s < 1$. In fact, $\nu_e(F) = \nu(p_{1-s}(\Sigma_s, e)F)$ for all bounded \mathcal{G}_s^o -measurable functions.
- (3) The process $\{\Sigma_s\}_{0 \leq s \leq 1}$ is an M -valued Brownian motion with respect to the filtered probability space $(W(M), \{\mathcal{G}_s^o\}, \mathcal{G}^o, \nu)$. Recall that this means for all $f \in (C^\infty(M))$, the process

$$(2.7) \quad M_s^f \equiv f(\Sigma_s) - f(\Sigma_0) - \frac{1}{2} \int_0^s \Delta f(\Sigma_{s'}) ds'$$

is a martingale.

- (4) The process $\{\Sigma_s\}_{0 \leq s \leq 1}$ is an M -valued semimartingale with respect to the filtered probability space $(W(M), \{\mathcal{G}_s^o\}, \mathcal{G}^o, \nu_e)$. Furthermore for all $f \in C^\infty(M)$, the process

$$(2.8) \quad N_s^f \equiv f(\Sigma_s) - f(\Sigma_0) - \frac{1}{2} \int_0^s \Delta f(\Sigma_{s'}) ds' - \int_0^s df \langle \nabla \log p_{1-s'}(\Sigma_{s'}, e) \rangle ds'$$

is a martingale.

Remark 2.4. Notice that the measure ν is concentrated on $W_o(M)$ and the measure ν_e is concentrated on the set of pinned paths in $W_o(M)$ which end at $e \in M$.

Proof. Most of the results in this theorem are standard. Item 4 goes under the theory of Doob's "h-transformation"; see Doob [12]. To my knowledge the proof that N_s^f is a martingale for $s \in [0, 1]$ including $s = 1$ was first given by Bismut in [4]. For a more general treatment of this issue see Yor [56], and the related articles in the same lecture notes. The reader may also wish to consult Jones and Leandre [25].

For the sake of completeness I will sketch the proof of this theorem. There are of course a number of different proofs for each part.

The first step is the proof of existence of ν and ν_e . I will only prove the existence of ν_e since the argument for ν is similar. Using Kolmogorov's extension theorem (see for example [53]) it is easy to conclude there exists an M -valued process X_s on some probability space (Ω, \mathcal{F}, P) such that finite dimensional distributions of $\{X_s\}$ agree with those proposed for ν_e in (2.6). If we can show that the process $\{X_s\}$ has a continuous version (which will still be called X_s), then the law of the process $\{X_s\}$ will be the desired measure ν_e . To this end we will apply Kolmogorov's lemma, see for example [44, 46] or [53]. In order to use this lemma it is necessary to estimate $P(d(X_s, X_t)^k)$, where $0 \leq s, t \leq 1$, and k is a positive integer. (The following argument is analogous to the proof of Proposition 2.1 in Epperson and Lohrenz [15].) We consider the case where $\frac{1}{4} \leq s \leq t < 1$, the other cases can be handled

similarly. For s and t in this range we have

(2.9)

$$\begin{aligned}
 P(d(X_s, X_t)^k) &= \int_{M^2} p_s(o, x)p_{t-s}(x, y)p_{1-t}(y, e)d(x, y)^k dx dy \\
 &\leq C \int_{M^2} (t-s)^{-d/2}e^{-cd(x, y)^2/(t-s)}p_{1-t}(y, e)d(x, y)^k dx dy,
 \end{aligned}$$

where we have used the Gaussian upper bound in (2.1) twice. Now concentrate on estimating the x -integral. By compactness of M , it is easy to see that there is an $\epsilon > 0$ (independent of $m \in M$) such that the (geodesic) exponential map (exp) restricted to T_mM is a diffeomorphism on $B_m(\epsilon)$ —the ball in T_mM centered at $0_m \in T_mM$ of radius ϵ . For $v \in T_mM$, let $\rho_m(v)$ denote the density of the pullback of the Riemannian volume form on M by \exp_m with respect to the volume form on T_mM . Clearly $\rho_m(v)$ is a continuous function on the ϵ -neighborhood of the 0-section of TM . Therefore, by reducing ϵ if necessary we may assume there are constants $0 < a < b < \infty$ such that $a \leq \rho_m(v) \leq b$ for all $m \in M$ and $v \in B_m(\epsilon)$. Using these comments and passing to polar coordinates one finds (for $\delta > 0$) the estimate

$$\begin{aligned}
 (2.10) \quad &\int_M \delta^{-d/2} \cdot e^{-cd(x, y)^2/\delta} d(x, y)^k dx \\
 &\leq \int_{B_y(\epsilon)} \delta^{-d/2} e^{-cd(x, y)^2/\delta} d(x, y)^k dx + \delta^{-d/2} O(e^{-c\epsilon^2}) \\
 &\leq K \int_0^\epsilon \delta^{-d/2} e^{-cr^2/\delta} r^k r^{d-1} dr + \delta^{-d/2} O(e^{-c\epsilon^2/\delta}) \\
 &\leq C(k)\delta^{k/2}
 \end{aligned}$$

where the constant $C(k)$ only depends on k . Combining 2.9 and 2.10 with $\delta = t - s$ yields upon integrating out the y -variable (using the symmetry of $p_s(x, y)$ and the fact that $\int_M p_s(x, y) dy = 1$ for all $x \in M$)

$$(2.11) \quad P(d(X_s, X_t)^k) \leq C(k)|t - s|^{k/2}.$$

By similar arguments this last estimate is easily seen to be valid for all s and t in $[0, 1]$. It is now an easy matter to apply Kolmogorov’s Lemma to conclude there is a continuous version of $\{X_s\}$ and in fact this version is α -Hölder continuous for all $\alpha < 1/2$. This proves the existence of ν_e and also item 1 in the theorem. The same techniques also work for the measure ν . The unicity of the measures ν and ν_e satisfying (2.5) and (2.6) respectively follows easily from the fact that “smooth cylinder functions” of the form $F = f(\Sigma_{s_1}, \dots, \Sigma_{s_k})$ generate the σ -algebra \mathcal{G}^o .

It is clear that $\nu_e(F) = \nu(p_{1-s}(\Sigma_s \cdot, e)F)$ when F is a smooth cylinder function on $W(M)$ of the form $F = f(\Sigma_{s_1}, \dots, \Sigma_{s_k})$ with $0 \leq s_1, \dots, s_k \leq s < 1$. Since such functions generate \mathcal{G}_s^o , it follows that $\nu_e = p_{1-s}(\Sigma_s, \cdot) \cdot \nu$ on \mathcal{G}_s^o for all $s < 1$. This proves item 2.

Item 3 in the theorem is well known from the theory of Markov processes, see [53]. Indeed, using the the Markovian property of ν one computes for $s > t$,

F a bounded measurable \mathcal{G}_t^o -measurable function, and $f \in (C^\infty(M))$:

$$\begin{aligned} \frac{d}{ds} \nu(f(\Sigma_s)F) &= \frac{d}{ds} \nu((e^{(s-t)\Delta/2} f)(\Sigma_t)F) \\ &= \frac{1}{2} \nu(e^{(s-t)\Delta/2} \Delta f)(\Sigma_t)F = \nu \left(\frac{1}{2} \Delta f(\Sigma_s)F \right). \end{aligned}$$

Integrating this last expression from t to s shows that

$$\nu([M_t^f - M_s^f]F) = \nu \left(\left\{ f(\Sigma_t) - f(\Sigma_s) - \int_s^t \frac{1}{2} \Delta f(\Sigma_{s'}) ds' \right\} F \right) = 0,$$

which shows that M^f is a martingale.

More generally, the computation done above can be extended to the case that f is a function of $(s, x) \in [0, 1) \times M$. One then finds for $s > t$ that

$$(2.12) \quad \frac{d}{ds} \nu(f(s, \Sigma_s) \cdot F) = \nu \left(\left(\frac{\partial}{\partial s} + \frac{1}{2} \Delta \right) f(s, \Sigma_s) \cdot F \right).$$

Now let $h(s, x) \equiv p_{1-s}(x, e)$. Then h satisfies $\partial h / \partial s + \frac{1}{2} \Delta h = 0$ on $[0, 1) \times M$. Hence using (2.12), for $s > t$ one computes

$$\begin{aligned} (2.13) \quad \frac{d}{ds} \nu_e(f(\Sigma_s)F) &= \frac{d}{ds} \nu(h(s, \Sigma_s) f(\Sigma_s)F) = \nu \left(\left(\frac{\partial}{\partial s} + \frac{1}{2} \Delta \right) [h(s, \Sigma_s) f(\Sigma_s)]F \right) \\ &= \nu \left(\left[\nabla h(s, \Sigma_s) \cdot \nabla f(\Sigma_s) + h(s, \Sigma_s) \frac{1}{2} \Delta f(\Sigma_s) \right] F \right) \\ &= \nu_e \left(\left[\nabla \log h(s, \Sigma_s) \cdot \nabla f(\Sigma_s) + \frac{1}{2} \Delta f(\Sigma_s) \right] F \right). \end{aligned}$$

It now follows by integrating (2.13), as was done for M^f , that N_s^f defined in equation (2.8) is a martingale for $s < 1$. Because of the estimate (2.2) and item 1 of this theorem, it is clear that the integral in (2.8) is absolutely convergent ν_e -a.e. even for $s = 1$. Therefore N_s^f is a local martingale for $s \in [0, 1]$. We have already seen that $\nu_e(d(\Sigma_{1-s}, e)^k) \leq C(k)(1-s)^{k/2}$; see equation (2.11). Using this estimate and the gradient estimate (2.2), one easily shows that $|N_s^f|$ is bounded by a fixed function (g) which is independent of s and satisfies $\nu_e(|g|^k) \leq C(k, f) \int_0^1 (1-s)^{-k/2} < \infty$ provided that $k < 2$. Therefore, N_s^f is in fact a uniformly integrable martingale on $[0, 1]$. \square

2.4. Filtration technicalities. For technical reasons when doing stochastic calculus it is convenient to work with filtered probabilities spaces $(\Omega, \{\mathcal{F}_s\}, \mathcal{F}, P)$ which satisfy the “usual hypothesis.” That is \mathcal{F} should be complete with respect to P , \mathcal{F}_s should contain all P -null sets, and the filtration $\{\mathcal{F}_s\}$ should be right continuous. Since the two measures ν and ν_e are not equivalent with respect to one another on \mathcal{G}^o , it is not possible to complete $(W(M), \{\mathcal{G}_s^o\}, \mathcal{G}^o)$ in such a way that the usual hypothesis will hold simultaneously for both measures ν and ν_e . However for our purposes, it will suffice to have the usual hypothesis hold with respect to the measure $\bar{\nu} \equiv \nu + \nu_e$. We recall the following standard augmentation process.

Definition 2.5. Let $(\Omega, \{\mathcal{F}_s\}, \mathcal{F}, P)$ be a filtered measure space. The completion of \mathcal{F} with respect to P is denoted by \mathcal{F}^P . The measure P extended to \mathcal{F}^P will still be denoted by P . Let \mathcal{N}^P denote the P -null sets in \mathcal{F}^P and $\mathcal{F}_s^P \equiv \sigma(\mathcal{F}_s \cup \mathcal{N}^P)$, the augmented filtration.

In order to simplify notation we write \mathcal{N} for $\mathcal{N}^{\bar{\nu}}$, \mathcal{G} for $(\mathcal{G}^0)^{\bar{\nu}}$, and \mathcal{G}_s for $(\mathcal{G}_s^0)^{\bar{\nu}}$. Since ν and ν_e are absolutely continuous with respect to $\bar{\nu}$, it follows that both ν and ν_e have natural extensions to \mathcal{G} . These extensions will still be denoted by ν and ν_e respectively.

Proposition 2.6. *Using the above notation:*

- (1) *The σ -algebra \mathcal{G}_s may alternatively be described as $\mathcal{G}_s = \{A \subset W(M) : \exists B \in \mathcal{G}_s^0 \ni A \Delta B \in \mathcal{N}\}$, where $A \Delta B \equiv (A \setminus B) \cup (B \setminus A)$.*
- (2) *The measures ν , ν_e , and $\bar{\nu}$ are equivalent on \mathcal{G}_s for all $s < 1$.*
- (3) *The σ -algebras \mathcal{G}_s for $s < 1$ are complete with respect to both ν and ν_e .*
- (4) *The filtration $\{\mathcal{G}_s\}_{0 \leq s < 1}$ is right continuous.*

Proof. The first three items in the theorem are routine and left to the reader. The last item is well known in Markov theory and I will only briefly sketch the argument.

Let $s < t$ be in $[0, 1]$, and assume that Z is a bounded smooth cylinder function of the form $Z = f(\Sigma_{s_1}, \dots, \Sigma_{s_k}, \Sigma_{s_{k+1}}, \dots, \Sigma_{s_{k+n}})$, where $0 = s_1 < s_2 < \dots < s_k = s < s_{k+1} < \dots < s_{k+n} = 1$. I claim that $\nu(Z|\mathcal{G}_{s+}^0) = \nu(Z|\mathcal{G}_s^0)$ ν -a.s. To see this choose $t \in (s, s_{k+1})$, and compute $\nu(Z|\mathcal{G}_t^0)$ using the Markov property to get

$$\nu(Z|\mathcal{G}_t^0) = g(\Sigma_{s_1}, \dots, \Sigma_{s_k}, \Sigma_t; t),$$

where

$$g(x_1, \dots, x_k, y_0; t) \equiv \int_{M^n} f(x_1, \dots, x_k, y_1, \dots, y_n) \prod_{i=1}^n (p_{\Delta s_i}(y_{i-1}, y_i) dy_i),$$

$\Delta s_i \equiv s_{i+k} - s_{i+k-1}$ for $i > 1$, and $\Delta s_1 \equiv s_{k+1} - t$. Because the heat kernel is smoothing the function $g(x_1, \dots, x_k, y_0; t)$ is still smooth in all of its variables. Therefore

$$\nu(Z|\mathcal{G}_{s+}^0) = \lim_{t \downarrow s} \nu(Z|\mathcal{G}_t^0) = g(\Sigma_{s_1}, \dots, \Sigma_{s_k}, \Sigma_s; s) = \nu(Z|\mathcal{G}_s^0),$$

for all smooth cylinder functions Z . By a monotone class argument one concludes that $\nu(Z|\mathcal{G}_{s+}^0) = \nu(Z|\mathcal{G}_s^0)$ ν -a.s., where Z is any bounded \mathcal{G}^0 -measurable function. From this observation and item 1 of this proposition, it follows that $\nu(Z|\mathcal{G}_{s+}) = \nu(Z|\mathcal{G}_s)$ ν -a.s. for all bounded \mathcal{G} -measurable functions Z and $s < 1$. So if $s < 1$ and Z is a bounded \mathcal{G}_{s+} -measurable function, then $Z = \nu(Z|\mathcal{G}_{s+}) = \nu(Z|\mathcal{G}_s)$. Hence Z is equal to a \mathcal{G}_s -measurable function up to a ν -null set in \mathcal{G}_{s+} . By item 2 one concludes that Z is equal to a \mathcal{G}_s -measurable function up to a $\bar{\nu}$ -null set. Therefore, Z is in fact \mathcal{G}_s -measurable, because $(W(M), \{\mathcal{G}_s\}, \mathcal{G}, \bar{\nu})$ satisfies the usual hypothesis. \square

2.5. The stochastic development map. The reader who finds this section too abbreviated should consult §8 of Driver [13]. In this section I will only remark

here on the minor modifications needed to generalize the discussion in [13] to cover the applications needed in this article.

We will need some more notations. For $s \in [0, 1]$ let $B_s: W(\mathbb{R}^d) \rightarrow \mathbb{R}^d$ denote the coordinate map $B_s(\omega) \equiv \omega(s)$, and \mathcal{F}_s^o be the σ -algebra on $W(\mathbb{R}^d)$ generated by B_t for $t \leq s$. (Again \mathcal{F}_s^o will be called the raw filtration and $\mathcal{F}^o \equiv \mathcal{F}_1^o$ will be called the raw σ -algebra on $W(\mathbb{R}^d)$.)

Definition 2.7. Relative to the filtered probability space $(W(M), \{\mathcal{G}_s\}, \mathcal{G}, \bar{\nu})$, let $\hat{\Psi}: W(M) \rightarrow W(\mathbb{R}^d)$ be a fixed version of $\int \theta \langle \delta H_s \rangle$, where H_s is the stochastic horizontal lift of $\{\Sigma_s\}$ starting at u_o and δH denotes the Stratonovich differential of H .

Remarks 2.8. (1) Because the coordinate process $\{\Sigma_s\}$ is a semimartingale relative to both of the measures ν_e and ν it follows that $\{\Sigma_s\}$ is a semimartingale relative to the measure $\bar{\nu}$ also. This follows easily from Theorem 3, p. 45 of Protter [44] and the definition of an M -valued semimartingale.

(2) Because ν and ν_e are absolutely continuous relative to $\bar{\nu}$, it follows that $\hat{\Psi}$ is a version of $\int \theta \langle \delta H_s \rangle$ where all stochastic integrals and differential equations are taken with respect to either ν or ν_e .

Definition 2.9. The measures μ , μ_e , and $\bar{\mu}$ on $(W(\mathbb{R}^d), \mathcal{F}^o)$ are defined to be $\mu = \hat{\Psi}_* \nu$, $\mu_e = \hat{\Psi}_* \nu_e$, and $\bar{\mu} = \hat{\Psi}_* \bar{\nu} = \mu + \mu_e$. To simplify notation we write \mathcal{F} for $(\mathcal{F}^o)^\mu$, \mathcal{F}_s for $(\mathcal{F}_s^o)^\mu$, and \mathcal{N} for \mathcal{N}^μ .

It is well known that $\mu = \hat{\Psi}_* \nu$ is nothing other than the standard Wiener measure on $W(\mathbb{R}^d)$; see for example Emery [14, Proposition 8.26].

Proposition 2.10. Let $\hat{\Psi}: W(M) \rightarrow W(\mathbb{R}^d)$ be as above, then:

- (1) The measures μ , μ_e , and $\bar{\mu}$ are all equivalent on \mathcal{F}_s for each $s < 1$.
- (2) For each $s < 1$ the σ -algebra \mathcal{F}_s is complete with respect to μ , μ_e , and $\bar{\mu}$.
- (3) The filtration $\{\mathcal{F}_s\}$ is right continuous.
- (4) The map $\hat{\Psi}: W(M) \rightarrow W(\mathbb{R}^d)$ is $\mathcal{G}_s / \mathcal{F}_s$ -measurable for all $s \in [0, 1]$.
- (5) The coordinate process $\{B_s\}$ on $W(\mathbb{R}^d)$ ($B_s(\omega) \equiv \omega(s)$) is a semimartingale relative to the filtered measure space $(W(M), \{\mathcal{F}_s\}_{0 \leq s \leq 1}, \mathcal{F}, \mu^*)$, where μ^* is either μ , μ_e , or $\bar{\mu}$.

Proof. Items 1, 2, and 4 are easy and will be left to the reader. The proof of item 3 is the same as the proof of item 4 in Proposition 2.6. The proof of item 5 follows from the “good integrator” definition of a semimartingale (see [44, p. 44]). See Lemma 8.5. of Driver [13] for more details. \square

Remark 2.11. We will use Girsanov’s theorem in Theorem 2.13 to show that martingale part of B_s relative to $(W(M), \{\mathcal{F}_s\}_{0 \leq s \leq 1}, \mathcal{F}, \mu_e)$ is again a Brownian motion.

Theorem 2.12. Relative to the filtered measure space $(W(\mathbb{R}^d), \{\mathcal{F}_s\}, \mathcal{F}, \bar{\mu})$, let $\Psi: W(\mathbb{R}^d) \rightarrow W(M)$ be a fixed version of $\pi \circ I$, where I_s is the solution to the Stratonovich stochastic differential equation $dI_s = B \langle \delta B_s \rangle (I_s)$ with initial condition $I_o = u_o$. The function Ψ is $\mathcal{F}_s / \mathcal{G}_s$ -measurable for all $s \in [0, 1]$ and

the function $\Psi : W(\mathbb{R}^d) \rightarrow W(M)$ is a measure theoretic isomorphism with inverse $\hat{\Psi}$ in the following sense:

$$\nu\{\Psi \circ \hat{\Psi} \neq \text{id}\} = 0 \quad \text{and} \quad \mu\{\hat{\Psi} \circ \Psi \neq \text{id}\} = 0.$$

Proof. The proof is only a minor modification of the proof of Theorem 8.3 in [13]. \square

Theorem 2.13. Let Ψ and $\hat{\Psi}$ be as in Theorem 2.12 and set

$$b_s \equiv \hat{\Psi}_s = B_s \circ \hat{\Psi},$$

$$a_s \equiv \int_0^s H_{s'}^{-1} \nabla \log p_{1-s'}(\Sigma_{s'}, e) ds',$$

and

$$A_s \equiv \int_0^s I_{s'}^{-1} \nabla \log p_{1-s'}(\Psi_{s'}, e) ds'.$$

Then $\beta_s \equiv b_s - a_s$ and $\bar{\beta}_s \equiv B_s - A_s$ (for $s \in [0, 1]$) are \mathbb{R}^d -valued Brownian motions relative to the filtered probability spaces $(W(M), \{\mathcal{G}_s\}_{0 \leq s \leq 1}, \mathcal{G}, \nu_e)$ and $(W(\mathbb{R}^d), \{\mathcal{F}_s\}_{0 \leq s \leq 1}, \mathcal{F}, \mu_e)$ respectively.

Proof. In order to simplify notation set $z_s \equiv p_{1-s}(\Sigma_s, e)$. As has already been pointed out, it is well known that $b_s \equiv \hat{\Psi}_s$ is a Brownian motion relative the filtered probability space $(W(M), \{\mathcal{G}_s\}_{0 \leq s \leq 1}, \mathcal{G}, \nu)$. Since $\nu_e = p_{1-s}(\Sigma_s, e) \cdot \nu \equiv z_s \cdot \nu$ on \mathcal{G}_s for all $s < 1$, we know by Girsanov's theorem (see [44]) that $\beta_s \equiv b_s - \int_0^s z_s^{-1} d[z_s, b_s]$ is a ν_e -martingale for $s < 1$. We now need to compute $d z_s$. First recall that $\Sigma = \Psi \circ b$ and that $H = I \circ b$ ν -a.s., so that $dH = B\langle \delta b \rangle(H)$. See §8 of [13] for more details. To simplify notation let $h: [0, 1) \times O(M) \rightarrow \mathbb{R}$ be given by $h(s, u) = p_{1-s}(\pi(u), e)$. Notice because $p_s(x, e)$ is a solution to the heat equation on M that $(\partial/\partial s + \frac{1}{2} \sum_{i=1}^d B_i^2)h \equiv 0$, where $B_i \equiv B\langle e_i \rangle$ with $\{e_i\}_{i=1}^d$ the standard basis on \mathbb{R}^d . Because $z_s = h(s, H_s)$ and the above remarks we find

$$d z_s = B\langle db_s \rangle(H_s)h(s, \cdot) = H_s^{-1} \nabla p_{1-s}(\Sigma_s, e) \cdot db_s,$$

where we have used $B\langle a \rangle(u)h(s, \cdot) = u^{-1} \nabla p_{1-s}(\pi(u), e) \cdot a$. (We are writing $a \cdot b$ for the standard dot product on \mathbb{R}^d .) Thus

$$z_s^{-1} d z_s db_s = H_s^{-1} \nabla \log p_{1-s}(\Sigma_s, e) ds$$

and hence, with $a_s \equiv \int_0^s H_{s'}^{-1} \nabla p_{1-s'}(\Sigma_{s'}, e) ds'$, it follows that $\beta_s \equiv b_s - a_s$ is a ν_e -martingale at least for $s < 1$. Because ν and ν_e are equivalent on \mathcal{G}_s for $s < 1$, it follows that β_s and b_s have the same quadratic variations, i.e. $[\beta_s^i, \beta_s^j] = \delta_{ij}s$. Therefore by Levi's theorem (see for example [44]) β_s is an \mathbb{R}^d -valued Brownian motion relative ν_e . We already know a priori that b_s is a ν_e -semimartingale, so that the decomposition of b into $\beta + a$ must be valid for $s = 1$ also.

Alternatively, by the same techniques as in the proof of (2.8) it is clear that the integrand in the definition of a_s is ν_e -a.s. integrable for $0 \leq s \leq 1$. In particular a_s is a continuous process of bounded variation (relative to ν_e) and since β_s is a Brownian motion for $s < 1$ it follows that $\beta_1 \equiv \lim_{s \uparrow 1} \beta_s$ exists

ν_e -a.s.. Therefore, by continuity, the set $\{b_s = \beta_s + a_s \forall s \in [0, 1]\}$ has full ν_e -measure.

The proof of the decomposition of B_s is similar. The key observations is that on \mathcal{F}_s for any $s < 1$ the measure μ_e is given by $\mu_e = p_{1-s}(\Psi_s, e) \cdot \mu$. Using this fact, the desired decomposition is again a straightforward application of Girsanov's theorem. \square

3. THE FLOW AND QUASI-INVARIANCE

Let \mathbf{H} denote the set of absolutely continuous functions $h : [0,1] \rightarrow \mathbb{R}^d$ such that $h(0) = 0$ and $h' \in L^2(ds)$. As in [13] one can consider the following "vector field" (X^h) on $W_o(M)$ given by $X^h(\sigma)(s) \equiv H_s(\sigma)h(s)$, where $\sigma \in W_o(M)$ and $H_s(\sigma)$ denotes the stochastic horizontal lift of $\sigma|_{[0,s]}$ to $O(M)$ starting at u_o . One should note that this vector field is really an equivalence class of vector fields with any particular representative depending on the particular representative chosen for $H_s(\cdot)$. We now recall the Theorem 8.5 from Driver [13].

Theorem 3.1. *For each $h \in \mathbf{H} \cap C^1$ there exists a function $S^h : W(M) \rightarrow W(M)$ having the following properties:*

- (1) S^h is $\mathcal{G}_s^\nu / \mathcal{G}_s^{\nu^*}$ -measurable for all $s \in [0, 1]$.
- (2) There are versions of S^{th} and its horizontal lifts $H(S^{th})$ such that for ν almost every $\omega \in W(M)$, the functions $f(t, s) \equiv S_s^{th}(\omega)$, $\partial f(t, s) / \partial t$, and $(t, s) \rightarrow H_s(S^{th}(\omega))$ are jointly continuous. Furthermore for ν -a.e. ω , $S_s^{th}(\omega)$ satisfies

$$(3.1) \quad \frac{d}{dt} S_s^{th}(\omega) = H_s(S_s^{th}(\omega))h(s) \quad \text{with } S_s^{0h}(\omega) = \omega(s).$$

- (3) The functions $\{S^h\}_{h \in \mathbf{H} \cap C^1}$ satisfy the group property $S^{th} \circ S^{th} = S^{(t+\tau)h}$ up to ν -indistinguishability, for all $t, \tau \in \mathbb{R}$.
- (4) The measures $\nu^h \equiv S_*^h \nu$ and ν are equivalent. Furthermore for $s \in [0, 1]$, the Radon-Nikodym $Z_s^h = d\nu^h / d\nu|_{\mathcal{G}_s^\nu}$ of $\nu^h|_{\mathcal{G}_s^\nu}$ with respect to $\nu|_{\mathcal{G}_s^\nu}$ is

$$(3.2) \quad Z_s^h = \exp \left[- \int_0^s \alpha^{-h} \cdot O^{-h} db - \frac{1}{2} \int_0^s |\alpha^{-h}(s)|^2 ds \right],$$

In (3.2) (α^h, O^h) is a $\mathbb{R}^d \times O(d)$ -valued continuous \mathcal{G}_s^ν -adapted process and b is the ν -Brownian motion $b \equiv \hat{\Psi}$. The process (α^h, O^h) can be defined by the equation

$$(3.3) \quad \hat{\Psi} \circ S_s^h = \int_0^s O_{s'}^h db_{s'} + \int_0^s \alpha_{s'}^h ds'.$$

- (5) There is a constant C independent of $h \in \mathbf{H} \cap C^1$ such that ν -a.s.

$$(3.4) \quad |\alpha_s^h| \leq C \{ |h'(s)| + |h(s)| \}.$$

Remark 3.2. The estimate (3.4) is not explicitly stated in Theorem 8.5. of [13]. However, this estimate follows easily from Lemma 6.1. of [13] using $\alpha_o = 0$,

the assumption that ∇ is TSS, and the fact that α^{th} solves equation (6.6) of [13].

Now let H_o denote the set of $h \in H$ such that $h(1) = 0$. In the next proposition we choose a nice version of the flow S^h for $h \in H_o \cap C^1$.

Proposition 3.3. *Keep the same notation as in Theorem 3.1. For h restricted to lie in $H_o \cap C^1$ it is possible to choose a versions of S^h in Theorem 3.1 such that*

- (1) S^h is $\mathcal{G}_s/\mathcal{G}_s^o$ -measurable for all $s \in [0, 1]$ (we will see later that S^h is in fact $\mathcal{G}_s/\mathcal{G}_s$ -measurable),
- (2) ν_e -a.s. $d(S_s^h, \Sigma_s) \leq |h(s)|$ and in particular $\lim_{s \uparrow 1} S_s^h = e$,
- (3) the process (O^h, α^h) may be chosen to be \mathcal{G}_s -adapted for $s \in [0, 1]$.

Proof. By Proposition 2.6, for each $s_o \in [0, 1)$ the filtered probability space $(W(M), \{\mathcal{G}_s\}_{s \leq s_o}, \mathcal{G}_{s_o}, \nu)$ satisfies the usual hypothesis. Therefore one can construct a function ${}^{s_o}S^h$ which is $\mathcal{G}_s/\mathcal{G}_s^o$ -measurable for all $s \leq s_o$ and satisfies items 2–5 in Theorem 3.1 for $s \leq s_o$. (This is Theorem 3.1 with $[0, 1]$ replaced by $[0, s_o]$.) Now choose a sequence $\{s_n\} \subset [0, 1)$ such that s_n increases to 1 as $n \rightarrow \infty$. Let ${}^nS^h = {}^{s_n}S^h$, with ${}^{s_n}S^h$ described above. For any pair of positive integers (m, n) set

$$\Omega_{m,n} = \{\omega \in W(M) : {}^nS_s^h(\omega) = {}^mS_s^h(\omega) \forall s \in [0, s_n \wedge s_m]\}.$$

Because of the uniqueness of the functions S^h in Theorem 3.1 (see [13, Corollary 6.3]) it follows that $\nu(\Omega_{m,n}) = 1$. Clearly $\Omega_{m,n}$ is $\mathcal{G}_{s_n \vee s_m}$ -measurable, and hence it also follows that $\nu_e(\Omega_{m,n}) = \nu_e(1)$ for all (m, n) . Let $\Omega_o \equiv \bigcap_{m,n=1}^\infty \Omega_{m,n}$. Then $\nu(\Omega_o) = \nu_e(\Omega_o)/\nu_e(1) = 1$ and in particular $\Omega_o \in \mathcal{G}_o$. For $s < 1$ define

$$(3.5) \quad S_s^h(\omega) = \begin{cases} \lim_{n \rightarrow \infty} {}^nS_s^h(\omega) & \text{for } \omega \in \Omega_o, \\ \omega_o & \text{for } \omega \notin \Omega_o, \end{cases}$$

where ω_o is a fixed path in $W(M)$ for which $\omega_o(0) = o$ and $\omega_o(1) = e$. It is easy to see that S^h in (3.5) is $\mathcal{G}_s/\mathcal{G}_s^o$ -measurable for all $s < 1$.

Because of equation (3.1), the fact that $H_s(S^{th}) \in O(M)$, and the definition of the Riemannian distance d on M it easily follows for each $s \in [0, 1]$ that

$$(3.6) \quad |d(S_s^h, \Sigma_s)| \leq |h(s)| \text{ holds } \nu\text{-a. s.}$$

Now for $s < 1$ the set $\{|d(S_s^h, \Sigma_s)| \leq |h(s)|\}$ is \mathcal{G}_s -measurable. Since ν and ν_e are equivalent on \mathcal{G}_s , it follows that (3.6) holds ν_e -a.e. also. Because both sides of the inequality in (3.6) are continuous it follows that the set

$$(3.7) \quad \{|d(S_s^h, \Sigma_s)| \leq |h(s)| \forall s < 1\}$$

has full measure with respect to both ν_e and ν . Because of the assumption that $h \in H_o \cap C^1$ ($h(1) = 0$) and the above comments it follows that the set

$$(3.8) \quad \Omega_1 \equiv \left\{ \omega : \lim_{s \rightarrow 1} d(S_s^h(\omega), \omega(s)) = 0 \right\}$$

has full measure with respect to both ν and ν_e . So on Ω_1 we may extend the definition of S^h (to $s = 1$) by setting $S_1^h \equiv \Sigma_1$ on Ω_1 . For $\omega \notin \Omega_1$ redefine

$S_s^h(\omega) = \omega_o(s)$ where ω_o is a fixed path in $W(M)$ such that $\omega_o(0) = o$ and $\omega_o(1) = e$. It is easily seen that this function S^h has all of the properties asserted in the theorem. (Notice that $\Sigma_1 = e$ ν_e -a.s.)

By similar reasoning, one may show that the process (O_s^h, α_s^h) can be chosen to be \mathcal{G}_s -adapted for all $s < 1$. The key point is that (O^h, α^h) are unique up to ν -indistinguishability (see Theorem 6.1 in [13]) and so the same arguments as above still apply. \square

We now come to the main theorem of this paper.

Theorem 3.4. *Let $h \in \mathbf{H}_o \cap C^1$ and S^h be as above. For notational simplicity let $\nu_e^h \equiv S_*^h \nu_e$ on \mathcal{G}^o . Then ν_e and ν_e^h are equivalent on \mathcal{G}^o . More precisely $\nu_e^h = Z^h \cdot \nu_e$, where*

$$(3.9) \quad Z^h = \exp \left[- \int_0^1 \alpha^{-h} \cdot O^{-h} db - \frac{1}{2} \int_0^1 |\alpha^{-h}(s)|^2 ds \right]$$

which is the same expression as in (3.2) with $s = 1$. However, one should now interpret the stochastic integrals relative to the filtered probability space $(W(M), \{\mathcal{G}_s\}, \mathcal{G}, \nu_e)$ or even better $(W(M), \{\mathcal{G}_s\}, \mathcal{G}, \bar{\nu})$. Furthermore, the process S^h is $\mathcal{G}_s/\mathcal{G}_s$ -measurable for all $s \in [0, 1]$ and $S^{th} \circ S^{\tau h} = S^{(t+\tau)h}$ $\bar{\nu}$ -a.e. for all $t, \tau \in \mathbb{R}$.

Proof. Recall relative to ν_e , the process b is no longer a Brownian motion but has the decomposition $b = \beta + a$ given in Theorem 2.13. Now by using item 2 in Theorem 2.3 one easily shows $\nu_e^h = z_s \cdot \nu_e$ on \mathcal{G}_s^o for $s < 1$, where

$$(3.10) \quad z_s \equiv \frac{p_{1-s}(S_s^{-h}, e)}{p_{1-s}(\Sigma_s, e)} Z_s^h \equiv w_s \cdot Z_s^h \equiv w_s \cdot (x_s \cdot y_s),$$

$$(3.11) \quad w_s \equiv \frac{p_{1-s}(S_s^{-h}, e)}{p_{1-s}(\Sigma_s, e)},$$

$$(3.12) \quad x_s \equiv \exp \left[- \int_0^s \alpha^{-h} \cdot O^{-h} d\beta - \frac{1}{2} \int_0^s |\alpha^{-h}(s)|^2 ds \right],$$

and

$$(3.13) \quad \begin{aligned} y_s &\equiv \exp \left[- \int_0^s \alpha^{-h} \cdot O^{-h} da \right] \\ &= \exp \left[- \int_0^s \alpha^{-h}(s') \cdot O^{-h}(s') H_{s'}^{-1} \nabla \log p_{1-s'}(\Sigma_{s'}, e) ds' \right]. \end{aligned}$$

In order to show $\nu_e^h = Z^h \cdot \nu_e$, it suffices to show that z_s converges to $Z^h = Z_1^h$ in $L^1(\nu_e)$ as $s \rightarrow 1$. For then it would follow that $\nu_e^h(F) = \nu_e(Z_1^h F)$ where F is any \mathcal{G}_s^o -bounded measurable function and $s \in [0, 1]$. By a monotone class argument one can then easily conclude that $\nu_e^h(F) = \nu_e(Z_1^h F)$ for all bounded $\mathcal{G}_1^o = \sigma(\bigcup_{s < 1} \mathcal{G}_s^o)$ -measurable functions. The assertion that $Z_1^h = L^1\text{-}\lim_{s \uparrow 1} z_s$ is an easy consequence Hölder's inequality and Lemmas 3.5–3.7 below.

We now know on \mathcal{G}^o that $S_*^h \nu = Z^h \cdot \nu$ and $S_*^h \nu_e = Z^h \cdot \nu_e$, so it immediately follows that $S_*^h \bar{\nu} = Z^h \cdot \bar{\nu}$ on \mathcal{G}^o also. It is a trivial exercise to

show that $(S^h)^{-1}(\mathcal{N}^{\bar{\nu}}) \subset \mathcal{N}^{\bar{\nu}}$ which, along with the fact that S^h is $\mathcal{G}_s/\mathcal{G}_s^o$ -measurable for all s , implies that S^h is $\mathcal{G}_s/\mathcal{G}_s$ -measurable for all s . Therefore, given $s \in [0, 1)$, the set $A_s \equiv \{S_s^{th} \circ S^{\tau h} \neq S_s^{(t+\tau)h}\}$ is \mathcal{G}_s -measurable. Because of Theorem 3.1, item 3 and Proposition 2.6, item 2, it follows that $\bar{\nu}(A_s) = 0$ for all $s \in [0, 1)$. Finally because of the continuity of S_s^h it follows that $\bar{\nu}(\{S^{th} \circ S^{\tau h} \neq S^{(t+\tau)h}\}) = \bar{\nu}(\bigcap_{s \in [0, 1]} A_s) = 0$. This proves the remaining assertions of the theorem. \square

Lemma 3.5. ν_e -a.s. the process w_s in equation (3.11) is bounded and $\lim_{s \uparrow 1} w_s = 1$.

Lemma 3.6. The process x_s in equation (3.12) converges to x_1 in $L^p(\nu_e)$ for all $p < \infty$.

Lemma 3.7. The process y_s in equation (3.13) converges to y_1 in $L^p(\nu_e)$ for all $p < \infty$.

Proof of Lemma 3.5. By the estimate (2.3) and Proposition 3.3 there exists a constant C such that ν_e -a.s.

$$(3.14) \quad |w_s| = \left| \frac{p_{1-s}(S_s^{-h}, e)}{p_{1-s}(\Sigma_s, e)} \right| \leq \exp \frac{Cd(S_s^{-h}, \Sigma_s)}{1-s} \leq \exp \frac{C|h(s)|}{1-s}.$$

Since by assumption $h(1) = 0$ and h is C^1 , it follows that $|h(s)/(1-s)$ remains bounded for all $s \in [0, 1)$, and hence so does $|w_s|$.

From the asymptotic expansion (2.4) one learns that

$$(3.15) \quad w_s = \exp \left(\frac{-1}{2(1-s)} [d(S_s^{-h}, e)^2 - d(\Sigma_s, e)^2] \right) \{1 + O(1-s)\}$$

for s sufficiently close to 1. Now by repeated use of the triangle inequality and Proposition 3.3 one has

$$(3.16) \quad \begin{aligned} |d(S_s^{-h}, e)^2 - d(\Sigma_s, e)^2| &\leq |d(S_s^{-h}, e) - d(\Sigma_s, e)| \cdot |d(S_s^{-h}, e) + d(\Sigma_s, e)| \\ &\leq d(S_s^{-h}, \Sigma_s) \cdot |d(S_s^{-h}, e) + d(\Sigma_s, e)| \\ &\leq |h(s)| \cdot \{|h(s)| + 2d(\Sigma_s, e)\}. \end{aligned}$$

Since $|h(s)/(1-s)$ remains bounded and because ν_e -a.s. $\Sigma_s \rightarrow e$ as $s \uparrow 1$, it follows from (3.15) and (3.16) that ν_e -a.s. $\lim_{s \uparrow 1} w_s = 1$. \square

Proof of Lemma 3.6. Let $\beta = b - a$ be the ν_e -Brownian motion defined in Theorem 2.13 and define the ν_e -martingale M_s by $M_s \equiv -\int_0^s \alpha^{-h} \cdot O^{-h} d\beta$. Then one easily shows that $x_s = \exp\{M_s - \frac{1}{2}[M, M]_s\}$. Because of equation (3.4), there is a constant $C = C(h)$ such that $[M, M]_1 \leq C$ ν_e -a.s. It is now an easy application of Novikov's criterion to show that x_s is a martingale and $\nu_e(x_s^r) \leq e^{C(r^2-r)/2}$ for all $r \in \mathbb{R}$; see Remark 8.1. of [13] for more details. By Doob's inequality, for $p > 1$, $x^* \equiv \sup_{s \in [0, 1]} |x_s|$ satisfies $\nu_e(x^{*p}) \leq (p/(p-1))^p e^{C(r^2-r)/2}$. The lemma is now easily proved using the dominated convergence theorem. \square

In order to prove Lemma 3.7 we will need the following:

Lemma 3.8. *There is an $\epsilon > 0$ such that*

$$(3.17) \quad \nu_e(e^{\epsilon \int_0^1 \frac{d(\Sigma_s, e)}{1-s} ds})^2 < \infty.$$

Proof. Since M is compact it follows that $\int_0^{1/2} d(\Sigma_s, e)/(1-s) ds \leq K \log 2 < \infty$ where $K = \sup_{m \in M} d(m, e)$. Therefore it suffices to show that there is an $\epsilon > 0$ such that

$$(3.18) \quad \nu_e(e^{\epsilon \int_{1/2}^1 \frac{d(\Sigma_s, e)}{1-s} ds})^2 < \infty.$$

Now let $\kappa: W(M) \rightarrow W(M)$ denote the map defined by $\kappa(\omega)(s) \equiv \omega(1-s)$ where $s \in [0, 1]$. Then if $\nu_{e,o}$ is defined in the same way as ν_e except with o and e interchanged it follows trivially from (2.6) that $\nu_e = \kappa_* \nu_{e,o}$. Therefore (3.18) may be rewritten as

$$(3.19) \quad \kappa_* \nu_{e,o}(e^{\epsilon \int_{1/2}^1 \frac{d(\Sigma_s, e)}{1-s} ds})^2 = \nu_{e,o}(e^{\epsilon \int_0^{1/2} \frac{d(\Sigma_s, e)}{s} ds})^2.$$

We may now write the right member of (3.19) as

$$(3.20) \quad \hat{\nu}(p_{1/2}(\Sigma_{1/2}, o) \cdot e^{\epsilon \int_0^{1/2} \frac{d(\Sigma_s, e)}{s} ds})^2,$$

where $\hat{\nu}$ is the Wiener measure on $(W(M), \mathcal{G}^o)$ which is concentrated on the paths $(W_e(M))$ starting at $e \in M$. Since M is compact $p_{1/2}(\cdot, o)$ is bounded, and so it now suffices to show that

$$(3.21) \quad \exists \epsilon > 0 \ni \hat{\nu}(e^{\epsilon \int_0^{1/2} \frac{d(\Sigma_s, e)}{s} ds})^2 < \infty.$$

With the aid of the Whitney embedding theorem, one may choose C^∞ -functions $\{D_i\}_{i=1}^N$ such that $D_i(e) = 0$ and $d(x, e) \leq \sum_{i=1}^N |D_i(x)|$ for all x in M . (Indeed, let M be embedded in \mathbb{R}^N and set $D_i(x) = C[y_i(x) - y_i(0)]$ where C is a suitably large constant and y_i is the i th standard coordinate function on \mathbb{R}^N .) Using these remarks, in order to prove (3.21) it suffices to show for each C^∞ -function $D: M \rightarrow \mathbb{R}$ satisfying $D(e) = 0$ that

$$(3.22) \quad \exists \epsilon > 0 \ni \hat{\nu}(e^{\epsilon \int_0^{1/2} \frac{|D(\Sigma_s)|}{s} ds})^2 < \infty.$$

Now fix such a function D , then using the stochastic development in §2.5 one finds that

$$(3.23) \quad D(\Sigma_s) = \int_0^s H_s^{-1} \nabla D(\Sigma_s) \cdot db_s + \int_0^s \frac{1}{2} \Delta D(\Sigma_s) ds,$$

where b_s is an \mathbb{R}^d -Brownian motion and H_s is a continuous $O(M)$ valued process. Therefore $|D(\Sigma_s)| \leq Ks + |\int_0^s H_s^{-1} \nabla D(\Sigma_s) \cdot db_s|$, where K is now a bound on $\frac{1}{2} \Delta D$. Set $N_s \equiv \int_0^s H_s^{-1} \nabla D(\Sigma_s) \cdot db_s$, so that N_s is a continuous martingale. Notice that the quadratic variation satisfies $[N, N]_s \leq Cs$, where C is a bound for $|\nabla D|$. Putting these comments together we now see it suffices

to show that

$$(3.24) \quad \exists \epsilon > 0 \ni E(e^{\epsilon [\int_0^{1/2} \frac{|N_s|}{s} ds]^2}) < \infty,$$

where N is a continuous martingale such that $N_0 = 0$ and $[N, N]_s \leq Cs$ for some constant C .

To verify (3.24) we use the same technique employed to prove Lemma 9.3. in [13]. By the Dambis-Dubins-Schwarz Theorem (see [46, Theorem 1.7, p. 171] or [48, Chapter IV, §34]) there is a continuous martingale \tilde{N} and a Brownian motion \tilde{B} on some probability space $((\Omega, \tilde{\mathcal{F}}, \tilde{P}))$ such that N and \tilde{N} have the same law, and $\tilde{N} = \tilde{B}_{[\tilde{N}, \tilde{N}]}$. Given an $\alpha \in (0, 1/2)$ set $|f|_\alpha = \sup_{s, s' \in [0, C]} |f(s) - f(s')|/|s - s'|^\alpha$, where f is any real valued continuous function on $[0, C]$ such that $f(0) = 0$. Then

$$|\tilde{N}_s| = |\tilde{B}([\tilde{M}, \tilde{M}]_s)| \leq |\tilde{B}|_\alpha \cdot [\tilde{M}, \tilde{M}]_s^\alpha \leq |\tilde{B}|_\alpha \cdot C^\alpha s^\alpha.$$

It now follows by elementary calculus that $\int_0^{1/2} |\tilde{N}_s|/s ds \leq \alpha^{-1} C^\alpha |\tilde{B}|_\alpha$. Since the expectation in (3.24) satisfies

$$E(e^{\epsilon [\int_0^{1/2} \frac{|N_s|}{s} ds]^2}) = \tilde{E}(e^{\epsilon [\int_0^{1/2} \frac{|\tilde{N}_s|}{s} ds]^2}) \leq \tilde{E}(e^{\epsilon \alpha^{-2} C^{2\alpha} |\tilde{B}|_\alpha^2}),$$

it suffices to show that there is an $\epsilon > 0$ such that $\tilde{E}e^{\epsilon |\tilde{B}|_\alpha^2} < \infty$. But this follows from Fernique's theorem (see [29, pp. 159–160] or [24, p. 402]), since the Hölder seminorms $|\cdot|_\alpha$ for $\alpha < 1/2$ are measurable seminorms for the standard Wiener measure. \square

Proof of Lemma 3.7. First notice that the integrand in the right-hand member of (3.13) can be estimated (using the estimates (3.4), (2.2), and the assumption that $h \in \mathbf{H}_o \cap C^1$) by

$$(3.25) \quad |\alpha^{-h}(s)| \cdot |\nabla \log p_{1-s}(\Sigma_s, e)| \leq K[|d(\Sigma_s, e)|/(1-s) + (1-s)^{-1/2}],$$

where K is a suitably large constant. Because of item 1 in Theorem 2.3 we know ν_e -a.s. that $|d(\Sigma_s, e)| \leq C(1-s)^{1/4}$ for some random constant C . Therefore the right member of (3.25) is in $L^1([0, 1], ds)$ ν_e -a.s. This shows ν_e -a.s. that y_1 exists and $\lim_{s \uparrow 1} y_s = y_1$.

From the definition of y_s in (3.13) and the bounds in (3.25) it follows that

$$(3.26) \quad |y_s| \leq \exp K \left(\int_0^1 \left| \frac{d(\Sigma_u, e)}{1-u} \right| du + 2 \right).$$

By Lemma 3.8, one easily shows that all powers of the dominating function on the right side of (3.26) are integrable. Therefore, the lemma is now proved with the aid of the dominated convergence theorem. \square

3.1. Integration by parts. We conclude this paper with an integration by parts formula which is completely analogous to Theorem 9.1 in Driver [13]. In fact, it is possible to give essentially the same proof but we will instead prove the result using Theorem 9.1 in [13]. Before stating the integration by parts formula we will need some more notation.

Definition 3.9. For $H \in O(M)$ and $a \in \mathbb{R}^d$, define the *Ricci tensor* by

$$\text{Ric}_H \langle a \rangle \equiv \sum_{i=1}^d \Omega_H \langle a, e_i \rangle e_i \in \mathbb{R}^d.$$

Definition 3.10. For $H \in O(M)$ and $a \in \mathbb{R}^d$, set

$$\widehat{\Theta}_H \langle a \rangle \equiv \sum_{i=1}^d \Theta'_H \langle e_i, a, e_i \rangle,$$

where $\{e_i\}_{i=1}^d$ is an orthonormal basis for \mathbb{R}^d , and

$$\Theta'_H \langle a, b, c \rangle \equiv B \langle a \rangle (H)(u \rightarrow \Theta_u \langle b, c \rangle).$$

Let \mathcal{H}_s^o be the raw filtration on $W(O(M))$, i.e., the filtration on $W(O(M))$ generated by the coordinate functions $\{U_t\}_{t \in [0, s]}$ where $U_t(u) = u(t)$ for all $u \in W(O(M))$. As above, let $H: W(M) \rightarrow W(O(M))$ be a fixed version (relative to the filtered measure space $(W(M), \{\mathcal{G}_s\}, \mathcal{G}, \bar{\nu})$) of the horizontal lift of the process $\{\Sigma_s\}$ such that $H_0 = u_o$. It follows automatically that H is $\mathcal{G}_s/\mathcal{H}_s^o$ -measurable for all $s \in [0, 1]$ and that H is a version of the horizontal lift of $\{\Sigma_s\}$ relative to both measures ν_e and ν .

Definition 3.11 (*h-derivative*). Let $f: W(M) \rightarrow \mathbb{R}$ be a C^2 -cylinder function, that is $f(\Sigma) = F(\Sigma_{s_1}, \dots, \Sigma_{s_k})$ for some $0 \leq s_1 \leq s_2 \leq \dots \leq s_k \leq 1$ and some C^2 -function $F: M^k \rightarrow \mathbb{R}$. (Note: Because ν_e -a.s. $\Sigma_1 = e$, we can and **always** do assume that $s_k < 1$.) Then the *h-derivative* of f is the function $\partial_h f: W(M) \rightarrow \mathbb{R}$ given by

$$(3.27) \quad \partial_h f(\Sigma) = \sum_{i=1}^k f_i(\Sigma) \langle H_{s_i}(\Sigma) h(s_i) \rangle,$$

where for $v \in TM$, $f_i(\Sigma) \langle v \rangle \equiv v(F(\Sigma_{s_1}, \dots, \Sigma_{s_{i-1}}, \cdot, \Sigma_{s_{i+1}}, \dots, \Sigma_{s_k}))$.

Remark 3.12. As noted in [13], $\partial_h f$ is only defined up to ν -equivalence and hence ν_e -equivalence. It is also shown in [13] that $\partial_h f$ can be defined by $\partial_h f = (d/dt)|_0 f \circ S^{th}$, where the derivative is taken in ν -probability.

Theorem 3.13. Let $h \in \mathbf{H}_o \cap C^1$, $b \equiv \widehat{\Psi} \circ \Sigma$, and H be the horizontal lift as above. Then with respect to the $L^2(\nu_e)$ -inner product the adjoint (∂_h^*) of ∂_h contains the set of C^2 -cylinder functions (hence is densely defined). Furthermore if f is a C^2 -cylinder function, then

$$(3.28) \quad \partial_h^* f = -\partial_h f + z(h) f,$$

where $z(h) \equiv \int_0^1 [\frac{1}{2} \text{Ric}_H \langle h \rangle + \frac{1}{2} \widehat{\Theta}_H \langle h \rangle + h'] \cdot db$.

Proof. For each C^2 -cylinder function (f) , let $\partial_h^* f$ be given by equation (3.28). It suffices to show that for all C^2 -cylinder functions that

$$\nu_e(\partial_h f \cdot g) = \nu_e(f \cdot \partial_h^* g).$$

According to Theorem 9.1 of Driver [13] the above displayed equation is valid if ν_e is replaced by ν . Therefore if f and g are two C^2 -cylinder functions and s is sufficiently close to 1 we have

$$\begin{aligned} \nu_e(\partial_h f \cdot g) &= \nu(\partial_h f \cdot (gp_{1-s}(\Sigma_s, e))) \\ (3.29) \quad &= \nu(f[-\partial_h + z(h)](gp_{1-s}(\Sigma_s, e))) \\ &= \nu(f\partial_h^* g \cdot p_{1-s}(\Sigma_s, e)) + \nu_e(fg \cdot \partial_h \ln(p_{1-s}(\Sigma_s, e))). \end{aligned}$$

So in order to finish the proof it suffices to show that $J \equiv \lim_{s \uparrow 1} \nu(f\partial_h^* g \cdot p_{1-s}(\Sigma_s, e)) = \nu_e(f\partial_h^* g)$, and $L \equiv \lim_{s \uparrow 1} \nu_e(fg \cdot \partial_h \ln(p_{1-s}(\Sigma_s, e))) = 0$. We start with J .

First notice that $\partial_h g$ is still \mathcal{E}_s^o -measurable for s sufficiently close to 1, and hence $\nu(f\partial_h g \cdot p_{1-s}(\Sigma_s, e)) = \nu_e(f\partial_h g)$ for s close to 1. So to show $J = \nu_e(f\partial_h^* g)$ it suffices to show $\lim_{s \uparrow 1} \nu(fgz(h)p_{1-s}(\Sigma_s, e)) = \nu_e(fgz(h))$. To this end set

$$z_s \equiv \int_0^s \left[\frac{1}{2} \text{Ric}_H \langle h \rangle + \frac{1}{2} \widehat{\Theta}_H \langle h \rangle + h' \right] \cdot db,$$

where $b = \widehat{\Psi}$ as in Theorem 2.13. Now b is a ν -Brownian motion so that z_s is a ν -martingale. Since $fgp_{1-s}(\Sigma_s, e)$ is \mathcal{E}_s^o -measurable for s close to 1, it follows for s near 1 that

$$\nu(fgz(h)p_{1-s}(\Sigma_s, e)) = \nu(fgz_s p_{1-s}(\Sigma_s, e)) = \nu_e(fgz_s).$$

From this last equation and the fact that f and g are bounded, it now suffices to show that $z_s \rightarrow z_1 = z(h)$ in $L^1(\nu_e)$ as $s \uparrow 1$.

To verify that $z_s \rightarrow z_1$, let $b_s = \beta_s + a_s$ be the decomposition in Theorem 2.13 and decompose $z_s = x_s + y_s$ where

$$x_s \equiv \int_0^s \left[\frac{1}{2} \text{Ric}_H \langle h \rangle + \frac{1}{2} \widehat{\Theta}_H \langle h \rangle + h' \right] \cdot d\beta$$

and

$$y_s \equiv \int_0^s \left[\frac{1}{2} \text{Ric}_H \langle h \rangle + \frac{1}{2} \widehat{\Theta}_H \langle h \rangle + h' \right] \cdot da.$$

Since β is a ν_e -Brownian motion, it is clear that x_s is an L^2 -martingale and in particular $x_s \rightarrow x_1$ in L^1 as $s \uparrow 1$. We now must show that y_s converges to y_1 in L^1 as $s \uparrow 1$. To simplify notation, set $R = [\frac{1}{2} \text{Ric}_H \langle h \rangle + \frac{1}{2} \widehat{\Theta}_H \langle h \rangle + h']$ and notice that R is uniformly bounded since $h \in \mathbf{H}_o \cap C^1$, and M is compact. With this notation and the definition of a , y_s can be written as

$$y_s = \int_0^s R_{s'} H_{s'}^{-1} \nabla \log p_{1-s'}(\Sigma_{s'}, e) ds'.$$

Using this last equation and the estimate (2.2) one easily shows (with $K = K(h') > 0$ sufficiently large) that:

$$(3.30) \quad |y_1 - y_s| \leq K \int_s^1 \left[\frac{d(\Sigma_{s'}, e)}{(1-s')} + \sqrt{1-s'} \right] ds'.$$

By (2.11) it follows that $\nu_e(d(\Sigma_{s'}, e)) \leq C(1)\sqrt{1-s'}$, which combined with (3.30) easily shows that $\nu_e(|y_1 - y_s|) \leq 2K(C(1) + 1)\sqrt{1-s}$. Hence, y_s con-

verges in $L^1(\nu_e)$ to y_1 as desired. Therefore, $z_s = x_s + y_s$ converges to z_1 in $L^1(\nu_e)$ as claimed and we have shown that $J = \nu_e(f\partial_h^*g)$.

Now the “ L ” term is easily shown to be zero as follows. First notice that $|\partial_h \ln(p_{1-s}(\Sigma_s, e))| \leq |\nabla \ln(p_{1-s}(\Sigma_s, e))||h(s)|$. Hence using (2.2) and (2.11) as above one shows for sufficiently large constants K and C that

$$L \leq K \limsup_{s \uparrow 1} \nu_e(|\nabla \ln(p_{1-s}(\Sigma_s, e))||h(s)|) \leq C \limsup_{s \uparrow 1} [|h(s)|/\sqrt{1-s}].$$

Since by assumption $h(1) = 0$ and h is C^1 , the last limit is clearly 0 and hence $L = 0$. \square

ACKNOWLEDGMENTS

It is a pleasure to acknowledge M. Cranston, M. Gage, W. Kendall, P. Li., and A. Setti for their efforts in trying to pointing me to the correct source for the gradient estimate used in this paper. I would especially like to thank R. Hamilton for sharing his gradient estimate results in [23] with me. It is also a pleasure to thank P. Fitzsimmons, R. Williams, and M. Yor for a number of useful conversations dealing with probability theory and “ h -transforms.”

REFERENCES

1. H. Airault and P. Malliavin, *Integration on loop group. II; Heat equation for the Wiener measure*, J. Funct. Anal. **112** (1992), 71–109.
2. S. Albeverio and R. Hoegh-Krohn, *The energy representation of Sobolev Lie groups*, Compositio Math. **36** (1978), 37–52.
3. Jean-Michel Bismut, *Mecanique aleatoire*, Lecture Notes in Math., vol. 866 (A. Dold and B. Eckmann, eds.), Springer-Verlag, Berlin, Heidelberg, New York, 1981.
4. ———, *Large deviations and the Malliavin calculus*, Birkhäuser, Boston and Basel, 1984.
5. P. Brémaud and M. Yor, *Changes of filtrations and of probability measures*, Z. Wahrsch. verw. Gebiete **45** (1978) 269–295.
6. R. H. Cameron, *The first variation of an indefinite Wiener integral*, Proc. Amer. Math. Soc. **2** (1951), 914–924.
7. R. H. Cameron and W. T. Martin, *Transformations of Wiener integrals under translations*, Ann. of Math. (2) **45** (1944), 386–386.
8. ———, *Transformations of Wiener integrals under a general class of linear transformations*, Trans. Amer. Math. Soc. **58** (1945), 184–219.
9. ———, *The transformation of Wiener integrals by non-linear transformations*, Trans. Amer. Math. Soc. **66** (1949), 253–283.
10. J. Cheeger and S. T. Yau, *A lower bound for the heat kernel*, Comm. Pure Appl. Math. **34** (1981), 465–480.
11. E. B. Davies, *Heat kernels and spectral theory*, Cambridge Univ. Press, Cambridge and New York, 1990.
12. J. L. Doob, *Classical potential theory and its probabilistic counterpart*, Springer-Verlag, Berlin, Heidelberg, New York, 1983.
13. B. Driver, *A Cameron-Martin type quasi-invariance theorem for Brownian motion on a compact Riemannian manifold*, J. Funct. Anal. **110** (1992), 272–376.
14. Michel Emery, *Stochastic calculus in manifolds*, Springer-Verlag, Berlin, Heidelberg, New York, 1989.
15. Epperson and Lohrenz, *Brownian motion on the path space of a compact Lie group*, Rice Univ. preprint, 1991.
16. I. B. Frenkel, *Orbital theory for affine Lie algebras*, Invent. Math. **77** (1984), 301–352.

17. I. V. Girsanov, *On transforming a certain class of stochastic processes by absolutely continuous substitution of measures*, Theory Probab. Appl. **5** (1960), 285–301.
18. P. Gilkey, *Invariance theory, the heat equation, and the Atiyah-Singer index theorem*, Publish or Perish, Wilmington, DE, 1984.
19. L. Gross, *Integration and nonlinear transformations in Hilbert space*, Trans. Amer. Math. Soc. **94** (1960), 404–440.
20. ———, *Abstract Wiener spaces*, Proc. 5th Berkeley Sympos. Math. Stat. Prob. 2, 1965, pp. 31–42.
21. ———, *Logarithmic Sobolev inequalities for the heat kernel of a Lie group*, White Noise Analysis : Mathematics and Applications, (T. Hida et al., eds.) World Scientific, Singapore and Teaneck, NJ, 1990, pp. 108–130.
22. ———, *Logarithmic Sobolev inequalities on loop groups*, J. Funct. Anal. **102** (1991), 268–313.
23. Richard Hamilton, *A matrix Harnack estimate for the heat equation*, Comm. Anal. & Geom. **1** (1993), 113–126.
24. N. Ikeda and S. Watanabe, *Stochastic differential equations and diffusion processes*, North-Holland, Amsterdam, 1981.
25. J. D. S. Jones and R. Leandre, *L^p -Chen forms on loop spaces*, Stochastic Analysis, Barlow and Bingham, eds., Cambridge Univ. Press, Cambridge and New York, 1991, pp. 103–162.
26. S. Kobayashi and K. Nomizu, *Foundations of differential geometry*, Vol. 1, Interscience, New York, 1963.
27. H.-H. Kuo, *Integration theory on infinite-dimensional manifolds*, Trans. Amer. Math. Soc. **159** (1971) 57–78.
28. ———, *Diffusion and Brownian motion on infinite dimensional manifolds*, Trans. Amer. Math. Soc. **169** (1972) 439–457.
29. ———, *Gaussian measures in Banach spaces*, Lecture Notes in Math., vol. 463, Springer-Verlag, Berlin and New York, 1975.
30. S. Kusuoka, *The nonlinear transformation of Gaussian measure on Banach space and its absolute continuity*. I, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **29** (1982), 567–597.
31. ———, *On the foundations of Wiener-Riemannian manifolds*, Stochastic Analysis, Path Integration and Dynamics: Emanations from “Summer Stochastics” Warwick 1987 (K. D. Elworthy and J.-C. Zambrini, eds.), Wiley, New York, 1989, pp. 130–165.
32. ———, *Analysis on Wiener spaces*. I: *Nonlinear maps*, R.I.M.S. report 670, 1989.
33. ———, *Analysis on Wiener spaces*. II: *Differential forms*, R.I.M.S. report 705, 1990.
34. P. Li and S. T. Yau, *On the parabolic kernel of the Schrödinger operator*, Acta Math. **156** (1986), 153–201.
35. M.-P. Malliavin, and M. Malliavin, *Integration on loop groups*. I: *Quasi invariant measures*, J. Funct. Anal. **93** (1990), 207–237.
36. ———, *Integration on loop group*. III. Asymptotic Peter-Weyl orthogonality, J. Func. Anal. **108** (1992), 13–46.
37. ———, *La representation reguliere des groupes de lacets*, Proc. Conf. Trientenaire 1990, Hamburg Math. Sem., in the Third Centenary Anniversary of Hamburg University.
38. ———, *An infinitesimally quasi invariant measure on the group of diffeomorphisms of the circle*, Proc. Hashibara Forum (M. Kashiwara and T. Miwa, eds.), 1991.
39. P. Malliavin, *Naturality of quasi-invariance of some measures*, Proc. Lisbonne Conf., Birkhäuser (A. B. Cruzeiro, ed.).
40. ———, *Hypoellipticity in infinite dimension*, Diffusion Process and Related Problems in Analysis, Vol I. (M. A. Pinsky, ed.), Birkhäuser, 1991.
41. ———, *Diffusion on the loops*, Proc. Conf. in the Honour of Antoni Zygmund, Chicago 1981–1983 (W. Beckner and A. Calderon, eds.), Vol. 2, Wadsworth Internat. Group, 1983, pp. 764–782.
42. G. Maruyama, *Notes on Wiener integrals*, Kodai Math. Sem. Rep. **3** (1950), 41–44.

43. J. R. Norris, *Path integral formulae for heat kernels and their derivatives*, Probab. Theory Related Fields **94** (1993), 525–541.
44. P. Protter, *Stochastic integration and differential equations; a new approach*, Springer-Verlag, Berlin, Heidelberg, New York, 1990.
45. R. Ramer, *On nonlinear transformations of Gaussian measures*, J. Funct. Anal. **15** (1974), 166–187.
46. D. Revuz and M. Yor, *Continuous martingales and Brownian motion*, Springer-Verlag, Berlin, Heidelberg, New York, 1991.
47. John Roe, *Elliptic operators, topology, and asymptotic methods*, Pitman Res. Notes Math. Ser., no. 179, Wiley, New York, 1988.
48. L. C. G. Rogers and D. Williams, *Diffusions, Markov processes, and martingales*. Vol. 2 : *Ito calculus*, Wiley, New York, 1987.
49. A. Setti, *Eigenvalue estimates for the Laplacian with lower order terms on a complete Riemannian manifold*, Cornell Univ. preprint, 1990.
50. C. Stricker and M. Yor, *Calcul stochastique dépendant d'un parametre*, Z. Wahrsch. Verw. Gebiete **45** (1978), 109–103.
51. D. W. Stroock, *Lectures on topics in stochastic differential equations*, Lectures given at Tata Institute of Fundamental Research, Bombay, 1982, Springer-Verlag, Berlin, Heidelberg, New York, 1982.
52. ———, *Lectures on stochastic analysis: diffusion theory*, Cambridge Univ. Press, Cambridge and New York, 1987.
53. D. W. Stroock and Varadhhan, *Multidimensional diffusion processes*, Springer-Verlag, Berlin, Heidelberg, New York, 1979.
54. N. Th. Varopoulos, *Small time Gaussian estimates of heat diffusion kernels*. I: *The semigroup technique*, Bull. Sci. Math. (2) **113** (1989), 253–277.
55. ———, *Small time Gaussian estimates of heat diffusion kernels*. II: *The theory of large deviations*, J. Funct. Anal. **93** (1990), 1–33.
56. Marc Yor, *Grossissement de filtrations et absolue continuité de noyaux*, Lecture Notes in Math, vol. 1118 (Th. Jeulin and M. Yor, eds.), Springer-Verlag, Berlin and New York, 1985.

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