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A TRANVERSALITY THEOREM FOR HOLOMORPHIC MAPPINGS AND STABILITY OF EISENMAN-KOBAYASHI MEASURES

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Abstract. We show that Thom's Transversality Theorem is valid for holomorphic mappings from Stein manifolds. More precisely, given such a mapping $f : S \to M$ from a Stein manifold S to a complex manifold M and given an analytic subset A of the jet space $J^k(S, M)$, f can be approximated in neighborhoods of compacts by holomorphic mappings whose k-jet extensions are transversal to A. As an application the stability of Eisenman-Kobayshi intrinsic k-measures with respect to deleting analytic subsets of codimension $> k$ is proven. This is a generalization of the Campbell-Howard-Ochiai-Ogawa theorem on stability of Kobayashi pseudodistances.

1. Introduction

1.1. Let X and M be connected complex manifolds. Denote by $\mathrm{Hol}\,(X, M)$ the space of holomorphic mappings $X \to M$ and by $J^k(X, M)$ the space of k-jets of holomorphic mappings $X \to M$. We say that $A \subset J^k(X, M)$ is a stratified analytic subset if A is a closed analytic subset in $J^k(X, M)$ (regarded as a complex manifold) with a stratification which satisfies Whitney's condition (a). Recall that for every closed analytic subset such stratification always exists [W]. The symbol $j_k(f) \pitchfork A$ where $f \in \mathrm{Hol}\,(X, M)$ means that the k-jet extension $j_k(f) : X \to J^k(X, M)$ of f is transversal to each stratum of the given stratification of A.

The main result of this paper is the following analytic version of Thom's Transversality Theorem [T]. (To our surprise we did not find this fact in the literature, and it must be formulated.)

1.2. Transversality Theorem. *Let X and M be as above. Suppose that X is Stein and M is endowed with a Hermitian metric h. Let $A_i \subset J^{k_i}(X, M)$ be a sequence of stratified analytic subsets where k_i is a nonnegative integer and $i = 1, 2, \ldots$. Let $f \in \mathrm{Hol}\,(X, M)$ and let Y be a closed analytic subset of X such that $j^{k_i}(f)\big|_Y \pitchfork A_i$ for every i (where the restriction is regarded as a mapping from Y to $J^{k_i}(X, M)$). Then for each compact subset $K \subset X$, every natural k, and every $\varepsilon > 0$ there exist an open neighborhood Ω of K in X and a holomorphic mapping $\tilde{f} = f_{\varepsilon,K,k} : \Omega \to M$ such that*

(1) $j^{k_i}(\tilde{f}) \pitchfork A_i$, $i = 1, 2, \ldots$;

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(2) $j^k(\tilde{f})\big|_{Y\cap\Omega} = j^k(f)\big|_{Y\cap\Omega}$;
(3) $\mathrm{dist}_h(f\big|_\Omega, \tilde{f}) < \varepsilon$ *uniformly in* Ω.

In the algebraic setting this theorem can be strengthened due to the Demailly-Lempert-Shiffman Approximation Theorem [DLS].

1.3. Corollary. *Let the assumption of Theorem* 1.2 *hold. Suppose also that* X *is an affine algebraic variety,* M *is a smooth quasiprojective variety,* Y *is a closed algebraic subvariety of* X*, and for some* $k > 0$ *the mapping* $j^k(f)\big|_Y : Y \to J^k(X, M)$ *is a regular mapping. Then for each compact* $K \subset X$*, every* $\varepsilon > 0$*, and every natural* n *there exist a neighborhood* Ω *of* K *in* X *and a Nash algebraic mapping* $\tilde{f} : \Omega \to M$ *so that conditions* (1) *for* $i = 1, ..., n$*,* (2)*,* (3) *from Theorem* 1.2 *hold.*

Recall that $\tilde{f}$ being *Nash algebraic* means that the graph of $\tilde{f}$ is contained in an algebraic subvariety $N \subset X \times M$ where $\dim N = \dim X$; the same holds for the image of $\tilde{f}$.

1.4. Let M be a connected complex manifold of complex dimension $m > 0$. Then for any $k = 1, ..., m$ the Eisenman k-measure E^k_M is intrinsicly defined on the set of decomposable k-wedge vectors in $\Lambda^k TM$ and it has the following basic properties:

-every mapping of the k-wedge vector bundles of complex manifolds generated by a holomorphic mapping does not increase Eisenman measures;

-E^k_M coincides with the Bergman volume in the case when M is the unit ball B^k in $\mathbb{C}^k$;

Moreover, the Eisenman k-measure is the largest measure on decomposable k-wedge vectors with the two above properties. Further information on Eisenman measures and their applications may be found in [E], [P], [Ko 1], [Ko 2], [LZ], [PS], [GW], [Ka 2]. Mention also that E^1_M is nothing but the Kobayashi-Royden infinitesimal pseudometric K_M on TM, and E^m_M is the Eisenman-Kobayashi pseudovolume form on $\Lambda^m TM$.

By Campbell-Howard-Ochiai-Ogawa's theorem [CHO], [CO] the Kobayashi pseudodistance k_M of M does not change under deleting analytic subsets of M of codimension at least 2. From the Transversality Theorem 1.2 we obtain a generalization of this result.

1.5. Corollary. *Let* A *be a closed analytic subset of* M *of codimension at least* $k+1$*. Then* $E^k_{M\setminus A} = E^k_M\big|_{M\setminus A}$*. In particular, for* $k = 1$ *we have* $K_{M\setminus A} = K_M\big|_{M\setminus A}$ *if* $\mathrm{codim}_M A \geq 2$*.*

By Royden's Theorem [Ro] the Kobayashi pseudodistance k_M is the integrated form of K_M. Therefore, $k_{M\setminus A} = k_M|_{M\setminus A}$ if $\mathrm{codim}_M A \geq 2$ which is the Campbell-Howard-Ochiai-Ogawa Stability Theorem.

1.6. It is easily seen that the Eisenman-Kobayashi volume of the unit ball does change after deleting the origin (see [Ra] for qualified estimates). Thus in general condition $\mathrm{codim}_M A \geq k + 1$ of Corollary 1.5 cannot be made weaker. However, by a result of [Ka 1] if A is a smooth closed submanifold of M of codimension $\ell \geq 2$ then in some cases deleting A affects the Eisenman k-measure E^k_M exactly in the same way as blowing up $\sigma_A : M_A \to M$ of M with centrum at A. More precisely, if either $\ell \geq \left[\frac{k-1}{2}\right] + 2$ or the $(\ell - 1)$-st Chern class of the normal bundle of A is trivial then $\sigma^*_A E^k_{M_A\setminus A} = E^k_{M_A}\big|_{M_A\setminus E}$ where E is the exceptional divisor of σ_A in M_A. In other words, the Eisenman k-measure does not change under deleting of a smooth divisor which can be contracted onto a smooth submanifold of codimension $\geq \left[\frac{k-1}{2}\right] + 2$. In particular, deleting of the exceptional divisor of a blow-up with centrum at a point does not affect any of Eisenman k-measures. This stability result was a motivation for the present paper.

1.7. Returning to the Transversality Theorem note that in general an approximation of a given holomorphic mapping from a Stein manifold X by transversal ones (which does exists in neighborhoods of compacts, by Theorem 1.2) cannot be found on the whole X due to certain rigidity phenomena. This can be shown by simple examples.

1.8. Examples. a) Consider a smooth complex surface S with a (-1)-curve $E \cong \mathbb{P}^1$. Let $f : \mathbb{C} \to \mathbb{P}^1 \cong E \hookrightarrow S$ be an embedding. Put $A = f(0) \in S$. Then f cannot be uniformly approximated by holomorphic mappings $\tilde{f} : \mathbb{C} \to S$ transversal to A, since after blowing E down the image of $\tilde{f}(\mathbb{C})$ would be contained in a ball with center at the image of E, but not coincide with this point. Thus, $\tilde{f}$ itself must be constant which is absurd. Moreover, if S is a blow-up of a hyperbolic surface (for instance, S is the blow-up of the unit ball B^2 in $\mathbb{C}^2$ at the origin) then the above f cannot be approximated (not necessarily uniformly) on compacts by $\tilde{f} \in \text{Hol}\,(\mathbb{C}, S)$ transversal to A, by the same reason.

b) Let $\pi : S \to C$ be a morphism of a smooth projective surface S' onto a curve C whose generic fibers are pairwise non-isomorphic curves of genus $g \geq 2$. Fix a generic fiber $F = \pi^{-1}(c_0)$ over a point $c_0 \in C$ and two distinct points $A, B \in F$. Put $R = F - \{B\}$. Then R is an open Riemann surface, i.e. it is Stein. The identical embedding $f : R \hookrightarrow F \hookrightarrow S$ cannot be approximated uniformly by holomorphic mappings $\tilde{f} : R \to S$ transversal to A. Indeed, $\pi \circ \tilde{f} \equiv \text{ const } = c \neq c_0$ and, therefore, $\tilde{f}(R) \subset F_c = \pi^{-1}(c)$. Since c can be assumed to be sufficiently close to c_0 the fiber F_c is a hyperbolic curve which is not isomorphic to F. But, by the big Picard theorem, the mapping $\tilde{f} : R \to F_c$ can be extended to an isomorphism $F \to F_c$ which is impossible.

c) Consider the Stein domain $D = \{z = (x, y) \in \mathbb{C}^2 \mid \mid x \mid < 1, \mid xy \mid < 1\}$. Clearly, the image of every non-constant holomorphic mapping $\mathbb{C} \to D$ is contained in the y-axis. Hence the identical embedding of this axis into D cannot be approximated on compacts by holomorphic mappings $\mathbb{C} \to D$ transversal to the origin.

d) In [BG] a family of surfaces in $\mathbb{P}^3$ was constructed which gives a smooth proper morphism $\pi : M \to \Delta$ of a threefold M onto the unit disc so that all fibers of π except for the central one $F_0 = \pi^{-1}(0)$ are hyperbolic, and F_0 is a Fermat surface of degree 50. It is known [G] that any entire curve $f : \mathbb{C} \to F_0$, $f \not\equiv$ const, is contained in a finite union of projective lines in F_0. Thus the same is true for every entire curve in M due to the Liouville theorem and the hyperbolicity of nonzero fibers. Therefore, none of these curves can be pushed away from a point on it by means of small deformations.

2. Preliminaries

Let, as above, X and M be connected complex manifolds. By $J^k_x(X, M)$ we denote the stalk at x of k-jets of holomorphic mappings $X \to M$. If V is an open subset in X then the jet space $J^k(V, M)$ can be viewed as a domain in $J^k(X, M)$. For a holomorphic vector bundle $\zeta = (T, \pi, X)$ we treat k-jets of its holomorphic sections as k-jets of holomorphic mappings $X \to T$.

Let A be a closed analytic subset of a complex manifold N and Σ be an analytic stratification of A which satisfies Whitney's condition (a). By $A_{\Sigma,l}$ we denote the union of all strata of Σ of dimension $\leq l$. It is known that $A_{\Sigma,l}$ is a closed analytic subset of N. By Whitney's condition (a), if $f \in \mathrm{Hol}\,(X,N)$ is transversal to $A_{\Sigma,l}$ then f is transversal to A in a neighborhood of $A_{\Sigma,l}$ as well as any $\tilde{f} \in \mathrm{Hol}\,(X,N)$ which is sufficiently uniformly close to f in $C^1(X,N)$-topology.

3. Some lemmas on holomorphic vector fields and collective transversality

As in the smooth case the proof of the Transversality Theorem is based on the Collective Transversality Lemma (see [T], [AVG]) which is the main result of this section. To prove it we need several simple lemmas on holomorphic vector fields. In the first of them we show that fixing the k-jet of a vector field at a point yields fixing the k-jet of the associated phase flow at this point.

3.1. Lemma. *Let ν be a germ of a holomorphic vector field at the origin $\overline{0}_r \in \mathbb{C}^r$ with the phase flow $\varphi_{\nu,t}$, where t is the complex time in a neighborhood of the origin in $\mathbb{C}$. If $j^k(\nu)(\overline{0}_r) = \overline{0}$ for some $k \geq 0$ then for any t with $\mid t \mid \ll 1$ one has $j^k(\varphi_{\nu,t})(\overline{0}_r) = j^k(\mathrm{id}_{\mathbb{C}^r})(\overline{0}_r)$.*

Proof. Since the phase flow $\varphi_{\nu,t} : (\mathbb{C}^r \times \mathbb{C}, \overline{0}_r \times 0) \to (\mathbb{C}^r, \overline{0}_r)$ is holomorphic, it has a convergent power series expansion

$$\varphi_{\nu,t}(\overline{x}) = \overline{x} + t\nu(\overline{x}) + \sum_{i\geq 2} t^i \nu_i(\overline{x}) \tag{1}$$

in some neighborhood of the origin $\overline{0}_r \times 0 \in \mathbb{C}^r \times \mathbb{C}$, where $\overline{x} \in \mathbb{C}^r$ and $\nu_i(\overline{x})$, $i = 2, ...,$ are germs of holomorphic vector fields at $\overline{0}_r \in \mathbb{C}^r$. Therefore,

$$j^k(\varphi_{\nu,t})(\overline{x}') = j^k(\mathrm{id}_{\mathbb{C}^r})(\overline{x}') + tj^k(\nu)(\overline{x}') + \sum_{i\geq 2} t^i j^k(\nu_i)(\overline{x}'). \tag{2}$$

Thus we have to show that the condition $j^k(\nu)(\overline{0}_r) = \overline{0}$ implies that $j^k(\nu_i)(\overline{0}_r) = \overline{0}$ for all $i \geq 2$. Note that $\varphi_{\nu,t+s} = \varphi_{\nu,s} \circ \varphi_{\nu,t}$ when $\mid s \mid$ and $\mid t \mid$ are small enough. Put $\nu_1(\overline{x}) = \nu(\overline{x})$ and compare the two convergent expansions:

$$\varphi_{\nu,t+s}(\overline{x}) = \overline{x} + \sum_{i\geq 1} (t+s)^i \nu_i(x) \tag{3}$$

and

$$\begin{aligned} &\varphi_{\nu,s} \circ \varphi_{\nu,t}(\overline{x}) \\ &\quad = \overline{x} + \sum_{i\geq 1} t^i \nu_i(\overline{x}) + s\nu_1\left(\overline{x} + \sum_{i\geq 1} t^i \nu_i(\overline{x})\right) + \sum_{i\geq 2} s^i \nu_i \left(\overline{x} + \sum_{n\geq 1} t^n \nu_n(\overline{x})\right). \end{aligned} \tag{4}$$

Suppose by induction that $j_k(\nu_i)(\overline{0}_r) = \overline{0}$ for $i \leq l$ where $l \geq 1$. Assume that $j^k(\nu_{l+1})(\overline{0}_r) \neq \overline{0}$. By (3), we obtain

$$\begin{aligned} j^k(\varphi_{\nu,t+s})(\overline{0}_r) = j_k(\mathrm{id}_{\mathbb{C}^r})(\overline{0}_r) + (t+s)^{l+1} j^k(\nu_{l+1})(\overline{0}_r) \\ + (\text{higher order terms in } (t+s)). \end{aligned}$$

At the same time (4) implies

$$j^k(\varphi_{\nu,s}\circ\varphi_{\nu,t})(\overline{0}_r) = j^k(\mathrm{id}_{\mathbb{C}^r})(\overline{0}_r) + (t^{l+1}+s^{l+1})j^k(\nu_{l+1})(\overline{0}_r)$$
$$+ \text{(higher order terms in } t \text{ and } s).$$

Since $l+1 \geq 2$ this leads to contradiction. ■

3.2. Definition. Let $T \to N$ be a holomorphic vector bundle on a complex manifold N, $\mathcal{O}(T)$ be the sheaf of germs of holomorphic sections of T, and p be a point of N. A linear subspace $V \subset \mathcal{O}_p(T)$ will be called *k-sufficient* if the set of k-jets of germs from V coincides with the whole stalk $J^k_p(T)$, i.e. for any germ $\nu \in \mathcal{O}_p(T)$ there exists a germ $\mu \in V$ such that $j^k(\mu)(p) = j^k(\nu)(p)$. A linear subspace $W \subset H^0(N, \mathcal{O}(T))$ will be called *k-sufficient at p* if the subspace V of germs at p of sections from W is k-sufficient.

3.3. Denote by $\mathcal{T}_s$ the ${}_s\mathcal{O}$-module of germs of holomorphic vector fields at the origin $\overline{0}_s \in \mathbb{C}^s$. If $V \subset \mathcal{T}_s$ is a finite dimensional subspace then all germs from V can be represented by vector fields holomorphic in a common neighborhood $U \ni \overline{0}_s$ in $\mathbb{C}^s$. Moreover, one can choose U so that all phase flows $\varphi_{\nu,t}$ are defined correctly on U for every $\nu \in V$ and sufficiently small $|t|$. Furthermore, we may suppose that $\varphi_{\nu,t}$ is defined correctly on U for every $t \in \overline{\Delta} = \{t \in \mathbb{C} \mid |t| < 1\}$ as soon as $\nu \in \omega$, where ω is a neighborhood of the zero germ $\overline{0}_V$ in V.

Consider the holomorphic mapping $\rho_V : \omega \ni \nu \longmapsto \varphi_{\nu,1} \in \mathrm{Hol}\,(U, \mathbb{C}^s)$ and its k-jet extension at the origin $\rho_{V,k} : \omega \ni \nu \longmapsto j^k(\varphi_{\nu,1})(\overline{0}_s) \in J^k_{\overline{0}_s}((\mathbb{C}^s, \overline{0}_s), \mathbb{C}^s)$.

3.4. Lemma. *Let a finite dimensional subspace $V \subset \mathcal{T}_s$ be k-sufficient. Then the differential $d\rho_{V,k}$ at the origin $\overline{0}_V \in \omega \subset V$ is surjective.*

Proof. Since $\varphi_{t_1\nu,t_2} = \varphi_{\nu,t_1t_2}$ we have $\varphi_{t\nu,1} = \varphi_{\nu,t}$. Applying expansions (1) and (2) one obtains that for $\nu \in V$

$$\rho_{V,k}(t\nu) = j^k(\mathrm{id}_{\mathbb{C}^s})(\overline{0}_s) + tj^k(\nu)(\overline{0}_s) + \sum_{i\geq 2} t^i j^k(\nu_i)(\overline{0}_s).$$

Therefore, the derivative $\partial\rho_{V,k}/\partial\nu$ at the direction ν at the origin $\overline{0}_V \in V$ is equal to $j^k(\nu)(\overline{0}_s)$. Since V is k-sufficient the image of the differential $d\rho_{V,k}(\overline{0}_V)$ coincides with the whole stalk $J^k_{\overline{0}_s}(T\mathbb{C}^s) = J^k_{\overline{0}_s}((\mathbb{C}^s, \overline{0}_s), \mathbb{C}^s)$. ■

3.5. For $r \leq s$ we regard $\mathbb{C}^r$ as a coordinate subspace in $\mathbb{C}^s : \mathbb{C}^r = \{\overline{x}' \in \mathbb{C}^s | \overline{x}' = (x_1, ..., x_r, 0, ..., 0)\}$. The restriction to this subspace defines for any nonnegative integer k the projection $\pi^k_{s,r} : J^k((\mathbb{C}^s, \overline{0}_s), \mathbb{C}^s) \to J^k((\mathbb{C}^r, \overline{0}_r), \mathbb{C}^s)$.

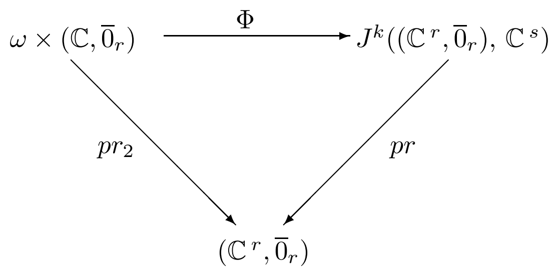
Let V and ω be as in 3.3. Consider the holomorphic mapping

$$\Phi = \Phi_{V,r,k} : \omega \times (\mathbb{C}^r, \overline{0}_r) \to J^k((\mathbb{C}^r, \overline{0}_r), \mathbb{C}^s)$$

given by the formula

$$\Phi(\nu, \overline{x}') = \pi^k_{s,r} \circ j^k(\varphi_{\nu,1})(\overline{x}').$$

We have the following commutative diagram:



where pr_2 and pr are the canonical projections. Since the restriction

$$\Phi\big|_{(pr_2)^{-1}(\overline{0}_r)} : \omega \to J^k_{\overline{0}_r}((\mathbb{C}^r, \overline{0}_r), \mathbb{C}^s) = pr^{-1}(\overline{0}_r)$$

coincides with the composition $\pi^k_{s,r} \circ \rho_{V,k}$, the above diagram and Lemma 3.4 imply the following fact.

3.6. Corollary. *Let a finite dimensional subspace $V \subset \mathcal{T}_s$ be k-sufficient. Then the differential $d\Phi$ at the origin $\overline{0}_V \times \overline{0}_r \in V \times \mathbb{C}^r$ is surjective.*

3.7. Lemma. *Let Y be a closed analytic subset of a Stein manifold X and let $x_0 \in X \backslash Y$. Suppose that T is a holomorphic vector bundle on X. Then for every nonnegative integer k there exists a holomorphic section σ of T with a given k-jet ν at x_0 and such that $j^k(\sigma)\big|_Y \equiv \overline{0}$.*

Proof. Let I_Y (resp. I_{x_0}) be the ideal sheaf of Y (resp. x_0) in $\mathcal{O}_X$. Put

$$\mathcal{T} = \mathcal{O}_X(T), \quad \mathcal{T}' = \mathcal{T} \otimes_{\mathcal{O}_X} (I_Y)^k, \quad \mathcal{T}'' = \mathcal{T}' \otimes_{\mathcal{O}_X} (I_{x_0})^{k+1}.$$

Note that $\mathcal{T}'$ (resp. $\mathcal{T}''$) can be regarded as a subset of $\mathcal{T}$ (resp. $\mathcal{T}'$). Put $J = \mathcal{T}'/\mathcal{T}''$. Then $J\big|_{X \backslash \{x_0\}}$ is trivial. It is also clear that $J\big|_{\{x_0\}}$ coincides with the stalk $J^k_{x_0}(T)$ of k-jets of germs at x_0 of holomorphic sections of T. Thus every k-jet ν at x_0 may be viewed as a global holomorphic section σ_0 of J. By the Cartan Theorem B applied to the coherent sheaf $\mathcal{T}''$, we have $H^1(X, \mathcal{T}'') = 0$. Hence the exact cohomology sequence implies that the section σ_0 is the image of a holomorphic section $\sigma \in H^0(X, \mathcal{T}')$. Clearly, σ is a section of T with the desired properties. ■

3.8. Corollary. *Under the assumption of Lemma 3.7 there exists a finite dimensional subspace $V \subset H^0(X, \mathcal{O}(T))$ such that $j^k(\sigma)\big|_Y \equiv \overline{0}$ for all $\sigma \in V$ and V is k-sufficient at x_0, i.e. $J^k_{x_0}(T) = J^k_{x_0}(V)$.*

Proof. Choose a basis $v_1, ..., v_N$ of $J^k_{x_0}(T)$. By Lemma 3.7, for each $i = 1, ..., N$ there exists $\nu_i \in H^0(X, \mathcal{O}(T))$ such that $j^k(\nu_i)\big|_Y \equiv 0$ and $j^k(\nu_i)(x_0) = v_i$. Put $V = \text{span}\,(\nu_1, ..., \nu_N)$. ■

Denote by $\mathcal{T}_M = H^0(M, \mathcal{O}(TM))$ the space of global holomorphic vector fields on a complex manifold M.

3.9. Lemma. *Let X be a closed submanifold of a Stein manifold M, Y be a closed analytic subset of X, and $K \subset X$, $K' \subset K \backslash Y$ be compact subsets. Then there exist a neighborhood U of K in X (resp. U' of K' in $X \backslash Y$), a finite dimensional subspace $V \subset \mathcal{T}_M$, and a neighborhood ω of the origin $\overline{0}_V \in V$ such that the following conditions hold:*

(i) $j^k(v)\big|_Y \equiv \overline{0}$ *for each* $\nu \in V$;

(ii) *for every* $\nu \in \omega$ *the associated phase flow* $\varphi_{\nu,t}$ *is defined on* $U \times \overline{\Delta}$;

(iii) *if* Φ *is the holomorphic mapping* $\Phi : \omega \times U \ni (\nu, x) \longmapsto j^k(\varphi_{\nu,1})(x) \in J^k(U, M)$ *then the differential* $d\Phi$ *is surjective at each point* $(\nu, x') \in \omega \times U'$.

Proof. By Corollary 3.8, for each point $x' \in K'$ there exists a finite dimensional subspace $V_{x'} \subset \mathcal{J}_M$ which is k-sufficient at x' and satisfies the above condition (i). Of course, $V_{x'}$ is k-sufficient at each point of some small neighborhood $U'_{x'}$ of x' in X. Let $\{U'_{x'_i}\}_{i=1}^{N'}$ be a finite covering of K' by such neighborhoods. Put $U' = \bigcup_{i=1}^{N'} U'_{x'_i}$ and $V = \bigoplus_{i=1}^{N'} V_{x'_i}$. Then V is a finite dimensional subspace of $\mathcal{T}_M$ which satisfies (i) and is k-sufficient at each point $x' \in U'$.

Furthermore, for each point $x \in K$ there exist a neighborhood U_x in X and a neighborhood $\omega_x \ni \overline{0}_V$ in V such that for every $\nu \in \omega_x$ the associated phase flow $\varphi_{\nu,t}$ is holomorphic in $U_x \times \overline{\Delta}$. Choose a finite covering $\{U_{x_j}\}_{j=1}^{N}$ of the compact K in X and put $U = \bigcup_{j=1}^{N} U_{x_j}, \quad \omega = \bigcap_{j=1}^{N} \omega_{x_j}$. Then U and ω satisfy condition (ii). By Corollary 3.6, condition (iii) also holds. ■

Next we give an analytic version of the Collective Transversality Lemma [AVG].

3.10. Lemma. *Under the assumption of Lemma* 3.9 *for every stratified analytic subset* $A \subset J^k(X, M)$ *there exists a dense subset* ω_0 *of* ω *such that for each* $\nu \in \omega_0$ *the mapping*

$$j^k(\varphi_{\nu,1})\big|_{U'} : U' \to J^k(U', M) \subset J^k(X, M)$$

is transversal to A.

Proof. Denote by $(A_\sigma)_{\sigma\in\Sigma}$ the collection of strata of the given stratification Σ of A which satisfies Whitney's condition (a). Put $\tilde{A} = \Phi^{-1}(A) \subset \omega \times U'$. Since $d\Phi$ is surjective at each point $(\nu, x') \in \omega \times U'$ it is clear that $\{\tilde{A}_\sigma = \Phi^{-1}(A_\sigma)\}_{\sigma\in\Sigma}$ is an analytic stratification of $\tilde{A}$ which also satisfies Whitney's condition (a). Furthermore, the mapping $j^k(\varphi_{\nu,1})\big|_{U'} : U' \to J^k(X, M)$ is transversal to A iff the identical embedding $U'_\nu := \{\nu\} \times U' \hookrightarrow \omega \times U'$ is transversal to $\tilde{A}$. The latter means that for each $\sigma \in \Sigma$ the differential of the canonical projection $pr_1 : \omega \times U' \to \omega$ restricted to $\tilde{A}$ is surjective at each of the sets $U'_\nu \cap \tilde{A}_\sigma$. Put $\pi_\sigma = pr_1\big|_{\tilde{A}_\sigma} : \tilde{A}_\sigma \to \omega$ and $S_\sigma = \{x \in \tilde{A}_\sigma \big| \text{rank} \ \ d\pi_\sigma(x) < \dim \omega\}$. By Proposition I.3.8 in [Ch], the image $\pi_\sigma(S_\sigma) \subset \omega$ is contained in at most countable union of proper analytic subsets of ω (not necessarily closed). The same is valid for the "discriminant" set $D = \bigcup_{\sigma\in\Sigma} \pi_\sigma(S_\sigma) \subset \omega$. Therefore, the desired transversality condition holds for any ν in the dense subset $\omega_0 = \omega \backslash D$ of ω. ■

4. Proof of the Transversality Theorem

4.1. Lemma. *Let* X *be a closed submanifold of a Stein manifold* M, Y *be a closed analytic subset of* X, *and* K *be a compact in* X. *Suppose that* A *is a stratified analytic subset of* $J^k(X, M)$. *Let* f *be the identical embedding and let* $j^k(f)\big|_Y \pitchfork A$. *Then in a neighborhood* Ω *of* K *in* X *the mapping* f *can be uniformly approximated* (*with respect to a given Hermitian metric h on M*) *by holomorphic mappings* $\tilde{f}$: $\Omega \to M$ *such that* $j^k(\tilde{f}) \pitchfork A$ *and* $j^k(\tilde{f})\big|_{Y\cap\Omega} = j^k(f)\big|_{Y\cap\Omega}$.

Proof. Since the given stratification Σ of A satisfies Whitney's condition (a) and $j^k(f)\big|_Y \pitchfork A$, there exists a neighborhood W of Y in X for which $j^k(f)\big|_W \pitchfork A$. Put $K' = K\backslash W'$, where $W' \subset W$ is a smaller neighborhood of Y such that $\overline{W}' \subset W$. Let $U \supset K$, $U' \supset K'$, $V \subset \mathcal{T}_M$, and $\omega \subset V$ be the same as in Lemma 3.9. Let Ω be a neighborhood of K so that $\Omega \subset\subset (U' \cup W') \cap U$. Put $f_\nu = \varphi_{\nu,1}|_\Omega$ for $\nu \in \omega$. Since (A, Σ) satisfies Whitney's condition (a) and $j^k(f_\nu) \to j^k(f)|_\Omega$ as $\nu \to \overline{0}_V \in V$ there exists a smaller neighborhood $\omega' \subset \omega$ of the origin $\overline{0}_V \in V$ for which $j^k(f_\nu)\big|_{W'\cap\Omega} \pitchfork A$ for all $\nu \in \omega'$. By Lemma 3.10, we also have $j_k(f_\nu)\big|_{U'\cap\Omega} \pitchfork A$ for every ν in a dense subset ω_0' of ω'. So, $j^k(f_\nu)\big|_\Omega \pitchfork A$ for every $\nu \in \omega_0'$. Condition $j^k(\nu)|_Y \equiv \overline{0}$ and Lemma 3.1 imply that $j^k(f_\nu)|_{Y\cap\Omega} = j^k(\varphi_{\nu,1})|_{Y\cap\Omega} = j^k(\mathrm{id}\,_X)|_{Y\cap\Omega} = j^k(f)|_{Y\cap\Omega}$. We conclude the proof by noting that $\tilde{f} = f_\nu$ uniformly converges to $f|_\Omega = \mathrm{id}$ on Ω when $\nu \in \omega_0'$ approaches $\overline{0}_V$. ■

The rest of this section is a reduction of Theorem 1.2 to Lemma 4.1. The first step is to replace $f = \mathrm{id}\,_X$ in Lemma 4.1 by an arbitrary holomorphic mapping $f : X \to M$.

4.2. Lemma. *Let X be a Stein manifold, M be a complex manifold endowed by a Hermitian metric h, $f : X \to M$ be a holomorphic mapping. Let Y, K, A be the same as in Lemma* 4.1, *and as before $j^k(f) \pitchfork A$. Then the conclusion of Lemma* 4.1 *holds for such f.*

Proof. Let $F = (\mathrm{id}\,_X, f)$ be the embedding of $X \hookrightarrow X \times M$ onto the graph of f. In sequel we identify X with its image under F and consider F as the identical embedding. Denote by $pr_* : j^k(X, X \times M) = J^k(X, X) \times J^k(X, M) \to J^k(X, M)$ the forgetting projection. It is easily seen that the analytic subset $\tilde{A} = (pr_*)^{-1}(A) \subset J^k(X, X \times M)$ with the induced stratification is a stratified analytic subset, i.e. the Whitney condition (a) holds. Furthermore, the assumption $j_k(f)\big|_Y \pitchfork A$ yields $j^k(F)\big|_Y \pitchfork \tilde{A}$. Let g be any Hermitian metric on X and let $\tilde{h}$ be the Hermitian metric on $X \times M$ that is the Euclidian sum of g and h. By Siu's theorem [S], the Stein submanifold $X = F(X) \hookrightarrow X \times M$ admits a Stein neighborhood M' in $X \times M$. By Lemma 4.1 applied to the identical embedding F of $X = F(X)$ into the Stein manifold M', in a neighborhood Ω of a compact K the mapping F can be uniformly approximated (with respect to the metric $\tilde{h}$) by holomorphic mappings $\tilde{F} : \Omega \to M' \hookrightarrow X \times M$ such that $j^k(\tilde{F})|_{Y\cap\Omega} = j^k(F)|_{Y\cap\Omega}$ and $j^k(\tilde{F}) \pitchfork \tilde{A}$. Put $\tilde{f} = pr_M \circ \tilde{F}$ where $pr_M : X \times M \to M$ is the canonical projection. Clearly, $\tilde{f}$ uniformly approximates $f|_\Omega$ with respect to the metric h on M, $j^k(\tilde{f})\big|_{Y\cap\Omega} = j^k(f)\big|_{Y\cap\Omega}$, and $j^k(\tilde{f}) \pitchfork A$. Therefore, all conclusions of Lemma 4.1 hold. ■

4.3. *Proof of Theorem* 1.2. For any holomorphic submersion $\varphi : E \to B$ of complex manifolds which is a smooth fiber bundle and for every Hermitian metric h on B, using smooth partition of unity, one can construct a Hermitian metric $\tilde{h}$ on E so that $\varphi^* h \leq \tilde{h}$. By this simple remark, we may fix a sequence of Hermitian metrics

h_l on $J^l(X,M)$ such that all natural projections $\pi_{m,n} : J^m(X,M) \to J^n(X,M)$ (where $n < m$) and $\pi_l : J^l(X,M) \to M$ (M being endowed with a Hermitian metric h) are contractions with respect to these metrics. Let $U \subset X$ be an open subset and $g : U \to M$ be a holomorphic mapping. If $A \subset J^n(X,M)$ is a stratified analytic subset and for $m > n$ the preimage $\pi_{m,n}^{-1}(A)$ is endowed with the induced stratification, then it is easily seen that conditions $j^n(g) \pitchfork A$ and $j^m(g) \pitchfork \pi_{m,n}^{-1}(A)$ are equivalent. Hence, passing to appropriate preimages of the given stratified analytic subsets $A_i \subset J^{k_i}(X,M),\ i = 1,...,$ we may suppose that $k \le k_i < k_j$ for all $i < j,\ i,j = 1,....$ Next we fix a relatively compact neighborhood Ω of K in X which is a Runge domain in X. To see that Ω exists it is enough to take the intersection of the Stein manifold X embedded in $\mathbb{C}^N$ with a large ball B_R^N containing the image of K. Fix also a bigger Runge neighborhood Ω_0 (for instance, put $\Omega_0 = X \cap B_{R+1}^N$). Replacing K by the compact $L = \overline{\Omega}_0$, we will construct by induction a decreasing sequence ... $\subset\subset \Omega_{l+1} \subset\subset \Omega_l \subset\subset \cdots \subset\subset \Omega_1$ of neighborhoods of L in X and a sequence of holomorphic mappings $f_l : \Omega_l \to M$ which satisfy

(1_l) $j^{k_l}(f_l) \pitchfork A_l$,
(2_l) $j^k(f_l)\big|_{Y\cap\Omega_l} = j^k(f)\big|_{Y\cap\Omega_l}$,
(3_l) $\operatorname{dist}_{h_{k_l}}(j^{k_l}(f_l), j^{k_l}(f_{l-1})) < \delta_l \le 2^{-l}\varepsilon$ uniformly on Ω_l,

where $\delta_l > 0$ will be defined later on, and where for $l = 1$ in (3_l) $f_0 := f$. By our choice of Hermitian metrics h_i, given such a sequence $\{f_l\}$, we have

($3'$) $\operatorname{dist}_h(f_l, f_{l-1}) < 2^{-l}\varepsilon$

for all l uniformly in Ω_0. Therefore, $f_l \xrightarrow{l} \tilde{f}$ uniformly in Ω_0 where $\tilde{f} \in \mathrm{Hol}(\Omega_0, M)$ satisfies condition (2) and (3) of Theorem 1.2. To ensure condition (1) we need to choose $\{\delta_l\}$ in an appropriate way.

By the Cauchy integral formula, given an open subset $U \subset X$ and a sequence $g_i \in \mathrm{Hol}(U,M)$ convergent to $g \in \mathrm{Hol}(U,M)$ uniformly on a compact subset $R \subset U$, we have a uniform convergence on R of the l-jet extensions $j^l(g_i)$ to $j^l(g)$ for every natural l. Thus, by Lemma 4.2, there exists a neighborhood Ω_1 of L in X and a holomorphic mapping $f_1 : \Omega_1 \to M$ such that $j^{k_1}(f_1) \pitchfork A_1,\ j^{k_1}(f_1)\big|_{Y\cap\Omega_1} = j^{k_1}(f)\big|_{Y\cap\Omega_1}$, and $\operatorname{dist}_{h_{k_1}}(j^{k_1}(f_1),\ j^{k_1}(f)) < \varepsilon/2$. Hence f_1 satisfies $(1_1) - (3_1)$ with $\delta_1 = \varepsilon/2$. Passing to a smaller neighborhood of L we may suppose that Ω_1 is Stein. Let $\delta_i,\ i = 1,...,l,$ be already constructed so that f_i satisfies $(1_i) - (3_i)$ for $i \le l$. By Lemma 4.2 and the note above applied to the holomorphic mapping $f_l : \Omega_l \to M$, for every δ_{l+1} such that $0 < \delta_{l+1} < 2^{-(l+1)}\varepsilon$ there exist a smaller Stein neighborhood $\Omega_{l+1} \subset\subset \Omega_l$ of the compact L in X and a holomorphic mapping $f_{l+1} : \Omega_{l+1} \to M$ which satisfy $(1_{l+1}) - (3_{l+1})$. Now for each $i = 1,2,...,l$ we have $\operatorname{dist}_{h_{k_{i+1}}}(j^{k_{i+1}}(f_i)\ |_{\Omega_{l+1}},\ j^{k_{i+1}}(f_{l+1})) < \sum_{p=i+1}^{l+1} \delta_p$. Since $k_i < k_{i+1}$ and, by condition (1), $j^{k_i}(f_i) \pitchfork A_i$ there exists $\mu_i > 0$ such that for any $g \in \mathrm{Hol}(\Omega_0, M)$ the condition

$$\operatorname{dist}_{h_{k_{i+1}}}(j^{k_{i+1}}(g), j^{k_{i+1}}(f_i)) < \mu_i \tag{5}$$

in Ω_0 yields $j^{k_i}(g)\big|_{\Omega} \pitchfork A_i$. So, we will choose $\{\delta_i\}$ in such a way that

$$\sum_{p=i+1}^{\infty} \delta_p < \mu_i \tag{6}$$

for every $i = 1, ...$ By induction, we may suppose that the condition

$$\rho_i := \sum_{p=i+1}^{l} \delta_p < \mu_i/2, \quad i = 1, ..., l-1,$$

is already true. Choose $\delta_{l+1},\ 0 < \delta_{l+1} < 2^{-(l+1)}\varepsilon$ so that $\delta_{l+1} < \min\limits_{1\le i\le l-1} (\mu_i/2 - \rho_i)$ and $\delta_{l+1} < \mu_l/2$. This implies (6) for every $i = 1, ...$ Therefore, the limit mapping $g := \tilde{f} = \lim\limits_{l\to\infty} f_l \in \mathrm{Hol}\,(\Omega_0, M)$ satisfies (5). Hence $j^{k_i}(\tilde{f})\Big|_{\Omega} \pitchfork A_i$ which completes the proof of Theorem 1.2.

5. Stability of Eisenman intrinsic k-measures

5.1. *Proof of Corollary* 1.5. Denote by $\overline{e}_k$ the k-vector

$$\frac{\partial}{\partial z_1} \wedge \cdots \wedge \frac{\partial}{\partial z_k} \in \Lambda^k T_{\overline{0}_k} B^k$$

at the origin $\overline{0}_k \in B^k \subset \mathbb{C}^k$. Recall [E], [P], [Ko 1], [GW] that for a complex manifold M and a decomposable vector $\overline{v} = v_1 \wedge \cdots \wedge v_k \in \Lambda^k T_p M$, where $p \in M$ and $1 \le k \le m = \dim\, M$, the Eisenman k-measure of $\overline{v}$ is defined by the formula

$$E_M^k(p, \overline{v}) = \inf\{\lambda^2 \big| \lambda > 0 \text{ is such that there exists } f \in \mathrm{Hol}\,(B^k, M) \\ \text{with } f(\overline{0}_k) = p \text{ and } df(\lambda\overline{e}_k) = \overline{v}\}.$$

If A is a closed analytic subset of M of codimension at least $k+1$ and $p \in M\backslash A$ then, by Transversality Theorem 1.2, every holomorphic mapping $f : B^k \to M$ with $f(\overline{0}_k) = p$ and $df(\lambda\overline{e}_k) = \overline{v}$ can be approximated by holomorphic mappings of smaller balls $f_\varepsilon : B^k_{1-\varepsilon} \to M\backslash A$ with $f_\varepsilon(\overline{0}_k) = p$ and $df_\varepsilon(\lambda e_k) = v$. This observation implies Corollary 1.5 in the Introduction.

Indeed, consider the mapping $\tilde{f}_\varepsilon(z) = f_\varepsilon((1-\varepsilon)z),\ \tilde{f}_\varepsilon \in \mathrm{Hol}\,(B^k, M\backslash A),\ \tilde{f}_\varepsilon(\overline{0}_k) = p,\ d\tilde{f}_\varepsilon(\overline{e}_k) = (1-\varepsilon)\lambda\overline{v}$. By the above definition, we obtain

$$E^k_{M\backslash A}(p, \overline{v}) \le E^k_M(p, \overline{v}).$$

The opposite inequality holds since embeddings do not increase Eisenman measures.

In [PS] it was mentioned that the Campbell-Howard-Ochiai-Ogawa Stability Theorem is still true for any subset of M of Hausdorff $(2m-2)$-measure zero. Here we present the similar generalization of Corollary 1.5.

5.2. Proposition. *Let M be a connected complex manifold of complex dimension m and let A be its subset of Hausdorff $(2m-2k)$-measure 0 for some $k \in \{1, 2, ..., m\}$. Then $E^k_{M\backslash A} = E^k_M\big|_{M\backslash A}$.*

The proof is similar to the proof of Corollary 1.5, but instead of Theorem 4.3 we have to use the following modification of it.

5.3. Theorem. *Let M be a connected complex manifold of $\dim_{\mathbb{C}} M = m$ with a Hermitian metric h, and let A be its closed subset of Hausdorff $(2m-2k)$-measure zero for some $k \in \{1, ..., m\}$. Let X be a Stein manifold of $\dim_{\mathbb{C}} X = k$, Y be a closed analytic subset of X, and $f : X \to M$ be a holomorphic mapping such that $f(Y) \subset M\backslash A$. Then for any compact $K \subset X$ the mapping f can be uniformly (with respect to h) approximated in a neighborhood Ω of K by holomorphic mappings $\tilde{f} : \Omega \to M\backslash A$ such that $j^1(f)\big|_{Y\cap\Omega} = j^1(\tilde{f})\big|_{Y\cap\Omega}$.*

Proof. We follow the lines of the proof of Theorem 1.2. As in Lemma 4.2, replacing f by the embedding $F = (\mathrm{id}_X, f)$ onto the graph of f and replacing M by a Stein neighborhood of the graph in $X \times M$, we may suppose that M is Stein, X is a closed submanifold of M, and f is the identical embedding $X \hookrightarrow M$. Note that under this replacement the codimension of X in M is still equal to the dimension of vanishing Hausdorff measure of A (A being replaced by $pr_2^{-1}(A) \subset X \times M$). Since $f(Y) \cap A = \emptyset$ there exists a neighborhood W of $Y \cap K$ in X such that $f(W) \cap A = W \cap A = \emptyset$. Fix a smaller relatively compact neighborhood $W' \subset\subset W$ and $\varepsilon > 0$ such that for each $\tilde{f} \in \mathrm{Hol}\,(W', M)$ with $\mathrm{dist}_h(\tilde{f}, f|_{W'}) < \varepsilon$ uniformly in W' one has $\tilde{f}(W') \cap A = \emptyset$. Put $K' = K - W'$. By Lemma 3.9, there exist neighborhoods U of K in X, U' of K' in X, a finite dimensional subspace $V \subset \mathcal{T}_M$, and a neighborhood ω' of the origin $\overline{0}_V \in V$ such that the following conditions hold:

(i) $j^1(\nu)|_Y \equiv \overline{0}$ for every $\nu \in V$;

(ii) $\Phi \in \mathrm{Hol}\,(\omega' \times U, M)$ where $\Phi(\nu, z) = \varphi_{\nu,1}(z)$;

(iii) $\mathrm{dist}_h(\varphi_{\nu,1}|_U,\ f|_U) < \varepsilon$ uniformly in U;

(iv) the rank of $d\Phi$ is $m = \dim_{\mathbb{C}} M$ at each point $z \in U'$.

By Lemma 3.1, (i) and (ii) imply that $\varphi_{\nu,1}|_{U\cap Y} = f|_{U\cap Y} = \mathrm{id}\,|_{U\cap Y}$, and, by (iii), $\varphi_{\nu,1}(U \cap W') \subset M \backslash A$ for each $\nu \in \omega'$. By virtue of Theorem 2 from [Ch, Appendix, 2] condition (iv) implies that the mapping $\Phi|_{\omega' \times U'}$ may be locally treated as a projection. From Proposition 7 in [Ch, Appendix, 6] it easily follows that the preimage $A' = (\Phi|_{\omega' \times U'})^{-1}(A) \subset \omega' \times U'$ has Hausdorff $2l$-measure zero where $l = \dim_{\mathbb{C}} V$. Therefore, by Property 4 of Hausdorff measures as it was listed in [Ch, Appendix, 6], we have that the image A'' of A' under the projection $\omega' \times U' \to \omega'$ also has Hausdorff $2l$-measure zero. It follows that $\omega' \backslash A''$ is a dense subset of ω'. For each $\nu \in \omega' \backslash A''$ the image $\varphi_{\nu,1}(U') = \Phi\{\{\nu\} \times U'\} \subset M$ does not meet A. Thus for this ν we have $\varphi_{\nu,1}(U' \cup W') \cap A = \emptyset$. Take a neighborhood Ω of K in X such that $\Omega \subset (U' \cup W') \cap U$. Then $\tilde{f} := \varphi_{\nu,1}|_\Omega \in \mathrm{Hol}\,(\Omega, M)$, where $\nu \in \omega' \backslash A''$ and $\nu \to \overline{0}_V$, gives the desired approximation of f. ∎

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