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THE RELATIVE BURNSIDE MODULE AND THE STABLE MAPS BETWEEN CLASSIFYING SPACES OF COMPACT LIE GROUPS

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Dedicated to Professor Hirosi Toda

ABSTRACT. Tom Dieck's Burnside ring of compact Lie groups is generalized to the relative case: For any $G \rhd N$, a compact Lie group and its normal subgroup $A(G \rhd N)$ is defined to be an appropriate set of the equivalence classes of compact G-ENR's with free N-action, in such a way that $\psi : A(G \rhd N) \simeq \pi^0_{G/N}(S^0 ; B(N, G)_+)$, where $B(N, G)$ is the classifying space of principal (N, G)-bundle. Under the "product" situation, i.e. $G = F \times K$, $N = K$, $A(F \times K \rhd K)$ is also denoted by $A(F, K)$, as it turns out to be the usual $A(F, K)$ when both F and K are finite. Then a couple of applications are given to the study of stable maps between classifying spaces of compact Lie groups: a conceptual proof of Feshbach's double coset formula, and a density theorem on the map $\alpha^\wedge_p : A(L, H)^\wedge_p \to \{BL_+, BH_+\}^\wedge_p$ for any compact Lie groups L, K when p is odd. (Some restriction is applied to L when $p = 2$.) This latter result may be regarded as the pushout of Feshbach's density theorem and the theorem of May–Snaith–Zelewski, over the celebrated Carlsson solution of Segal's Burnside ring conjecture.

0. INTRODUCTION

Even before the arrival of Carlsson's celebrated paper [C], the Segal conjecture for finite groups was known to imply the complete calculation of the stable maps between classifying spaces of finite groups [AGM], [LMM].

More recently, Snaith [S2] got interested in the stable maps between classifying spaces of compact Lie groups, while studying his "Explicit Brauer Induction"problem (see [S2], which is a nice reference on this topic). Snaith's research led to the further generalization of the Segal conjecture by [MSZ]:

Theorem A [MSZ]. *Let F and J be finite groups and let K be a compact Lie group. Then α induces the isomorphism*

$$\alpha : A(F \times J, K)^\wedge_{I(J; F\times J)} \to \{B_F J_+, B_F K_+\}_F,$$

where $A(F \times J, K)$ is the Grothendieck $A(F \times J)$-module of principal $(F \times J, K)$-bundles over finite $(F \times J)$-sets, and $(?)^\wedge_{I(J; F\times J)}$ is the completion of

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$A(F \times J)$-module (?) at the ideal $I(J; F \times J)$ of elements of $A(F \times J)$ which restrict trivially to $A(H)$ for all $H \subset F \times J$ such that $H \cap J = \{e\}$.

For the notation of the equivariant bundle, see the beginning of §2. When $F = \{e\}$, this result gives the complete calculation of the stable maps from the classifying space of a finite group to the classifying space of a compact Lie group.

But, as soon as we allow the source to be the classifying space of a general compact Lie group, the situation becomes quite complicated: In the Segal conjecture situation, it has been known that the augmentation ideal does not work [M1] in general. The best positive answer was supplied by M. Feshbach [F2]:

Theorem B (Feshbach's density theorem [F2], [B]). *Let G be a compact Lie group. Suppose T is a maximal torus of dimension n and let $W = N_G G/T$. Then the map*

$$\alpha : A(G) \to \pi_s^0(BG_+)$$

has a dense image with respect to the skeletal filtration of the target at all odd prime p (i.e. when p-adically completed). Let $\rho : W \to Gl(n, \mathbb{Z})$ be a representation which gives rise to the action of W on $T \approx \mathbb{R}^n/\mathbb{Z}^n$. Then the Segal conjecture holds at 2 if ρ does not originate at a generalized quaternion group of order 2^n, $n \geq 4$. In particular the conjecture holds if $n < 8$ or if no subquotient of W is isomorphic to the quaternion group of order 16.

Meanwhile, the surprising paper of S. Bauer [B] found a counterexample to the 2-primary density conjecture (we were told that a preprint version of [F2] suggested a more complicated possible counterexample). Therefore, Feshbach's assumption in Theorem B turned out to be quite essential.

Now the purpose of this paper is to "relativise" tom Dieck's Burnside ring of compact Lie groups and apply it to the study of the stable maps between the classifying spaces of general compact Lie groups:

Let $G \triangleright N$ be a compact Lie group and its normal subgroup. Then

$$A(G \triangleright N)$$

is defined to be an appropriate set of the equivalence classes of compact G-ENR's with free N-action, in such a way that

(1) $A(G \triangleright \{e\}) = A(G)$, tom Dieck's Burnside ring of compact Lie group G [D].

(2) In the "product" case, i.e. $G = L \times K$, $N = K$, then $A(L \times K \triangleright K)$ is also denoted by $A(L, K)$. This is because, when F is finite, $A(F \times K \triangleright K)$ turns out to be the Grothendieck $A(F)$-module of principal (F, K)-bundles over finite F-sets, which was denoted by $A(F, K)$ in [MSZ].

(3)
$$\psi : A(G \triangleright N) \simeq \pi^0_{G/N}(S^0; B(N, G)_+),$$
where $B(N, G)$ is the classifying space of principal (N, G)-bundle (see §2 for the definition).

We have a couple of applications of the relative Burnside modules to the study of the stable maps between classifying spaces of compact Lie groups. The first is the compatibility of the pairings:

This result immediately gives a conceptual proof of Feshbach's double coset formula [F1] and explains the mysterious similarity between Feshbach's double coset formula and the product formula of tom Dieck's Burnside ring of compact Lie groups. Our second application concerns the density problem of the stable maps between the (nonequivariant) classifying spaces of compact Lie groups. In this case, as the skeletal filtration does not give a compact topology in general, the right question to ask is: *When L and K are compact Lie groups, does the composite*

$$A(L, H) \xrightarrow{\alpha} \{BL_+, BH_+\} \to \varprojlim_n \{BL_+^{(n)}, BH_+\}^\wedge_{finite}$$

have a dense image? Here $BL^{(n)}$ is the n-skeleton of BL and $A^\wedge_{finite} = \varprojlim_{|A/B|<\infty} A/B$ is the profinite completion of an abelian group A. Now such a problem can be easily reduced to the p-primary cases (see Appendix Local-Global), and our main result states

Theorem 3.9. *Let L and H be compact Lie groups. Then*

$$\alpha_p^\wedge : A(L, H)_p^\wedge \to \{BL_+, BH_+\}_p^\wedge$$

has a dense image with respect to the skeletal filtration of the target, if L satisfies the assumption of the Feshbach density theorem, i.e. if either one of the following two conditions holds:

(1) p is odd.

(2) $p = 2$ and $\rho : \Pi \to Gl(n, \mathbb{Z})$, given by the Weyl group action on the maximal torus, does not originate at a generalized quaternion group of order 2^n, $n \geq 4$. More generally, if all the irreducible $\mathbb{Q}\Pi'$-summands in the Π'-action on $T^* \otimes \mathbb{Q}$ remain irreducible after the 2-adic completion for any 2-subgroup $\Pi' \subset W$. In particular, this condition is satisfied, if $n < 8$ or if no subquotient of W is isomorphic to the quaternion group of order 16. Here Π is the 2-Sylow subgroup of the Weyl group of L and T is a maximal torus of L.

Of course, this result can be thought of as a pushout of the aforementioned two theorems of [MSZ] and [F2]. In fact, our proof uses [MSZ] and generalizes the basic idea of [B] (which simplified Feshbach's proof [F2]). But, it should be pointed out that the full generality of the relative Burnside module, i.e. that of non "product"case, was needed to prove Theorem 3.10, which is a "product"case result.

Now, as is shown in the Appendix, Carlsson's affirmative solution of the Segal conjecture [C] and its various generalizations [N], [F2], [MSZ] reduce the density problem of $A(P, K) \to \{BP_+, BK_+\}$ to a problem, which involves only relative Burnside modules of p-toral (i.e. an extension of a torus by a finite p-group) groups:

Reduction. *Let P and K be maximal p-tori of compact Lie groups L and H, respectively. Then the map*

$$\alpha_p^\wedge : A(L, H)_p^\wedge \to \{BL_+, BH_+\}_p^\wedge$$

has a dense image with respect to the skeletal filtration of the target, if

$$A(P, K)_p^\wedge \to \varprojlim_n A(F_n, K)_p^\wedge$$

has a dense image, where finite p-subgroups F_n 's of P form the following commutative diagram:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & T & \longrightarrow & P & \xrightarrow{\pi} & \Pi & \longrightarrow & 1 \\
 & & \cup\uparrow & & \cup\uparrow & & \| & & \\
1 & \longrightarrow & T_\infty \overset{\text{def.}}{=} F_\infty \cap T = (\mathbb{Z}/p^\infty)^r & \longrightarrow & F_\infty \overset{\text{def.}}{=} \varinjlim_n F_n & \longrightarrow & \Pi & \longrightarrow & 1 \\
 & & \cup\uparrow & & \cup\uparrow & & \| & & \\
1 & \longrightarrow & F_n \cap T = (\mathbb{Z}/p^n)^r & \longrightarrow & F_n & \xrightarrow{\pi} & \Pi & \longrightarrow & 1.
\end{array}$$

Here $\pi : P \to \Pi$ is the projection to the finite group of connected components (as P is p-toral Π should be a finite p-group), the left-upper map $(\mathbb{Z}/p^\infty)^r \to T$ is the inclusion of the subset consisting of all p-power order elements, and any two such F_∞ 's are conjugate to each other in P.

This paper is organized as follows:

In §1, we define $A(G \triangleright N)$ and various functors between the relative Burnside modules.

In §2, we begin with the isomorphism

$$\psi : A(L, K) \simeq \pi^0_L(S^0 ; B_L K_+),$$

and then we define

$$\alpha : A(F \times J, K)^\wedge_{I(J ; F \times J)} \to \{B_F J_+, B_F K_+\}_F$$

for general compact Lie groups. Then we state and prove Theorem 2.3, the "universal double coset formula."

In §3, we prove our main theorem by verifying the condition in the aforementioned Reduction. Our proof strongly depends upon our study of the general (not necessarily "product") relative Burnside modules and greatly influenced by Bauer's [B] simplified proof of Feshbach's density theorem [F2].

In Appendix, we prove Reduction. Here we also show how the profinite density problem (which was stated just before Theorem 3.9. in this introduction) is reduced to the p-primary situations.

For more on the related and background material, we refer our survey paper [LM2].

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Notation. Given a map $f : X \to Y$, its image is denoted by either one of

$$\operatorname{Im} X = f(X) = \{ f(x) \mid x \in X \}.$$

$G \triangleright N$ and $N \triangleleft G$ means N is a normal subgroup of G, and

$$N_G K = \{ g \in G \mid g^{-1} K g = K \},$$
$$C_G K = \{ g \in G \mid gk = kg \text{ for any } k \in K \},$$

as usual.

When X is a G-space, then for any $x \in X$ and a subgroup $H \subseteqq G$,

$$G_x = \{g \mid g \cdot x = x\},$$
$$X_H = \{x \mid G_x = H\},$$
$$X_{(H)} = \{x \mid G_x \text{ is conjugate to } H \text{ in } G\},$$
$$X^H = \{x \mid h \cdot x = x \text{ for any } h \in H\}.$$

Furthermore,

$$\{A, B\}_G = \pi_G^0(A, B)$$

stands for the (abelian) group of G stable homotopy classes from a G-spectrum A to a G-spectrum B.

1. $A(G \triangleright N)$ AND ITS PROPERTIES

Let G be a compact Lie group and let N be a normal closed subgroup of G. Then we define $A(G \triangleright N)$ to be the set of equivalence classes of compact G-ENR (for general properties of G-ENR, see [J], [Do1], [D2]) with a free N-action under the equivalence relation

$$X \sim Y \Longleftrightarrow \chi(X^S/N_N S) = \chi(Y^S/N_N S) \qquad \text{for any } S \subseteq G,$$

where $N_N S = N_G S \cap N$ acts freely on X^S and Y^S (recall N acts freely on both X and Y) so that the following diagrams commute:

$$\begin{array}{ccccccc}
N_N S \times X^S & \longrightarrow & X^S & \qquad & N_N S \times Y^S & \longrightarrow & Y^S \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
G \times X & \longrightarrow & X & & G \times Y & \longrightarrow & Y
\end{array}$$

Note that $A(G \triangleright \{e\}) = A(G)$, tom Dieck's Burnside ring of compact Lie group G [D]. We define $C^f(G \triangleright N)$ to be the set of the conjugacy (by elements of G) classes of closed subgroups (H) such that $N_G(HN)/HN \cong N_{G/N}(HN/N)$ is finite and $H \cap N = \{e\}$. In the "product"case, i.e. when $G = L \times K$ and $N = K$, we write $A(L, K) = A(G \triangleright N)$. Now given $H \subseteq L$ a closed subgroup and $\phi : H \to K$ a homomorphism, we put $(H, \phi) = \{ (m, \phi(m)) \in L \times K \mid m \in H \}$, a closed subgroup of $L \times K$. Then we immediately see that $C^f(L \times K \triangleright K) = C^f(L, K)$, the set of the conjugacy (by elements of $L \times K$) classes of those (H, ϕ) such that $(N_L H)/H$ is finite.

We will get a complete description of $A(G \triangleright N)$ (Theorem 1.2). For this purpose we provide an elementary lemma which enables us to deal with Euler characteristic of fixed-point sets and quotient spaces. Here and after, $F \xrightarrow{i} E \xrightarrow{p} B$ is said to be a G-fiber bundle, if it is a locally G-trivial (i.e. locally a product as G-spaces) fiber bundle such that both i and p are G-equivariant.

Lemma 1.1.

(i) *Let* $F \to E \to B$ *is a* G*-fiber bundle such that a closed normal subgroup* $N \triangleleft G$ *acts trivially on* B*, then* $F^N \to E^N \to B$ *is a* G/N*-fiber bundle.*
(ii) *Let* $F \to E \to B$ *is a* G*-fiber bundle such that* G *acts trivially on* B*, then* $F/G \to E/G \to B$ *is a fiber bundle.*

Proof. Both can be checked easily using the local triviality. □

Theorem 1.2. $A(G \triangleright N)$ *is a free abelian group with basis* $[G/H]$ *for each* $(H) \in C^f(G \triangleright N)$. *For any compact* G*-manifolds* X *with free* N*-action,*

$$[X] = \sum_{(H)} \chi_c(X_{(H)}/G)[G/H] \in A(G \triangleright N),$$

where (H) *runs over* $C^f(G \triangleright N)$.

Proof. (See 5.4.4 of [D].) The addition is given by the disjoint union, and the inverse is given by

$$-[X] = [X \times K],$$

where K is a compact G-ENR with trivial action such that $\chi(K) = -1$. (Note that $X \times K$ is a free N space since X is so.) We want to express any $[X]$, an element of $A(G \triangleright N)$, by elements of the form $[G/H]$ with $(H) \in C^f(G \triangleright N)$. We first note that any isotropy type of X is of the desired form (H) such that $H \cap N = \{e\}$. So by the additivity of the Euler-characteristic, we have

$$\chi(X^S/N_N S) = \sum_{(H) \text{ such that } H \cap N = \{e\}} \chi_c(X^S_{(H)}/N_N S),$$

where χ_c is the Euler characteristic with respect to the homology with compact support. Now recall the fiber bundle:

$$G/H \to X_{(H)} \to X_{(H)}/G.$$

We apply Lemma 1.1 in two ways: firstly Lemma 1.1(i) for $S \triangleleft N_G S \subseteq G$ to the above bundle; secondly Lemma 1.1(ii) for $N_N S \subseteq N_G S$ to the bundle obtained by the first step. Then, we have the following fiber bundle:

$$(G/H)^S/N_N S \to X^S_{(H)}/N_N S \to X_{(H)}/G.$$

From this, we get

$$\chi_c(X^S_{(H)}/N_N S) = \chi\left((G/H)^S/N_N S\right)\chi_c(X_{(H)}/G).$$

Thus, in $A(G \triangleright N)$

$$[X] = \sum_{(H) \text{ such that } H \cap N = \{e\}} \chi_c(X_{(H)}/G)[G/H],$$

where the summation is over finite terms because (a compact) G-ENR has only finitely many orbit types [J]. Now we should prove a couple of claims: (a) $[G/H] = 0$ if $N_G(HN)/HN$ is not finite; (b) $[G/H]$'s with $(N_G HN)/HN$ finite are linearly independent.

For (a), it is sufficient to show $\chi\left((G/H)^S/N_NS\right) = 0$ for any closed subgroup $S \subseteq G$. For this purpose, we study isotropy subgroups of $(G/H)^S/N_NS$, regarded as a compact N_GH-ENR under the usual action:

$$(N_GH) \times \left\{(G/H)^S/N_NS\right\} \to (G/H)^S/N_NS,$$
$$(n, N_NSgH) \mapsto N_NSgn^{-1}H.$$

If $n \in N_GH$ is contained in the isotropy group at N_NSgH, we get

$$\begin{aligned} &N_NSgH = N_NSgn^{-1}H \\ &\iff (g^{-1}N_NSg)H = (g^{-1}N_NSg)n^{-1}H \\ &\iff n^{-1} \in (g^{-1}N_NSg)(H). \end{aligned}$$

Since $n \in N_GH$, this implies that the isotropy group at N_NSgH is

$$(g^{-1}N_NSg)(H) \cap N_GH.$$

Therefore, from the additivity of the Euler characteristic, we only have to show

$$\chi(N_GH/(g^{-1}N_NSg)H \cap N_GH) = 0,$$

for any $g \in G$. Actually, as $(g^{-1}N_NSg)H \cap N_GH \subset HN \cap N_GH \lhd N_GH$, it is sufficient to show that the compact Lie group $N_GH/NH \cap N_GH$ is not finite (then there would be a free S^1 action on $N_GH/(g^{-1}N_NSg)H \cap N_GH$). To see this, look at the fiber bundle

$$N_GH/HN \cap N_GH \to N_G(HN)/HN \to N_G(HN)/(N_GH)(HN).$$

(Note: This is a fiber bundle because $(N_GH)(HN)$ is a subgroup and $N_GH/HN \cap N_GH \cong (N_GH)(HN)/HN$, for $N_GH \subset N_G(HN)$.) Then we immediately find out that the base space $N_G(HN)/(N_GH)(HN)$ is finite since $N_GH \supset C_G(HN)$. On the other hand, the total space $(N_GH)(HN)/HN$ is not finite from the assumption. Therefore the fiber $N_GH/NH \cap N_GH$ is not finite, as required.

For (b), we first note that

$$\begin{aligned} (G/H)^H/N_NH &= N_NH\backslash N_GH/H \\ &\cong N_GH/H(N_NH) = N_GH/HN \cap N_GH \end{aligned}$$

is finite when $(N_GHN)/HN$ is finite, by the preceding argument for (a). This, in particular, implies that $\chi\left((G/H)^H/N_NH\right) \neq 0$. Now suppose $[G/H]$'s, with N_GH/HN finite, are linearly dependent and we have a nontrivial linear relation

$$0 = \sum_{(H)\in C^f(G\rhd N)} a_{(H)}[G/H] \in A(G \rhd N).$$

Then, by taking (H') to be maximal among those such that $a_{(H)} \neq 0$, we get

$$\begin{aligned} 0 &= \chi\left((\textstyle\sum a_{(H)}[G/H])^{H'}/N_NH'\right) \\ &= a_{(H')}\chi\left(G/H')^{H'}/N_NH'\right). \end{aligned}$$

However, this is a contradiction, since the last term is nonzero since $a_{(H')} \neq 0$ and the above remark. This is a contradiction. □

Remark 1.3. For each $(S) \in C^f(G \triangleright N)$, we define the homomorphism

$$\chi^S : A(G \triangleright N) \to \mathbb{Z}$$

by

$$[X] \mapsto \chi(X^S/N_N S).$$

Then these completely characterize $A(G \triangleright N)$:

$$A(G \triangleright N) \xrightarrow{\prod_{(S)\in C^f(G\triangleright N)} \chi^S} \prod_{S\in C^f(G\triangleright N)} \mathbb{Z}$$

is injective. (Of course, such invariants were used to define $A(G \triangleright N)$, but here we restrict our attention to those $(S) \in C^f(G \triangleright N)$.) In fact, for any $\sum_{(H_i)\in C^f(G\triangleright N)} u_i[G/H_i] \in A(G \triangleright N)$, take H_j to be maximal among those such that $u_i \neq o$. Then

$$\chi^{H_j}\left(\sum u_i[G/H_i]\right) = u_j\left|N_G H_j/H_j N \cap N_G H_j\right| \neq 0.$$

Of course, when $N = \{e\}$, this is well known [tD].

Next we define various transformations:

Definition 1.4. (1) Let $G \supset H \supset N$ be a chain of compact Lie groups such that $G \triangleright N$. Then define

$$\mathrm{Res}_H^G : A(G \triangleright N) \to A(H \triangleright N)$$

by restricting the G action to the H action. When $G = L \times K$, $H = P \times K$, $N = K$, this is also denoted by

$$\mathrm{Res}_P^L : A(L, K) \to A(P, K)$$

for simplicity.

(2) Let N_1 and N_2 be closed normal subgroups of G such that $N_1 \cap N_2 = \{e\}$. Then the canonical quotient map $\pi : G \to G/N_1$ restricts to the isomorphism $\pi\,|_{N_2}: N_2 \to N_2/N_2 \cap N_1 \cong N_2N_1/N_1$. We define

$$\pi^* : A(G/N_1 \triangleright N_2N_1/N_1) \to A(G \triangleright N_2)$$

by regarding a compact G/N_1-ENR with a free N_2N_1/N_1 action X as a compact G-ENR with a free N_2 action through π.

(3) Let N_1 and N_2 be closed normal subgroup of a compact Lie group G such that $N_1 \supset N_2$. Then define

$$/N_2 : A(G \triangleright N_1) \to A(G/N_2 \triangleright N_1/N_2)$$

by $/N_2([C]) = [X/N_2]$.

(4) Let $G \triangleright N$ be a compact Lie group and its closed normal subgroup. Let H be a closed subgroup of G. Then define

$$\mathrm{Ind}_H^G : A(H \triangleright H \cap N) \to A(G \triangleright N)$$

by $\mathrm{Ind}_H^G([X]) = [G \times_H X]$. When $G = L \times K$, $H = P \times K$, $N = K$, this is also denoted by

$$\mathrm{Ind}_P^L : A(P, K) \to A(L, K)$$

for simplicity.

(5) Let N be a closed normal subgroup of G, which is a closed subgroup of the product of compact Lie groups $G_1 \times G_2$. In other words, $N \triangleleft G \subset G_1 \times G_2$.

Assume furthermore that $\operatorname{Im} p_1|_N \subseteq N_1$, $\operatorname{Ker} p_1|_N \subsetneqq N_2$, where $p_1 : G_1 \times G_2 \to G_1$ is the projection. Then we define the external product

$$\mathscr{E} : A(G_1 \rhd N_1) \otimes A(G_2 \rhd N_2) \to A(G \rhd N)$$

by $\mathscr{E}([X_1] \otimes [X_2]) = [X_1 \times X_2]$.

(6) Let F, K_1, K_2, K_3 be compact Lie groups. Then define the composition product

$$\mathscr{C} : A(F \times K_1, K_2) \otimes A(F \times K_2, K_3) \to A(F \times K_1, K_3)$$

by $\mathscr{C}([X_1] \times [X_2]) = [X_1 \times_{K_2} X_2]$.

Of course, we have to show the well-definedness of these transformations.

(1) *Well-definedness of* Res^G_H. This is trivial from our definition. □

(2) *Well-definedness of* π^*. For this, we use Theorem 1.2: We show for any $(H) \in C^f(G \rhd N_2)$ and a compact G/N_1-ENR with a free N_2N_1/N_1 action X, when regarded as a compact G-ENR, $\chi(X_{(H)}/G)$ is completely determined by $[X] \in A(G/N_1 \rhd N_2N_1/N_1)$. But this is trivial, because any isotropy subgroup H at an arbitrary point in X contains N_1 and $X_{(H)}/G = X_{(H/N_1)}/(G/N_1)$ whenever the left side is nonempty. □

(3) *Well-definedness of* $/N_2$. Just as in the proof of the well-definedness of (2), this follows from Theorem 1.2 and an observation that

$$(X/N_2)_{(\overline{H'})} / (G/N_2) = \coprod_{(H')\in C_f(G\rhd N_1)\ s.t.\ (H'N_2/N_2)=(\overline{H})\in C_f(G/N_2\rhd N_1/N_2)} X_{(H')}/G,$$

where X is a compact G-ENR with a free N_1-action (so the right-hand side is a finite disjoint union). □

To show the well-definedness of Ind^G_H, we need the following lemma:

Lemma 1.5. *Let G be a compact Lie group and let H and K be its closed subgroups. Then there is an isomorphism of finite sets:*

$$(G/H)^K/N_GK \cong i^{-1}((K)), \quad \text{with } \overline{gH} \mapsto (K^g) \in C(H),$$

where $i : C(H) \to C(G)$ is the map between the set of the conjugacy classes of H and G, induced by the inclusion $H \subseteq G$.

Proof. $(G/H)^K/N_GK$ is always finite by p. 87 of [Br]. The isomorphism follows from

$$\begin{aligned}
&(K^{g_1}) = (K^{g_2}) \in C(H) \\
&\iff K^{g_1} = K^{g_2h} \text{ for some } h \in H \\
&\iff N_GKg_1 = N_GKg_2h \text{ for some } h \in H \text{ and } g_1H \in (G/H)^K \\
&\iff \overline{g_1H} = \overline{g_2H} \in (G/H)^K/N_GK,
\end{aligned}$$

where the second $\iff$ follows from a sequence of equivalences:

$$K^g \subset H \Leftrightarrow KgH = gH \Leftrightarrow gH \in (G/H)^K. \quad □$$

(4) *Well-definedness of* Ind^G_H. Let X be a compact H-ENR with a free $H \cap N$ action. Then it is easy to see that

$$(G \times_H X)_{(K)} = \coprod_{(K')\in i^{-1}((K))} G \times_H (X_{(K')})$$

whenever the left side is nonempty, where $i : C(H) \to C(G)$ is the map between the sets of conjugacy classes induced by the inclusion $H \subseteq G$. From this, we first observe that $G \times_H X$ restricts to a free N action if X has a free $H \cap N$ action (since it would imply only those K' such that $\{e\} = K' \cap H \cap N = K' \cap N$ show up). Next, we use this to show that $\chi_c\left((G \times_H X)_{(K)}/G\right)$, with $(K) \in C^f(G \triangleright N)$, is completely determined by $[X] \in A(H \triangleright H \cap N)$; this would prove the well-definedness by Theorem 1.2. But, notice that

$$\begin{aligned}\chi_c\left((G \times_H X)_{(K)}/G\right) &= \sum_{(K')\in i^{-1}((K))} \chi_c\left((G \times_H (X_{(K')}))/G\right)\\ &= \sum_{(K')\in i^{-1}((K))} \chi_c\left(X_{(K')}/H\right),\end{aligned}$$

where $(K') \in C^f(H \triangleright H \cap N)$ because $(K) \in C^f(G \triangleright N)$ and $N_H(K'H)/K'N \subseteqq N_G(K'H)/K'N \subseteqq N_G(KN)/KN$, and we used Lemma 1.5 to guarantee that the sum is a finite sum (or, we could have used the fact that $G \times_H X$ is a compact G-ENR when X is a compact H-ENR). Those $\chi_c\left(X_{(K')}/H\right)$ with $(K') \in C^f(H \triangleright H \cap N)$ can be read off from the basis expression of $[X] \in A(H \triangleright H \cap N))$. This proves the well-definedness. □

(5) *Well-definedness of* $\mathscr{E}$. From the assumption on N, we have a short exact sequence

$$1 \to N_{N\cap N_2}S \to N_N S \to p_1(N_N S) \to 1$$

for any closed subgroup $S \subset G$. Then, as N acts freely on $X_1 \times X_2$ for a compact G_i-ENR X_i with a free N_i action ($i = 1, 2$), we get a fiber bundle

$$X_2^{p_2(S)}/N_{N\cap N_2}S \to X_1^{p_1(S)} \times X_2^{p_2(S)}/N_N S \to X_1^{p_1(S)}/p_1(N_N S).$$

From this, we get

$$\begin{aligned}\chi\left((X_1 \times X_2)^S/N_N S\right) &= \chi\left(X_1^{p_1(S)}/p_1(N_N S)\right)\chi\left(X_2^{p_2(S)}/N_{N\cap N_2}S\right)\\ &= \chi\left(X_1^{p_1(S)}/N_{N_1}p_1(S)\right)\cdot\chi\left(N_{N_1}p_1(S)/p_1(N_N S)\right)\\ &\quad\cdot\chi\left(X_2^{p_2(S)}/N_{N_2}p_2(S)\right)\cdot\chi\left(N_{N_2}p_2(S)/N_{N\cap N_2}S\right),\end{aligned}$$

which exhibits the well-definedness of $\mathscr{E}$ by the definition of the relative Burnside module. □

(6) *Well-definedness of* $\mathscr{C}$. Let $\triangle F \subset F \times F$ and $\triangle K_2 \subset K_2 \times K_2$ be the diagonal subgroups. Then the claim follows from an observation that $\mathscr{C}$ factorize as

$$\begin{aligned}&A(F \times K_1, K_2) \otimes A(F \times K_2, K_3) \xrightarrow{\mathscr{E}} A(\triangle F \times K_1, \triangle K_2 \times K_3)\\ &\xrightarrow{/\triangle K_2} A(\triangle F \times K_1, K_3) = A(F \times K_1, K_3).\end{aligned}$$ □

To conclude this section, we record two multiplicative properties of the relative Burnside modules, both of which generalize the multiplicative structure of tom Dieck's Burnside ring.

Theorem 1.6. (1) *$A(G \rhd N)$ is an $A(G/N)$-module, with respect to the action*

$$\mathscr{A} : A(G/N) \otimes A(G \rhd N) \to A(G \rhd N)$$
$$[Y] \otimes [X] \mapsto [Y \times X]$$

where Y is a compact G/N-ENR and X is a compact G-ENR with a free N-action.

(2) *When A is abelian, $A(L, A)$ becomes an $A(L)$-algebra, with respect to the product*

$$\mathscr{P} : A(L, A) \otimes A(L, A) \to A(L, A)$$

by $\mathscr{P}([X] \otimes [Y]) = [X \times Y/ \sim]$, where $(x_1, y_1) \sim (x_2, y_2) \iff \exists a \in A$ s.t. $(x_1, y_1) = (x_2 \cdot a, y_2 \cdot a^{-1})$.

Proof. (1) We only have to check the well-definedness of $\mathscr{A}$. But $\mathscr{A}$ is a special case of $\mathscr{E}$ by embedding $G \subset G \times G \rhd N$ diagonally. Another approach is to notice

$$\chi\left((Y \times X)^S / N_N S\right) = \chi(Y^{SN/N})\chi\left(X^S/N_N S\right)$$

for any closed subgroup $S \subset G$.

(2) Let A' be a left $A \times A$ and free right A action s.t. $(a_1 \times a_2) \cdot a' \cdot a = a_1 a_2 a' a$. Notice that this is a well-defined action because A is abelian. Then the well-definedness of $\mathscr{P}$ follows from the factorization of $\mathscr{P}$ as

$$A(L, A) \otimes A(L, A) \xrightarrow{\mathscr{E}} A(\triangle L, A \times A) \xrightarrow{\mathscr{E}(-\otimes[A'])} A(\triangle L, A),$$

where the second map is the composition with $[A'] \in A(A \times A, A)$. Then the associativity of $\mathscr{P}$ follows from that of A, and the identity element is given by $[A]$, with the trivial left L and the product (free) right A action. □

2. $A(G \rhd N)$ AND THE EQUIVARIANT STABLE HOMOTOPY THEORY

Let $G \rhd N$ be a compact Lie group and its normal subgroup as usual. We now recall the classifying space of the equivariant bundles. The general reference would be [LM][Ma].

Let $E(N, G)$ be a universal N-free G space; the G-homotopy type of $E(N, G)$ is characterized by the requirement that $(E(N, G))^H$ be contractible if $H \cap N = \{e\}$ and empty otherwise. We then put $B(N, G) = (E(N, G))/N$, which is a classifying space of principal (N, G)-bundle (which is defined to be the projection of a N-free G-space to its N-orbits). In the "product"case, i.e. when $G = L \times K$ and $N = K$, we write $E_L K = E(N, G)$ and $B_L K = B(N, G)$. We warn the reader that this notation is different from the more standard notation of the so-called (Γ, G)-bundle, i.e. Γ-equivariant G-principal bundle [D2]. This corresponds to the product case in our setting, and $B(\Gamma, G) = B_\Gamma G$, where $B(\Gamma, G)$ is the notation used in [D2] and not the one used here.

When A is abelian, $B_L A$ becomes of Hopf-L space with respect to the L-map

$$B_L(m) : B_L A \times B_L A \cong B_L(A \times A) \to B_L A,$$

where the product map $m : A \times A \to A$ is a homomorphism because A is abelian (see [LMSe]).

For a given compact G-manifold X with a free N-action, we have a G/N-fiber bundle $X/N \to \text{point}$, from which we get the transfer stable G/N-map

$$S^0 \to X/N_+.$$

On the other hand, the principal (N, G)-bundle $X \to X/N$ is classified by a G/N-map

$$X/N \to B(N, G).$$

Composing these, we get the stable G/N-map

$$S^0 \to B(N, G)_+.$$

We call this $\psi(X)$. We could construct $\phi(X)$ for a general compact G-ENR with a free N-action X, as was shown in [D2], [Do2], [Do3].

Proposition 2.1. *$\psi(X) \in \pi^0_{G/N}(S^0; B(N, G)_+)$ only depends upon $[X] \in A(G \triangleright N)$, and ϕ induces the $A(G/N)(\cong \pi^0_{G/N}(S^0, S^0))$-module isomorphism:*

$$\psi : A(G \triangleright N) \cong \pi^0_{G/N}(S^0; B(N, G)_+).$$

In particular, we have the $A(L)(\cong \pi^0_L(S^0, S^0))$-module isomorphism

$$\psi : A(L, K) \cong \pi^0_L(S^0; B_L K_+)$$

for the "product"case. Furthermore, when K is abelian, this is an isomorphism of $A(L)$-algebras, where the multiplicative structure of the target is induced from the Hopf-L structure of $B_L K$.

Proof. This follows from Theorem 1.2, Theorem 1.6, V.9.3 of [LMS], and Theorem 10 of [LMM]. □

Now we work with the "product"case and let $L = F \times J$, where F is a finite group. For a given compact $(F \times J) \times K$-manifold X with a free K-action, we can define another kind of a stable map. From the F-fiber bundle

$$(E_F J \times_J X)/K \simeq E_F J \times_J (X/K) \to E_F J/J \simeq B_F J,$$

we obtain the transfer stable F-map [LMS, IV§3]

$$B_F J_+ \to (E_F J \times_J X)/K_+.$$

On the other hand, the principal $(K, K \times F)$-bundle

$$E_F J \times_J X \to (E_F J \times_J X)/K$$

is classified by the classifying F-map

$$(E_F J \times_J X)/K \to B_F K.$$

Now, composing the transfer and the classifying map, we get the F-map

$$\alpha(X) : B_F J_+ \to B_F K_+.$$

The significance of α and the relationship between α and ψ are given by the following interpretation:

Theorem 2.2. *The above correspondence factors through* $A(F \times J, K)$, *and the induced correspondence*

$$\alpha : A(F \times J, K) \to \{B_F J_+, B_F K_+\}_F$$

can be thought as the composition of $\psi : A(F \times J, K) \simeq \pi^0_{F \times J}(S^0; B_{F\times J}K_+)$ *and the* 0*th term of the following composition*

$$\pi^*_{F\times J}(S^0; B_{F\times J}K_+) \to \pi^*_{F\times J}(E_F J_+; B_{F\times J}K_+) \\ \to \pi^*_{F\times J}(E_F J_+; B_F K_+) \to \pi^*_F(B_F J_+; B_F K_+).$$

Here the first map is induced by the projection $E_F J \to \text{point}$, *the second map is induced by* $B_{F\times J}K \simeq B_{\mathfrak{G}'}K \to B_{\mathfrak{G}}K \simeq B_F K$, *where* $\mathfrak{G}'$ *is the family of closed subgroups* $M \subseteq F \times J \times K$ *such that* $M \cap K = \{e\}$, $\mathfrak{G}$ *is the family of closed subgroups* $M \subseteq F \times J \times K$ *such that* $MJ \cap KJ = J$, *and* $B_{\mathfrak{G}'} \to B_{\mathfrak{G}}$ *is induced by the inclusion of the families* $\mathfrak{G}' \to \mathfrak{G}$. (*See the discussion after Theorem* 2.3 *of* [MSZ].)

Proof. This follows from the definitions of ψ and α, Proposition 2.1, and the definition of the transfer stable F-map [LMS, IV§3] which used the change of universe [LMS,II] argument. □

This was originally stated in [MSZ] with respect to their definition of $A(F \times J, K)$, i.e. the Grothendieck $A(F\times J)$-module of principal $(F \times J, K)$-bundle over finite $(F \times J)$-sets. But Theorem 1.2 shows their definition is the same as ours and our definition of α is now shown to generalize that of [MSZ]. In particular, Theorem A in §0, the main result of [MSZ], is equally valid with respect to our definitions of $A(F \times J, K)$ and α.

Theorem 2.3. *The paring* $\mathscr{C}$ *is compatible with the composition of classifying spaces, i.e. we have the following commutative diagram:*

$$\begin{array}{ccc} A(F\times K_1, K_2) \otimes A(F\times K_2, K_3) & \xrightarrow{\mathscr{C}} & A(F\times K_1, K_3) \\ \downarrow{\scriptstyle \alpha\otimes\alpha} & & \downarrow{\scriptstyle \alpha} \\ \{B_F K_{1+}, B_F K_{2+}\}_F \otimes \{B_F K_{2+}, B_F K_{3+}\}_F & \longrightarrow & \{B_F K_{1+}, B_F K_{3+}\}_F. \end{array}$$

Proof. This compatibility of $\mathscr{C}$ with the composition follows from the following $F \times K_1$-diagram (with the pullback square):

$$\begin{array}{ccccc} E_F K_1 \underset{K_1}{\times} \left((X \underset{K_2}{\times} Y)/K_3\right) = \left(E_F K_1 \underset{K_1}{\times} X\right) \underset{K_2}{\times} (Y/K_3) & \to & E_F K_2 \underset{K_2}{\times} (Y/K_3) & \to & B_F K_3 \\ \downarrow & & \downarrow & & \\ E_F K_1 \underset{K_1}{\times} (X/K_2) = (E_F K_1 \underset{K_1}{\times} X)/K_2 & \to & B_F K_2 & & \\ \downarrow & & & & \\ B_F K_1. & & & & \end{array}$$

□

Suppose H and K are closed subgroups of G; let $\xi(H, G) : B_F H_+ \to B_F G_+$ be the F-map induced by the inclusion $H \to G$, and $\tau(K, G) : B_F G_+ \to$

B_FK_+ be the stable F-transfer induced by the inclusion $K \to G$. Then Theorem 2.3 conceptually recovers Feshbach's double coset formula (whose equivariant generalization is due to [LMS]), which expresses the composite

$$\tau(K, G)\xi(H, G) : B_FH_+ \longrightarrow B_FG_+ \longrightarrow B_FK_+ .$$

Corollary 2.4 ([F1], [LMS]).

$$\tau(K, G)\xi(H, G) = \sum_{((H\cap K^g, c_{g^{-1}}))} \chi_c(G_{((H\cap K^g, c_{g^{-1}}))}/(H \times K))c_{g^{-1}}\xi(H\cap K^g, K^g)\tau(H\cap K^g, H)$$

where the sum corresponds to the following decomposition:

$$[G] = \sum_{((H\cap K^g, c_{g^{-1}}))} \chi_c(G_{((H\cap K^g, c_{g^{-1}}))}/(H \times K))[(H \times K)/(H\cap K^g, c_{g^{-1}})] \in A(H, K),$$

where G is regarded as a compact $H \times K$-ENR with a free K-action, by its left H right free K-action.

Proof. It is clear that $\xi(H, G) = \alpha([G])$, where this G is considered as a compact $F \times H \times G$-ENR with a free G-action, by the trivial F-action, the standard left multiplication H-action, and the standard right multiplication free G-action. Similarly, $\tau(K, G) = \alpha([G])$, where this G is considered as a compact $F \times G \times K$-ENR with a free K-action, by the trivial F-action, the standard left multiplication G-action, and the standard right multiplication free K-action.

Then, from Theorem 2. 3 with $K_1 = H$, $K_2 = G$, $K_3 = K$, the composition $\tau(K, G)\xi(H, G)$ is given by $\alpha([G \times_G G])$, where $G \times_G G$ is regarded as a compact $H \times K$ ENR with a free K-action, by the aforementioned left H right G action on the left factor G and the left G right K action on the right factor G. But, this is obviously the same as G with the standard left H right K action, as a compact $H \times K$ ENR with a free K-action. Therefore, by Theorem 1.2 ,

$$\tau(K, G)\xi(H, G) = \sum_{((H\cap K^g, c_{g^{-1}}))} \chi_c\left(G_{((H\cap K^g, c_{g^{-1}}))}/(H \times K)\right) \alpha\left([(H \times K)/(H\cap K^g, c_{g^{-1}})]\right).$$

On the other hand, $\alpha\left([(H \times K)/(H\cap K^g, c_{g^{-1}})]\right)$ is easily seen to correspond to the composite of the stable F-maps: $B_FH_+ \to B_F(H\cap K^g)_+ \to B_FK_+$, where the first map is the stable F-transfer $\tau(H, H\cap K^g)$ and the second map is induced by $c_{g^{-1}} : H\cap K^g \to K$. Of course, this completes the proof. □

3. The density of $\alpha : A(L, H) \to \{BL_+, BH_+\}$

In this section, we prove our main theorem. As was explained in the introduction (see Reduction), we only have to show the density of the map between the relative Burnside modules:

$$A(P, K)^\wedge_p \to \varprojlim A(F_n, K)^\wedge_p.$$

We prove this by "relativising" S. Bauer's approach to Feshbach's density theorem as follows:

(I) Construct a subgroup $A_f(F_\infty, K)$ of $\varprojlim A(F_n, K)$ generated by elemnts $[(F_\infty \times K)/(H_\infty, \phi)]$ constructed from each subgroup H_∞ of F_∞ $(= \bigcup_n F_n)$ and a (not necessarily continuous) group homomorphism $\phi : H_\infty \to K$.

(II) Show that $A_f(F_\infty, K)$ is dense in $\varprojlim A(F_n, K)^\wedge_p$.

(III) Show, under our assumption on P, that $[(F_\infty \times K)/(H_\infty, \phi)]$ is "replacable" by elements of the form $\varprojlim \mathrm{Res}^P_{F_n}([(P \times K)/(V, \psi)])$, where V is a closed subgroup of P.

Step (I): The definitions of $[(F_\infty \times K)/(H_\infty, \phi)]$ and $A_f(F_\infty, K)$. Let F_n be a p-group for each n, and $F_\infty = \cup_n F_n$ as before. Here we do not necessarily assume that F_∞ is a dense subgroup of a p-toral subgroup, but we assume that there is a surjective homomorphism $\pi_{F_\infty} : F_\infty \to \Pi$, where Π is a finite p-group such that $\mathrm{Ker}\, \pi_{F_\infty}$ is abelian. We further assume K is p-toral and let $\pi_K : K \to \Xi$ be the canonical quotient homomorphism by the normal torus. Then for any subgroup $H_\infty \subset F_\infty$, the element $[(F_\infty \times K)/(H_\infty, \phi)] = \prod_n (x_{F_n}) \in \varprojlim_n A(F_n, K)$ is defined by

$$x_{F_n} = \lim_{l \to \infty} \left(\mathrm{Res}^{F_l}_{F_n} [(F_l \times K)/(F_l \cap H_\infty, \phi|_{F_l \cap H_\infty})] \right) \in A(F_n, K),$$

where the limit is taken in the p-adic completion $A(F_n, K)^\wedge_p$ (but, still $x_{F_n} \in A(F_n, K)$). Then $A_f(F_\infty, K)$ is defined to be the additive subgroup of $\varprojlim_n A(F_n, K)$ generated by all such elements $[(F_\infty \times K)/(H_\infty, \phi)]$. Of course, we have to justify the definition:

Proposition 3.1. (1) *The above element*

$$[(F_\infty \times K)/(H_\infty, \phi)] \in \varprojlim_n A(F_n, K)$$

is well-defined.

(2) *Suppose F_∞ is a dense subgroup of a p-toral P such that $\pi_{F_\infty} : F_\infty \to \Pi$ is the restriction of the canonical quotient (by the maximal torus) map $\pi_P : P \to \Pi$. Suppose further that there is a closed subgroup H of P and a continuous homomorphism $\psi : H \to K$, such that $H \cap F_\infty = H_\infty$ is dense in H and $\psi|_{H_\infty} = \phi$. Then*

$$[(F_\infty \times K)/(H_\infty, \phi)] = \varprojlim_n \mathrm{Res}^P_{F_n} ([(P \times K)/(H, \psi)])$$

We begin with the key lemma:

Key Lemma. *Suppose H, K, and N are closed subgroups of a compact Lie groups G, such that $N \triangleleft NH = G$ and $K \subset H$. Then we have an isomorphism of compact K manifolds:*

$$N/H \cap N \longrightarrow G/H,$$
$$n(H \cap N) \mapsto nH,$$

where $N/H \cap N$ possesses the conjugate K action and G/H possesses the usual left K action:

$$K \times N/H \cap N \longrightarrow N/H \cap N, \qquad K \times G/H \longrightarrow G/H,$$
$$(k, n(H \cap N)) \mapsto knk^{-1}(H \cap N), \qquad (k, gH) \mapsto kgH.$$

Proof. This is just an easy verification. □

This Key Lemma is most effectively used in the following:

Lemma 3.2. (1) *Let $\{\Gamma_n\}_{n\in\mathbb{N}}$ be an increasing sequence of compact Lie groups with a common normal subgroup Π such that Γ_n/Π is a finite p-group for any $n \in \mathbb{N}$. Suppose F is a finite p-subgroup of $\Gamma_\infty = \bigcup_{n\in\mathbb{N}}\Gamma_n$ which intersects trivially with Π. Then for any closed subgroup S of some Γ_n,*

$$\chi\left((\Gamma_n/F)^S/N_\Pi S\right) \in \mathbb{Z}$$

converges to an integer (not merely to a p-adic integer) as n goes to ∞, with respect to the p-adic topology on $\mathbb{Z}$.

(2) *Under the situation of (1), suppose that there is a p-toral Γ, containing Γ_∞, such that $\Gamma \triangleright \Pi$. Suppose furthermore that the inclusion $\Gamma/\Pi \supset \Gamma_\infty/\Pi$ is one of a locally finite p-subgroup in a p-toral, such that any finite p-subgroup of Γ/Π is conjugate to a subgroup of Γ_∞/Π. Then*

$$\chi\left((\Gamma_n/F)^S/N_\Pi S\right) \in \mathbb{Z}$$

actually converges to

$$\chi\left((\Gamma/F)^S/N_\Pi S\right) \in \mathbb{Z}.$$

Proof. (1) Since F is finite, Lemma 1.5 implies that $(\Gamma_\infty/F)^S$ has finitely many, say k, $N_{\Gamma_\infty}S$-orbits. Therefore, for some $\gamma_i \in \Gamma_\infty$, $i = 1, 2, \dots, k$, we can write

$$\begin{aligned}
(\Gamma_\infty/F)^S/N_\Pi S &= \coprod_{1\le i\le k} (N_{\Gamma_\infty}S)\gamma_i F/N_\Pi S \\
&= \coprod_{1\le i\le k} \left(N_{\Gamma_\infty}S/(F^{\gamma_i^{-1}} \cap N_{\Gamma_\infty}S)\right)/N_\Pi S \\
&= \coprod_{1\le i\le k} (N_{\Gamma_\infty}S/N_\Pi S)\,/\left(N_\Pi S(F^{\gamma_i^{-1}} \cap N_{\Gamma_\infty}S)/N_\Pi S\right) \quad (\because N_\Pi S \triangleleft N_{\Gamma_\infty}S) \\
&= \coprod_{1\le i\le k} \varinjlim_n (N_{\Gamma_n}S/N_\Pi S)\,/\left(N_\Pi S(F^{\gamma_i^{-1}} \cap N_{\Gamma_n}S)/N_\Pi S\right).
\end{aligned}$$

Note that

$$N_{\Gamma_\infty}S/N_\Pi S \quad (\subset \Gamma_\infty/\Pi)$$

is a locally finite p-group and $N_\Pi S(F^{\gamma_i^{-1}} \cap N_{\Gamma_\infty}S)/N_\Pi S$ is its subgroup. Therefore, if

$$(N_{\Gamma_\infty}S/N_\Pi S)\,/\left(N_\Pi S(F^{\gamma_i^{-1}} \cap N_{\Gamma_\infty}S)/N_\Pi S\right)$$

is infinite,

$$\varinjlim_n \left|\,(N_{\Gamma_n}S/N_\Pi S)\,/\left(N_\Pi S(F^{\gamma_i^{-1}} \cap N_{\Gamma_n}S)/N_\Pi S\right)\,\right| = 0 \in \mathbb{Z}$$

with respect to the p-adic topology on $\mathbb{Z}$. This implies that

$$\chi\left((\Gamma_n/F)^S/N_\Pi S\right) = \left|(\Gamma_n/F)^S/N_\Pi S\right|$$

converges to the sum of those

$$\left| (N_{\Gamma_\infty}S/N_\Pi S) \,/\, \left(N_\Pi S(F^{\gamma_i^{-1}} \cap N_{\Gamma_\infty}S)/N_\Pi S\right) \right|$$

which are finite. Of course, such a sum is a natural number.

(2) As F is a finite p-group, we may assume that S is also a finite p-group without any loss of generality. We first prove that the natural map

$$(\Gamma_\infty/F)^S/N_{\Gamma_\infty}S \longrightarrow (\Gamma/F)^S/N_\Gamma S$$

is an isomorphism. Applying Lemma 1.5 repeatedly, we see it is enough to prove that $(\Gamma/\Gamma_\infty)^S/N_\Gamma S$ consists of a single element. For this, let T be the maximal torus of the p-toral Γ and $T_\infty = T \cap \Gamma_\infty$. Then the Key Lemma says that

$$(\Gamma/\Gamma_\infty)^S \cong (T/T_\infty)^S,$$

where the S-action on (T/T_∞) is the conjugate action. This right-hand side shows up in the cohomology long exact sequence:

$$0 \longrightarrow (T_\infty)^S \longrightarrow T^S \longrightarrow (T/T_\infty)^S \xrightarrow{\delta} H^1(S, T_\infty) \longrightarrow \cdots,$$

where δ is the connecting homomorphism. However, δ is a null homomorphism, because $(T/T_\infty)^S$ is a $\mathbb{Z}[\frac{1}{p}]$-module and $H^1(S, T_\infty)$ is $|S|$-torsion (recall that S is a finite p-group). Therefore, we find that

$$T^S \longrightarrow (T/T_\infty)^S$$

is an epimorphism. But, this fact and the Key Lemma (and its proof) implies that $(\Gamma/\Gamma_\infty)^S$ consists of a single $C_T S$ orbit. As $C_T S \subset N_\Gamma S$, this proves that $(\Gamma/\Gamma_\infty)^S/N_\Gamma S$ consists of a single element.

The preceding argument implies that, using $\gamma_i \in \Gamma_\infty$ $(i = 1, 2, \ldots, k)$ in (1) again, we can also write

$$(\Gamma/F)^S/N_\Pi S = \coprod_{1 \le i \le k} (N_\Gamma S/N_\Pi S) \,/\, \left(N_\Pi S(F^{\gamma_i^{-1}} \cap N_\Gamma S)/N_\Pi S\right).$$

Therefore, to prove

$$\varinjlim_n \chi\left((\Gamma_n/F)^S/N_\Pi S\right) = \chi\left((\Gamma/F)^S/N_\Pi S\right),$$

it suffices to show, for each i,

$$\varinjlim_n \chi\left((N_{\Gamma_n}S/N_\Pi S) \,/\, \left(N_\Pi S(F^{\gamma_i^{-1}} \cap N_{\Gamma_n}S)/N_\Pi S\right)\right)$$
$$= \chi\left((N_\Gamma S/N_\Pi S) \,/\, \left(N_\Pi S(F^{\gamma_i^{-1}} \cap N_\Gamma S)/N_\Pi S\right)\right),$$

where the limit is taken with respect to the p-adic topology. Now, to prove this, it suffices to show that $N_\Gamma S$ is a p-toral (then, it easily follows that

$$(N_{\Gamma_\infty}S/N_\Pi S) \,/\, \left(N_\Pi S(F^{\gamma_i^{-1}} \cap N_{\Gamma_\infty}S)/N_\Pi S\right)$$
$$\subseteqq (N_\Gamma S/N_\Pi S) \,/\, \left(N_\Pi S(F^{\gamma_i^{-1}} \cap N_\Gamma S)/N_\Pi S\right)$$

is a dense locally finite p-subgroup of a p-toral and then the claim about Euler characteristics can be shown as in (1)).

To prove $N_\Gamma S$ is a p-toral, we only have to show that $N_{(ST)}S/S = (ST/S)^S$ is a p-toral. Now the Key Lemma says this is nothing but $(T/S \cap T)^S$, where the S action on $(T/S \cap T)$ is a conjugate one. But such a conjugate invariant of a torus $(T/S \cap T) = T^n$, say, by a finite p-group S is is easily seen to be a p-toral. In fact, from the covering homotopy property, there is a short exact sequence of S modules $0 \to \mathbb{Z}^n \to \mathbb{R}^n \to T^n \to 0$. Then, from the associated cohomology long exact sequence, we get the following exact sequence:

$$0 \to (\mathbb{R}^n)^S/(\mathbb{Z}^n)^S \to (T^n)^S \to H^1(S, \mathbb{Z}^n),$$

where $H^1(S, \mathbb{Z}^n)$ is an abelian p-group, by the transfer argument, and $(\mathbb{R}^n)^S/(\mathbb{Z}^n)^S$ is a torus. Of course, we could have used the Tate cohomology to avoid using the covering homotopy property. □

Proof of Proposition 3.1(1). To show the well-definedness of $[(F_\infty \times K)/(H_\infty, \phi)]$ we prepare the following: Let G_∞ be the subgroup of $F_\infty \times K$, generated by (H_∞, ϕ) and $\operatorname{Ker}\pi_{F_\infty} \times \operatorname{Ker}\pi_K$. Let

$$N_\infty = H_\infty \cap \operatorname{Ker}\pi_{F_\infty} \cap \phi^{-1}(\operatorname{Ker}\pi_K),$$

then $N_\infty \triangleleft H_\infty$ with a finite p-group quotient

$$H_\infty/N_\infty \xrightarrow[\cong]{\pi F_\infty \times \pi_K} \pi_{F_\infty}(H_\infty) \times \pi_K(H_\infty) \subseteqq \Pi \times \Xi.$$

It should be noted that $\overline{(H_\infty, \phi)} \stackrel{\text{def}}{=} (H_\infty, \phi)/(N_\infty, \phi\,|_{N_\infty}) \cong H_\infty/N_\infty$ is a finite p-group, rather than infinite, and is a normal subgroup of G_∞ that intersects trivially with K.

Then consider the following system of groups and group homomorphisms:

$$\begin{aligned}
&G_n = G_\infty \cap (F_n \times K),\\
&N_n = N_\infty \cap F_n,\\
&\overline{G}_n = G_n/(N_n, \phi\,|_{N_n}),\\
&K' = K \cap G_\infty,\\
&\rho_n : G_n \to \overline{G}_n, \text{ the canonical projection},\\
&\overline{K'} = \rho_\infty(K') \subseteqq \overline{G}_\infty.
\end{aligned}$$

Since $\operatorname{Ker}\pi_K = K \cap \operatorname{Ker}\pi_K \subseteq K \cap G_n \subseteq K$ and $K/\operatorname{Ker}\pi_K = \Xi$ is finite, $K' = K \cap G_n$ for any sufficiently large n. For such n, $\overline{G_n}$ contains $\overline{K'}$ ($\cong K'$) as a matter of course. With these data, Definition 1.4 enables us to define

$$\begin{aligned}
\rho_n^* &: A(\overline{G}_n \triangleright \overline{K'}) \to A(G_n \triangleright K'),\\
\operatorname{Ind}_{G_n}^{F_n \times K} &: A(G_n \triangleright K') \to A(F_n \times K \triangleright K).
\end{aligned}$$

In fact, these form maps between systems of the relative Burnside modules:

$$\begin{array}{ccccc}
\downarrow & & \downarrow & & \downarrow \\
A(\overline{G}_{n+1} \rhd \overline{K'}) & \xrightarrow{\rho_{n+1}^*} & A(G_{n+1} \rhd K') & \xrightarrow{\mathrm{Ind}_{G_{n+1}}^{F_{n+1}\times K}} & A(F_{n+1} \times K \rhd K) \\
\mathrm{Res}_{\overline{G}_n}^{\overline{G}_{n+1}} \downarrow & & \mathrm{Res}_{G_n}^{G_{n+1}} \downarrow & & \mathrm{Res}_{F_n \times K}^{F_{n+1}\times K} \downarrow \\
A(\overline{G}_n \rhd \overline{K'}) & \xrightarrow{\rho_n^*} & A(G_n \rhd K') & \xrightarrow{\mathrm{Ind}_{G_n}^{F_n\times K}} & A(F_n \times K \rhd K) \\
\mathrm{Res}_{\overline{G}_{n-1}}^{\overline{G}_n} \downarrow & & \mathrm{Res}_{G_{n-1}}^{G_n} \downarrow & & \mathrm{Res}_{F_{n-1}\times K}^{F_n \times K} \downarrow \\
A(\overline{G}_{n-1} \rhd \overline{K'}) & \xrightarrow{\rho_{n-1}^*} & A(G_{n-1} \rhd K') & \xrightarrow{\mathrm{Ind}_{G_{n-1}}^{F_{n-1}\times K}} & A(F_{n-1} \times K \rhd K) \\
\downarrow & & \downarrow & & \downarrow
\end{array}$$

Here the commutativities of the left side squares are trivial to see. But, to guarantee the commutativities of the right side squares, we needed n to be sufficiently large so that $(F_{n+1} \times K)/G_{n+1} \cong (F_n \times K)/G_n$. The reason why $(F_n \times K)/G_n$ stabilizes can be seen from the diagram:

$$\begin{aligned}(F_n \times K)/G_n &= F_n \times K/(F_n \times K) \cap G_\infty \subseteq F_\infty \times K/G_\infty \\ &\twoheadleftarrow F_\infty \times K/\operatorname{Ker}\pi_{F_\infty} \times \operatorname{Ker}\pi_K \cong \Pi \times \Xi, \text{ a finite group}\end{aligned}$$

More precisely, $(F_n \times K)/G_n$ is non-decreasing and its limit $F_\infty \times K/G_\infty$ is finite, and so it should stabilize after sufficiently large n.

Now the point is

$$[(F_n \times K)/(H_n, \phi|_{H_n})] = \mathrm{Ind}_{G_n}^{F_n\times K} \cdot \rho_n^* \left([\overline{G}_n/\overline{(H_n, \phi|_{H_n})}]\right).$$

Therefore, to prove the well-definedness of $[(F_\infty \times K)/(H_\infty, \phi)] \in \varprojlim A(F_n, K)$, we just have to show the well-definedness of

$$[\overline{G}_\infty/\overline{(H_\infty, \phi)}] \in \varprojlim A(\overline{G}_n \rhd \overline{K \cap G_n}),$$

where $[\overline{G}_\infty/\overline{(H_\infty, \phi)}] = \prod_n (y_n) \in \varprojlim A(\overline{G}_n \rhd \overline{K \cap G_n})$ is given by

$$y_n = \lim_{l\to\infty} \left(\mathrm{Res}_{\overline{G}_n}^{\overline{G}_l} [\overline{G}_l/\overline{(H_l, \phi|_{H_l})}]\right) \in A(\overline{G}_n \rhd \overline{K \cap G_n}),$$

where the limit is taken in the p-adic completion $A(\overline{G}_n \rhd \overline{K \cap G_n})_p^\wedge$ (but, still $y_n \in A(\overline{G}_n \rhd \overline{K \cap G_n})$. The payoff of considering such a system of non-"product" relative Burnside modules is that $\overline{(H_\infty, \phi)}$ is a finite p-group; then the claim is an immediate consequence of Lemma 3.2(1) and Remark 1.3.

Proof of Proposition 3.1(2). The idea of the proof is quite similar to that of (1): Let G be the subgroup of $P \times K$, generated by (H, ψ) and $\operatorname{Ker}\pi_P \times \operatorname{Ker}\pi_K$. Let $N = H \cap \operatorname{Ker}\pi_P \cap \psi^{-1}(\operatorname{Ker}\pi_K)$, then $N \lhd H$ with a finite p-group quotient $H/N \xrightarrow[\cong]{\pi_P\times\pi_K} \pi_P(H) \times \pi_K(H) \subsetneqq \Pi \times \Xi$. It should be noted that

$\overline{(H, \psi)} \overset{\text{def}}{=} (H, \psi)/(N, \psi|_N) \cong H/N$ is a finite p-group, rather than infinite, and is a normal subgroup of G that intersects trivially with K. Furthermore, $\overline{(H, \psi)} \cong \overline{(H_\infty, \phi)}$ as H_∞ is dense in H. We can also form $\overline{G} = G/(N, \psi|_N)$ and we put $\rho : G \to \overline{G}$ to be the canonical quotient map. Notice also that $K \cap G = K'$ and $(P \times K)/G \cong (F_n \times K)/G_n$ for sufficiently large n. Both of these follow from the facts that H_∞ is dense in G (as H_∞ is dense in H) and that $\Pi \times \Xi \cong F_\infty \times K/\operatorname{Ker}\pi_{F_\infty} \times \operatorname{Ker}\pi_K$ is finite (see the proof of (1)). For such n, just as in (1), we have the following commutative diagram:

$$\begin{array}{ccccc}
A(\overline{G} \rhd \overline{K'}) & \xrightarrow{\rho^*} & A(G \rhd K') & \xrightarrow{\operatorname{Ind}_G^{P\times K}} & A(P \times K \rhd K) \\
\operatorname{Res}^{\overline{G}}_{\overline{G_n}} \downarrow & & \operatorname{Res}^G_{G_n} \downarrow & & \operatorname{Res}^{P\times K}_{F_n \times K} \downarrow \\
A(\overline{G}_n \rhd \overline{K'_n}) & \xrightarrow{\rho_n^*} & A(G_n \rhd K') & \xrightarrow{\operatorname{Ind}_{G_n}^{F_n\times K}} & A(F_n \times K \rhd K),
\end{array}$$

which leads to maps of inverse systems:

$$\begin{array}{ccccc}
A(\overline{G} \rhd \overline{K'}) & \xrightarrow{\rho^*} & A(G \rhd K') & \xrightarrow{\operatorname{Ind}_G^{P\times K}} & A(P \times K \rhd K) \\
\varprojlim_n \operatorname{Res}^{\overline{G}}_{\overline{G_n}} \downarrow & & \varprojlim_n \operatorname{Res}^G_{G_n} \downarrow & & \varprojlim_n \operatorname{Res}^{P\times K}_{F_n \times K} \downarrow \\
\varprojlim_n A(\overline{G}_n \rhd \overline{K'_n}) & \xrightarrow{\varprojlim_n \rho_n^*} & \varprojlim_n A(G_n \rhd K') & \xrightarrow{\varprojlim_n \operatorname{Ind}_{G_n}^{F_n\times K}} & \varprojlim_n A(F_n \times K \rhd K).
\end{array}$$

In this diagram, it is easy to see that

$$[(P \times K)/(H, \psi)] = \operatorname{Ind}_G^{P\times K} \rho^*([\overline{G}/\overline{(H, \psi)}]),$$
$$[(F_\infty \times K)/(H_\infty, \phi)] = \varprojlim_n \operatorname{Ind}_{G_n}^{F_n\times K} \cdot \varprojlim_n \rho_n^* \left([\overline{G}_\infty/\overline{(H_\infty, \phi)}]\right).$$

Therefore, we only have to show that

$$[\overline{G}_\infty/\overline{(H_\infty, \phi)}] = \varprojlim_n \operatorname{Res}^{\overline{G}}_{\overline{G_n}} \left([\overline{G}/\overline{(H, \psi)}]\right).$$

But, this is an immediate consequence of Lemma 3.2(2) and Remark 1.3. □

Step (II): $A_f(F_\infty, K)$ is dense in $\varprojlim A(F_n, K)^\wedge_p$. This is proved by the induction on $|\Pi|$, the order of the "connected components"of P: let $\{\Pi^\mu\}_{\mu\in\Lambda}$ be the set of all the maximal proper subgroups of $\Pi = \pi_{F_\infty}(F_\infty)$, and let $F_n^\mu = \pi_{F_\infty}^{-1}(\Pi^\mu) \cap F_n$ be its inverse images. Then define $\overline{A(F_n, K)}$ by the exact sequence

$$\bigoplus_{\mu\in\Lambda} A(F_n^\mu, K) \xrightarrow{\oplus_{\mu\in\Lambda} \operatorname{Ind}_{F_n^\mu}^{F_n}} A(F_n, K) \to \overline{A(F_n, K)} \to 0.$$

(Of course, $\overline{A(F_n, K)} = A(F_n, K)$, if $\Pi = \{e\}$.) We note that this sequence is actually that of systems as we vary n, and so induces

$$\bigoplus_{\mu\in\Lambda} \varprojlim_n A(F_n^\mu, K)^\wedge_p \xrightarrow{\oplus_{\mu\in\Lambda} \varprojlim_n \operatorname{Ind}_{F_n^\mu}^{F_n}} \varprojlim_n A(F_n, K)^\wedge_p \to \varprojlim_n \overline{A(F_n, K)}^\wedge_p \to 0.$$

Since

$$\left(\varprojlim_n \operatorname{Ind}_{F_n^\mu}^{F_n}\right)\left([(F_\infty^\mu \times K)/(H_\infty^\mu, \phi)]\right) = [(F_\infty \times K)/(H_\infty^\mu, \phi)]$$

it suffices to show the density of $A_f(F_\infty, K)$ in $\varprojlim_n \overline{A(F_n, K)}_p^\wedge$ in order to proceed the inductive argument.

For this purpose, we fix a finite subgroup $F \subseteqq F_\infty$ and we will show

$$\operatorname{Im} A_f(F_\infty, K)_p^\wedge = \operatorname{Im} \varprojlim A(F_n, K)_p^\wedge \subseteqq \overline{A(F, K)}_p^\wedge,$$

where $\overline{A(F, K)}_p^\wedge$ is non-trivial only when $\pi_{F_\infty}(F) = \Pi$. (Recall that $\operatorname{Im} X$ stands for the image of X by some given map.) Our strategy is to construct a subgroup S in $\overline{A(F, K)}_p^\wedge$ such that

$$\operatorname{Im} \varprojlim A(F_n, K)_p^\wedge \subseteqq \operatorname{Im} S \subseteqq \operatorname{Im} A_f(F_\infty, K)_p^\wedge \ (\subseteqq \overline{A(F, K)}_p^\wedge).$$

(Recall that $A_f(F_\infty, K)$ is the subgroup of $\varprojlim A(F_n, K)$ generated by those $[(F_\infty \times K)/(H_\infty, \phi)]$ studied in (I).)

The following lemma is used to define S:

Lemma 3.3. *Let $F \subset F_n$ be a subgroup and let $H_n \subset F_n$ be a subgroup such that $\pi_{F_\infty}(H_n) = \Pi$, together with a group homomorphism $\phi : H_n \to K$. Then, in $\overline{A(F, K)}$,*

$$\operatorname{Res}_F^{F_n}[(F_n \times K)/(H_n, \phi)] = p^l \cdot \sum_{(H_\lambda, \phi_\lambda) \in \mathscr{L} \subset \mathscr{H}} [(F \times K)/(H_\lambda, \phi_\lambda)]$$

for some nonnegative integer l, where $\phi_\lambda = \phi\,|_{H_\lambda}$, $\mathscr{H}$ is the set of the representatives (H, ϕ) of the $F \times K$ conjugacy classes of this form of subgroups in $F \times K$ satisfying $\pi_{F_\infty}(H) = \Pi$, and $\mathscr{L}$ is a subset in $\mathscr{H}$.

Proof. As $\pi_{F_\infty}(F_n) = \pi_{F_\infty}(H_n)$, any K-orbit of $F_n \times K/(H_n, \phi)$ has a representative of the form $(t_0 \times e)(H_n, \phi)$, where $t_0 \in T_n \overset{\text{def.}}{\equiv} F_n \cap T$, and e is the identity element (of K, in this setting). Then, when $F_n \times K/(H_n, \phi)$ is regarded as a compact $F \times K$-ENR, the isotropy subgroup $(F \times K)_{(t_0 \times e)(H_n, \phi)}$ is seen to be $(F \cap H_n^{t_0^{-1}}, \phi \cdot c_{t_0})\ (\subseteq F \times K)$, for

$$\begin{aligned}
(f \times k)(t_0 \times e)(H_n, \phi) = (t_0 \times e)(H_n, \phi) \\
&\Leftrightarrow (f^{t_0} \times k)(H_n, \phi) = (H_n, \phi) \\
&\Leftrightarrow (f^{t_0} \times k) \in (H_n, \phi) \\
&\Leftrightarrow f^{t_0} \in H_n \text{ and } k = \phi(f^{t_0}) \\
&\Leftrightarrow f \in H_n^{t_0^{-1}} \text{ and } k = \phi \cdot c_{t_0}(f) \\
&\Leftrightarrow (f \times k) \in (H_n^{t_0^{-1}}, \phi \cdot c_{t_0}).
\end{aligned}$$

We want to know when two such isotropy subgroups, which survive in $\overline{A(F, K)}$, are conjugate in $F \times K$. As $(F \cap H_n^{t_0^{-1}}, \phi \cdot c_{t_0})$ survives in $\overline{A(F, K)}$ iff $\pi_{F_\infty}(F \cap H_n^{t_0^{-1}}) = \pi_{F_\infty}(F) = \Pi$, we note that two such are conjugate in $F \times K$ iff they are conjugate in $(F \cap T) \times K$. Now we are going to show the following claim in several steps:

Sublemma 1. *$(F\times K)_{(t_0\times e)(H_n,\phi)}$ and $(F\times K)_{(t\times e)(H_n,\phi)}$ are conjugate in $(F\cap T)\times K$ if and only if $t\in t_0W(F\cap T)$.*

As a first step, we consider the problem: Given $t_0\in T_n$ s.t. $\pi_{F_\infty}(F\cap H_n^{t_0^{-1}})=\Pi$, determine all of those $t\in T_n$ s.t.

$$\left((F\times K)_{(t_0\times e)(H_n,\phi)}\right)^{(f\times k)}=(F\times K)_{(t\times e)(H_n,\phi)},$$

for some $f\in(F\cap T)$ and $k\in K$. Expressing these isotropy subgroups explicitly, this equality is nothing but the commutativity of the following diagram.

$$(*)\qquad \begin{array}{ccccc} F\cap H_n^{t_0^{-1}} & \xleftarrow[\cong]{c_{t_0^{-1}}} & F^{t_0}\cap H_n & \xrightarrow{\phi} & K \\ \cong\big\downarrow c_f & & & & \cong\big\downarrow c_k \\ F\cap H_n^{t^{-1}} & \xrightarrow[\cong]{c_t} & F^{t}\cap H_n & \xrightarrow{\phi} & K. \end{array}$$

As $c_t\cdot c_f\cdot c_{t_0^{-1}}(x)=x^{t_0^{-1}ft}=x^{t_0^{-1}tf}$,

$$\begin{aligned}(*)\text{ commutes} &\Leftrightarrow \phi\left(x^{t_0^{-1}tf}\right)=\phi(x)^k\\ &\Leftrightarrow \phi\left(x^{t_0^{-1}tf}\right)\phi(x)^{-1}=\phi(x)^k\phi(x)^{-1}\\ &\Leftrightarrow \phi\left(\left(t_0^{-1}tf\right)^{-1}\left({}^{x}\left(t_0^{-1}tf\right)\right)\right)=k^{-1}\left({}^{\phi(x)}k\right).\end{aligned}$$

Of course, the problem here is that ϕ is defined only on H_n, not on the whole $T_n=F_n\cap T$ where $t_0^{-1}tf$ belongs. (Otherwise, just put $k=\phi(t_0^{-1}tf)$.)

Now, we wish to study the both sides of the last equality more closely:

(i) *The map* $x\mapsto\phi\left(\left(t_0^{-1}tf\right)^{-1}\left({}^{x}\left(t_0^{-1}tf\right)\right)\right)$

We begin by claiming that the map

$$\begin{aligned} F^{t_0}\cap H_n &\longrightarrow H_n\cap T,\\ x &\mapsto \left(t_0^{-1}tf\right)^{-1}\left({}^{x}\left(t_0^{-1}tf\right)\right)\end{aligned}$$

factors through as

$$\begin{array}{ccccc} F^{t_0}\cap H_n & \twoheadrightarrow & F^{t_0}\cap H_n/F^{t_0}\cap H_n\cap T\cong\Pi & \longrightarrow & H_n\cap T,\\ x & \mapsto & x\left(F^{t_0}\cap H_n\cap T\right) & \mapsto & \left(t_0^{-1}tf\right)^{-1}\left({}^{x}\left(t_0^{-1}tf\right)\right).\end{array}$$

This is simply because both $F^{t_0}\cap H_n\cap T$ and $t_0^{-1}tf$ are contained in the abelian group T_n. Now, notice that this map from Π to $H_n\cap T$ is a cocycle in the sense of Galois cohomology (cf. [S]). Then the obvious question here is where does $t_0^{-1}tf$ belong in order that the left side part of the diagram $(*)$ to be well-defined. The answer is $t_0^{-1}tf\in N_{T_n}(H_n)$. This is because $t_0^{-1}tf\in C_{T_n}(H_n)$ (for T is abelian), $c_{t_0^{-1}tf}:F^{t_0}\cap H_n\xrightarrow{\cong}F^t\cap H_n$ (see $(*)$), and $\pi_{F_\infty}(F^{t_0}\cap H_n)=\pi_{F_\infty}(F^t\cap H_n)=\pi_{F_\infty}(H_n)$. Conversely, if $t_o^{-1}tf\in N_{T_n}(H_n)$, then we can recover the isomorphism $c_{t_0^{-1}tf} : F^{t_0} \cap H_n \xrightarrow{\cong} F^t \cap H_n$, since $c_{t_0^{-1}tf}(H_n) = H_n$ and $c_{t_0^{-1}tf}(F^{t_0}) = c_{t_0^{-1}ft}(F^{t_0}) = F^t$. Therefore, the necessarily and sufficient condition for the left side part of the diagram ($*$) to be well-defined is $t_0^{-1}tf \in N_{T_n}(H_n)$, as we claimed. Using the terminology of the Galois cohomology (cf. [S]), these data indicate that the map

$$\Pi \cong F^{t_0} \cap H_n / F^{t_0} \cap H_n \cap T \longrightarrow H_n \cap T \longrightarrow \phi(H_n \cap T) \subseteqq K$$

$$x\,(F^{t_0} \cap H_n \cap T) \mapsto \left(t_0^{-1}tf\right)^{-1} \left({}^x\left(t_0^{-1}tf\right)\right) \mapsto \phi\left(\left(t_0^{-1}tf\right)^{-1} \left({}^x\left(t_0^{-1}tf\right)\right)\right)$$

is a cocycle and its cohomologous class is nothing but the image of $t_0^{-1}tf \in N_{T_n}(H_n)$ under the composite

$$N_{T_n}(H_n) \twoheadrightarrow (T_n/H_n \cap T_n)^\Pi \xrightarrow{\delta} H^1(\Pi, H_n \cap T_n) \xrightarrow{\phi_*} H^1(\Pi, \phi(H_n \cap T_n)),$$

where the first map is the obvious one (recall $\pi_{F_\infty}(H_n) = \Pi$), δ is the connecting homomorphism, and ϕ_* is induced by ϕ which is Π-equivariant as ϕ is defined on the whole H_n (again, recall $\pi_{F_\infty}(H_n) = \Pi$).

(ii) *The map* $x \mapsto k^{-1}\left({}^{\phi(x)}k\right)$

Notice that this map has a form of a coboundary in the sense of the nonabelian Galois cohomology (see [S]). But the obvious question here is where does k belong so that the diagram ($*$) commutes. We claim the answer is $k \in C_K(\phi(F \cap H_n \cap T))$. Actually, this immediately follows from the fact that the left side part composite map in ($*$): $x \mapsto \phi\left(\left(t_0^{-1}tf\right)^{-1}\left({}^x\left(t_0^{-1}tf\right)\right)\right)$ factors through $\Pi \cong F^{t_0} \cap H_n / F^{t_0} \cap H_n \cap T$, as was discussed in the previous paragraph (here we used $F^{t_0} \cap H_n \cap T = F \cap H_n \cap T$).

Having studied these maps in (i) and (ii) using the Galois cohomology, we denote the kernel of the composite

$$\begin{array}{l} N_{T_n}(H_n) \twoheadrightarrow (T_n/H_n \cap T_n)^\Pi \xrightarrow{\delta} H^1(\Pi, H_n \cap T_n) \\ \qquad\qquad\qquad\qquad\qquad\qquad \downarrow \phi_* \\ \qquad\qquad\qquad\qquad H^1(\Pi, \phi(H_n \cap T_n)) \to H^1(\Pi, \phi(C_K(F \cap H_n \cap T_n))) \end{array}$$

by W. We warn the reader that W might not be a group, as

$$H^1(\Pi, C_K(\phi(F \cap H_n \cap T_n)))$$

is merely a set, the nonabelian Galois cohomology (notice that

$$C_K(\phi(F \cap H_n \cap T_n))$$

is not necessarily abelian). But we do emphasize that it does not depend upon t_0 (this is actually the key to our proof). Then combining those analyses of the two composites in the diagram ($*$) discussed in (i) and (ii), we see ($*$) commutes iff. $t_0^{-1}tf \in W$. In other words,

$$(F \times K)_{(t_0 \times e)(H_n, \phi)} \text{ and } (F \times K)_{(t \times e)(H_n, \phi)} \text{ are conjugate in } (F \cap T) \times K$$
$$\iff t_0^{-1}tf \in W \iff t \in t_0 W(F \cap T).$$

Of course, this is nothing but Sublemma 1. Now we are ready to prove a preliminary result toward Lemma 3.3:

Sublemma 2. *In* $\overline{A(F, K)}_p^{\wedge}$,

$$\operatorname{Res}_F^{F_n}[(F_n \times K)/(H_n, \phi)] = m \cdot \sum_{(H_\lambda, \phi_\lambda) \in \mathscr{L} \subset \mathscr{H}} [(F \times K)/(H_\lambda, \phi_\lambda)],$$

where $\phi_\lambda = \phi \mid_{H_\lambda}$, $\mathscr{H}$ *is the set of the representatives* (H, ϕ) *of the* $F \times K$ *conjugacy classes of this form of subgroups in* $F \times K$ *satisfying* $\pi_{F_\infty}(H) = \Pi$, *and* $\mathscr{L}$ *is a subset in* $\mathscr{H}$.

We begin by recalling that elements of the form $(t \times e)(H_n, \phi)$ form the set of representatives of K-orbits of such elements in $(F_n \times K)/(H_n, \phi)$ (see the beginning of this proof). Therefore, when $(F_n \times K)/(H_n, \phi)$ is regarded as a compact $F \times K$-ENR, the subset consisting of those points, whose isotropy subgroups are conjugate to $\left(F \cap H_n^{t_0^{-1}}, \phi \cdot c_{t_0}\right)$ such that $\pi_{F_\infty}(F \cap H_n^{t_0^{-1}}) = \Pi$, is given by

$$\begin{aligned} &(F_n \times K)/(H_n, \phi)_{((F\cap H_n^{t_0^{-1}}, \phi\cdot c_{t_0}))} \\ &= (t_0 W(F \cap T) \times K)(H_n, \phi)/(H_n, \phi) (\subseteq (F_n \times K)/(H_n, \phi)) \\ &\qquad\qquad (\because \text{Sublemma 1}) \\ &= ((t_0 W(F \cap T) H_n) \times K)/(H_n, \phi) (\subseteq (F_n \times K)/(H_n, \phi)), \end{aligned}$$

as an $F \times K$-space.

To apply Theorem 1.2, we notice that, for any $t \in t_0 W(F \cap T)$,

$$\begin{aligned} FtH_n &= (F \cap T)\left(F \cap H_n^{t^{-1}}\right) tH_n \\ &= (F \cap T)t(F^t \cap H_n)H_n = t(F \cap T)H_n. \end{aligned}$$

Using this, we get

$$\begin{aligned} &(F \times K) \Big\backslash \left[(F_n \times K)/(H_n, \phi)_{((F\cap H_n^{t_0^{-1}}, \phi\cdot c_{t_0}))}\right] \\ &= (F \times K)\backslash ((t_0 W(F \cap T) H_n) \times K)/(H_n, \phi) \\ &\cong F\backslash (t_0 W(F \cap T) H_n)/H_n \\ &\cong (t_0 W(F \cap T) H_n)/(F \cap T) H_n \qquad (\because \text{above remark}) \\ &\xrightarrow[t_0^{-1}\cdot]{\cong} (W(F \cap T) H_n)/(F \cap T) H_n \qquad (\subseteq F_n/(F \cap T) H_n). \end{aligned}$$

Notice that this is a finite set, as F_n is a finite group. So, let us denote the order of this finite set by m. At this stage, it is not clear that m is a p-power, as W is not necessarily a subgroup (whereas $(F \cap T) H_n$ is a subgroup) of a p-group F_n. But m is clearly independent of any particular choice of t_0, and Theorem 1.2 implies Sublemma 2.

Finally, we can complete the proof of Lemma 3.3 by showing

Sublemma 3. *In Sublemma* 2, m *is a* p*-power:* $m = p^l$, *for some nonnegative integer* l.

We first remark that Sublemma 3 holds for the case of the nonrelative situation, i.e. when $K = \{e\}$. This is because, in this case, $W = N_{T^n}(H_n)$ and

$$m = |N_{T^n}(H_n)(F \cap T) H_n/(F \cap T) H_n|,$$

which is of p-power order as $N_{T^n}(H_n)\,(F \cap T)\,H_n$ is a subgroup of the p-group F_n.

The reduction of the general relative case to this nonrelative case is given by the following commutative diagram:

$$\begin{array}{ccc} A(F_n\,,\,K) & \xrightarrow{\mathrm{Res}_F^{F_n}} & \overline{A(F\,,\,K)} \\ \downarrow & & \downarrow \\ A(F_n) & \xrightarrow{\mathrm{Res}_F^{F_n}} & \overline{A(F)}, \end{array}$$

where the vertical maps are given by

$$[X] \mapsto [X/K] = [X \times_K K] = \mu([X]\,,\,[K])$$

(recall that the well-definedness of μ was given in Theorem 2.3).
Suppose $[(F \times K)/(H_\lambda\,,\,\phi_\lambda)]$ shows up in $\mathrm{Res}_F^{F_n}[(F_n \times K)/(H_n\,,\,\phi)]$ with multiplicity m, as above. Then $[F/H_\lambda]$ also shows up in $\mathrm{Res}_F^{F_n}[F_n/H_n]$, with some p-power multiplicity (say p^{l_0}), as was shown above. Now, let m' be the number of those basis elements in $\overline{A(F\,,\,K)}$, which shows up in the basis expansion (in the sense of Theorem 1.2) of $\mathrm{Res}_F^{F_n}[(F_n \times K)/(H_n\,,\,\phi)]$ and is sent to $[F/H_\lambda]$ by the right-side vertical map. Then the commutativity of this diagram, applied to the element $[(F_n \times K)/(H_n\,,\,\phi)] \in A(F_n\,,\,K)$, reveals, by counting the coefficient of $[F/H_\lambda] \in \overline{A(F)}$ in two different ways,

$$m \cdot m' = p^{l_0}.$$

From this, we immediately see m divides p^{l_0}, and so we can write $m = p^l$ for some nonnegative integer l. This implies Sublemma 3, and thus the proof of Lemma 3.3 is finished. □

Definition 3.4.

(1) Under the situation of Lemma 3.3, the element $[(F_n \times K)/(H_n\,,\,\phi)]$ is called of *type* $(l\,,\,\mathscr{L})$ in F. If $\mathscr{L} = \varnothing$, it is called of *type* $\varnothing$ in F.

(2) For each type $(l\,,\,\mathscr{L})$, define the element $e_{(l\,,\,\mathscr{L})}$ by

$$e_{(l\,,\,\mathscr{L})} = p^l \cdot \sum_{(H\,,\,\phi) \in \mathscr{L} \subset \mathscr{H}} [(F \times K)/(H\,,\,\phi)] \in \overline{A(F\,,\,K)}_p^\wedge.$$

(3) Let $\mathscr{T}$ denote the set of types $(k\,,\,\mathscr{L})$ in F, with nonempty $\mathscr{L} \subseteqq \mathscr{H}$, s.t. $e_{(k\,,\,\mathscr{L})}$ is the image of some basis element $[(F_n \times K)/(H_n\,,\,\phi_n)] \in A(F_n\,,\,K)$ for infinitely many n's.

(4) Define S to be the $\mathbb{Z}_p^\wedge$-submodule of $\overline{A(F\,,\,K)}_p^\wedge$, generated by those $e_{(l\,,\,\mathscr{L})}$ such that $(l\,,\,\mathscr{L}) \in \mathscr{T}$.

(5) Let $(l\,,\,\mathscr{L})$ be an element in $\mathscr{T}$. Suppose

$$e_{(l\,,\,\mathscr{L})} = p^{l'} \sum_{i=1}^{j} e_{(l_i\,,\,\mathscr{L}_i)},$$

where $e_{(l_i\,,\,\mathscr{L}_i)} \in \mathscr{T}$ for $1 \leqq i \leqq j$, l' is a nonnegative integer (*resp.* positive integer), and $j \geqq 2$ (*resp.* $j = 1$). Then $(l\,,\,\mathscr{L})$ is said to be **decomposable**.

Otherwise, it is said to be **indecomposable**, and the set of indecomposable types is denoted by $\widehat{\mathcal{T}}$. Of course, S is actually generated by those $e_{(l,\mathcal{L})}$ such that $(l,\mathcal{L})\in\widehat{\mathcal{T}}$.

We are now going to prove the following main result of Step (II):

Proposition 3.5. *In* $\overline{A(F,K)}_p^\wedge$,

$$\operatorname{Im}\varprojlim A(F_n,K)_p^\wedge \subseteqq S \subseteqq \left(\operatorname{Im}A_f(F_\infty,K)\right)_p^\wedge .$$

Consequently,

$$\left(\operatorname{Im}A_f(F_\infty,K)\right)_p^\wedge = \operatorname{Im}\varprojlim A(F_n,K)_p^\wedge \subseteqq \overline{A(F,K)}_p^\wedge .$$

Proof of Proposition 3.5. We will show the two inclusion relations separately:

Proof of $\operatorname{Im}\varprojlim A(F_n,K)_p^\wedge \subseteqq S$.

We only have to show, for any natural number k,

$$\operatorname{Im}\varprojlim A(F_n,K)_p^\wedge \subseteqq \operatorname{Im}S \qquad (\subseteqq \overline{A(F,K)}/p^k\overline{A(F,K)}),$$

where two Im's here stand for the images in $\overline{A(F,K)}/p^k\overline{A(F,K)}$. Notice that, for each type $(l,\mathcal{L})\notin\mathcal{T}$, there is some natural number N_0 such that $e_{(l,\mathcal{L})}$ is never in the image of any basis element $[(F_n\times K)/(H_n,\phi_n)]\in A(F_n,K)$ for any $n\geqq N_0$. Now the point is that there are only finitely many of those types $(l,\mathcal{L})\notin\mathcal{T}$ such that $l<k$ (of course, it is the condition $l<k$, not the property $\notin\mathcal{T}$, which gives the finiteness property). Therefore, there is some natural number N such that, for any $n\geqq N$, the image of any basis element $[(F_n\times K)/(H_n,\phi_n)]\in A(F_n,K)$ in $\overline{A(F,K)}/p^k\overline{A(F,K)}$ is either 0 or an element of the form $e_{(l,\mathcal{L})}$ with $(l,\mathcal{L})\in\mathcal{T}$ and $l<k$. This immediately implies the desired inclusion relationship.

Proof of $S\subseteqq \left(\operatorname{Im}A_f(F_\infty,K)\right)_p^\wedge$. We fix a basis element $e_{(l,\mathcal{L})}\in S$ with $(l,\mathcal{L})\in\widehat{\mathcal{T}}$.

We first note that there are only finitely many ways such that $e_{(l,\mathcal{L})}$ can be expressed as

$$(\spadesuit)\qquad e_{(l,\mathcal{L})}=p^{l'}\sum_{i=1}^{j}e_{(l_i,\mathcal{L}_i)}$$

with $\mathcal{L}_i\neq\varnothing$. This is because $l_i\leqq l$ and $\mathcal{L}_i\subseteqq\mathcal{L}$ for any $1\leqq i\leqq j$.

Suppose $[(F_n\times K)/(H_n,\phi_n)]$ is of this fixed type $(l,\mathcal{L})$ in F, and suppose F_m is a subgroup of F_n, containing F. Then, by Lemma 3.3, we get an expression of the form

$$(\heartsuit)\quad \operatorname{Res}^{F_n}_{F_m}[(F_n\times K)/(H_n,\phi_n)]=p^{l'}\sum_{i=1}^{j'}[(F_m\times K)/(H_m^i,\phi_m^i)]\in\overline{A(F_m,K)}_p^\wedge .$$

Without loss of generality, we may assume, for some j with $1\leqq j\leqq j'$, $[(F_m\times K)/(H_m^i,\phi_m^i)]$ is of type $(l_i,\mathcal{L}_i)$ with $\mathcal{L}_i\neq\varnothing$ for $1\leqq i\leqq j$, and, if $j<j'$, of type $\varnothing$ for $j+1\leqq i\leqq j'$. Then we come up with the expression $(\spadesuit)$.

We now claim that there exists sufficiently large N such that, for any $n \geqq m \geqq N$, the only expression (♠) we get by the procedure (♡) is the trivial one: $l' = 0$ and $j = 1$, i.e.

$$e_{(l,\mathscr{L})} = e_{(l,\mathscr{L})}.$$

Suppose this is not the case. Then there is at least one nontrivial way of expressing

$$e_{(l,\mathscr{L})} = p^{l'} \sum_{i=1}^{j} e_{(l_i,\mathscr{L}_i)},$$

which is produced by (♡) for infinitely many pairs $n \geq m$ with m arbitrary large (recall that there are only finitely many ways of such an expression as (♠)). This implies that these $e_{(l_i,\mathscr{L}_i)}$ come from $A(F_m, K)$ for infinitely many m's, and so $(l_i, \mathscr{L}_i) \in \mathscr{T}$. Of course, this would imply that $(l, \mathscr{L})$ is decomposable, i.e. $(l, \mathscr{L}) \notin \widehat{\mathscr{T}}$, which is a contradiction.

We now set, for each n, $C_n \subsetneqq C^f(F_n, K)$ to be the set of those $((H_n, \phi)) \in C^f(F_n, K)$ such that $[(F_n \times K)/(H_n, \phi)]$ is of type $(l, \mathscr{L})$ in F. Then the above claim indicates that $\{C_n\}_{n\geq N}$ is an inverse system by the correspondence

$$\begin{gathered} C_n \to C_m \\ ((H_n, \phi_n)) \mapsto ((H_m, \phi_m)), \end{gathered}$$

where

$$\begin{aligned} \mathrm{Res}^{F_n}_{F_m}[(F_n \times K)/(H_n, \phi_n)] &= [(F_m \times K)/(H_m, \phi_m)] \\ &\quad \left(+ \sum_{i=2}^{j'} [(F_m \times K)/(H^i_m, \phi^i_m)] \right) \in \overline{A(F_m, K)}^{\wedge}_{p}. \end{aligned}$$

Here (H_m, ϕ_m) is of the type $(l, \mathscr{L})$ in F, and in the second summation, which might not show up, every term (H^i_m, ϕ^i_m) $(2 \leqq i \leqq j')$ is of type $\varnothing$. In particular, this shows each C_n $(n \geqq N)$ is a nonempty (finite) set. Thus, as $\{C_n\}_{n\geq N}$ is seen to be an inverse system of nonempty finite sets,

$$\varprojlim_{n \geq N} C_n \neq \varnothing.$$

Therefore, after applying appropriate conjugations, we can find a system of type $(l, \mathscr{L})$ subgroups $\{(H_n, \phi_n)\}_{n\geq N}$ such that

$$\begin{gathered} (H_n, \phi_n) \subset F_n \times K, \\ H_{n+1} \cap F_n = H_n, \\ \phi_{n+1}|_{H_n} = \phi_n \end{gathered}$$

for all $n \geq N$. Now, just set

$$H_\infty = \bigcup_n H_n, \quad \phi_\infty = \bigcup_n \phi_n (\text{so } \phi_\infty|_{H_n} = \phi_n).$$

Then, from Proposition 3.1(1) and Lemma 3.2(1), we get immediately

$$\left(\varinjlim \mathrm{Res}^{F_n}_{F} \right) ([(F_\infty \times K)/(H_\infty, \phi_\infty)] = e_{(l,\mathscr{L})} \in \overline{A(F_m, K)}.$$

Since $e_{(l,\mathscr{L})}$ is a basis element of S, this completes the proof. □

Step (III): Representation Theory. Subgroups of F_∞ are studied by the integral representation theory, through the Pontryagin duality. This was first observed by Feshbach [F2]. We now recall the elementary property of the Pontryagin duality.

(1) For a locally compact Lie group G, the Pontryagin dual G^* is defined by

$$G^* = \mathrm{Hom}_{\mathrm{Continuous}}(G, S^1).$$

(2) $G^{**} = G$.

(3) For each closed subgroup $H \subset G$, define H_* as a subgroup of G^* consisting of those $\phi : G \to S^1$ with $\phi(H) = 1$.

(4) $H_* \cong (G/H)^*$ and $H_{**} = H$.

(5) T^*, the Pontryagin dual of the torus T with Π-action, is a finitely generated $\mathbb{Z}\Pi$-module.

(6) Let T^*_∞ be the subgroup consisting of elements of p-power order in the torus T with Π-action, as usual. Then T^*_∞, the Pontryagin dual of T_∞ with Π-action, is the p-completion of T^*.

Now the representation theory we need is the following:

Lemma 3.6. (1) *Let H_* be a finitely generated $\mathbb{Z}_p\Pi$-submodule of T^*_∞, such that $H_* \otimes \mathbb{Q}_p$ is isomorphic to the p-completion of a $\mathbb{Q}\Pi$-module. Then for every n there is an automorphism α_n of T^*_∞, inducing the identity on $T^*_\infty/p^n \cdot T^*_\infty$, such that $\alpha_n(H_n)$ is the p-completion of a $\mathbb{Z}\Pi$-submodule K^n_* of T^*.*

(2) *Let F be a finite group and let S and T be finitely generated $\mathbb{Z}F$-modules which are free over $\mathbb{Z}$. Then*

$$\mathrm{Hom}_{\mathbb{Z}F}(S, T) \to \mathrm{Hom}_{\mathbb{Z}_p^\wedge F}(S_p^\wedge, T_p^\wedge)$$

has a dense image.

Proof. (1) was essentially used in [F2], but was singled out explicitly in [B]. (2) is proved in 2-e of [R]. Of course, the general reference of these results is [CR]. □

This is used in the following:

Proposition 3.7. *Suppose that all the irreducible $\mathbb{Q}\Pi'$-summands in the Π'-action on $T^* \otimes \mathbb{Q}$ remain irreducible after the p-adic completion for any subgroup $\Pi' \subset \Pi$. Then for any $[(F_\infty \times K)/(H_\infty, \phi_\infty)] \in A_f(F_\infty \times K)$ and $n \in \mathbb{N}$ such that $\pi_P(H_n) = \pi_P(H_\infty) \subseteqq \Pi$, there is a closed subgroup $H \subset P$ and a continuous homomorphism $\psi : H \to K$ such that*

$$\left(\varprojlim_m \mathrm{Res}^{F_m}_{F_n}\right)([(F_\infty \times K)/(H_\infty, \phi_\infty)]) = \mathrm{Res}^P_{F_n}([(P \times K)/(H, \psi)]) \in A(F_n, K).$$

Proof. Let $P' = P \cap \pi_P^{-1}(\pi_P(H_\infty))$ and $F'_n = F_n \cap \pi_P^{-1}(\pi_P(H_\infty))$ for any n.

Then we have the following commutative diagram:

$$\begin{array}{ccccc}
A(P', K) & \xrightarrow{\underleftarrow{\lim}_m \operatorname{Res}^{P'}_{F'_m}} & \underleftarrow{\lim}_m A(F'_m, K) & \xrightarrow{\underleftarrow{\lim}_m \operatorname{Res}^{F'_m}_{F'_n}} & A(F'_n, K) \\
\Big\downarrow \operatorname{Ind}^P_{P'} & & \Big\downarrow \underleftarrow{\lim}_m \operatorname{Ind}^{F_m}_{F'_m} & & \Big\downarrow \operatorname{Ind}^{F_n}_{F'_n} \\
A(P, K) & \xrightarrow{\underleftarrow{\lim}_m \operatorname{Res}^{P}_{F_m}} & \underleftarrow{\lim}_m A(F_m, K) & \xrightarrow{\underleftarrow{\lim}_m \operatorname{Res}^{F_m}_{F_n}} & A(F_n, K).
\end{array}$$

Notice that

$$\left(\underleftarrow{\lim}_m \operatorname{Ind}^{F_m}_{F'_m}\right)([(F'_\infty \times K)/(H_\infty, \phi_\infty)]) = [(F_\infty \times K)/(H_\infty, \phi_\infty)].$$

Therefore, the commutativity of the diagram implies that it suffices to find a closed subgroup $H \subsetneqq P'$ and a continuous homomorphism $\psi : H \to K$ s.t.

$$\left(\underleftarrow{\lim}_m \operatorname{Res}^{F'_m}_{F'_n}\right)([(F'_\infty \times K)/(H_\infty, \phi_\infty)]) = \operatorname{Res}^{P'}_{F'_n}((P' \times K)/(H, \psi)) \in A(F'_n, K).$$

For this, we apply two claims of Lemma 3.6 in two steps: (i) Find an automorphism α of $F'_\infty = \bigcup_{n\in\mathbb{N}} F'_n$ fixing F'_n such that $\alpha(H_\infty) = H \cap F'_\infty$ for for some closed subgroup $H \subset P'$; (ii) Find a continuous homomorphism $\psi : H \to K$ such that $\psi|_{\alpha(H_m)} = \phi_\infty \cdot \alpha^{-1}|_{\alpha(H_m)}$ for some sufficiently large m so that

$$\left(\underleftarrow{\lim}_m \operatorname{Res}^{F'_m}_{F'_n}\right)\left([(F'_\infty \times K)/\left(\alpha(H_\infty), \phi_\infty \cdot \alpha^{-1}\right)]\right) = \operatorname{Res}^{P'}_{F'_n}((P' \times K)/(H, \psi)) \in A(F'_n, K).$$

These would prove the claim by Proposition 3.1(2) and the following commutative diagram:

$$\begin{array}{ccccc}
A(P', K) & \xrightarrow{\underleftarrow{\lim}_m \operatorname{Res}^{P'}_{F'_m}} & \underleftarrow{\lim}_m A(F'_m, K) & \xrightarrow{\alpha^*} & \underleftarrow{\lim}_m A(F'_m, K) \\
\Big\| & & \Big\downarrow \underleftarrow{\lim}_m \operatorname{Res}^{F'_m}_{F'_n} & & \Big\downarrow \underleftarrow{\lim}_m \operatorname{Res}^{F'_m}_{F'_n} \\
A(P'.K) & \xrightarrow{\operatorname{Res}^{P'}_{F'_n}} & A(F'_n, K) & = & A(F'_n, K),
\end{array}$$

where

$$\alpha_*\left([(F'_\infty \times K)/\left(\alpha(H_\infty), \phi_\infty \cdot \alpha^{-1}\right)]\right) = ([(F'_\infty \times K)/(H_\infty, \phi_\infty)]).$$

For (i), it suffices to find a $\Pi' \overset{\text{def.}}{=} \pi_P(H_\infty)$ ($\subsetneqq \Pi$)-invariant closed subgroup A such that

$$\begin{array}{ccccc}
T & \overset{\supset}{\longleftarrow} & & & T_\infty \\
\cup\big\uparrow & & & & \cup\big\uparrow \\
A & \overset{\supset}{\longleftarrow} & A \cap T_\infty & \underset{\alpha^{-1}}{\overset{\cong}{\to}} & H_\infty \cap T
\end{array}$$

commutes for some automorphism α of F_∞, fixing F_n. This is because H is taken to be the closed subgroup of P', generated by A and $\alpha(H_n)$. Now, through the Pontryagin duality, this is equivalent to the commutativity of the following diagram:

$$\begin{array}{ccc} T^* & \xrightarrow{\subset} & T^*_\infty \\ \cup\uparrow & & \cup\uparrow \\ A_* & \xrightarrow{\subset} & A_{*p}^{\wedge} \underset{\alpha_*}{\overset{\cong}{\to}} (H_\infty \cap T)_*, \end{array}$$

where α_* is a $\mathbb{Z}_p^\wedge \Pi'$-automorphism of T^*_∞, inducing an identity on $T^*_\infty / p^n \cdot T^*_\infty$, To find such a submodule A_* and an automorphism α_*, it suffices to show $(H_\infty \cap T)_* \otimes \mathbb{Q}$ is the p-completion of a $\mathbb{Q}\Pi'$-module, thanks to Lemma 3.6(1). In this situation we can write $T \otimes \mathbb{Q} = \bigoplus_i V_i$ for some irreducible $\mathbb{Q}\Pi'$-modules V_i's. But each $V_{ip}^\wedge$ is still an irreducible $\mathbb{Q}_p\Pi'$-module by the assumption. Therefore $(H_\infty \cap T)_* \otimes \mathbb{Q}$, which is a $\mathbb{Q}\Pi'$-submodule of $T^*_\infty \otimes \mathbb{Q} = \bigoplus_i V_{ip}^\wedge$, is isomorphic to a finite combination of the irreducible summands $V_{ip}^\wedge$'s. As each $V_{ip}^\wedge$ is the p-completion of V_i, this immediately implies that $(H_\infty \cap T)_* \otimes \mathbb{Q}$ is the p-completion of a $\mathbb{Q}\Pi'$-submodule, as desired.

For (ii), we first notice that the above H and $\alpha(H_\infty)$ are a p-toral and its dense locally p-subgroup. Therefore, we may apply the following Lemma 3.8 to obtain the isomorphism $A(F'_m, \alpha(H_\infty)) \overset{\cong}{\to} A(F'_m, H)$ for any m. These isomorphisms are combined to give an isomorphism $\varprojlim_m A(F'_m, \alpha(H_\infty)) \overset{\cong}{\to} \varprojlim_m A(F'_m, H)$, so we can talk about the element

$$[F'_\infty \times \alpha(H_\infty)/(\alpha(H_\infty), \text{identity})] \in \varprojlim_m A\left(F'_m, \alpha(H_\infty)\right)$$

(see Proposition 3.1(1)) and the isomorphism

$$\begin{aligned}(\phi_\infty \cdot \alpha^{-1})_* &\left([F'_\infty \times \alpha(H_\infty)/(\alpha(H_\infty), \text{identity})]\right) \\ &= [(F'_\infty \times K)/\left(\alpha(H_\infty), \phi_\infty \cdot \alpha^{-1}\right)].\end{aligned}$$

For this important element, we also get a finite expansion

$$\begin{aligned}&\left(\varprojlim_m \operatorname{Res}^{F'_m}_{F'_n}\right) \left([F'_\infty \times \alpha(H_\infty)/(\alpha(H_\infty), \text{identity})]\right) \\ &\qquad = \sum_i n_i \left[F'_n \times \alpha(H_\infty) \Big/ \left(F'_n \cap \alpha(H_\infty)^{f_i}, c_{f_i^{-1}}\right)\right],\end{aligned}$$

where $n_i \in \mathbb{N}$ and $f_i \in F'_\infty$. Then, these two equalities and the commutativity of the diagram

$$\begin{array}{ccc} \varprojlim_m A(F'_m, \alpha(H_\infty)) & \xrightarrow{\phi_\infty \cdot \alpha^{-1}} & \varprojlim_m A(F'_m, K) \\ \varprojlim_m \operatorname{Res}^{F'_m}_{F'_n} \downarrow & & \varprojlim_m \operatorname{Res}^{F'_m}_{F'_n} \downarrow \\ A(F'_n, \alpha(H_\infty)) & \xrightarrow{\phi_\infty \cdot \alpha^{-1}} & A(F'_n, K) \end{array}$$

immediately imply

$$\left(\varprojlim_m \operatorname{Res}^{F'_m}_{F'_n}\right)\left([F'_\infty \times K/(\alpha(H_\infty), \phi_\infty \cdot \alpha^{-1})]\right)$$
$$= \sum_i n_i \left[F'_n \times K\Big/\left(F'_n \cap \alpha(H_\infty)^{f_i}, \phi_\infty \cdot \alpha^{-1} \cdot c_{f_i^{-1}}\right)\right] \in A(F'n, K).$$

Here $\phi_\infty \cdot \alpha^{-1} \cdot c_{f_i^{-1}}$ factors through as

$$F'_n \cap \alpha(H_\infty)^{f_i} \xrightarrow{c_{f_i^{-1}}} F'^{f_i^{-1}}_n \cap \alpha(H_\infty) \xrightarrow{\phi_\infty \cdot \alpha^{-1}} K.$$

But, as there are only finitely many these f_i's of F'_∞, there is some sufficiently large $m \in \mathbb{N}$ s.t. $F'^{f_i^{-1}}_n \subseteqq F'_m$ for all i. Thus,

$$F'^{f_i^{-1}}_n \cap \alpha(H_\infty) \subseteqq F'_m \cap \alpha(H_\infty) = \alpha(F'_m) \cap \alpha(H_\infty) = \alpha(H_m)$$

for all these f_i's. Therefore, these indicate the following: As far as $\phi_\infty \cdot \alpha^{-1}$ concerns,

$$\left(\varprojlim_m \operatorname{Res}^{F'_m}_{F'_n}\right)\left([F'_\infty \times K/(\alpha(H_\infty), \phi_\infty \cdot \alpha^{-1})]\right) \in A(F'_n, K)$$

depends only upon $\phi_\infty \cdot \alpha^{-1} |_{\alpha(H_m)}$.

Now, by Lemma 3.6 (2) and the Pontryagin duality argument as in (i), we can find a continuous homomorphism $\psi : H \to K$ such that $\psi |_{\alpha(H_m)} = \phi_\infty \cdot \alpha^{-1} |_{\alpha(H_m)}$. Of course, the above argument implies

$$\left(\varprojlim_m \operatorname{Res}^{F'_m}_{F'_n}\right)\left([F'_\infty \times K/(\alpha(H_\infty), \phi_\infty \cdot \alpha^{-1})]\right)$$
$$= \left(\varprojlim_m \operatorname{Res}^{F'_m}_{F'_n}\right)\left([F'_\infty \times K/(\alpha(H_\infty), \psi |_{\alpha(H_\infty)})]\right) \in A(F'_n, K).$$

On the other hand, Proposition 3.1(2) implies

$$[F'_\infty \times K/(\alpha(H_\infty), \psi |_{\alpha(H_\infty)})] = \varprojlim_n \operatorname{Res}^{P'}_{F'_n}\left([(P' \times K)/(H, \psi)]\right).$$

Now the claim (ii) follows immediately, and thus we have finished the proof of Proposition 3.7. □

In the course of the above proof, we used the following:

Lemma 3.8. *Let F be a finite p-group and let Q be a p-toral which is built by the extension*

$$1 \to T \to Q \to \Omega \to 1,$$

where T is a torus and the connected component of Q and Ω is a p-group. Suppose $Q_\infty \overset{\text{def.}}{=} \bigcup_n Q_n$ is a dense locally p-subgroup of Q such that

$$\begin{array}{ccccccccc}
1 & \longrightarrow & T & \longrightarrow & Q & \longrightarrow & \Omega & \longrightarrow & 1 \\
 & & \cup\uparrow & & \cup\uparrow & & \| & & \\
1 & \longrightarrow & T_\infty \overset{\text{def.}}{=} Q_\infty \cap T = (\mathbb{Z}/p^\infty)^r & \longrightarrow & Q_\infty & \longrightarrow & \Omega & \longrightarrow & 1 \\
 & & \cup\uparrow & & \cup\uparrow & & \| & & \\
1 & \longrightarrow & Q_n \cap T = (\mathbb{Z}/p^n)^r & \longrightarrow & Q_n & \xrightarrow{\pi} & \Omega & \longrightarrow & 1,
\end{array}$$

where r is the rank of T, and so the left-upper map $(\mathbb{Z}/p^\infty)^r \to T$ is the inclusion of the subset consisting of all p-power order elements.

Then the canonical map

$$A(F, Q_\infty) \overset{\text{def.}}{=} \varinjlim_n A(F, Q_n) \longrightarrow A(F, Q)$$

is an isomorphism.

Proof. We begin by showing the surjectivity of this map. Notice that any basis element of $A(F, Q)$ is of the form $[F \times Q/(H, \phi)]$. As $\phi(H)$ is a finite p-group of Q, we can find another dense locally p-subgroup Q'_∞ such that $\phi(H) \subset Q'_\infty$. But Q'_∞ is conjugate to Q_∞ in Q [F2]. So the surjectivity follows.

To prove the injectivity, it suffices to show any two p-subgroups of Q_∞, which are conjugate in Q (then automatically they are conjugate in T), are conjugate in Q_∞. For this, let us suppose S is a p-group such that both S and S^t are subgroups of Q_∞, where $t \in T$. Then, as $S^t \subset Q_\infty$,

$$tQ_\infty \in (Q/Q_\infty)^S.$$

Also, as $S \subset Q_\infty$, the Key Lemma implies

$$(Q/Q_\infty)^S = (T/T_\infty)^S,$$

where $(Q/Q_\infty)^S$ is taken with respect to the standard left S-multiplication on Q/Q_∞, and $(T/T_\infty)^S$ is taken with respect to the conjugate S-action on T/T_∞. The reason we prefer $(T/T_\infty)^S$ to $(Q/Q_\infty)^S$ is the accessibility of the connecting homomorphism

$$\delta : (T/T_\infty)^S \longrightarrow H^1(S, T_\infty),$$

which we claim to be trivial. This is follows from the following: (i) $(T/T_\infty)^S$ is a $\mathbb{Z}[\frac{1}{p}]$-module; (ii) $H^1(S, T_\infty)$ is $|S|$-torsion. So the composite

$$tQ_\infty \in (Q/Q_\infty)^S = (T/T_\infty)^S \xrightarrow{\delta} H^1(S, T_\infty)$$

is trivial. Therefore, the cocycle

$$\delta(tQ_\infty) : s \mapsto t^{-1}({}^s t), \qquad s \in S,$$

is expressed as a principal crossed homomorphism

$$s \mapsto u^{-1}({}^s u), \qquad s \in S,$$

for some $u \in T_\infty$. In particular, this implies $S^t = S^u$. Thus, S and S^t are conjugate by an element $u \in T_\infty \subset Q_\infty$, as was desired. □

Now we are finally ready to prove our main theorem:

Theorem 3.9. *Let L and K be compact Lie groups, then*

$$\alpha_p^\wedge : A(L, K)_p^\wedge \to \{BL_+, BK_+\}_p^\wedge,$$

is dense with respect to the skeletal filtration of the target, if L satisfies the assumption of the Feshbach density theorem, i.e. if either one of the following two conditions holds:

(1) p is odd.

(2) $p = 2$ and $\rho : \Pi \to Gl(n, \mathbb{Z})$, given by the action of a 2-Sylow subgroup Π of the Weyl group on the maximal torus, does not originate at a generalized quaternion group of order 2^n, $n \geq 4$. Or if all the irreducible $\mathbb{Q}\Pi'$-summands in the Π'-action on $T^* \otimes \mathbb{Q}$ remain irreducible after the 2-adic completion for any 2-subgroup $\Pi' \subset W$. In particular, this condition holds if $n < 8$ or if no subquotient of W is isomorphic to the quaternion group of order 16. Here Π is the 2-Sylow subgroup of the Weyl group of L and T is a maximal torus of L.

Proof. As was noted in the introduction, the problem is reduced to the density problem of the map

$$\varprojlim_n \mathrm{Res}^P_{F_n} : A(P, K)_p^\wedge \to \varprojlim_n A(F_n, K)_p^\wedge.$$

But, this is an immediate consequence of Step (I), Step (II), and Step (III) (Proposition 3.7). (See the beginning of §3.) □

Corollary 3.10. *If L is a central extension of a finite group by a torus:*

$$0 \to T \to L \to G \to 1,$$

then the homomorphism

$$\alpha_p^\wedge : A(L, K)_{I(L)}^\wedge \to \{BL_+, BK_+\}$$

is injective and has a dense image.

Proof. As usual, we may reduce to the local situation. Then the density part follows from Theorem 3.9. To prove the injectivity, let $P \subset L$ be a finite index subgroup corresponding to a p-Sylow subgroup of G. We only have to show the injectivity of

$$\varprojlim_n \mathrm{Res}^P_{F_n} : A(P, K)_p^\wedge \to \varprojlim_n A(F_n, K)_p^\wedge.$$

Now the point is $U \supset T$ if $(U, \psi) \in C^f(P, K)$. This is because the G-action on T is trivial. From this, we immediately see

$$\mathrm{Res}^P_{F_n}([(P \times K)/(U, \psi)]) = [(F_n \times K)/(U \cap F_n, \psi|_{U \cap F_n})].$$

(Of course, n is sufficiently large so that F_n surjects onto a p-Sylow subgroup of G.) Now the injectivity follows quite easily. □

Corollary 3.11.

$$A(L)_{I(L)}^\wedge \to \pi_S^0(BL_+) \qquad \textit{is an isomorphism}$$

$\iff$ *L is a central extension of a finite group G by a torus:*

$$0 \to T \to L \to G \to 1. \quad \square$$

Proof. $\Longrightarrow$ was shown in [M1]. $\Longleftarrow$ follows from Corollary 3.10 and the fact that $A(L)$ is finitely generated. $\square$

Remark 3.12. This result was announced in the introduction of [M1].

Appendix

In this appendix, we will prove a couple of (not surprising) claims stated (and used) in the introduction.

Let P and K be maximal p-tori of L and H, respectively, and let $I(P, K)$ be the kernel of $\mathrm{Res}^P_{\{e\}} : A(P, K) \to A(\{e\}, K) \cong \mathbb{Z}$. Furthermore, let F_i's be increasing finite p-subgroups of P, such that $\bigcup_i F_i \subset P$ is dense (see Reduction in §0 and [F2], [M1], [N]). Then, consider the following various density properties:

(1) Density of $A(L, H) \to \varprojlim_n \{BL_+^{(n)}, BH_+\}^\wedge_{finite}$,

(2_p) Density of $A(L, H) \to \varprojlim_n \{BL_+^{(n)}, BH_+\}^\wedge_p$,

(3_p) Density of $A(P, K) \to \varprojlim_n \{BP_+^{(n)}, BK_+\}^\wedge_p$,

(4_p) Density of $I(P, K) \to \varprojlim_n \{BP^{(n)}, BK_+\}^\wedge_p$,

(5_p) Density of $I(P, K) \to \varprojlim_i \{BF_i, BK_+\}^\wedge_p$,

(6_p) Density of $A(P, K) \to \varprojlim_i \{BF_{i+}, BK_+\}^\wedge_p$,

(7_p) Density of $A(P, K) \to \varprojlim_i A(BF_i, BK)^\wedge_p$.

Here all the maps are canonical ones induced by α, $X^{(n)}$ stands for the n-skeleton of X, and the topology of the target is the inverse limit topology of the profinite topology (resp. p-adic topology) on the relevant $\{-, -\}$ groups in (1) (resp. (2_p) – (7_p)). Now, we are going to show

Local-Global. (2_p) *for any* $p \implies$ (1).

Reduction. (7_p) $\implies$ (2_p).

Of course, Theorem 3.9 claims (7_p) for any odd prime p.

Proof of Local-Global. This is standard. Given a natural number N, write $N = \prod_{p|N} p^{\nu_p(N)}$. Then consider the commutative diagram

$$\begin{array}{ccccc}
A(L, H) & \longrightarrow & \varprojlim_n \{BL_+^{(n)}, BH_+\}^\wedge_{finite} & \longrightarrow & \dfrac{\{BL_+^{(n)}, BH_+\}}{N\{BL_+^{(n)}, BH_+\}} \\
\Big\| & & \Big\downarrow & & \Big\downarrow \pi_p \\
A(L, H) & \longrightarrow & \varprojlim_n \{BL_+^{(n)}, BH_+\}^\wedge_p & \longrightarrow & \dfrac{\{BL_+^{(n)}, BH_+\}}{p^{\nu_p(N)}\{BL_+^{(n)}, BH_+\}}
\end{array},$$

where π_p is the canonical projection, and the middle vertical map exists because $\{BL_+^{(n)}, BH_+\}$ is finitely generated. Let $x \in \{BL_+^{(n)}, BH_+\}/N\{BL_+^{(n)}, BH_+\}$ be an element which comes from $\varprojlim_n \{BL_+^{(n)}, BH_+\}^\wedge_{finite}$, and let x_p be its image in $\{BL_+^{(n)}, BH_+\}/p^{\nu_p(N)}\{BL_+^{(n)}, BH_+\}$. Then, by (2_p), there is some $a_p \in A(L, H)$ which goes to x_p. To slightly modify a_p, choose some integer N_p s.t. $N_p \equiv 1 \pmod{p^{\nu_p(N)}}$, $N_p \equiv 0 \pmod{\prod_{q\neq p} q^{\nu_q(N)}}$. Then set $a'_p = N_p a_p$. Notice that a'_p also goes to x_p just like a_p, but its real advantage is that it goes to 0 under the composite

$$\begin{aligned}A(L, H) &\to \{BL_+^{(n)}, BH_+\}/N\{BL_+^{(n)}, BH_+\}\\ &\xrightarrow{\pi_q} \{BL_+^{(n)}, BH_+\}/q^{\nu_q(N)}\{BL_+^{(n)}, BH_+\}\end{aligned}$$

for any $q \neq p$. Finally, set $a = \sum_{p|N} a'_p \in A(L, H)$. Clearly, a goes to x, for $\prod_{p|N} \pi_p$ is an isomorphism. □

Proof of reduction. We are going to show $(7_p) \iff (6_p) \iff (5_p) \iff (4_p) \iff (3_p) \implies (2_p)$.

$(7_p) \iff (6_p)$ This follows from the topological isomorphism

$$A(BF_i, BK)_p^\wedge \to \{BF_{i+}, BK_+\}_p^\wedge,$$

proved in [MSZ].

$(6_p) \iff (5_p)$ This is because $A(P, K) \to \varprojlim_i \{BF_{i+}, BK_+\}_p^\wedge$ is the direct sum of $I(P, K) \to \varprojlim_n \{BP^{(n)}, BK_+\}_p^\wedge$ and the canonical map $\mathbb{Z} \cong A(\{e\}, K) \to \{S^0, BK_+\}_p^\wedge \cong \mathbb{Z}_p^\wedge$.

$(5_p) \iff (4_p)$ For any spectrum E, we follow [Bo] to define its p-adic completion $E_p^\wedge$ to be the function spectrum $F(\Sigma^{-1}M(\mathbb{Z}/p^\infty), E)$, where $M(A)$ is the Moore spectrum of an abelian group A s.t. $\pi_0(M(A)) = A$, $\pi_i(M(A)) = 0$ if $i \neq 0$. Suppose $\pi_i E$ is finitely generated for any i, then $\pi_*(E_p^\wedge) \cong (\pi_* E)_p^\wedge$ (Proposition 2.5 [Bo]). Notice that the p-adic topology of the right-hand side is induced from the filtration associated with the expression $\underrightarrow{\text{holim}}_m \Sigma^{-1}M(\mathbb{Z}/p^m) = \Sigma^{-1}M(\mathbb{Z}/p^\infty)$ of the left-hand side.

Consider a spectrum $X = \underrightarrow{\text{holim}}_n X_n$ with each X_n a finite spectrum. We equip $(E_p^\wedge)^q(X)$ with the induced topology with respect to the canonical isomorphism $(E_p^\wedge)^q(X) = \varprojlim_n (E_p^\wedge)^q(X_n)$. Thus, using Proposition 2.5 [Bo] again, we have a topological isomorphism $(E_p^\wedge)^q(X) = \varprojlim_n E^q(X_n)_p^\wedge$. In particular, for $X = BP$ and $X_n = BP^{(n)}$, we have

$$(E_p^\wedge)^q(BP) = \varprojlim_n E^q(BP^{(n)})_p^\wedge.$$

But, as the canonical map $BF_\infty \cong \underrightarrow{\text{holim}}_n BF_n^{(n)} \to \underrightarrow{\text{holim}}_n BP^{(n)} = BP$ induces an isomorphism of mod-p homology, we have the following sequence of topological isomorphisms:

$$\begin{aligned}(E_p^\wedge)^q(BP) &= (E_p^\wedge)^q(BF_\infty) = \varprojlim_n (E_p^\wedge)^q(BF_n^{(n)})\\ &= \varprojlim_{i,n} (E_p^\wedge)^q(BF_i^{(n)}) = \varprojlim_i \varprojlim_n (E_p^\wedge)^q(BF_i^{(n)})\\ &= \varprojlim_i \varprojlim_n E^q(BF_i^{(n)}) = \varprojlim_i E^q(BF_i),\end{aligned}$$

where the topology of $E^q(BF_i)$ is the skeletal topology, and the fifth equality follows from the fact that the image of $E^q(BF_i^{(n)}) \to E^q(BF_i^{(n-1)})$ is a finite p-group. Now, we specialize ourselves to the case $E = \Sigma^\infty BK_+$ and $q = 0$. Then the main theorem of [MSZ] implies that $\{BF_i, BK_+\}$ is topologically isomorphic to the p-adic completion of the finitely generated free abelian group $I(F_i, K)$. In summary, we have the topological isomorphisms

$$\varprojlim_n \{BP^{(n)}, BK_+\}_p^\wedge \cong \{BP, (BK_+)_p^\wedge\} \cong \varprojlim_i \{BF_i, BK_+\}_p^\wedge,$$

which clearly imply $(5_p) \iff (4_p)$. (Compare with [F2].)

$(4_p) \iff (3_p)$ This is the same as $(6_p) \iff (5_p)$.

$(3_p) \implies (2_p)$ Given a compact Lie group G, and its closed subgroups H_1 and H_2, we set $\xi(H_1, G) : BH_{1+} \to BG_+$ and $\tau(H_2, G) : BG_+ \to BH_{2+}$ to be the induced map of the inclusion $H_1 \subset G$ and the stable transfer map associated with the fiber bundle $BH_2 \to BG$, respectively. Furthermore, let $[_{H_1}G_{H_2}] \in A(H_1, H_2)$ be an element, which is represented by G considered as a compact $H_1 \times H_2$ manifold with respect to the left (free) H_1 and the right H_2 actions given by the multiplication of G.

With these notations, Theorem 2.2 and Corollary 2.3 imply the following diagram commutes:

$$\begin{array}{ccccc}
A(L, H) & \xrightarrow{k_1} & A(P, K) & \xrightarrow{k_2} & A(L, H) \\
\downarrow & & \downarrow & & \downarrow \\
\varprojlim_n\{BL_+^{(n)}, BH_+\}_p^\wedge & \xrightarrow{l_1} & \varprojlim_n\{BP_+^{(n)}, BK_+\}_p^\wedge & \xrightarrow{l_2} & \varprojlim_n\{BL_+^{(n)}, BH_+\}_p^\wedge \\
\| & & \| & & \| \\
\varprojlim_n\{BL_+^{(n)\wedge}{}_p, BH_+{}_p^\wedge\} & \xrightarrow{l_1} & \varprojlim_n\{BP_+^{(n)\wedge}{}_p, BK_+{}_p^\wedge\} & \xrightarrow{l_2} & \varprojlim_n\{BL_+^{(n)\wedge}{}_p, BH_+{}_p^\wedge\},
\end{array}$$

where

$$k_1 = \mathscr{C}([_PL_L] \otimes - \otimes [_HH_K]), \qquad k_2 = \mathscr{C}([_LL_P] \otimes - \otimes [_KH_H]),$$
$$l_1 = \tau(K, H) \circ - \circ \xi(P, L), \qquad l_2 = \xi(K, H) \circ - \circ \tau(P, L).$$

But both the composites

$$\xi(P, L)\tau(P, L) : BL_+^{(n)} \to BP_+^{(n)} \to BL_+^{(n)},$$
$$\xi(K, H)\tau(K, H) : BH_+ \to BK_+ \to BH_+,$$

induce isomorphisms of the mod-p homology, since both $\chi(L/P)$ and $\chi(H/K)$ are finite and relatively prime to p. This immediately implies that the composite $l_2 \circ l_1$ is an isomorphism. Now, $(3_p) \implies (2_p)$ follows immediately by the diagram chase in the above commutative diagram. This completes the proof of Reduction. □

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