TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 348, Number 1, January 1996

SUBGROUPS OF $\mathrm{GL}(n^2, \mathbf{C})$ CONTAINING $\mathrm{PSU}(n)$

V. P. PLATONOV AND D. Ž. ĐOKOVIĆ

ABSTRACT. Let $\mathrm{PSU}(n)$ be the image of the unitary group $\mathrm{U}(n)$ under the representation $x \to axa^{-1}$ on the space $M_n(\mathbf{C})$ of n by n complex matrices. We classify all connected Lie subgroups of $\mathrm{GL}(n^2, \mathbf{C})$ containing $\mathrm{PSU}(n)$. We use this result to obtain a description of all abstract overgroups of $\mathrm{PSU}(n)$ in $\mathrm{GL}(n^2, \mathbf{C})$.

We apply this classification to solve the problem of describing all invertible linear transformations of $M_n(\mathbf{C})$ which preserve the set of normal matrices. Our results can be applied to solve many other problems of similar nature.

1. INTRODUCTION

The problem of classification of subgroups of the general linear group $\mathrm{GL}(m, \mathbf{C})$ has a long history but, even for Lie subgroups, it is too difficult. For semisimple subgroups, this problem is essentially equivalent to the description of all finite-dimensional representations of such groups. If we are interested only in subgroups having some specific property, then even for semisimple subgroups the problem may be very difficult.

About 20 years ago it was realized, in connection with some problems of invariant theory and linear algebra, that it is important to obtain the classification of subgroups of $\mathrm{GL}(m, \mathbf{C})$ containing a given natural classical subgroup. A well-known problem in this direction is the problem about the description of all algebraic subgroups of $\mathrm{GL}(n^2, K)$, K an algebraically closed field of characteristic 0, containing the image of $\mathrm{SL}(n, K)$ under the adjoint representation. This problem was solved in our paper [5]. In the present paper we shall solve the problem of describing all abstract subgroups of $\mathrm{GL}(n^2, \mathbf{C})$ containing the image of $\mathrm{SU}(n)$ under the adjoint representation.

In more detail, let M_n be the algebra of n by n complex matrices. The group of all automorphisms of the vector space M_n will be denoted by $\mathrm{GL}(n^2, \mathbf{C})$. We shall consider M_n as a left module over the algebra $M_n \otimes_{\mathbf{C}} M_n^{\mathrm{opp}}$ such that $(a \otimes b)(x) = axb$. In this way we can identify this algebra with the algebra of all complex linear operators on M_n.

The group $\mathrm{GL}(n, \mathbf{C})$ acts on M_n by similarity transformations $x \to axa^{-1}$, $x \in M_n$, $a \in \mathrm{GL}(n, \mathbf{C})$. The image of $\mathrm{GL}(n, \mathbf{C})$ in $\mathrm{GL}(n^2, \mathbf{C})$, under this representation, will be denoted by G. The image of the unitary group $\mathrm{U}(n) \subset \mathrm{GL}(n, \mathbf{C})$ in G will

---

Received by the editors August 6, 1994.

1991 *Mathematics Subject Classification.* Primary 20G20, 15A30.

The first author was supported in part by NSERC Grant A-6197 and the Alexander von Humboldt Foundation.

The second author was supported in part by NSERC Grant A-5285.

be denoted by G_0. It is clear that $G \cong \mathrm{PSL}(n, \mathbf{C})$, $G_0 \cong \mathrm{PSU}(n)$, and that G_0 is a compact real form of G.

Our main objective is to obtain a description of all abstract subgroups of $\mathrm{GL}(n^2, \mathbf{C})$ containing G_0. While this problem is quite natural, it is worthwhile to point out some important applications of this classification.

In our papers [1], [5] we have described all connected algebraic subgroups of $\mathrm{SL}(n^2, K)$ containing $\mathrm{PSL}(n, K)$ for an arbitrary algebraically closed field K of characteristic 0, and we have applied these results to solve many linear preserver problems (concerning these problems see the survey volume [6]). We shall use here this classification for $K = \mathbf{C}$ to obtain the description of all abstract overgroups of G_0 in $\mathrm{GL}(n^2, \mathbf{C})$. The list of these overgroups is not as short as the one in [5], but it is not too long. The main result shows that any overgroup of G_0 in $\mathrm{GL}(n^2, \mathbf{C})$ is "almost" a Lie subgroup. The deviation from Lie subgroups is expressed in terms of arbitrary subgroups of a 2-dimensional complex torus T, the centralizer of G in $\mathrm{GL}(n^2, \mathbf{C})$. In any case, our list of overgroups can be used effectively to solve many new linear preserver problems. We give here only one example, namely the linear preserver problem for normal matrices in M_n.

2. Connected complex Lie overgroups of $\mathrm{PSL}(n, \mathbf{C})$

Since the list of connected complex Lie overgroups of G plays an important role in our proofs, we shall present this list below in Theorem A.

We have $M_n = M_n^0 \oplus \mathbf{C} \cdot 1$ where M_n^0 is the subspace of M_n defined by $\mathrm{tr}(x) = 0$ and 1 denotes the identity matrix. The subgroup of $\mathrm{GL}(n^2, \mathbf{C})$ which fixes 1 and leaves invariant M_n^0 will be denoted by $\mathrm{GL}(n^2 - 1, \mathbf{C})$.

The symmetric bilinear form $\mathrm{tr}(xy)$ on $M_n \times M_n$ is non-degenerate. The subgroup of $\mathrm{GL}(n^2, \mathbf{C})$ that preserves this form will be denoted by $\mathrm{O}(n^2, \mathbf{C})$. By $\mathrm{O}(n^2-1, \mathbf{C})$ we denote its subgroup consisting of operators which fix 1. As $M_n^0 \perp \mathbf{C} \cdot 1$, we have $\mathrm{O}(n^2 - 1, \mathbf{C}) = \mathrm{O}(n^2, \mathbf{C}) \cap \mathrm{GL}(n^2 - 1, \mathbf{C})$. We remark that if $n = 2$, then $G = \mathrm{SO}(n^2 - 1, \mathbf{C})$.

By P (resp. Q) we denote the subgroup of $\mathrm{GL}(n^2, \mathbf{C})$ which acts trivially on M_n^0 and M_n/M_n^0 (resp. $\mathbf{C} \cdot 1$ and $M_n/\mathbf{C} \cdot 1$). By T we denote the 2-dimensional subtorus of $\mathrm{GL}(n^2, \mathbf{C})$ which acts by scalar transformations on M_n^0 and $\mathbf{C} \cdot 1$. This torus centralizes $\mathrm{GL}(n^2 - 1, \mathbf{C})$ and normalizes P and Q. The center of $\mathrm{GL}(n^2, \mathbf{C})$ will be denoted by Z. We also introduce the 1-dimensional torus $T_1 = T \cap \mathrm{SL}(n^2, \mathbf{C})$.

If we fix a basis of M_n consisting of a basis of M_n^0 and the identity matrix 1, then P, Q, T can be identified with the groups of all matrices of the form

$$\begin{pmatrix} I_{n^2-1} & x \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} I_{n^2-1} & 0 \\ y & 1 \end{pmatrix}, \quad \begin{pmatrix} \alpha I_{n^2-1} & 0 \\ 0 & \beta \end{pmatrix},$$

respectively, where x is arbitrary column vector, y arbitrary row vector, and α and β arbitrary nonzero scalars.

Since G consists of all operators $a \otimes a^{-1}$ with $a \in \mathrm{SL}(n, \mathbf{C})$, we have

$$\mathrm{SL}(n, \mathbf{C}) \otimes \mathrm{SL}(n, \mathbf{C})^{\mathrm{opp}} = \{a \otimes b : a, b \in \mathrm{SL}(n, \mathbf{C})\} \supset G.$$

We denote by τ the transposition map on M_n and write x' instead of $\tau(x)$.

When $n = 4$ we have constructed in [5] an overgroup $\Lambda \cong \mathrm{SL}(6, \mathbf{C})/\langle -1 \rangle$ of G. We recall the definition of Λ. Let K_4 be the space of skew-symmetric matrices in M_4 and $\mathrm{GL}(6, \mathbf{C})$ the group of invertible linear transformations of K_4. Define the homomorphism

$$\theta : \mathrm{SL}(4,\mathbf{C}) \to \mathrm{SL}(6,\mathbf{C})$$

by $\theta(a)(x) = axa'$, $x \in K_4$. We fix a basis of K_4 :

$$\begin{array}{lll} v_1 = e_{12} - e_{21}, & v_2 = e_{13} - e_{31}, & v_3 = e_{14} - e_{41}, \\ v_4 = e_{23} - e_{32}, & v_5 = e_{24} - e_{42}, & v_6 = e_{34} - e_{43}, \end{array}$$

where $\{e_{ij}\}$ is the standard basis of M_4, and define the involution $f \in \mathrm{GL}(6,\mathbf{C})$ by

$$\begin{array}{lll} f(v_1) = v_6, & f(v_2) = -v_5, & f(v_3) = v_4, \\ f(v_4) = v_3, & f(v_5) = -v_2, & f(v_6) = v_1. \end{array}$$

It was shown in [5] that there is an isomorphism

$$\varphi : K_4 \wedge K_4 \to M_4^0$$

of SL(4, **C**)-modules which is characterized by

$$\varphi(x \wedge y) = xf(y) - yf(x), \quad x, y \in K_4.$$

By transferring the natural action of SL(6, **C**) on $K_4 \wedge K_4$ to M_4^0, via the isomorphism φ, we obtain a homomorphism

$$\rho : \mathrm{SL}(6,\mathbf{C}) \to \mathrm{SL}(15,\mathbf{C}),$$

with $\mathrm{Ker}(\rho) = \langle -1 \rangle$. Explicitly we have

$$\rho(a)(xf(y) - yf(x)) = a(x)f(a(y)) - a(y)f(a(x)) \tag{1}$$

for $x, y \in K_4$ and $a \in \mathrm{SL}(6,\mathbf{C})$. The group Λ is the image of ρ, and G is the image of $\rho \circ \theta$.

Theorem A A. *Let* Γ *be a proper connected complex Lie subgroup of* $\mathrm{SL}(n^2,\mathbf{C})$ *containing* $G = \mathrm{PSL}(n,\mathbf{C})$.

If Γ *is reducible, then* $\Gamma = H$, HP, HQ, HT_1, HPT_1, *or* HQT_1, *where* $H = G$, $\mathrm{SO}(n^2-1,\mathbf{C})$, $\mathrm{SL}(n^2-1,\mathbf{C})$, *or* Λ $(n = 4)$.

If Γ *is irreducible, then* Γ *is a* T*-conjugate of* $\mathrm{SL}(n,\mathbf{C}) \otimes \mathrm{SL}(n,\mathbf{C})^{\mathrm{opp}}$ *or* $\mathrm{SO}(n^2,\mathbf{C})$.

Remark. In the case when Γ is algebraic, this theorem is the complex version of [5, Theorem A]. For complex Lie subgroups Γ, the assertion of the theorem follows easily from the above special case.

3. Connected compact overgroups of PSU(n) in SL(n^2, C)

For the description of connected Lie overgroups of $G_0 = \mathrm{PSU}(n)$ in $\mathrm{SL}(n^2,\mathbf{C})$, we need some additional notations. The subgroup of $\mathrm{GL}(n^2,\mathbf{C})$, which preserves the space of all hermitian matrices in M_n, will be denoted by $\mathrm{GL}(n^2,\mathbf{R})$. By x^* we denote the conjugate transpose of a matrix x. The involution $*$ on M_n induces an involution, also denoted by $*$, on $M_n \otimes M_n^{\mathrm{opp}}$. Namely for $z \in M_n \otimes M_n^{\mathrm{opp}}$, viewed as a linear transformation on M_n, we define z^* by :

$$z^*(x) = (z(x^*))^*, \quad x \in M_n. \tag{2}$$

When $z = a \otimes b$ we have

$$\begin{aligned} (a \otimes b)^*(x) &= ((a \otimes b)(x^*))^* = (ax^*b)^* = b^*xa^* \\ &= (b^* \otimes a^*)(x), \quad x \in M_n, \end{aligned}$$

i.e.,

$$(a \otimes b)^* = b^* \otimes a^*.$$

Note that $z^* = z$ iff z preserves the space of hermitian matrices in M_n. It follows that

$$\mathrm{GL}(n^2, \mathbf{R}) = \{z \in \mathrm{GL}(n^2, \mathbf{C}) : z^* = z\}.$$

It is also straightforward to check that G is $*$-invariant and that

$$G_0 = G \cap \mathrm{GL}(n^2, \mathbf{R}).$$

The subgroup of $\mathrm{GL}(n^2, \mathbf{C})$ which preserves the positive definite hermitian form $\mathrm{tr}(x^*y)$ will be denoted by $\mathrm{U}(n^2)$. We also set

$$\begin{aligned}
&\mathrm{SL}(n^2, \mathbf{R}) = \mathrm{SL}(n^2, \mathbf{C}) \cap \mathrm{GL}(n^2, \mathbf{R}),\\
&\mathrm{SU}(n^2) = \mathrm{SL}(n^2, \mathbf{C}) \cap \mathrm{U}(n^2),\\
&\mathrm{O}(n^2) = \mathrm{O}(n^2, \mathbf{C}) \cap \mathrm{GL}(n^2, \mathbf{R}),\\
&\mathrm{SO}(n^2) = \mathrm{SO}(n^2, \mathbf{C}) \cap \mathrm{GL}(n^2, \mathbf{R}),\\
&\mathrm{U}(n^2-1) = \mathrm{GL}(n^2-1, \mathbf{C}) \cap \mathrm{U}(n^2),\\
&\mathrm{SU}(n^2-1) = \mathrm{SL}(n^2-1, \mathbf{C}) \cap \mathrm{U}(n^2),\\
&\mathrm{O}(n^2-1) = \mathrm{O}(n^2-1, \mathbf{C}) \cap \mathrm{GL}(n^2, \mathbf{R}),\\
&\mathrm{SO}(n^2-1) = \mathrm{O}(n^2-1) \cap \mathrm{SL}(n^2, \mathbf{R}),\\
&P_0 = P \cap \mathrm{GL}(n^2, \mathbf{R}),\\
&Q_0 = Q \cap \mathrm{GL}(n^2, \mathbf{R}).
\end{aligned}$$

By $\mathrm{U}(n^2-1,1)$ we denote the subgroup of $\mathrm{GL}(n^2, \mathbf{C})$, which preserves the non-degenerate hermitian form

$$\mathrm{tr}(x^*y) - \mathrm{tr}(x^*)\,\mathrm{tr}(y)$$

of Witt index 1. By $\mathrm{O}(n^2-1,1)$ we denote the subgroup of $\mathrm{GL}(n^2, \mathbf{R})$ which preserves the non-degenerate symmetric bilinear form

$$\mathrm{tr}(xy) - \mathrm{tr}(x)\,\mathrm{tr}(y)$$

of Witt index 1 on the space of hermitian matrices in M_n. We set

$$\begin{aligned}
&\mathrm{SU}(n^2-1,1) = \mathrm{U}(n^2-1,1) \cap \mathrm{SL}(n^2, \mathbf{C}),\\
&\mathrm{SO}(n^2-1,1) = \mathrm{O}(n^2-1,1) \cap \mathrm{SL}(n^2, \mathbf{R}).
\end{aligned}$$

We also introduce notation for two connected Lie subgroups of the torus T_1. Namely, U_1 will denote the circle group in T_1 and R_1 the subgroup of T_1 which acts as multiplication by positive scalars on M_n^0 (and $\mathbf{C} \cdot 1$).

When $n = 4$, we introduce two real forms Λ_0 and Λ_1 of Λ. Let $\mathrm{U}(6)$ be the subgroup of $\mathrm{GL}(6, \mathbf{C})$ that preserves the positive definite hermitian form $\mathrm{tr}(x^*y)$ on $K_4 \times K_4$. Then $\Lambda_0 \cong \mathrm{SU}(6)/\langle -1 \rangle$ is the image of $\mathrm{SU}(6)$ under the homomorphism ρ. The subgroup $\mathrm{SU}(4)$ of $\mathrm{SL}(4, \mathbf{C})$ perserves the above hermitian form, and consequently its image under the homomorphism θ is contained in $\mathrm{SU}(6)$. As the image of $\mathrm{SU}(4)$ under $\rho \circ \theta$ is G_0, we have $G_0 \subset \Lambda_0$.

Let $\sigma : K_4 \to K_4$ be the anti-linear involution defined by $\sigma(x) = f(\bar{x}) = \overline{f(x)}$, where bar denotes complex conjugation. We have $\sigma(x) = x$ iff $f(x) = \bar{x}$. The map σ induces on $\mathrm{SL}(6, \mathbf{C})$ an anti-holomorphic involution and provides this group with a structure of an algebraic group defined over $\mathbf{R}$. We shall denote by $\mathrm{SL}(6, \mathbf{R})$ its group of real points. Thus an element of $\mathrm{SL}(6, \mathbf{C})$ belongs to $\mathrm{SL}(6, \mathbf{R})$ iff it commutes with σ, or, equivalently, iff it preserves the real subspace of K_4 defined by $f(x) = \bar{x}$. We define $\Lambda_1 \cong \mathrm{SL}(6, \mathbf{R})/\langle -1 \rangle$ to be the image of $\mathrm{SL}(6, \mathbf{R})$ under ρ. If $a \in \mathrm{GL}(6, \mathbf{C})$, i.e., a is an invertible linear transformation on K_4, then we define

$$a^\sigma := \sigma \circ a \circ \sigma.$$

Then $\mathrm{SL}(6,\mathbf{C})$ is σ-invariant and

$$\mathrm{SL}(6,\mathbf{R}) = \{a \in \mathrm{SL}(6,\mathbf{C}) : a^\sigma = a\}.$$

For $x \in K_4$ we have $x^* = -\bar{x}$. By using (1) and (2), we obtain

$$\begin{aligned}\rho(a)^*(xf(y) - yf(x)) &= [\rho(a)((xf(y) - yf(x))^*)]^* \\ &= [\rho(a)(f(\bar{y})\bar{x} - f(\bar{x})\bar{y})]^* \\ &= [\rho(a)(\sigma(y)f(\sigma(x)) - \sigma(x)f(\sigma(y)))]^* \\ &= [a(\sigma(y))f(a(\sigma(x))) - a(\sigma(x))f(a(\sigma(y)))]^* \\ &= a^\sigma(x)f(a^\sigma(y)) - a^\sigma(y)f(a^\sigma(x)) \\ &= \rho(a^\sigma)(xf(y) - yf(x)).\end{aligned}$$

Since this holds for all $x, y \in K_4$, we have

$$\rho(a^\sigma) = \rho(a)^*, \quad a \in \mathrm{SL}(6,\mathbf{C}).$$

This implies that

$$\Lambda_1 = \Lambda \cap \mathrm{GL}(n^2,\mathbf{R}), \quad n = 4.$$

Consequently $G_0 \subset \Lambda_1$, and so G_0 is a maximal compact subgroup of Λ_1.

The connected compact overgroups of G_0 in $\mathrm{SL}(n^2,\mathbf{C})$ are described in the next theorem.

Theorem 1. *Let Γ_0 be a connected compact subgroup of $\mathrm{SL}(n^2,\mathbf{C})$ containing G_0. If Γ_0 is reducible, then $\Gamma_0 = H_0$ or H_0U_1 where*

$$H_0 = G_0,\ \mathrm{SO}(n^2-1),\ \mathrm{SU}(n^2-1),\ \text{or } \Lambda_0 \text{ if } n = 4. \tag{3}$$

If Γ_0 is irreducible, then it is a T-conjugate of one of the groups:

$$\mathrm{SO}(n^2),\ \mathrm{SU}(n)\otimes\mathrm{SU}(n)^{\mathrm{opp}},\ \mathrm{SU}(n^2). \tag{4}$$

Proof. The complexification, Γ, of Γ_0 is either $\mathrm{SL}(n^2,\mathbf{C})$ or one of the groups listed in Theorem A. If $\Gamma \neq \mathrm{SL}(n^2,\mathbf{C})$, we claim that Γ_0 is the unique maximal compact subgroup of Γ containing G_0.

In order to prove this claim, we start by choosing a positive definite Γ_0-invariant hermitian form $h : M_n \times M_n \to \mathbf{C}$. As M_n^0 is an irreducible G_0-module, there is only one (up to a scalar factor) positive definite G_0-invariant hermitian form on $M_n^0 \times M_n^0$. Therefore we may assume that

$$h(x,y) = \mathrm{tr}(x^*y), \qquad x, y \in M_n^0.$$

If $\Gamma = H$ (see Theorem A) is simple and reducible, it follows that $\Gamma_0 = \Gamma \cap \mathrm{SU}(n^2-1)$. Thus our claim holds in that case. Clearly, it also holds if $\Gamma = HT_1$.

If Γ is irreducible, then by using Theorem A we see that it suffices to prove our claim when Γ is $\mathrm{SO}(n^2,\mathbf{C})$ or $\mathrm{SL}(n,\mathbf{C})\otimes\mathrm{SL}(n,\mathbf{C})^{\mathrm{opp}}$. We consider these two cases separately.

Case 1. $\Gamma = \mathrm{SO}(n^2,\mathbf{C})$. There is a unique anti-linear automorphism $u : M_n \to M_n$ such that $h(x,y) = \mathrm{tr}(u(x)y)$ for all $x, y \in M_n$. For $x, y \in M_n^0$, we have

$$\mathrm{tr}(x^*y) = h(x,y) = \mathrm{tr}(u(x)y),$$

i.e., $\mathrm{tr}((u(x) - x^*)y) = 0$. Consequently, $u(x) = x^* + \psi(x)\cdot 1$ for some anti-linear function $\psi : M_n^0 \to \mathbf{C}$. As $h(x,y)$ and $\mathrm{tr}(xy)$ are both Γ_0-invariant, u commutes with the action of Γ_0, i.e., we have $u(axa^{-1}) = au(x)a^{-1}$ for all $a \in \mathrm{SU}(n)$ and $x \in M_n$. This implies that $\psi(axa^{-1}) = \psi(x)$ for all $x \in M_n^0$ and $a \in \mathrm{SU}(n)$. Thus ψ is a homomorphism of $\mathrm{SU}(n)$-modules, and so $\psi = 0$. Hence $u(x) = x^*$ for all $x \in M_n^0$.

For $x \in M_n^0$, $h(x,1) = \mathrm{tr}(u(x)) = \mathrm{tr}(x^*) = 0$ and $\mathrm{tr}(u(1)x) = h(1,x) = 0$. This implies that $u(1) = \lambda \cdot 1$ for some $\lambda \in \mathbf{C}^*$. As $n\lambda = \mathrm{tr}(u(1)) = h(1,1) > 0$, we have $\lambda > 0$.

The restrictions of $\mathrm{tr}(xy)$ and $\mathrm{tr}(x^*y)$ to the space of Hermitian n by n matrices of trace 0 coincide. We choose an orthonormal basis B_0 for this space. Then B_0 together with $(1/\sqrt{n}) \cdot 1$ provides an orthonormal basis B for the form $\mathrm{tr}(xy)$. The matrix of $\mathrm{tr}(xy)$ with respect to B is the identity matrix while that of h is

$$J = \begin{pmatrix} I_{n^2-1} & 0 \\ 0 & \lambda \end{pmatrix}.$$

With respect to this basis, $\mathrm{SO}(n^2, \mathbf{C})$ consists of all complex matrices x satisfying $xx' = I_{n^2}$, and $\mathrm{SU}(h)$ consists of all complex matrices x satisfying $x^*Jx = J$. The matrices $x \in \Gamma_0$ satisfy both of these conditions, and so $\bar{x}J = Jx$. Since Γ_0 is irreducible, the last condition easily implies that $\lambda = 1$. Consequently $\Gamma_0 = \mathrm{SO}(n^2)$.

Case 2. $\Gamma = \mathrm{SL}(n, \mathbf{C}) \otimes \mathrm{SL}(n, \mathbf{C})^{\mathrm{opp}}$. In this case we must have $\Gamma_0 = A \otimes B$ where A and B are maximal compact subgroups of $\mathrm{SL}(n, \mathbf{C})$, i.e., they are conjugates of $\mathrm{SU}(n)$. Since

$$\Gamma_0 \supset G_0 = \{a \otimes a^{-1} : a \in \mathrm{SU}(n)\},$$

it follows that $A = B = \mathrm{SU}(n)$. This completes the proof of our claim.

It remains to prove the assertion of the theorem in the case $\Gamma = \mathrm{SL}(n^2, \mathbf{C})$. As above, we find that Γ_0 preserves a positive definite Hermitian form $h(x,y)$ such that $h(x,y) = \mathrm{tr}(x^*y)$ for $x, y \in M_n^0$. As h is also G_0-invariant, the h-orthogonal complement of M_n^0 in M_n is a G_0-submodule. Hence this orthogonal complement must be $\mathbf{C} \cdot 1$. Let $\lambda := h(1,1) > 0$ and $\mu = \sqrt{\lambda/n}$. Define $t \in T$ by $t(x) = x$ for all $x \in M_n^0$ and $t(1) = \mu \cdot 1$. Then $h(x,y) = \mathrm{tr}(t(x)^* t(y))$ holds for all $x, y \in M_n$. It follows that $\Gamma_0 = t^{-1} \cdot \mathrm{SU}(n^2) \cdot t$. □

4. Abstract overgroups of $\mathrm{PSU}(n)$ in $\mathrm{GL}(n^2, \mathbf{C})$

We first describe all connected Lie overgroups of G_0 in $\mathrm{SL}(n^2, \mathbf{C})$. For that purpose we need the following two lemmas.

Lemma 1. *Let $\mathfrak{g}$ be a finite-dimensional semisimple complex Lie algebra and $\mathfrak{g}_0$ a real Lie subalgebra of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{g}_0 + i\mathfrak{g}_0$. Then $\mathfrak{a} = \mathfrak{g}_0 \cap i\mathfrak{g}_0$ is an ideal of $\mathfrak{g}$, $\mathfrak{g} = \mathfrak{a} \times \mathfrak{b}$, and $\mathfrak{g}_0 = \mathfrak{a} \times \mathfrak{b}_0$, where $\mathfrak{b}_0$ is a real form of $\mathfrak{b}$.*

Proof. It is clear that $\mathfrak{a}$ is a complex subspace of $\mathfrak{g}$. It is also evident that $\mathfrak{a}$ is an ideal of $\mathfrak{g}_0$. Since $\mathfrak{a} = \mathfrak{g}_0 + i\mathfrak{g}_0$ and

$$[\mathfrak{a}, i\mathfrak{g}_0] = [i\mathfrak{a}, \mathfrak{g}_0] = [\mathfrak{a}, \mathfrak{g}_0] \subset \mathfrak{a},$$

it follows that $\mathfrak{a}$ is an ideal of $\mathfrak{g}$. Hence $\mathfrak{g} = \mathfrak{a} \times \mathfrak{b}$ for some ideal $\mathfrak{b}$ of $\mathfrak{g}$. As $\mathfrak{g}_0 \supset \mathfrak{a}$, we have $\mathfrak{g}_0 = \mathfrak{a} \times \mathfrak{b}_0$ for some real subalgebra $\mathfrak{b}_0$ of $\mathfrak{b}$. Since $\mathfrak{b}_0 + i\mathfrak{b}_0 = \mathfrak{b}$ and $\mathfrak{b}_0 \cap i\mathfrak{b}_0 = 0$, $\mathfrak{b}_0$ is a real form of $\mathfrak{b}$. □

Lemma 2. *Let $\mathfrak{g}$ be a finite-dimensional simple complex Lie algebra and $\mathfrak{g}_1$, $\mathfrak{g}_2$ two isomorphic real forms of $\mathfrak{g}$. If $\mathfrak{g}_1 \cap \mathfrak{g}_2$ contains a maximal compact Lie subalgebra $\mathfrak{k}$ of $\mathfrak{g}_1$ and $\mathfrak{k}$ is semisimple, then $\mathfrak{g}_1 = \mathfrak{g}_2$.*

Proof. Let $\mathfrak{g}_1 = \mathfrak{k} \oplus \mathfrak{p}_1$ and $\mathfrak{g}_2 = \mathfrak{k} \oplus \mathfrak{p}_2$ be Cartan decompositions. Then $\mathfrak{p}_1 \perp \mathfrak{k}$ and $\mathfrak{p}_2 \perp \mathfrak{k}$ with respect to the Killing form of $\mathfrak{g}$. Consequently $\mathfrak{p}_2 \subset \mathfrak{p} := \mathfrak{p}_1 + i\mathfrak{p}_1$. As $\mathfrak{g}_1$ is absolutely simple and $\mathfrak{k}$ is semisimple, $\mathfrak{p}_1$ is an absolutely simple $\mathfrak{k}$-module, see [2, Proposition (8.8.3)]. This implies that $\mathfrak{p}_2 = \alpha\mathfrak{p}_1$ for some complex number α with $|\alpha| = 1$. As $[\mathfrak{p}_2, \mathfrak{p}_2] \subset \mathfrak{k}$, we have $\alpha^2 \in \mathbf{R}$, and so $\mathfrak{p}_2 = \mathfrak{p}_1$ or $\mathfrak{p}_2 = i\mathfrak{p}_1$. Since $\mathfrak{k} + i\mathfrak{p}_1$ is a compact real form of $\mathfrak{g}$ and $\mathfrak{g}_1 \cong \mathfrak{g}_2$, we must have $\mathfrak{p}_2 = \mathfrak{p}_1$, i.e., $\mathfrak{g}_2 = \mathfrak{g}_1$. □

Theorem 2. *Let* Γ *be a proper connected Lie subgroup of* $\mathrm{SL}(n^2, \mathbf{C})$ *containing* G_0. *If* Γ *is reducible, then either*

$$\Gamma = HR,\ HPR,\ HQR, \tag{5}$$

where

$$H = \begin{array}{l} G_0,\ G,\ \mathrm{SO}(n^2-1),\ \mathrm{SO}(n^2-1,\mathbf{C}),\ \mathrm{SU}(n^2-1), \\ \mathrm{SL}(n^2-1,\mathbf{R}),\ \mathrm{SL}(n^2-1,\mathbf{C}),\ \text{or } \Lambda_0,\ \Lambda_1,\ \Lambda\ (n=4), \end{array} \tag{6}$$

and R *is a connected Lie subgroup of* T_1 *(which may be trivial), or* Γ *is a* U_1*-conjugate of one of the groups :*

$$HP_0,\ HQ_0,\ HP_0R_1,\ HQ_0R_1 \tag{7}$$

where

$$H = G_0,\ \mathrm{SO}(n^2-1),\ \mathrm{SL}(n^2-1,\mathbf{R}),\ \text{or } \Lambda_0,\ \Lambda_1\ (n=4). \tag{8}$$

If Γ *is irreducible, then* Γ *is a* T*-conjugate of one of the groups:*

$$\begin{array}{l} \mathrm{SL}(n^2,\mathbf{R}),\ \mathrm{SU}(n^2),\ \mathrm{SU}(n^2-1,1), \\ \mathrm{SO}(n^2,\mathbf{C}),\ \mathrm{SO}(n^2),\ \mathrm{SO}(n^2-1,1)^0, \\ \mathrm{SL}(n,\mathbf{C}) \otimes \mathrm{SL}(n,\mathbf{C})^{\mathrm{opp}},\ \mathrm{SU}(n) \otimes \mathrm{SL}(n,\mathbf{C})^{\mathrm{opp}}, \\ \mathrm{SL}(n,\mathbf{C}) \otimes \mathrm{SU}(n)^{\mathrm{opp}},\ \mathrm{SU}(n) \otimes \mathrm{SU}(n)^{\mathrm{opp}}, \\ \{a \otimes a^* : a \in \mathrm{SL}(n,\mathbf{C})\}. \end{array} \tag{9}$$

Proof. In view of Theorem A and Theorem 1, we may assume that Γ is neither a complex Lie group nor a compact group. Let $L \subset \mathrm{SL}(n^2, \mathbf{C})$ be the smallest complex Lie subgroup containing Γ, and $\Gamma_0 \supset G_0$ a maximal compact subgroup of Γ. Then L is either $\mathrm{SL}(n^2, \mathbf{C})$ or one of the groups listed in Theorem A. We shall consider the various possibilities for L separately. If L is reducible and semisimple, there are four cases to consider.

Case 1. $L = \mathrm{SL}(n^2-1, \mathbf{C})$. By Lemma 1, Γ is a real form of L. For the list of real forms of almost simple complex Lie groups see [4]. By Theorem 1, $\Gamma_0 = G_0$, $\mathrm{SO}(n^2-1)$, or Λ_0 if $n = 4$. Hence Γ is not isomorphic to $\mathrm{SU}(p,q)$ for $p+q = n^2-1$, $p \geq q \geq 1$. If $\Gamma \cong \mathrm{SL}(n^2-1, \mathbf{R})$, then Lemma 2 implies that $\Gamma = \mathrm{SL}(n^2-1, \mathbf{R})$. The real form $\mathrm{SL}((n^2-1)/2, \mathbf{H})$, n odd, is ruled out because its maximal compact subgroup is $\mathrm{Sp}((n^2-1)/2)$.

Case 2. $L = \mathrm{SO}(n^2-1, \mathbf{C})$. By Lemma 1, Γ is a real form of L. By Theorem 1, $\Gamma_0 = G_0$ or Λ_0 if $n = 4$. The real forms $\mathrm{SO}(p,q)^0$, $p+q = n^2-1$, $p \geq q \geq 1$, are ruled out because their maximal compact subgroups are not isomorphic to G_0 or Λ_0 if $n = 4$. The real form $\mathrm{SO}^*(n^2-1)$, n odd, is also ruled out by the same argument.

Case 3. $L = G$. By Lemma 1, Γ must be a real form of G. Since $\Gamma \supset G_0$ and we assume that Γ is not compact, this case cannot occur.

Case 4. $L = \Lambda$ $(n = 4)$. By Lemma 1, Γ is a real form of Λ. Hence Γ is isomorphic to $\mathrm{SU}(p, q)/\langle -1\rangle$, $p + q = 6$, $p \geq q \geq 1$, $\mathrm{SL}(3, \mathbf{H})/\langle -1\rangle$, or Λ_1. By inspecting the maximal compact subgroups of these real forms we see that $\Gamma \cong \Lambda_1$, and so $\Gamma = \Lambda_1$ by Lemma 2.

Next we assume that L is reducible and reductive, but not semisimple. Then the maximal semisimple subgroup H of Γ contains G_0. By taking into account the previous cases, we conclude that H is one of the groups listed in (6). Since T_1 is the centralizer of G_0 in $\mathrm{SL}(n^2, \mathbf{C})$, we have $\Gamma = HR$ where R is a non-trivial connected Lie subgroup of T_1.

When L is irreducible, there are three cases to consider.

Case 5. $L = \mathrm{SL}(n^2, \mathbf{C})$. By Lemma 1, Γ is a real form of L. Assume first that $\Gamma \cong \mathrm{SL}(n^2, \mathbf{R})$. Then, by Theorem 1, $\Gamma_0 = t \cdot \mathrm{SO}(n^2) \cdot t^{-1}$ for some $t \in T$. By Lemma 2 we have $\Gamma = t \cdot \mathrm{SL}(n^2, \mathbf{R}) \cdot t^{-1}$. Next assume that $\Gamma \cong \mathrm{SU}(p, q)$ where $p + q = n^2, p \geq q \geq 1$. Theorem 1 implies that $p = n^2 - 1$ and $q = 1$. By the same argument as above, we conclude that $\Gamma = t \cdot \mathrm{SU}(n^2 - 1, 1) \cdot t^{-1}$ for some $t \in T$. The real form $\mathrm{SL}(n^2/2, \mathbf{H})$, n even, is ruled out because its maximal compact subgroup $\mathrm{Sp}(n^2/2)$ does not occur in Theorem 1.

Case 6. $L = t \cdot \mathrm{SO}(n^2, \mathbf{C}) \cdot t^{-1}$, $t \in T$. Without any loss of generality, we may assume that $t = 1$. By Lemma 1, Γ is isomorphic to $\mathrm{SO}(p, q)^0$, $p+q = n^2$, $p \geq q \geq 1$, or $\mathrm{SO}^*(n^2)$, n even. The latter case is ruled out because a maximal compact subgroup of $\mathrm{SO}^*(n^2)$ is isomorphic to $\mathrm{U}(n^2/2)$. In the former case, Theorem 1 implies that $p = n^2 - 1$, $q = 1$, and that $\Gamma_0 = \mathrm{SO}(n^2 - 1)$. By Lemma 2, we conclude that $\Gamma = \mathrm{SO}(n^2 - 1, 1)^0$.

Case 7. $L = t \cdot (\mathrm{SL}(n\mathbf{C}) \otimes \mathrm{SL}(n, \mathbf{C})^{\mathrm{opp}}) \cdot t^{-1}$, $t \in T_1$. Again we may assume that $t = 1$. Since

$$\Gamma \supset G_0 = \{a \otimes a^{-1} : a \in \mathrm{SU}(n)\},$$

Lemma 1 implies that Γ is either one of the groups

$$\mathrm{SL}(n, \mathbf{C}) \otimes \mathrm{SU}(n)^{\mathrm{opp}},\ \mathrm{SU}(n) \otimes \mathrm{SL}(n, \mathbf{C})^{\mathrm{opp}},$$

or a real form of L. In the latter case Γ is either $\mathrm{SU}(n) \otimes \mathrm{SU}(n)^{\mathrm{opp}}$ or the group $\{a \otimes a^* : a \in \mathrm{SL}(n, \mathbf{C})\} \cong \mathrm{PSL}_n(\mathbf{C})$.

It remains to consider the cases where L is not reductive. By Theorem A, L is reducible and has the form HP, HQ, HPT_1, HQT_1 with $H = G$, $\mathrm{SO}(n^2 - 1, \mathbf{C})$, $\mathrm{SL}(n^2 - 1, \mathbf{C})$ or Λ if $n = 4$.

If $\Gamma \supset P$ (or Q), then Γ is a semidirect product of P (or Q) and some reducible reductive group $F \supset G_0$. We have shown that $F = HR$ where H is one of the groups in (6) and R is a connected Lie subgroup of T_1. Hence Γ is one of the groups in (5).

Let $\mathfrak{p}$ (resp $\mathfrak{p}_0$) be the Lie algebra of P (resp. P_0). Then $\mathfrak{p} = \mathfrak{p}_0 \oplus i\mathfrak{p}_0$ and $\mathfrak{p}_0$ is a simple G_0-module. Furthermore every simple G_0-submodule of $\mathfrak{p}$ has the form $\alpha\mathfrak{p}_0$ for some $\alpha \in \mathbf{C}^*$, and so it is a U_1-conjugate of $\mathfrak{p}_0$.

Assume that $L \supset P$ but $\Gamma \not\supset P$. Let $\mathfrak{r}$ be the solvable radical of the Lie algebra of Γ. If $\mathfrak{t}_1$ is the Lie algebra of the torus T_1, then $\mathfrak{r} \subset \mathfrak{p} + \mathfrak{t}_1$. As $\mathfrak{t}_1$ is a trivial G_0-module, we must have $\mathfrak{r} = (\mathfrak{r} \cap \mathfrak{p}) \oplus (\mathfrak{r} \cap \mathfrak{t}_1)$. As $L \supset P$, we have $\mathfrak{r} \cap \mathfrak{p} \neq 0$. As $\Gamma \not\supset P$, we have $\mathfrak{r} \cap \mathfrak{p} \neq \mathfrak{p}$. Therefore $\mathfrak{r} \cap \mathfrak{p} = \alpha\mathfrak{p}_0$ for some $\alpha \in \mathbf{C}^*$. By replacing Γ with a suitable U_1-conjugate, we can assume that $\mathfrak{r} \cap \mathfrak{p} = \mathfrak{p}_0$. It follows that $\Gamma \cap P = P_0$ and that Γ is a semidirect product of P_0 and a reducible reductive group $F \supset G_0$. By using the description of such F, we conclude that Γ is one of the groups in (7). □

Corollary 1. *Let Γ be a proper connected Lie subgroup of $\mathrm{GL}(n^2, \mathbf{C})$ containing G_0. If Γ is reducible, Γ is either one of the groups (5) where R is now a connected Lie subgroup of T, or a U_1-conjugate of one of the groups HP_0R, HQ_0R, where H is as in (8) and R is a connected Lie subgroup of R_1Z. If Γ is irreducible, then $\Gamma = HR$ where H is $\mathrm{SL}(n^2, \mathbf{C})$ or a T-conjugate of a group in (9) and R is a connected Lie subgroup of Z.*

For applications, in particular to linear preserver problems, it is important to have a description of all (abstract) overgroups of G_0 in $\mathrm{GL}(n^2, \mathbf{C})$. Let L be a Lie group and Γ an abstract subgroup of L. Denote by Σ the collection of all connected Lie subgroups of L which are contained in Γ. Since arcwise connected subgroups of L are Lie subgroups [3], Σ has a unique largest element, say Γ^0. Thus Γ^0 contains all subgroups in Σ. By construction, Γ^0 is a normal subgroup of Γ. Consequently we have the following theorem.

Theorem 3. *Let Γ be an abstract subgroup of $\mathrm{GL}(n^2, \mathbf{C})$ containing G_0, and let Γ^0 be the largest connected Lie subgroup contained in Γ. Then*

$$G_0 \subset \Gamma^0 \subset \Gamma \subset N(\Gamma^0),$$

where $N(\Gamma^0)$ is the normalizer of Γ^0 in $\mathrm{GL}(n^2, \mathbf{C})$.

Now the description of all overgroups of G_0 is reduced to the computation of the normalizers of all possible connected Lie overgroups of G_0.

Theorem 4. *Let Γ be a connected Lie subgroup of $\mathrm{GL}(n^2, \mathbf{C})$ containing G_0, and let $N(\Gamma)$ be the normalizer of Γ in $\mathrm{GL}(n^2, \mathbf{C})$. As in Corollary 1, let H be a maximal semisimple subgroup of Γ containing G_0. Then*

- (i) *$N(\Gamma) = \Gamma T$ if H is one of the groups $\mathrm{SL}(n^2 - 1, \mathbf{C})$, $\mathrm{SU}(n^2 - 1)$, or Λ, Λ_0 $(n = 4)$;*
- (ii) *$N(\Gamma) = \Gamma T\langle\tau\rangle$ if H is one of the groups G, G_0, or Λ_1 $(n = 4)$;*
- (iii) *$N(\Gamma) = \mathrm{O}(n^2 - 1)\Gamma T$ if H is one of the groups $\mathrm{SO}(n^2 - 1, \mathbf{C})$, $\mathrm{SO}(n^2 - 1)$, or $\mathrm{SL}(n^2 - 1, \mathbf{R})$;*
- (iv) *$N(\Gamma) = \Gamma Z$ if H is one of the groups $\mathrm{SL}(n^2, \mathbf{C})$, $\mathrm{SU}(n^2)$, $\mathrm{SU}(n^2 - 1, 1)$, $\mathrm{SU}(n) \otimes \mathrm{SL}(n, \mathbf{C})^{\mathrm{opp}}$, or $\mathrm{SL}(n, \mathbf{C}) \otimes \mathrm{SU}(n)^{\mathrm{opp}}$;*
- (v) *$N(\Gamma) = \Gamma Z\langle\tau\rangle$ if H is one of the groups $\mathrm{SL}(n, \mathbf{C}) \otimes \mathrm{SL}(n, \mathbf{C})^{\mathrm{opp}}$, $\mathrm{SU}(n) \otimes \mathrm{SU}(n)^{\mathrm{opp}}$, or $\{a \otimes a^* : a \in \mathrm{SL}(n, \mathbf{C})\}$;*
- (vi) *$N(\Gamma) = \mathrm{O}(n^2 - 1)\Gamma Z$ if H is one of the groups $\mathrm{SO}(n^2, \mathbf{C})$, $\mathrm{SO}(n^2)$, $\mathrm{SO}(n^2 - 1, 1)^0$, or $\mathrm{SL}(n^2, \mathbf{R})$.*

Proof. All normalizers will be taken in $\mathrm{GL}(n^2, \mathbf{C})$. As the maximal semisimple subgroups of Γ are all conjugate in Γ, we have $N(\Gamma) \subset \Gamma N(H)$. Let H_0 be a maximal compact subgroup of H containing G_0. As the maximal compact subgroups of H are all conjugate in H, we have $N(H) \subset HN(H_0)$. Hence

$$N(\Gamma) \subset \Gamma N(H_0). \tag{10}$$

Let $L \subset \mathrm{SL}(n^2, \mathbf{C})$ be the complexification of H_0. By [5, Theorem B], we have

$$N(L) = \begin{cases} GT\langle\tau\rangle & \text{if } L = G, \\ LT & \text{if } L = \mathrm{SL}(n^2-1, \mathbf{C}) \text{ or } \Lambda \ (n=4), \\ \mathrm{O}(n^2-1, \mathbf{C})T & \text{if } L = \mathrm{SO}(n^2-1, \mathbf{C}), \\ \mathrm{O}(n^2, \mathbf{C})Z & \text{if } L = \mathrm{SO}(n^2, \mathbf{C}), \\ LZ\langle\tau\rangle & \text{if } L = \mathrm{SL}(n, \mathbf{C}) \otimes \mathrm{SL}(n, \mathbf{C})^{\mathrm{opp}}. \end{cases}$$

Since H_0 is its own normalizer in H, see [4, p. 275], and τ normalizes G_0 and $\mathrm{SU}(n) \otimes \mathrm{SU}(n)^{\mathrm{opp}}$, it follows easily that

$$N(H_0) = \begin{cases} G_0T\langle\tau\rangle & \text{if } H_0 = G_0, \\ H_0T & \text{if } H_0 = \mathrm{SU}(n^2-1) \text{ or } \Lambda_0 \ (n=4), \\ \mathrm{O}(n^2-1)T & \text{if } H_0 = \mathrm{SO}(n^2-1), \\ \mathrm{O}(n^2)Z & \text{if } H_0 = \mathrm{SO}(n^2), \\ H_0Z\langle\tau\rangle & \text{if } H_0 = \mathrm{SU}(n) \otimes \mathrm{SU}(n)^{\mathrm{opp}}, \\ H_0Z & \text{if } H_0 = \mathrm{SU}(n^2). \end{cases}$$

We can now determine $N(\Gamma)$ by using (10), Theorem 2 and Corollary 1. For each H_0 and all possible H's and Γ's, it is easy to check that the identity component of $N(H_0)$ normalizes H and Γ. We shall now consider the cases where $N(H_0)$ is not connected.

If $H_0 = G_0$, then H is one of the groups G_0, G, Λ_1 $(n = 4)$, or $\{a \otimes a^* : a \in \mathrm{SL}(n, \mathbf{C})\}$. If $n = 4$, τ normalizes Λ, see [5], and we have shown in Section 5 that $\Lambda_1 = \Lambda \cap \mathrm{GL}(n^2, \mathbf{R})$. It follows that τ normalizes Λ. Thus in all cases τ normalizes H (and Γ).

If $H_0 = \mathrm{SO}(n^2-1)$, then H is one of the groups H_0, $\mathrm{SL}(n^2-1, \mathbf{R})$, $\mathrm{SO}(n^2-1, \mathbf{C})$, or $\mathrm{SO}(n^2-1, 1)^0$. In all these cases $\mathrm{O}(n^2-1)$ normalizes H (and Γ).

If $H_0 = \mathrm{SO}(n^2)$, then either $H = H_0$ or $\mathrm{SO}(n^2, \mathbf{C})$. In both cases $\mathrm{O}(n^2)$ normalizes H (and Γ).

Finally if $H_0 = \mathrm{SU}(n) \otimes \mathrm{SU}(n)^{\mathrm{opp}}$, then H is one of the groups H_0, $\mathrm{SL}(n, \mathbf{C}) \otimes \mathrm{SL}(n, \mathbf{C})^{\mathrm{opp}}$, $\mathrm{SU}(n) \otimes \mathrm{SL}(n, \mathbf{C})^{\mathrm{opp}}$, or $\mathrm{SL}(n, \mathbf{C}) \otimes \mathrm{SU}(n)^{\mathrm{opp}}$. Since $\tau \circ (a \otimes b) \circ \tau = b' \otimes a'$, only the first two of these groups are normalized by τ.

In view of (10), this completes the proof of the theorem. ☐

This theorem shows that $N(\Gamma)$ is not much larger than Γ. For instance we have $N(\Gamma)^0 = \Gamma T$ or ΓZ, and $[N(\Gamma) : N(\Gamma)^0]$ is 1 or 2.

5. Linear preserver problem for normal matrices

Let $X \subset M_n$ be the set of normal matrices, i.e., matrices x satisfying $xx^* = x^*x$. Since X is a (real) algebraic set, the subset

$$G_X := \{a \in \mathrm{GL}(n^2, \mathbf{C}) : a(X) \subset X\}$$

is in fact a subgroup of $\mathrm{GL}(n^2, \mathbf{C})$, see [5]. It is also evident that G_X is closed and $G_X \supset G_0$. We shall illustrate our method for solving a wide class of linear preserver problems by applying it to determine the group G_X. The groups Q, T and the element τ, used in the next theorem, have been defined in previous sections.

Theorem 5. *If $X \subset M_n$ is the set of normal matrices, then $G_X = HQT\langle\tau\rangle$ where $H = G_0$ for $n > 2$ and $H = \mathrm{SL}(3, \mathbf{R})$ for $n = 2$. If $n > 2$, this means that any nonsingular linear transformation u of M_n such that $u(X) \subset X$ has the form*

$$u(x) = \alpha a x a^{-1} + \psi(x) \cdot 1,$$

where $\alpha \in \mathbf{C}^*$, $a \in U(n)$, and $\psi : M_n \to \mathbf{C}$ is a linear function.

Proof. Let $u \in G_X$ and $a = u(1)$ where 1 is the identity matrix. Since $1 + X = X$ and $u(X) = X$, we have $a + X = X$. Hence

$$(a + x)(a + x)^* = (a + x)^*(a + x), \quad x \in X,$$

i.e., $ax^* - x^*a = a^*x - xa^*$ for all $x \in X$. By replacing x with ix, we infer that $ax^* = x^*a$ for all $x \in X$. As X spans M_n, we conclude that a is a central element of M_n, i.e., $u(1) = a = \lambda \cdot 1$ for some $\lambda \in \mathbf{C}^*$. Hence the subspace $\mathbf{C} \cdot 1$ is G_X-invariant and G_X is reducible. It is easy to check that $G_X \supset QT$.

By Theorem 2, the identity component, G_X^0, of G_X has the form $G_X^0 = HQT$ where H is one of the groups listed in (6).

If $n = 2$ we claim that $H = \mathrm{SL}(3, \mathbf{R})$. Let $u \in \mathrm{SL}(3, \mathbf{R})$, which means that $u(1) = 1$, u preserves the space of hermitian matrices of trace 0, and $\det(u) = 1$. Each $x \in X$ can be written uniquely as $x = \lambda \cdot 1 + x_0$ with $x_0 \in X$ and $\mathrm{tr}(x_0) = 0$. Furthermore, $x_0 = a + ib$ where a and b are commuting hermitian matrices. As a and b have real traces and $\mathrm{tr}(x_0) = 0$, we must have $\mathrm{tr}(a) = \mathrm{tr}(b) = 0$. Since a and b are unitarily diagonalizable, it is easy to see that $ab = ba$ implies that a and b are linearly dependent over $\mathbf{R}$. Since $u(x_0) = u(a) + iu(b)$, it follows that $u(x_0)$, and also $u(x)$, is a normal matrix. As $\mathrm{SL}(3, \mathbf{C}) \not\subset G_X$, our claim follows by inspecting the list (6).

From now on let $n > 2$. We claim that $G_X \not\supset \mathrm{SO}(n^2 - 1)$. Indeed let x and y be the diagonal matrices

$$x = \mathrm{diag}(1, -1, 0, \ldots, 0), \quad y = \mathrm{diag}(1, 1, -2, 0, \ldots, 0),$$

and let

$$z = \begin{pmatrix} 0 & \sqrt{3} \\ \sqrt{3} & 0 \end{pmatrix} \oplus 0_{n-2}.$$

These are hermitian matrices of trace 0. As $\mathrm{tr}(xy) = \mathrm{tr}(xz) = 0$ and $\mathrm{tr}(y^2) = \mathrm{tr}(z^2) = 6$, there exists $u \in \mathrm{SO}(n^2 - 1)$ such that $u(x) = x$ and $u(y) = z$. As $xy = yx$, $x + iy$ is normal. On the other hand $xz \neq zx$, and so $u(x + iy) = x + iz$ is not normal. Hence $u \notin G_X$, and our claim is proved.

It is clear that $G_X \not\supset G$. By inspecting the list (6), we infer that $H = G_0$ if $n \neq 4$. We claim that this is also true for $n = 4$. It suffices to show that Λ_0 and Λ_1 are not contained in G_X. Let $\{v_i\}$ be the basis of K_4 introduced in Section 2. Since $\mathrm{tr}(v_1^* v_3) = \mathrm{tr}(v_1^* v_6) = 0$ and $\mathrm{tr}(v_3^* v_3) = \mathrm{tr}(v_6^* v_6) = 2$, there exists $a \in \mathrm{SU}(6)$ such that $a(v_1) = v_1$ and $a(v_6) = v_3$. As $v_1 f(v_6) - v_6 f(v_1) = v_1^2 - v_6^2$ is normal, while

$$\rho(a)(v_1 f(v_6) - v_6 f(v_1)) = v_1 f(v_3) - v_3 f(v_1) = -v_1 v_4 - v_3 v_6$$

is not, we infer that $\rho(a) \notin G_X$, and so $G_X \not\supset \Lambda_0$.

The matrices

$$\begin{aligned} w_1 &= v_1 + v_6, & w_2 &= v_2 - v_5, & w_3 &= v_3 + v_4, \\ w_4 &= i(v_1 - v_6), & w_5 &= i(v_2 + v_5), & w_6 &= i(v_3 - v_4), \end{aligned}$$

form a basis of the real subspace of K_4 defined by the equation $f(x) = \bar{x}$. If x and y belong to this subspace, then it is easy to check that $xf(y) - yf(x)$ is a skew-hermitian matrix. Let $a \in \mathrm{SL}(6, \mathbf{R})$ be such that

$$a(w_1) = w_1, \quad a(w_2) = w_1 + w_2, \quad a(w_4) = w_4, \quad a(w_5) = w_6.$$

Since

$$\begin{aligned} w_1 f(w_4) - w_4 f(w_1) &= -w_1 w_4 - w_4 w_1 = -2 w_1 w_4 \\ &= 2i \cdot \mathrm{diag}(1, 1, -1, -1), \\ w_2 f(w_5) - w_5 f(w_2) &= -w_2 w_5 - w_5 w_2 = -2 w_2 w_5 \\ &= 2i \cdot \mathrm{diag}(1, -1, 1, -1), \end{aligned}$$

the matrix

$$x = [w_1 f(w_4) - w_4 f(w_1)] + i[w_2 f(w_5) - w_5 f(w_2)]$$

is normal. On the other hand, it is easy to check that the (1,3) entry of the matrix

$$(w_1 + w_2) f(w_6) - w_6 f(w_1 + w_2) = -(w_1 + w_2) w_6 - w_6 (w_1 + w_2)$$

is not 0. Consequently this matrix does not commute with $w_1 f(w_4) - w_4 f(w_1)$, and so the matrix

$$\rho(a)(x) = [w_1 f(w_4) - w_4 f(w_1)] + i[(w_1 + w_2) f(w_6) - w_6 f(w_1 + w_2)]$$

is not normal. Thus $\rho(a) \notin G_X$, and so $G_X \not\supset \Lambda_1$.

Since $\tau \in G_X$, the assertion of the theorem follows from Theorem 4. □

References

1. D. Ž. Đoković and V. P. Platonov, *Algebraic groups and linear preserver problems*, C. R. Acad. Sci. Paris Sér. I Math. **317** (1993), 925–930. MR **94i:**20080
2. M. Goto and F. Grosshans, *Semisimple Lie algebras*, Dekker, New York, 1978. MR **58:**28084
3. M. Goto, *On an arcwise connected subgroup of a Lie group*, Proc. Amer. Math. Soc. **20** (1969), 157–162. MR **38:**2244
4. S. Helgason, *Differential geometry, Lie groups, and symmetric spaces*, Academic Press, New York, 1978. MR **80k:**53081
5. V. P. Platonov and D. Ž. Đoković, *Linear preserver problems and algebraic groups*, Preprint 94-029, Discrete Structures in Math., University of Bielefeld, 1994.
6. *A survey of linear preserver problems*, Linear and Multilinear Algebra **33** (1992), 1–130.

Department of Pure Mathematics, University of Waterloo, Waterloo, Ontario, N2L 3G1 Canada

E-mail address: dragomir@herod.uwaterloo.ca