

QUASIADDITIVITY AND MEASURE PROPERTY OF
CAPACITY AND THE TANGENTIAL BOUNDARY
BEHAVIOR OF HARMONIC FUNCTIONS

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Dedicated to Professor F.-Y. Maeda on the occasion of his sixtieth birthday

ABSTRACT. We show that if a set E is dispersely decomposed into subsets, then the capacity of E is comparable to the summation of the capacities of the subsets. From this fact it is derived that the Lebesgue measure of a certain expanded set is estimated by the capacity of E . These properties hold for classical capacities, L^p -capacities and energy capacities of general kernels. The estimation is applied to the boundary behavior of harmonic functions. We introduce a boundary thin set and show a fine limit type boundary behavior of harmonic functions. We show that a thin set does not meet essentially Nagel-Stein and Nagel-Rudin-Shapiro type approaching regions at almost all boundary points.

1. INTRODUCTION

Let $K(r) \not\equiv 0$ be a nonnegative nonincreasing lower semicontinuous (l. s. c.) function for $r > 0$. We assume that

$$\lim_{r \rightarrow 0} K(r) = \infty \text{ and } \lim_{r \rightarrow \infty} K(r) = 0.$$

For $x \in \mathbb{R}^N$ we define $K(x) = K(|x|)$, and assume that $K(x)$ is locally integrable on \mathbb{R}^N , in other words

$$\int_0^\infty K(t)t^{N-1}dt < \infty.$$

We define the capacity C_K by

$$C_K(E) = \inf\{\|\mu\| : K * \mu \geq 1 \text{ on } E\}.$$

Let $K_\alpha(r) = r^{\alpha-N}$ for $0 < \alpha < N$. This is the Riesz kernel of order α . If $K(r) = K_\alpha(r)$, then we write C_α for C_K and call it α -capacity.

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Let $h(r)$ be a positive nondecreasing function for $r > 0$ and $h(0) = 0$. Such a function is called a measure function. We define the content M_h by

$$M_h(E) = \inf\left\{\sum h(r_j) : E \subset \bigcup B(x_j, r_j)\right\},$$

where $B(x, r)$ stands for the open ball with center at x and radius r . If $h(r) = r^\beta$, then we write M_β for M_h and call it β -content.

It is well known that the capacity C_α and the content M_β are closely related:

(i) If $\beta > N - \alpha$, then $C_\alpha(E) = 0 \implies M_\beta(E) = 0$.

(ii) If $\beta \leq N - \alpha$, then $M_\beta(E) = 0 \implies C_\alpha(E) = 0$.

Since C_α is countably subadditive and homogeneous of degree $N - \alpha$, it follows that

$$C_\alpha(E) \leq AM_{N-\alpha}(E),$$

which readily yields (ii). It is noteworthy that we cannot put $\beta = N - \alpha$ in (i); there is a set E such that $C_\alpha(E) = 0$ and $M_{N-\alpha}(E) > 0$. Thus the C_α -capacity does not majorize the $(N - \alpha)$ -dimensional measure.

One of the main purposes of this paper is to compare C_α with a certain $(N - \alpha)$ -dimensional quantity. Hereafter we shall use the following notation. By the symbol A we denote an absolute positive constant whose value is unimportant and may change from line to line. We shall say that two positive quantities f and g are comparable, written $f \approx g$, if and only if there exists a constant A such that $A^{-1}g \leq f \leq Ag$. By $|E|$ we denote the Lebesgue measure of E . For a proper subset E we put $\delta_E(x) = \text{dist}(x, E^c)$. Let us consider the set

$$\tilde{E}_\alpha = \bigcup_{x \in E} B(x, \delta_E(x)^{(N-\alpha)/N}).$$

It is easy to see that if $E = B(0, r)$ and $r > 0$ is small, then \tilde{E}_α is a ball with radius comparable to $r^{(N-\alpha)/N}$, so that

$$|\tilde{E}_\alpha| \approx r^{N-\alpha} \text{ for small } r > 0.$$

Thus $|\tilde{E}_\alpha|$ is, in a sense, an $(N - \alpha)$ -dimensional quantity. We shall show the following theorem.

Theorem 1. *Let U be a bounded set. Suppose $E \subset U$ and let \tilde{E}_α be as above. Then*

$$|\tilde{E}_\alpha| \leq AC_\alpha(E),$$

where $A > 0$ depends only on N, α and U .

Let g_α be the Bessel kernel. The Riesz and the Bessel kernels have the same asymptotics as $r \rightarrow 0$. However, $g_\alpha(r)$ decreases rapidly as $r \rightarrow \infty$ and hence g_α is integrable on \mathbb{R}^N . It is well known that

$$C_\alpha(E) \approx C_{g_\alpha}(E) \text{ for } E \subset U,$$

where U is a bounded set. Hence Theorem 1 follows from Theorem 2 below. Let us consider the capacity of a ball. Suppose $K * \mu \geq 1$ on $B(0, r)$. Then $|B(0, r)| \leq \|K\|_1 \|\mu\|$ by Fubini's theorem, so that

$$(1) \quad C_K(B(0, r)) \geq \|K\|_1^{-1} |B(0, r)|$$

by definition. We define a positive function $\eta(r)$ by $|B(0, \eta(r))| = C_K(B(0, r))$ and let $\eta^*(r) = \max\{\eta(r), 2r\}$. It follows from (1) that

$$(2) \quad \eta(r) \approx \eta^*(r) \geq Ar.$$

We put

$$\tilde{E}_K = \bigcup_{x \in E} B(x, \eta^*(\delta_E(x))).$$

This is a set enlarged from E if E is open.

Theorem 2. *Let K be an integrable kernel. Then*

$$|\tilde{E}_K| \leq AC_K(E),$$

where $A > 0$ depends only on N and K .

Theorem 2 comes from the following quasiadditive property of C_K : if a set E is decomposed into pieces E_j dispersely, then $C_K(E)$ is comparable to $\sum C_K(E_j)$. More precisely we have the following theorem.

Theorem 3. *Let K be an integrable kernel. Suppose $\{B(x_j, \eta^*(r_j))\}$ is disjoint and E is an analytic subset of $\bigcup B(x_j, r_j)$. Then*

$$C_K(E) \leq \sum C_K(E \cap B(x_j, r_j)) \leq AC_K(E),$$

where $A > 0$ depends only on N and K .

Theorem 2 and Theorem 3 have counterparts in L^p -capacity theory. Let $1 < p < \infty$. We define

$$C_{K,p}(E) = \inf\{\|f\|_p^p : K * f \geq 1 \text{ on } E\}.$$

Let us consider the capacity of a ball. Suppose $K * f \geq 1$ on $B(0, r)$. Then $\|\chi_{B(0,r)}\|_p \leq \|K\|_1 \|f\|_p$, so that

$$(3) \quad C_{K,p}(B(0, r)) \geq \|K\|_1^{-p} |B(0, r)|$$

by definition. We define a positive function $\eta_p(r)$ by

$$|B(0, \eta_p(r))| = C_{K,p}(B(0, r))$$

and let $\eta_p^*(r) = \max\{\eta_p(r), 2r\}$. It follows from (3) that

$$(4) \quad \eta_p(r) \approx \eta_p^*(r) \geq Ar.$$

We put

$$\tilde{E}_{K,p} = \bigcup_{x \in E} B(x, \eta_p^*(\delta_E(x))).$$

Theorem 4. *Let K be an integrable kernel. Then*

$$|\tilde{E}_{K,p}| \leq AC_{K,p}(E),$$

where $A > 0$ depends only on N , K and p .

Theorem 5. *Let K be an integrable kernel. Suppose $\{B(x_j, \eta_p^*(r_j))\}$ is disjoint and E is an analytic subset of $\bigcup B(x_j, r_j)$. Then*

$$C_{K,p}(E) \leq \sum C_{K,p}(E \cap B(x_j, r_j)) \leq AC_{K,p}(E),$$

where $A > 0$ depends only on N , K and p .

We note that the quasiadditivity in Theorem 3 and Theorem 5 is different from the quasiadditivity considered in [3] and [4]. Theorem 2 and Theorem 4 have an application to the tangential boundary behavior of harmonic functions. We shall later generalize these theorems, in connection with Nagel-Stein approach regions ([8]). We shall introduce a notion of “thin sets” and combine it with the generalized version of Theorem 2 and Theorem 4 to obtain precise description of the tangential boundary behavior of harmonic functions given as the Poisson integral of certain potentials.

The plan of this paper is as follows. We prove Theorem 3 and Theorem 5 in Section 2 and Section 3, respectively. In Section 4 we prove Theorem 2 and Theorem 4. In Section 5 we introduce the Nagel-Stein approach region and generalize Theorem 2 and Theorem 4. The boundary behavior of harmonic function will be considered in Sections 6–10. We shall introduce a new notion of boundary thinness in Section 6.

Hereafter, we let K be an integrable kernel, unless otherwise specified.

2. PROOF OF THEOREM 3

We shall need to give a lower estimate for the capacity. To this end we use the dual definition of C_K .

Theorem A. *Let E be an analytic set. Then*

$$C_K(E) = \sup\{\|\mu\| : \mu \text{ is concentrated on } E, K * \mu \leq 1 \text{ on } \mathbb{R}^N\}.$$

We prepare an elementary lemma.

Lemma 1. *Let $0 < 2r \leq R$. Suppose $x \notin B(x_0, R)$ and let $\rho = \text{dist}(x, B(x_0, r))$. Then*

$$|B(x, \rho) \cap B(x_0, R)| \geq A|B(x_0, R)|,$$

where A depends only on the dimension.

Proof. Let x_1 be the point on the line segment connecting x_0 and x such that $|x_1 - x_0| = \frac{3}{4}R$. It is easy to see that

$$\rho = |x - x_0| - r = |x - x_1| + \frac{3}{4}R - r \geq |x - x_1| + \frac{1}{4}R.$$

We observe that

$$B(x_1, \frac{1}{4}R) \subset B(x, \rho) \cap B(x_0, R),$$

since if $y \in (x_1, \frac{1}{4}R)$, then

$$|x - y| < |x - x_1| + \frac{1}{4}R \leq \rho \text{ and } |x_0 - y| < |x_0 - x_1| + \frac{1}{4}R = R.$$

Thus the required inequality follows.

Proof of Theorem 3. The countable subadditivity yields

$$C_K(E) \leq \sum C_K(E \cap B(x_j, r_j)).$$

Let us prove $\sum C_K(E \cap B(x_j, r_j)) \leq AC_K(E)$. For simplicity we write E_j for $E \cap B(x_j, r_j)$. Let ε be an arbitrary positive number. By Theorem A we can find a measure μ_j such that

$$\begin{aligned} \mu_j &\text{ is concentrated on } E_j, \\ K * \mu_j &\leq 1 \text{ on } \mathbb{R}^N, \\ \|\mu_j\| &\geq C_K(E_j) - 2^{-j}\varepsilon. \end{aligned}$$

Let $\mu = \sum \mu_j$. Observe that μ is concentrated on E . We claim

$$(5) \quad K * \mu \leq A \text{ on } \mathbb{R}^N.$$

If we have (5), then Theorem A yields

$$C_K(E) \geq A\|\mu\| = A \sum \|\mu_j\| \geq A \sum C_K(E_j) - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, the required inequality follows.

For the proof of (5) we put

$$d\mu'_j = \frac{\|\mu_j\|}{|B(x_j, \eta^*(r_j))|} \chi_{B(x_j, \eta^*(r_j))} dx$$

and $\mu' = \sum \mu'_j$. This kind of measure was introduced by Borichev [5]. Since

$$\|\mu_j\| \leq C_K(E_j) \leq C_K(B(x_j, r_j)) = |B(x_j, \eta(r_j))| \leq |B(x_j, \eta^*(r_j))|,$$

it follows that

$$d\mu' \leq \sum \chi_{B(x_j, \eta^*(r_j))} dx \leq dx,$$

where we have used the disjointness of $B(x_j, \eta^*(r_j))$ in the last inequality. Hence

$$(6) \quad K * \mu' \leq K * 1 = \int_{\mathbb{R}^N} K dx < \infty \text{ on } \mathbb{R}^N.$$

Let us compare $K * \mu$ and $K * \mu'$. Suppose first $x \notin B(x_j, \eta^*(r_j))$. We apply Lemma 1 with $x_0 = x_j$, $R = \eta^*(r_j)$ and $r = r_j$. Let $\rho_j = \text{dist}(x, B(x_j, r_j))$. We have

$$\begin{aligned} K * \mu'_j(x) &\geq \int_{B(x, \rho_j) \cap B(x_j, \eta^*(r_j))} K(x - y) d\mu'_j(y) \\ &\geq K(\rho_j) \mu'_j(B(x, \rho_j) \cap B(x_j, \eta^*(r_j))) \\ &= K(\rho_j) \|\mu_j\| \frac{|B(x, \rho_j) \cap B(x_j, \eta^*(r_j))|}{|B(x_j, \eta^*(r_j))|} \\ &\geq AK(\rho_j) \|\mu_j\|. \end{aligned}$$

Obviously, $K * \mu_j(x) \leq K(\rho_j) \|\mu_j\|$, whence

$$(7) \quad K * \mu_j(x) \leq AK * \mu'_j(x).$$

Now suppose $x \in \bigcup B(x_j, \eta^*(r_j))$. Then (7) holds for all j . Hence we have from (6)

$$K * \mu(x) = \sum K * \mu_j(x) \leq A \sum K * \mu'_j(x) \leq AK * \mu'(x) \leq A.$$

Suppose $x \in \bigcup B(x_j, \eta^*(r_j))$, say $x \in B(x_{j_0}, \eta^*(r_{j_0}))$ for some j_0 . Then, by the disjointness of $\{B(x_j, \eta^*(r_j))\}$ we have $x \notin \bigcup_{j \neq j_0} B(x_j, \eta^*(r_j))$. Hence (6) and (7) yield

$$\sum_{j \neq j_0} K * \mu_j(x) \leq A \sum_{j \neq j_0} K * \mu'_j(x) \leq AK * \mu'(x) \leq A.$$

Since μ_{j_0} is an equilibrium measure, it follows that $K * \mu_{j_0} \leq 1$ on \mathbb{R}^N . Thus

$$K * \mu(x) = K * \mu_{j_0}(x) + \sum_{j \neq j_0} K * \mu_j(x) \leq A.$$

Therefore (5) follows. The theorem is proved.

3. PROOF OF THEOREM 5

First we prepare fundamental facts for $C_{K,p}$. We have the dual definition ([6, Theorem 14]). Let $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem B. *Let E be an analytic set. Then*

$$C_{K,p}(E) = \sup\{\|\mu\|^p : \mu \text{ is concentrated on } E, \|K * \mu\|_q \leq 1\}.$$

Proof of Theorem 5. The proof will be carried out in a fashion similar to that of Theorem 3. It is sufficient to show that $\sum C_{K,p}(E \cap B(x_j, r_j)) \leq AC_{K,p}(E)$ in case the left hand side is positive. For simplicity we write E_j for $E \cap B(x_j, r_j)$. Let ε be an arbitrary number such that

$$0 < \varepsilon < \sum C_{K,p}(E_j).$$

By Theorem B we can find a measure μ_j such that

$$\begin{aligned} \mu_j &\text{ is concentrated on } E_j, \\ \|K * \mu_j\|_q &= 1, \\ C_{K,p}(E_j) - 2^{-j}\varepsilon &\leq \|\mu_j\|^p \leq C_{K,p}(E_j). \end{aligned}$$

Let $\mu_j^* = C_{K,p}(E_j)^{1/q} \mu_j$ and let $\mu^* = \sum \mu_j^*$. We observe that

$$(8) \quad C_{K,p}(E_j) - 2^{-j}\varepsilon \leq \|\mu_j^*\| \leq \|K * \mu_j^*\|_q^q = C_{K,p}(E_j).$$

We claim

$$(9) \quad \|K * \mu^*\|_q^q \leq A \sum C_{K,p}(E_j).$$

If we have (9), then

$$\tilde{\mu} = \left(\sum C_{K,p}(E_j) \right)^{-1/q} \mu^*$$

satisfies $\|K * \tilde{\mu}\|_q \leq A$. Obviously $\tilde{\mu}$ is concentrated on E and by (8)

$$\|\tilde{\mu}\| \geq \left(\sum C_{K,p}(E_j) \right)^{-1/q} \left(\sum C_{K,p}(E_j) - \varepsilon \right).$$

Since ε is arbitrary, it follows from Theorem B that

$$C_{K,p}(E) \geq A \sum C_{K,p}(E_j).$$

Thus the required inequality follows.

For the proof of (9) we put

$$d\mu'_j = \frac{\|\mu_j^*\|}{|B(x_j, \eta_p^*(r_j))|} \chi_{B(x_j, \eta_p^*(r_j))} dx$$

and $\mu' = \sum \mu'_j$. By (8)

$$(10) \quad \|\mu'\| = \|\mu^*\| \leq \sum C_{K,p}(E_j).$$

Let $d\mu' = f dx$. Then, we observe from the disjointness of $B(x_j, \eta_p^*(r_j))$ that

$$f = \sum \chi_{B(x_j, \eta_p^*(r_j))} \leq 1.$$

Hence by (10)

$$(11) \quad \begin{aligned} \|K * \mu'\|_q^q &= \|K * f\|_q^q \leq \|K\|_1^q \|f\|_q^q = A \|f\|_q^q \\ &\leq A \|f\|_\infty^{q-1} \|f\|_1 \leq A \|f\|_1 = A \|\mu'\| \leq A \sum C_{K,p}(E_j). \end{aligned}$$

Let us compare $K * \mu^*$ and $K * \mu'$. Applying Lemma 1, we obtain

$$(12) \quad K * \mu_j^*(x) \leq A K * \mu'_j(x) \text{ for } x \notin B(x_j, \eta_p^*(r_j)).$$

Thus

$$K * \mu^*(x) \leq AK * \mu'(x) \text{ for } x \notin \bigcup B(x_j, \eta_p^*(r_j)).$$

Suppose $x \in \bigcup B(x_j, \eta_p^*(r_j))$, say $x \in B(x_{j_0}, \eta_p^*(r_{j_0}))$ for some j_0 . Then, by the disjointness of $\{B(x_j, \eta_p^*(r_j))\}$ we have $x \notin \bigcup_{j \neq j_0} B(x_j, \eta_p^*(r_j))$. Hence (12) yields

$$K * \mu^*(x) \leq K * \mu_{j_0}^*(x) + A \sum_{j \neq j_0} K * \mu_j'(x) \leq K * \mu_{j_0}^*(x) + AK * \mu'(x).$$

It follows from (8) and (11) that

$$\begin{aligned} \|K * \mu^*\|_q^q &= \int_{\mathbb{R}^N \setminus \bigcup B(x_j, \eta_p^*(r_j))} (K * \mu^*)^q dx + \int_{\bigcup B(x_j, \eta_p^*(r_j))} (K * \mu^*)^q dx \\ &\leq A \|K * \mu'\|_q^q + A \sum \int_{B(x_j, \eta_p^*(r_j))} (K * \mu_j^*)^q dx \\ &\leq A \sum C_{K,p}(E_j) + A \sum \|K * \mu_j^*\|_q^q \\ &\leq A \sum C_{K,p}(E_j). \end{aligned}$$

Therefore (9) follows. The theorem is proved.

4. PROOF OF THEOREM 2 AND THEOREM 4

Let $C_{K,1}(E) = C_K(E)$, $\eta_1(r) = \eta(r)$, $\eta_1^*(r) = \eta^*(r)$ and $\tilde{E}_{K,1} = \tilde{E}_K$. Hereafter we let $1 \leq p < \infty$. Then we can prove Theorem 2 and Theorem 4 in the same way with the help of Theorem 3 and Theorem 5.

Proof of Theorem 2 and Theorem 4. Take an arbitrary compact subset F of $\tilde{E}_{K,p}$. By the usual covering lemma we can find $x_j \in E$ such that

$$\begin{aligned} F &\subset \bigcup B(x_j, 5\eta_p^*(r_j)), \\ \{B(x_j, \eta_p^*(r_j))\} &\text{ is disjoint,} \\ r_j &= \delta_E(x_j). \end{aligned}$$

Let $E' = \bigcup B(x_j, r_j)$. By definition this is a subset of E . We apply Theorem 3 and Theorem 5 for $B(x_j, r_j)$ and E' . We obtain

$$\sum C_{K,p}(B(x_j, r_j)) \leq AC_{K,p}(E') \leq AC_{K,p}(E).$$

On the other hand we have

$$|F| \leq \sum |B(x_j, 5\eta_p^*(r_j))| = A \sum |B(x_j, \eta_p^*(r_j))|.$$

It follows from (2) and (4) that

$$(13) \quad |B(0, \eta_p^*(r))| \approx |B(0, \eta_p(r))| = C_{K,p}(B(0, r)).$$

Hence

$$|F| \leq AC_{K,p}(E).$$

Since F is an arbitrary compact subset of $\tilde{E}_{K,p}$, the required inequality follows. The theorems are proved.

5. GENERALIZATION

In this section we let $1 \leq p < \infty$ and use the convention for $C_{K,p}(E)$, $\eta_p(r)$, $\eta_p^*(r)$ and $\tilde{E}_{K,p}$ as in the last section. Let Ω be a set in \mathbb{R}_+^{N+1} with $\overline{\Omega} \cap \partial\mathbb{R}_+^{N+1} = \{0\}$. For simplicity we assume that $\Omega \supset \{(0, y) : y > 0\}$. Put $\Omega(y) = \{x : (x, y) \in \Omega\}$. We say that Ω satisfies the Nagel-Stein condition (abbreviated to (NS)), if

- (i) $|\Omega(y)| \leq Ay^N$ with $A = A(\Omega)$;
- (ii) there is $\alpha > 0$ such that

$$(x_1, y_1) \in \Omega \text{ and } |x - x_1| < \alpha(y - y_1) \implies (x, y) \in \Omega.$$

For E we put

$$\tilde{E}_{K,p;\Omega} = \bigcup_{x \in E} (x - \Omega(\eta_p^*(\delta_E(x)))) .$$

Theorem 6. *Let Ω satisfy (NS). Then*

$$|\tilde{E}_{K,p;\Omega}| \leq AC_{K,p}(E),$$

where $A > 0$ depends only on N, K, p and Ω .

Proof. The proof will be carried out in the same way as in the proof of Theorem 2 and Theorem 4 if we use the covering lemma due to Nagel-Stein [8, pp. 90-92]. But, for the completeness we give a proof.

Let $Q(y) = \bigcup_{x \in \Omega(y)} B(x, y)$. Obviously, $\Omega(y) \subset Q(y)$. In particular,

$$\tilde{E}_{K,p;\Omega} \subset \bigcup_{x \in E} (x - Q(\eta_p^*(\delta_E(x)))) .$$

For a moment we fix $y > 0$. By [8, Lemma 1 (d)] we find points $u_1, \dots, u_M \in \Omega(y)$ so that $Q(y) \subset \bigcup_{j=1}^M B(u_j, 3y)$. Note that the number M is independent of y . We put

$$\tilde{Q}(y) = \bigcup_{j,k=1}^M \{u_j - u_k + \bigcup_{i=1}^M B(u_i, 9y)\}.$$

We observe that

$$(14) \quad |\tilde{Q}(y)| \leq Ay^N,$$

where $A > 0$ is independent of y .

Take an arbitrary compact subset F of $\tilde{E}_{K,p;\Omega}$. We follow the same argument as in [8, pp. 90-92] and find points $x_1, \dots, x_J \in E$ such that

$$(15) \quad F \subset \bigcup_{j=1}^J x_j - \tilde{Q}(y_j), \quad y_j = \eta_p^*(\delta_E(x_j));$$

$$\{x_j - Q(y_j)\}_{j=1}^J \text{ is disjoint.}$$

We assumed that $\Omega \supset \{(0, y) : y > 0\}$ and hence $Q(y) \supset B(0, y)$. Hence, in particular, $\{B(x_j, y_j)\}_{j=1}^J$ is disjoint. Let

$$E' = \bigcup_{j=1}^J B(x_j, \delta_E(x_j)).$$

Obviously, E' is a subset of E . We apply Theorem 3 and Theorem 5 to E' to obtain

$$\sum C_{K,p}(B(x_j, \delta_E(x_j))) \leq AC_{K,p}(E') \leq AC_{K,p}(E).$$

On the other hand, by (13), (14) and (15) we have

$$|F| \leq A \sum y_j^N = A \sum |B(x_j, \eta_p^*(\delta_E(x_j)))| \leq A \sum C_{K,p}(B(x_j, \delta_E(x_j))).$$

Hence

$$|F| \leq AC_{K,p}(E).$$

Since F is an arbitrary compact subset of $\tilde{E}_{K,p,\Omega}$, the required inequality follows. The theorem is proved.

Remark 1. Let Ω be as in Theorem 6. Then

$$\left| \bigcup_{x \in E} (x + \Omega(\eta_p^*(\delta_E(x)))) \right| \leq AC_{K,p}(E).$$

We record counterparts of Theorems 1–6 for energy capacity. The energy capacity for the kernel K is defined by

$$e_K(E) = \sup\{\|\mu\|^2 : \mu \text{ is concentrated on } E, \int K * \mu d\mu \leq 1\}.$$

We define $\eta_e(r)$ by

$$|B(0, \eta_e(r))| = e_K(B(0, r))$$

and let $\eta_e^*(r) = \max\{\eta_e(r), 2r\}$.

Theorem 7. *Let Ω satisfy (NS). Then*

$$\left| \bigcup_{x \in E} (x - \Omega(\eta_e^*(\delta_E(x)))) \right| \leq Ae_K(E).$$

where $A > 0$ is independent of E . In particular,

$$\left| \bigcup_{x \in E} B(x, \eta_e^*(\delta_E(x))) \right| \leq Ae_K(E).$$

Theorem 8. *Suppose $\{B(x_j, \eta_e^*(r_j))\}$ is disjoint and E is an analytic subset of $\bigcup B(x_j, r_j)$. Then*

$$e_K(E) \leq \sum e_K(E \cap B(x_j, r_j)) \leq Ae_K(E),$$

where $A > 0$ is independent of E .

We can prove Theorem 7 in exactly the same way as in the proof of Theorem 2 and Theorem 4, if we invoke Theorem 8. The proof of Theorem 8 is somewhat a mixture of those of Theorem 3 and Theorem 5. The details are left to the reader.

6. BOUNDARY BEHAVIOR OF HARMONIC FUNCTIONS

In what follows we are interested in the boundary behavior of harmonic functions in \mathbb{R}_+^{N+1} . Hereafter we let $1 \leq p < \infty$. Following the idea in [2] and [5] we introduce the notion of thinness at the boundary. For a set $E \subset \mathbb{R}_+^{N+1}$ we put $E_t = \{(x, y) : 0 < y < t\}$ and $E^* = \bigcup_{(x,y) \in E} B(x, y)$. We recall that $B(x, y)$ is the N -dimensional ball with center at x and radius y , so that the set E^* is a set on the boundary $\mathbb{R}^N = \partial\mathbb{R}_+^{N+1}$. We shall combine the above notation and write

$$E_t^* = \bigcup_{(x,y) \in E, 0 < y < t} B(x, y).$$

Definition. Let $E \subset \mathbb{R}_+^{N+1}$. We say that E is $C_{K,p}$ -thin at $\partial\mathbb{R}_+^{N+1}$ if

$$\lim_{t \rightarrow 0} C_{K,p}(E_t^*) = 0.$$

Remark 2. If E is $C_{K,p}$ -thin at $\partial\mathbb{R}_+^{N+1}$, then the essential projection of E

$$\{x : \text{for any } t > 0 \text{ there is a positive number } y < t \text{ such that } (x, y) \in E\}$$

is of $C_{K,p}$ -capacity 0, and hence of measure 0.

We shall show the following theorems.

Theorem 9. Suppose Ω satisfies (NS). Let

$$\Omega_{K,p} = \{(x, y) : x \in \Omega(\eta_p^*(y))\}.$$

If E is $C_{K,p}$ -thin at $\partial\mathbb{R}_+^{N+1}$, then

$$\left| \bigcap_{t > 0} \{x : (x + \Omega_{K,p}) \cap E_t \neq \emptyset\} \right| = 0.$$

In other words, for almost all $x \in \partial\mathbb{R}_+^{N+1}$, $x + \Omega_{K,p}$ lies eventually outside E , i.e., there is $t = t_x > 0$ such that $E_t \cap (x + \Omega_{K,p}) = \emptyset$.

Remark 3. It is not so difficult to see that $\eta_p^*(r)/r \rightarrow \infty$ as $r \rightarrow 0$ (cf. [1]). Hence $\Omega_{K,p}$ is a tangential region.

For a function f on $\partial\mathbb{R}_+^{N+1}$ we denote by $PI(f)$ its Poisson integral. More precisely, let $\varphi(x) = A_N(|x|^2 + 1)^{-(N+1)/2}$ and $\varphi_y(x) = y^{-N}\varphi(x/y)$, where $A_N > 0$ is such that $\|\varphi\|_1 = 1$. Then we see that

$$PI(f)(x, y) = \varphi_y * f(x).$$

Theorem 10. Let $\Omega \subset \mathbb{R}_+^{N+1}$ and suppose $\bar{\Omega} \cap \partial\mathbb{R}_+^{N+1} = \{0\}$. If $p = 1$, then we assume that

$$(16) \quad K(r) \approx r^{-N} \int_0^r K(t)t^{N-1}dt \text{ for small } r > 0.$$

Suppose $f \in L^p(\mathbb{R}^N)$. Then there is a set $E \subset \mathbb{R}_+^{N+1}$ such that E is $C_{K,p}$ -thin at $\partial\mathbb{R}_+^{N+1}$ and that

$$(17) \quad \lim_{P \rightarrow x, P \in (x+\Omega) \setminus E} PI(K * f)(P) = K * f(x)$$

for $C_{K,p}$ -a.e. $x \in \partial\mathbb{R}_+^{N+1}$, i.e. there is a set $F \subset \partial\mathbb{R}_+^{N+1}$ such that $C_{K,p}(F) = 0$ and (17) holds for $x \in \partial\mathbb{R}_+^{N+1} \setminus F$.

As a corollary to Theorem 9 and Theorem 10 we have the following theorem. This is a generalization of [7, Theorem 2.9].

Theorem 11. Let $\Omega \subset \mathbb{R}_+^{N+1}$ and suppose Ω satisfies (NS). If $p = 1$, then we assume that (16) holds. Suppose $f \in L^p(\mathbb{R}^N)$. Then

$$\lim_{P \rightarrow x, P \in x + \Omega_{K,p}} PI(K * f)(P) = K * f(x)$$

for almost all $x \in \partial\mathbb{R}_+^{N+1}$.

We note that Theorem 10 gives a sharp characterization of exceptional sets.

Proposition 1. Suppose E is $C_{K,p}$ -thin at $\partial\mathbb{R}_+^{N+1}$. Then there is $f \in L^p(\mathbb{R}^N)$ such that

$$(18) \quad \inf_{(x,y) \in E, 0 < y < t} PI(K * f)(x, y) \rightarrow \infty \text{ as } t \rightarrow 0.$$

Finally we observe that the region $\Omega \subset \mathbb{R}_+^{N+1}$ in Theorem 10 is necessary. In fact, we have

Proposition 2. Let g be an integrable function and let $x_0 \in \partial\mathbb{R}_+^{N+1}$. Suppose there is a set E such that the essential projection of E is of measure 0 and

$$(19) \quad \lim_{P \rightarrow x_0, P \notin E} PI(g)(P) = \alpha.$$

Then

$$(20) \quad \text{ess lim}_{x \rightarrow x_0} g(x) = \alpha.$$

Remark 2 and Proposition 2 assert that if $h(P) = PI(K * f)(P)$ approaches a limit as P approaches $x_0 \in \partial\mathbb{R}_+^{N+1}$ outside a set $C_{K,p}$ -thin at $\partial\mathbb{R}_+^{N+1}$, then $K * f$ must be essentially continuous at x_0 . Unless a point has positive $C_{K,p}$ -capacity, there exists $f \in L^p(\mathbb{R}^N)$ such that $K * f$ is essentially discontinuous at every point. In fact, assume that a point has null $C_{K,p}$ -capacity. Let E be a countable dense subset of \mathbb{R}^N . Then we see that $C_{K,p}(E) = 0$, so that there is a nonnegative function $f \in L^p(\mathbb{R}^N)$ such that $K * f = \infty$ on E . Since $K * f$ is lower semicontinuous, it follows that $\text{ess lim sup}_{x \rightarrow x_0} K * f(x) = \infty$ for every $x_0 \in \mathbb{R}^N$. At almost every point, $K * f$ is discontinuous even in the extended sense, since $\text{ess lim inf}_{x \rightarrow x_0} K * f(x) < \infty$ for a.e. $x_0 \in \mathbb{R}^N$ by the local integrability of $K * f$.

7. PROOF OF THEOREM 9

We begin by an elementary geometrical observation.

Lemma 2. Let $\Omega \subset \mathbb{R}_+^{N+1}$ and suppose $\Omega(y)$ is increasing, i.e., if $y_1 < y_2$, then $\Omega(y_1) \subset \Omega(y_2)$. Let $E \subset \mathbb{R}_+^{N+1}$ and let E^* be as in Section 7. Then

$$(21) \quad \{x \in \mathbb{R}^N : (x + \Omega) \cap E \neq \emptyset\} \subset \bigcup_{x \in E^*} (x - \Omega(\delta_{E^*}(x))),$$

where $\delta_{E^*}(x) = \text{dist}(x, E^{*c})$.

Proof. We take an arbitrary point x_0 from the set in the left hand side of (21).

Then $(x_0 + \Omega) \cap E \neq \emptyset$. Assume that $(x, y) \in (x_0 + \Omega) \cap E$. This implies

- (i) $x \in x_0 + \Omega(y)$ and hence $x_0 \in x - \Omega(y)$;
- (ii) $(x, y) \in E$ and hence $B(x, y) \subset E^*$ by definition, so that $\delta_{E^*}(x) \geq y$.

By the monotonicity of $\Omega(y)$ we have

$$x_0 \in x - \Omega(\delta_{E^*}(x)),$$

which yields (21). The lemma is proved.

Proof of Theorem 9. We apply Lemma 2 for $\Omega_{K,p}$ replacing Ω . Note that $\Omega_{K,p}(y) = \Omega(\eta_p^*(y))$. Then (21) becomes

$$\begin{aligned} \{x \in \mathbb{R}^N : (x + \Omega_{K,p}) \cap E \neq \emptyset\} &\subset \bigcup_{x \in E^*} (x - \Omega_{K,p}(\delta_{E^*}(x))) \\ &= \bigcup_{x \in E^*} (x - \Omega(\eta_p^*(\delta_{E^*}(x)))). \end{aligned}$$

Hence Theorem 6 yields

$$|\{x \in \mathbb{R}^N : (x + \Omega_{K,p}) \cap E \neq \emptyset\}| \leq AC_{K,p}(E^*).$$

Apply this inequality for E_t replacing E . Then the definition of thinness implies that

$$|\{x : (x + \Omega_{K,p}) \cap E_t \neq \emptyset\}| \leq AC_{K,p}(E_t^*) \rightarrow 0 \text{ as } t \rightarrow 0.$$

Thus the theorem follows.

8. PROOF OF THEOREM 10 IN CASE $1 < p < \infty$

In this section we let $1 < p < \infty$.

Lemma 3. *Let f be a nonnegative function in $L^p_+(\mathbb{R}^N)$ and let $\varepsilon > 0$. Put*

$$\begin{aligned} E(f, \varepsilon) &= \{(x, y) \in \mathbb{R}^{N+1}_+ : PI(K * f)(x, y) > \varepsilon\}, \\ E^*(f, \varepsilon) &= \bigcup_{(x,y) \in E(f,\varepsilon)} B(x, y). \end{aligned}$$

Then

$$C_{K,p}(E^*(f, \varepsilon)) \leq A \left(\frac{\|f\|_p}{\varepsilon} \right)^p,$$

where A is independent of f and ε .

Proof. Let

$$E'(f, \varepsilon) = \bigcup_{(x,y) \in E(f,\varepsilon)} B(x, y) \times \{y\}.$$

Then the projection of $E'(f, \varepsilon)$ is equal to $E^*(f, \varepsilon)$. Since $PI(K * f)$ is a positive harmonic function, it follows from the Harnack inequality that

$$(22) \quad PI(K * f) \geq A\varepsilon \text{ on } E'(f, \varepsilon),$$

where A depends only on the dimension. Let us put

$$F(x) = \sup_{y>0} \varphi_y * f(x),$$

where φ_y is the Poisson kernel as defined in Section 7. Then, the maximal inequality shows that

$$(23) \quad \|F\|_p \leq A\|f\|_p$$

([9, Theorem 3.7]). By definition

$$K * F(x) \geq K * (\varphi_y * f)(x) = \varphi_y * (K * f)(x) = PI(K * f)(x, y) \text{ for all } y > 0.$$

Hence by (22)

$$K * F \geq A\varepsilon \text{ on } E^*(f, \varepsilon).$$

Thus, by (23),

$$C_{K,p}(E^*(f, \varepsilon)) \leq A \left(\frac{\|F\|_p}{\varepsilon} \right)^p \leq A \left(\frac{\|f\|_p}{\varepsilon} \right)^p.$$

The lemma follows.

In [2] we proved a theorem similar to Theorem 10 by using oscillation. Here, we present a more direct proof without using oscillation. By $\mathbf{B}(P, r)$ we denote the $(N + 1)$ -dimensional open ball with center at P and radius r .

Lemma 4. *Let $f \in L^p(\mathbb{R}^N)$ and let $\delta > 0$. Then there exist $E \subset \mathbb{R}_+^{N+1}$ and $F \subset \partial\mathbb{R}_+^{N+1}$ such that*

- (i) $C_{K,p}(E^*) < \delta$ and $C_{K,p}(F) < \delta$;
- (ii) for each $\varepsilon > 0$ there is $r > 0$ such that

$$(24) \quad \sup_{x \in \partial\mathbb{R}_+^{N+1} \setminus F} \left(\sup_{P \in \mathbf{B}(x, r) \cap \mathbb{R}_+^{N+1} \setminus E} |PI(K * f)(P) - K * f(x)| \right) < \varepsilon.$$

Proof. In view of the definition of $C_{K,p}$ and Lemma 3, we can choose a sequence $g_j \in C_0^\infty(\mathbb{R}^N)$ such that

$$C_{K,p}(F_j) < 2^{-j}\delta \text{ with } F_j = \{x : |K * g_j(x) - K * f(x)| \geq 2^{-j}\},$$

$$C_{K,p}(E_j^*) < 2^{-j}\delta \text{ with } E_j = \{P : |PI(K * g_j)(P) - PI(K * f)(P)| \geq 2^{-j}\}.$$

Let $F = \bigcup_{j=1}^\infty F_j$ and $E = \bigcup_{j=1}^\infty E_j$. Then $C_{K,p}(F) < \delta$ and $C_{K,p}(E^*) < \delta$. Moreover we observe that

$$K * g_j \rightarrow K * f \text{ uniformly on } \partial\mathbb{R}_+^{N+1} \setminus F,$$

$$PI(K * g_j) \rightarrow PI(K * f) \text{ uniformly on } \mathbb{R}_+^{N+1} \setminus E.$$

Hence we can find j_0 such that

$$\sup_{x \in \partial\mathbb{R}_+^{N+1} \setminus F} |K * g_{j_0}(x) - K * f(x)| < \frac{\varepsilon}{3},$$

$$\sup_{P \in \mathbb{R}_+^{N+1} \setminus E} |PI(K * g_{j_0})(P) - PI(K * f)(P)| < \frac{\varepsilon}{3}.$$

Since $PI(K * g_{j_0})$ is uniformly continuous on \mathbb{R}_+^{N+1} and has boundary values $K * g_{j_0}$, we can find $r > 0$ such that

$$\sup_{x \in \partial \mathbb{R}_+^{N+1}} \left(\sup_{P \in \mathbf{B}(x,r) \cap \mathbb{R}_+^{N+1}} |PI(K * g_{j_0})(P) - K * g_{j_0}(x)| \right) < \frac{\varepsilon}{3}.$$

The above three inequalities altogether imply (24). The lemma is proved

We prepare a geometrical observation for Ω . We assume that $\bar{\Omega} \cap \partial \mathbb{R}_+^{N+1} = \{0\}$, which implies that for each $r > 0$ there is $t > 0$ such that

$$\Omega \cap \{(x, y) : 0 < y < t\} \subset \mathbf{B}(0, r),$$

or equivalently

$$\Omega \setminus \mathbf{B}(0, r) \subset \{(x, y) : y \geq t\}.$$

Hence, if we have a decreasing sequence $r_j \downarrow 0$, then we can find $t_j \downarrow 0$ such that

$$(25) \quad t_j < r_{j+1}$$

and $\Omega \cap \{(x, y) : 0 < y < t_j\} \subset \mathbf{B}(0, r_{j+1})$, or equivalently

$$\Omega \setminus \mathbf{B}(0, r_{j+1}) \subset \{(x, y) : y \geq t_j\}.$$

Observe that

$$\Omega \cap \mathbf{B}(0, r_i) = \bigcup_{j=i}^{\infty} \Omega \cap \mathbf{B}(0, r_j) \setminus \mathbf{B}(0, r_{j+1}) \subset \bigcup_{j=i}^{\infty} \mathbf{B}(0, r_j) \cap \{(x, y) : y \geq t_j\}$$

By translation we have for $x \in \partial \mathbb{R}_+^{N+1}$

$$(26) \quad (x + \Omega) \cap \mathbf{B}(x, r_i) \subset \bigcup_{j=i}^{\infty} \mathbf{B}(x, r_j) \cap \{(x, y) : y \geq t_j\}.$$

Proof of Theorem 10. Take $\varepsilon_j \downarrow 0$. By Lemma 4 there are $E_j \subset \mathbb{R}_+^{N+1}$, $F_j \subset \partial \mathbb{R}_+^{N+1}$ and $r_j \downarrow 0$ such that

$$(27) \quad \sum_j C_{K,p}(E_j^*) < \infty \text{ and } \sum_j C_{K,p}(F_j) < \infty,$$

$$(28) \quad \sup_{x \in \partial \mathbb{R}_+^{N+1} \setminus F_j} \left(\sup_{P \in \mathbf{B}(x,r_j) \setminus E_j} |PI(K * f)(P) - K * f(x)| \right) < \varepsilon_j.$$

We choose $t_j \downarrow 0$ so that (25) and (26) hold. Let $E'_j = E_j \cap \{(x, y) : y \geq t_j\}$ and put $E = \bigcup_j E'_j$. Then (27) yields

$$C_{K,p}(E_{i_i}^*) \leq C_{K,p}\left(\bigcup_{j=i}^{\infty} E_j^*\right) \leq \sum_{j=i}^{\infty} C_{K,p}(E_j^*) \rightarrow 0 \text{ as } i \rightarrow \infty,$$

so that E is $C_{K,p}$ -thin at $\partial\mathbb{R}_+^{N+1}$. Again by (27) we see that $F = \bigcap_{i=1}^\infty \bigcup_{j=i}^\infty F_j$ satisfies $C_{K,p}(F) = 0$. Take $x \in \partial\mathbb{R}_+^{N+1} \setminus F$. We shall show that (17) holds. By definition there is $j_0 = j_0(x)$ such that $x \notin F_j$ for all $j \geq j_0$. By (28)

$$\sup_{P \in \mathbf{B}(x, r_j) \setminus E_j} |PI(K * f)(P) - K * f(x)| < \varepsilon_j$$

for all $j \geq j_0$. Let $i \geq j_0$. In view of (26) and the definition of E

$$\begin{aligned} & \sup_{P \in (x+\Omega) \cap \mathbf{B}(x, r_i) \setminus E} |PI(K * f)(P) - K * f(x)| \\ & \leq \sup_{j \geq i} \left(\sup_{P \in \mathbf{B}(x, r_j) \cap \{(x, y) : y \geq t_j\} \setminus E} |PI(K * f)(P) - K * f(x)| \right) \\ & \leq \sup_{j \geq i} \left(\sup_{P \in \mathbf{B}(x, r_j) \cap \{(x, y) : y \geq t_j\} \setminus E_j} |PI(K * f)(P) - K * f(x)| \right) \\ & \leq \sup_{j \geq i} \varepsilon_j = \varepsilon_i \rightarrow 0 \text{ as } i \rightarrow \infty. \end{aligned}$$

This implies (17). The theorem is proved.

9. PROOF OF THEOREM 10 IN CASE $p = 1$

In this section we let $p = 1$. In this case we cannot use the maximal inequality in the proof of Lemma 3. Instead, we estimate the kernel $K_y(x) = K * \varphi_y(x)$. Let

$$\kappa(r) = K\left(\frac{r}{2}\right) + \frac{1}{r+1} r^{-N} \int_0^r K(t) t^{N-1} dt$$

and let $\kappa(x) = \kappa(|x|)$ for $x \in \mathbb{R}^N$ as usual. The following lemma is a substitute of Lemma 3.

Lemma 5. *Let $f \in L^1_+(\mathbb{R}^N)$ and let $\varepsilon > 0$. Put*

$$\begin{aligned} E_1(f, \varepsilon) &= \{(x, y) \in \mathbb{R}_+^{N+1} : 0 < y < 1, PI(K * f)(x, y) > \varepsilon\}, \\ E_1^*(f, \varepsilon) &= \bigcup_{(x, y) \in E_1(f, \varepsilon)} B(x, y). \end{aligned}$$

Then

$$C_\kappa(E_1^*(f, \varepsilon)) \leq A \frac{\|f\|_1}{\varepsilon},$$

where A is independent of f and ε .

Proof. First we claim

$$K_y(x) \leq A\kappa(x) \text{ for } 0 < y < 1.$$

In fact we have

$$\begin{aligned} K_y(x) &= A_N \int_{\mathbb{R}^N} K(x-z) \frac{y}{(|z|^2 + y^2)^{(N+1)/2}} dz \\ &\leq A_N \int_{|x-z| \geq |x|/2} K\left(\frac{|x|}{2}\right) \frac{y}{(|z|^2 + y^2)^{(N+1)/2}} dz \\ &\quad + A_N \int_{|x-z| < |x|/2} K(x-z) \frac{y}{((|x|/2)^2 + y^2)^{(N+1)/2}} dz \\ &\leq K\left(\frac{|x|}{2}\right) + A \frac{1}{|x|/y + 1} \frac{1}{|x|^N} \int_{|z| < |x|/2} K(z) dz \\ &\leq A\kappa(x). \end{aligned}$$

In view of the claim and the Harnack inequality we have

$$\kappa * f(x) \geq A\varepsilon \text{ on } E_1^*(f, \varepsilon),$$

whence the required inequality follows.

We observe that if (16) holds, then $\kappa(r) \approx K(r)$ for small $r > 0$, so that

$$C_K(E) \approx C_\kappa(E) \text{ for } E \subset U,$$

where U is a bounded set. From this fact and Lemma 5 we can prove Theorem 10 in the same way as in Section 8 with a slight modification.

10. PROOF OF PROPOSITION 1 AND PROPOSITION 2

Proof of Proposition 1. By definition there is $t_j \downarrow 0$ such that

$$C_{K,p}(E_{t_j}^*) < 2^{-j}.$$

We find nonnegative functions $f_j \in L^p(\mathbb{R}^N) \cap C_0^\infty(\mathbb{R}^N)$ such that

$$K * f_j \geq 1 \text{ on } E_{t_j}^*$$

and $\|f_j\|_p^p < 2^{-j}$. Let $f = \sum f_j$. Then $f \in L^p(\mathbb{R}^N)$. We show that this f satisfies (18). Since $K * f_j$ is continuous, it follows that $PI(K * f_j)$ has continuous boundary values and

$$PI(K * f_j) \geq 1 \text{ on } E_{t_j}^*.$$

Observe that if $(x, y) \in E_{t_j}$, then $B(x, y) \subset E_{t_j}^*$. Since the harmonic measure of $B(x, y)$ evaluated at (x, y) is greater than a positive constant A depending only on the dimension, it follows that

$$PI(K * f_j) \geq A \text{ on } E_{t_j}.$$

Therefore

$$PI(K * f) \geq Aj \text{ on } E_{t_j}.$$

This implies (18). The proposition is proved.

Proof of Proposition 2. We prove the proposition by contradiction. Suppose (20) does not hold. Then there is $\varepsilon_0 > 0$ such that

$$(29) \quad |\{x \in B(x_0, r) : |g(x) - \alpha| > 2\varepsilon_0\}| > 0$$

for all $r > 0$. By (19) we see that there is $r_0 > 0$ such that

$$(30) \quad |PI(g)(P) - \alpha| < \varepsilon_0 \text{ for } P \in B(x_0, r_0) \cap \mathbb{R}_+^{N+1} \setminus E.$$

Let us consider the boundary values of $PI(g)$ on $B(x_0, r_0)$. It is well known that $PI(g)$ has nontangential boundary values g a.e. on $\partial\mathbb{R}_+^{N+1}$. Hence, by (29), we can find a set $F \subset B(x_0, r_0)$ such that $|F| > 0$ and

$$|\text{nt} \lim_{P \rightarrow x} PI(g)(P) - \alpha| > 2\varepsilon_0 \text{ for } x \in F,$$

where $\text{nt} \lim_{P \rightarrow x} PI(g)(P)$ means the nontangential limit of $PI(g)$ at x . In particular, there is a positive function $t(x)$ on F such that

$$|PI(g)(x, y) - \alpha| > \varepsilon_0 \text{ for } x \in F \text{ and } 0 < y < t(x),$$

or, in other words

$$|PI(g) - \alpha| > \varepsilon_0 \text{ on } A = \bigcup_{x \in F} I(x),$$

where $I(x) = \{x\} \times (0, t(x))$. By (30) we see that $A \cap B(x_0, r_0) \subset E$. Obviously, F is the essential projection of $A \cap B(x_0, r_0)$, Hence the essential projection of E has positive measure. This is a contradiction.

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