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A PERIODIC POINT FREE HOMEOMORPHISM OF A TREE-LIKE CONTINUUM

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ABSTRACT. An example of a homeomorphism without periodic points is constructed on a tree-like continuum.

1. INTRODUCTION

By a *continuum* we understand a compact and connected metric space. A continuum is *tree-like* if it is the inverse limit of an inverse sequence of trees (simply connected finite graphs). A continuum is *cell-like*, if it is the intersection of a nested sequence of cells. Each one-dimensional cell-like continuum is tree-like.

Suppose X is a continuum and f is a map of X into itself. A point $x \in X$ is a fixed point of f, if $f(x) = x$. If n is a positive integer, then by f^n we understand the n-th iteration of f. A point $x \in X$ is a periodic point of period n, if $f^n(x) = x$.

In 1935, K. Borsuk [3] constructed an example of a cell-like continuum in $\mathbf{R}^3$ admitting a fixed point free homeomorphism. R. H. Bing [2] gave a two-dimensional version of Borsuk's example and asked whether a tree-like continuum without the fixed point property could be constructed. D. P. Bellamy [1] answered this question affirmatively presenting in 1978 his spectacular example.

The Borsuk continuum admits a homeomorphism without periodic points. In 1972, T. Ingram [7, Problem 34] (see also [10, Problem 35] and [6, Problem 2]) asked whether each map of a tree-like continuum into itself must have a periodic point. The Bellamy map and other fixed-point free maps on tree-like continua constructed subsequently (see [12], [13], [14] and [11]) have periodic points. In this paper we construct a tree-like continuum X and a homeomorphism $\tilde{h}$ of X onto itself without periodic points.

Questions. The Borsuk map can be extended to a periodic point free homeomorphism of $\mathbf{R}^3$ onto itself. Is there a periodic point free homeomorphism of $\mathbf{R}^3$ with an invariant tree-like continuum? The Borsuk map is homotopic to the identity. It is not known whether each homotopic to the identity map of a tree-like continuum must have a fixed point (see [10, Problem 27]). Recently, C. Hagopian [9] proved that at least some levels of the homotopy have fixed points. Is it true that each homotopic to the identity map of a tree-like continuum must have a periodic point?

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Outline of Construction. Before reading this outline, one should realize that definitions included in it are not precise and should not taken literally. The precise definition can be found elsewhere in the paper.

By a slight modification of the original Bellamy's construction, the author [11] obtained, for each positive integer j, a tree-like continuum and a map without periodic points of periods smaller than or equal to j. Applying to those maps a technique by J. B. Fugate and L. B. Mohler [8], we get a tree-like continuum $\widetilde{B}_j$ and a homeomorphism $\tilde{f}_j$ of $\widetilde{B}_j$ onto itself such that $\tilde{f}_j$ does not have periodic points of periods smaller than or equal to j. Using $\widetilde{B}_j$ and $\tilde{f}_j$ as building blocks, we construct a commutative diagram

$$\begin{array}{ccccccc}
X_1 & \xleftarrow{\tilde{\sigma}_1} & X_2 & \xleftarrow{\tilde{\sigma}_2} & X_3 & \xleftarrow{\tilde{\sigma}_3} & \dots \\
\downarrow h_1 & & \downarrow h_2 & & \downarrow h_3 & & \\
X_1 & \xleftarrow{\tilde{\sigma}_1} & X_2 & \xleftarrow{\tilde{\sigma}_2} & X_3 & \xleftarrow{\tilde{\sigma}_3} & \dots
\end{array}$$

where $X_1, X_2, X_3, \dots$ are tree-like continua, $\tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\sigma}_3, \dots$ are continuous maps and $h_1, h_2, h_3, \dots$ are homeomorphisms such that h_j does not have periodic points of periods smaller than or equal to j. The continuum X is the inverse limit of the system $\{X_j, \tilde{\sigma}_j\}$ and $\tilde{h}$ is the map induced by the sequence $h_1, h_2, h_3, \dots$. We start with $X_1 = \widetilde{B}_1$ and $h_1 = \tilde{f}_1$. The map h_1 does not have fixed points. To get X_2, we replace each periodic point of h_1 by a copy of $\widetilde{B}_2$. The bonding map $\tilde{\sigma}_1$ is the projection mapping each copy of $\widetilde{B}_2$ onto the point it replaces. The homeomorphism h_2 maps the copy of $\widetilde{B}_2$ replacing a point x onto the copy of $\widetilde{B}_2$ replacing $h_1(x)$; h_2 restricted to each such copy of $\widetilde{B}_2$ is equal to $\tilde{f}_2^{\,-1}$. Since each periodic point of h_2 belongs to a copy of $\widetilde{B}_2$, h_2 has neither fixed points nor periodic points of period 2. To get X_3, we replace each periodic point of h_2 by a copy of $\widetilde{B}_3$. The homeomorphism h_3 restricted to each copy of $\widetilde{B}_3$ maps it onto the appropriate copy of $\widetilde{B}_3$ and is equal to $\tilde{f}_3$. We continue this procedure defining h_j to be either $\tilde{f}_j$ on each added copy of $\widetilde{B}_j$, if j is odd, or $\tilde{f}_j^{\,-1}$, if j is even.

To have topology in X_j reasonably well-defined, and to have a chance for continuity of h_j we should replace much more than periodic points. The continuum $\widetilde{B}_j$ looks almost like the product of the Cantor set by the interval $I = [0, 1]$ with β_j denoting the projection onto I (see Proposition 2.11 for a more precise statement). When we replace a point $x \in \widetilde{B}_j$ by a copy of $\widetilde{B}_{j+1}$, we should do the same for each point $y \in \widetilde{B}_j$ such that $\beta_j(y) = \beta_j(x)$. Then, of course, we should replace all of the images and preimages of such points. But even that is not enough.

To outline further elements of the construction, we need a simplified model. For the sake of this discussion, let us pretend that $\widetilde{B}_j = I$ and that $\tilde{f}_j$ is equal to the roof-top map (g_2 in the next section). Even though 0 is a fixed point of g_2, we do not need not worry about it, because 0 does not correspond to a periodic point in 'true' $\widetilde{B}_j$. We must remove all other periodic points of g_2. To simplify our task further, we start with $X_1 = I$ and attempt to construct X_2 by removing only the other fixed point $c = \frac{2}{3}$. Let A_c denote the arc $\{c\} \times I$. Let $\tilde{\Phi}_c : I \setminus \{c\} \to I$ be a map such that $R'_c = \left\{ \left(t, \tilde{\Phi}_c(t) \right) \mid 0 \le t < c \right\}$ and $R''_c = \left\{ \left(t, \tilde{\Phi}_c(t) \right) \mid c < t \le 1 \right\}$ are two rays approaching A_c the same way as the $\sin 1/x$-curve approaches its limiting interval. Let $X_2 = R'_c \cup A_c \cup R''_c$. To define h_2, we have to overcome the following two difficulties. Since $g_2(\frac{1}{3}) = \frac{2}{3}$, we want to replace the point $\frac{1}{3}$ by an arc. Then we have to do the same for $\frac{1}{6}$, then for $\frac{1}{12}$ and so on. The second difficulty is to define h_2 on A_c so that it is continuous in a neighborhood of A_c. Suppose that $h_2(c,x) = (h_1(c), g_2(x))$ for each $x \in I$. Since g_2 stretches I and then folds it onto itself, each of the 'basic' arcs forming R'_c and R''_c which is sufficiently close to A_c would have to be mapped onto two such arcs on the opposite sides of A_c. This would force points to be mapped closer to A. Since $c = \frac{2}{3}$ is a repelling fixed point of $h_1 = g_2$, we need the property $h_2(c, g_2(x)) = (h_1(c), x)$ for each $x \in I$. This is not possible because g_2 cannot be reversed. But, it will become possible in the main construction, where we will have the homeomorphism $\tilde{f}_2$ instead g_2.

In the simplified model still, let $C(i)$ denote the set of points in X_i which have to be replaced by I to get X_{i+1}. If $c \in C(i)$, then points in the replacing it $A_c = \{c\} \times I$ will be denoted here by (c,t), where $t \in I$. If $x \in X_i \backslash C(i)$, then the corresponding to it point of X_{i+1} will be denoted by x too. For reasons outlined above, $C(1)$ consists of those points $c \in I$ which are mapped by some iteration of g_2 to a non-zero periodic point of g_2. Suppose c and c' belong to $C(1)$. Then (c,c') belongs to $C(2)$. Since the 'basic' arcs forming R'_c and R''_c limit on A_c, each change of A_c should be followed by the corresponding changes of these arcs. Therefore, if $x \in X_1 = I$ is such that $\tilde{\Phi}_c(x) = c'$, then $x \in C(2)$. The set $C(3)$ has to accommodate two levels of converging rays, $C(4)$ three levels and so on. We have to be careful in our construction of each set $C(i)$ and each function $\tilde{\Phi}_c$ to guarantee that X_j is a one-dimensional compactum and that the function h_n is continuous. The remaining required properties will follow more or less automatically.

The continua $\widetilde{B}_j$ are defined in Section 2 of this paper. The set $C(i)$ and the function Φ_c are defined analytically in Section 4. The definition involving continued fractions is prepared in Section 3. Finally, in Section 5, we define X_j and h_j and conclude the proof of the example.

Notation. All spaces considered in this paper are metric. If points x and y belong to a metric space, then by $|x-y|$ we will denote the distance between x and y. The set of real numbers and the unit interval $[0,1]$ will be denoted by $\mathbf{R}$ and I, respectively. The remaining part of the notation will be introduced as needed throughout the rest of the paper. To make it more accessible, we indicate here where the key symbols are defined.

$\approx$, 4.22	$\alpha(j,c)$, 4.2, 4.40	$\mathcal{A}$, 2.1
$a_j(c)$, 4.2	β_j, 2.10	$\widetilde{B}_j$, 2.8
$C(n)$, 4.2	$\delta_n(c,x)$, 4.36	$\delta'(c,x)$, 4.31
$\tilde{\delta}_n(c,b)$, $\tilde{\delta}'(c,b)$, 5.11	$\Delta(n)$, 5.8	$D(i)$, 4.22
$\epsilon(\underline{a})$, 3.15	$\epsilon(c)$, 4.6, $\epsilon(d)$, 4.22	$\eta(\underline{a},\epsilon)$, 3.12
$\eta(c)$, 4.6, $\eta(d)$, 4.22	E, 3.13	$\tilde{f}_j$, 2.8
$\gamma[\ldots]$, 3.3	γ_c, 4.9	g_n, 2.4
h, 5.16	h_n, 5.28	$\widetilde{H}_n$, 5.3
$I(c,t)$, 4.19	$\lambda(d,t)$, 4.44	$\lambda'(d,t)$, 4.43
Λ, 4.38	L, 4.38, 4.40	len, 4.1, 4.40, 5.2
μ, 3.13	$M(c)$, 4.6, $M(d)$, 4.22	$\omega[\ldots]$, 3.3
$\tilde{\omega}_c$, 4.16	$\tilde{\varphi}_j$, 2.14	Φ'_c, Φ_c, 4.16
$\widehat{\Phi}_z$, 4.41	$\tilde{\Phi}_z$, 5.1	π_n, 5.21

P_n, 4.35
r_j, 4.1, 5.2
σ_d^n, 5.9
$\theta(c,k,t)$, 4.19
v, 2.2
X_n, 5.21
$\widetilde{P}_n$, 5.4
σ_n, 5.4
s_j, 4.1, 5.2
T, 4.1
w_i, 3.2
$X_n(d)$, 5.23
$\widetilde{Q}$, 5.2
$\tilde{\sigma}_n$, 5.21
τ, 5.3
u_c, 4.6, u_d, 4.22
$\xi_-(c)$, $\xi_+(c)$, 4.16

2. Bellamy's continua

In this section we define the continuum $\widetilde{B}_j$, which is used as a basic building block of the construction.

2.1. **Definition.** Let $\mathcal{A}$ denote the set of those rational numbers which can be represented as p/q, where p and q are relatively prime positive integers and q is not a power of 2.

2.2. **Definition.** Let $v : \mathbf{R} \to I$ be defined by:

$$v(t) = \begin{cases} t - 2m, & \text{if } m \text{ is an integer and } 2m \le t \le 2m+1, \\ 2m - t, & \text{if } m \text{ is an integer and } 2m-1 \le t \le 2m. \end{cases}$$

2.3. **Proposition.** *If x is not an integer, then exactly one of the numbers $x - v(x)$ and $x + v(x)$ is an even integer.*

2.4. **Definition.** For each positive integer n let $g_n : I \to I$ be defined by $g_n(t) = v(nt)$.

2.5. **Proposition.** *Suppose x is a point of $(0,1]$ and i is a positive integer such that ${g_2}^i(x) = x$. Then $x \in \mathcal{A}$.*

Proof. By [11, Proposition 2], ${g_2}^i(x) = v\left(2^i x\right)$. There is an integer m such that either $v\left(2^i x\right) = 2^i x - 2m$ or $v\left(2^i x\right) = 2m - 2^i x$. If $v\left(2^i x\right) = 2^i x - 2m$, then $x = 2m \big/ \left(2^i - 1\right) \in \mathcal{A}$. If $v\left(2^i x\right) = 2m - 2^i x$, then $x = 2m \big/ \left(2^i + 1\right) \in \mathcal{A}$.

2.6. **Definition.** For each positive integer n, let S_n be the inverse limit of the inverse system of copies of I with every bonding map equal to g_n. Let p_n^k be the projection of S_n onto the k-th element of the inverse system. Let e_n denote the point $(0,0,\dots)$ and let d_n denote the point $\left(1, 1/n, 1/n^2, 1/n^3, \dots\right)$. Let J_n denote the arc in S_n between e_n and d_n. Let g denote the map from S_n onto itself induced by g_2, i.e. $g((x_0, x_1, \dots)) = (g_2(x_0), g_2(x_1), \dots)$.

For each positive integer j, let $n(j) = 2\left(4^1 - 1\right)\left(4^2 - 1\right)\dots\left(4^j - 1\right)$. Note that if k is a positive integer less than or equal to j, then $e_{n(j)}$ is the only fixed point of $g^k : S_{n(j)} \to S_{n(j)}$ [11, Proposition 6].

2.7. **Definition.** By a slight variation of the original Bellamy's construction [1], it was proven in [11] that there is a tree-like continuum B_j and there is a continuous map $f_j : B_j \to B_j$ without periodic points of periods less than or equal to j. Roughly speaking, B_j was obtained by replacing $J_{n(j)}$ in $S_{n(j)}$ by a cone over some zero-dimensional set Z_j. More precisely, there is a continuous map q_j (this map was denoted by q in [11]) of B_j onto $S_{n(j)}$ with the following properties:

(1) ${q_j}^{-1}(x)$ is a one-point set for each $x \in \left(S_{n(j)} \backslash J_{n(j)}\right) \cup \left\{d_{n(j)}\right\}$.
(2) The set $Z_j = {q_j}^{-1}\left(e_{n(j)}\right)$ is zero dimensional.

(3) ${q_j}^{-1}\left(J_{n(j)}\right)$ is a cone over Z_j. ${q_j}^{-1}\left(J_{n(j)}\right)$ is nowhere dense in B_j. If $\tilde{d}$ denotes the vertex of the cone and, for each $z \in Z_j$, A_z denotes the arc between z and $\tilde{d}$, then q_j restricted to A_z is a homeomorphism onto $J_{n(j)}$.
(4) $q_j \circ f_j = g \circ q_j$.
(5) $f_j\left(Z_j\right) = Z_j$ and f_j does not have periodic points on Z_j.

2.8. **Definition.** The map f_j is not a homeomorphism. To replace it by one, we will use the technique presented by J. B. Fugate and L. B. Mohler in [8]. Let $\widetilde{B}_j$ be the inverse limit of the inverse system of copies of B_j with the bonding maps equal to f_j. Let $\tilde{p}_j^k$ be the projection of $\widetilde{B}_j$ onto the k-th element of the inverse system. Let $\tilde{f}_j$ denote the right shift on $\widetilde{B}_j$, i.e. $\tilde{f}_j\left(\left(b_0, b_1, b_2, \dots\right)\right) = \left(f_j\left(b_0\right), b_0, b_1, b_2, \dots\right)$.

Since f_j does not have periodic points of periods smaller than or equal to j, the same is true for $\tilde{f}_j$.

2.9. **Proposition.** *$\tilde{f}_j$ does not have periodic points of periods smaller than or equal to j.*

2.10. **Definition.** Let $\beta_j : \widetilde{B}_j \to I$ be the map $p^0_{n(j)} \circ q_j \circ \tilde{p}^0_j$.

2.11. **Proposition.** *The set ${\beta_j}^{-1}((0,1))$ is homeomorphic to the product of the Cantor set with the interval $(0,1)$. The map β_j restricted to ${\beta_j}^{-1}((0,1))$ is the projection of the product onto $(0,1)$.*

Proof. Let K denote ${p^0_{n(j)}}^{-1}\left(\frac{1}{2}\right)$. Observe that K is homeomorphic to the Cantor set. For each integer $i \geq 0$, let I_i denote the interval I without the points $0/2^i, 1/2^i, \dots, 2^i/2^i$. Let $T_i = {p^0_{n(j)}}^{-1}\left(I_i\right)$. Let κ_1 and κ_2 denote the projections of $I_0 \times K$ onto I_0 and K, respectively. There is a homeomorphism χ mapping T_0 onto $I_0 \times K$ such that $\kappa_1 \circ \chi = p^0_{n(j)} \mid T_0$ and $\kappa_2 \circ \chi$ is the identity on K. Let $\sigma : I_1 \times K \to I_0 \times K$ denote $\chi \circ g \circ \chi^{-1}$. Note that $\kappa_1 \circ \sigma = g_2 \circ \kappa_1$, $\sigma^{-1}\left(I_i \times K\right) = I_{i+1} \times K$, $g^{-1}\left(T_i\right) = T_{i+1}$ and $\chi\left(T_i\right) = I_i \times K$.

Let $\widetilde{K} = {q_j}^{-1}(K)$ and let $\widetilde{T}_i = {q_j}^{-1}\left(T_i\right)$. Observe that $\widetilde{K}$ is homeomorphic to the Cantor set. Let $\tilde{\kappa}_1$ and $\tilde{\kappa}_2$ denote the projections of $I_0 \times \widetilde{K}$ onto I_0 and $\widetilde{K}$, respectively. Let $\tilde{q} : I_0 \times \widetilde{K} \to I_0 \times K$ denote the map defined by $\tilde{q}(t,c) = \left(t, q_j(c)\right)$. By (1), (2) and (3) of Definition 2.7 there is a homeomorphism $\tilde{\chi}$ mapping $\widetilde{T}_0$ onto $I_0 \times \widetilde{K}$ such that $\tilde{\kappa}_1 \circ \tilde{\chi} = p^0_{n(j)} \circ q_j \mid \widetilde{T}_0$ and $\tilde{\kappa}_2 \circ \tilde{\chi}$ is the identity on $\widetilde{K}$. Notice that $\chi \circ q_j = \tilde{q} \circ \tilde{\chi}$. Let $\tilde{\sigma} : I_1 \times \widetilde{K} \to I_0 \times \widetilde{K}$ denote $\tilde{\chi} \circ f_j \circ \tilde{\chi}^{-1}$. It follows from (4) of Definition 2.7, that $\tilde{\kappa}_1 \circ \tilde{\sigma} = g_2 \circ \tilde{\kappa}_1$. Observe also that $\tilde{\sigma}^{-1}\left(I_i \times \widetilde{K}\right) = I_{i+1} \times \widetilde{K}$, ${f_j}^{-1}\left(\widetilde{T}_i\right) = \widetilde{T}_{i+1}$, $\tilde{\chi}\left(T_i\right) = \tilde{I}_i \times \widetilde{K}$ and $f_j \circ \tilde{\chi}^{-1} = \tilde{\chi}^{-1} \circ \tilde{\sigma}$. It follows that ${\beta_j}^{-1}\left(I_0\right) = \tilde{p}_j^{0-1}\left(\widetilde{T}_0\right)$ is homeomorphic to the inverse sequence $I_0 \times \widetilde{K}, I_0 \times \widetilde{K}, \dots$ with the restrictions of $\tilde{\sigma}$ as the bonding maps. Since $\tilde{\kappa}_1 \circ \tilde{\sigma} = g_2 \circ \tilde{\kappa}_1$, ${\beta_j}^{-1}\left(I_0\right)$ is homeomorphic to the Cartesian product of I_0 and the Cantor set.

The next proposition follows readily from the definitions.

2.12. **Proposition.** *$\beta_j \circ \tilde{f}_j = g_2 \circ \beta_j$.*

The next proposition is a simple consequence of 2.5 and 2.12.

2.13. **Proposition.** *If $x \in \widetilde{B}_j$ and i is a positive integer such that $\tilde{f}_j{}^i(x) = x$, then $\beta_j(x) \in \mathcal{A}$.*

2.14. **Definition.** Let $\varphi_j : [0,\infty) \to S_{n(j)}$ be the map defined by the formula $\varphi_j(t) = (t_0, t_1, t_2, \dots)$, where $t_i = v\left(t \Big/ (n(j))^i\right)$. Observe that $\varphi_j(0) = e_{n(j)}$, φ_j is a injection and $g \circ \varphi_j(t) = \varphi_j(2t)$ for each $t \in [0,\infty)$. Observe also that if $t > 1$, then $\varphi(t) \notin J_{n(j)}$.

Let $z_j^0, z_j^1, \dots$ be a sequence of points of Z_j such that $f_j\left(z_j^{i+1}\right) = z_j^i$ for each $i = 0, 1, \dots$. Observe that, for each $i = 0, 1, \dots$, there is exactly one map $\varphi_j^i : [0,\infty) \to B_j$ such that $\varphi_j^i(0) = z_j^i$ and $q_j \circ \varphi_j^i(t) = \varphi_j\left(t2^{-i}\right)$ for each $t \in [0,\infty)$. Let $\tilde{\varphi}_j : [0,\infty) \to \widetilde{B}_j$ be defined by $\tilde{\varphi}_j(t) = \left(\varphi_j^0(t), \varphi_j^1(t), \dots\right)$. Observe that $\tilde{\varphi}_j$ is a well defined continuous map.

2.15. **Proposition.** *$\beta_j \circ \tilde{f}_j{}^{-1} \circ \tilde{\varphi}_j(2t) = \beta_j \circ \tilde{\varphi}_j(t)$ for each $t \geq 0$ and*

$$\lim_{t\to\infty} \left|\tilde{f}_j{}^{-1} \circ \tilde{\varphi}_j(2t) - \tilde{\varphi}_j(t)\right| = 0.$$

Proof. Let i be a positive integer. Since $\tilde{p}_j^i \circ \tilde{\varphi}_j(t) = \varphi_j^i(t)$, $q_j \circ \tilde{p}_j^i \circ \tilde{\varphi}_j(t) = \varphi_j\left(t2^{-i}\right)$. Observe that $\tilde{p}_j^i \circ \tilde{f}_j{}^{-1} \circ \tilde{\varphi}_j(2t) = \varphi_j^{i+1}(2t)$. So

$$(*) \qquad q_j \circ \tilde{p}_j^i \circ \tilde{f}_j{}^{-1} \circ \tilde{\varphi}_j(2t) = \varphi_j\left(t2^{-i}\right) = q_j \circ \tilde{p}_j^i \circ \tilde{\varphi}_j(t).$$

The first part of the proposition follows from (*) for $i = 0$.

To prove the remaining part of the proposition it is enough to observe that for each positive integer i, if t is sufficiently large, then the projection $\tilde{p}_j^i$ agrees at the points $\tilde{f}_j{}^{-1} \circ \tilde{\varphi}_j(2t)$ and $\tilde{\varphi}(t)$. If $t > 2^i$, then $\varphi_j\left(t2^{-i}\right) \notin J_{n(j)}$ and it follows from (*) and (1) of 2.7 that $\tilde{p}_j^i \circ \tilde{\varphi}_j(t) = \tilde{p}_j^i \circ \tilde{f}_j{}^{-1} \circ \tilde{\varphi}_j(2t)$.

2.16. **Proposition.** *$\beta_j \circ \tilde{f}_j \circ \tilde{\varphi}_j(t) = \beta_j \circ \tilde{\varphi}_j(2t)$ for each $t \geq 0$ and*

$$\lim_{t\to\infty} \left|\tilde{f}_j \circ \tilde{\varphi}_j(t) - \tilde{\varphi}_j(2t)\right| = 0.$$

Proof. Since $\beta_j \circ \tilde{\varphi}_j(2t) = \beta_j \circ \tilde{f}_j \circ \tilde{f}_j{}^{-1} \circ \tilde{\varphi}_j(2t)$, it follows from Proposition 2.12 that $\beta_j \circ \tilde{\varphi}_j(2t) = g_2 \circ \beta_j \circ \tilde{f}_j{}^{-1} \circ \tilde{\varphi}_j(2t)$. By Proposition 2.15, $g_2 \circ \beta_j \circ \tilde{f}_j{}^{-1} \circ \tilde{\varphi}_j(2t) = g_2 \circ \beta_j \circ \tilde{\varphi}_j(t)$. Since, by Proposition 2.12, $g_2 \circ \beta_j \circ \tilde{\varphi}_j(t) = \beta_j \circ \tilde{f}_j \circ \tilde{\varphi}_j(t)$, we have that $\beta_j \circ \tilde{\varphi}_j(2t) = \beta_j \circ \tilde{f}_j \circ \tilde{\varphi}_j(t)$. The remaining part of the proposition follows from Proposition 2.15.

3. Continued fractions

3.1. **Definition.** If $n_1, n_2, \dots, n_k, d_1, d_2, \dots, d_k$ are real numbers, then the continued fraction

$$\cfrac{n_1}{d_1 + \cfrac{n_2}{d_2 + \cfrac{\cdots}{\cdots + \cfrac{n_{k-1}}{d_{k-1} + \cfrac{n_k}{d_k}}}}}$$

is denoted by $\operatorname{cf}(n_1, \dots, n_k; d_1, \dots, d_k)$. We will say that $\operatorname{cf}(n_1, \dots, n_k; d_1, \dots, d_k)$ is well defined if the denominators on all levels are not 0.

3.2. **Definition.** Let Ω be an infinite set of positive real numbers < 1 such that $P(\omega_1, \omega_2, \ldots, \omega_n) \neq 0$ for any non-zero polynomial $P(x_1, x_2, \ldots, x_n)$ with rational coefficients and every set of different elements $\omega_1, \omega_2, \ldots, \omega_n$ of Ω. Let $w_0, w_1, \ldots$ be a non-repetitive sequence of elements of Ω.

3.3. **Definition.** For any two positive integers i and j and any real numbers $a_i, \ldots, a_j$, let $\omega[i, j; a_i, \ldots, a_j]$ denote cf $(w_i, \ldots, w_j; a_i, \ldots, a_j)$. Note that in the above definition i may be equal to j or either of the integers may be greater than the other. If $i \leq j$, then by $\gamma[i, j; a_i, \ldots, a_j]$ we will denote a_i when $i = j$ and $a_i + \omega[i, j-1; a_{i+1}, \ldots, a_j]$ when $i < j$. We will say that $\omega[i, j; a_i, \ldots, a_j]$ (or $\gamma[i, j; a_i, \ldots, a_j]$) is well defined if the continued fraction in its definition is well defined.

The proof of the next three propositions follows readily from the definitions.

3.4. **Proposition.** *Suppose that i, j and k are positive integers such that $i \leq j \leq k$. Suppose $a_i, \ldots, a_k$ be real numbers such that $\gamma[i, k; a_i, \ldots, a_k]$ is well defined. Let b_j denote $\gamma[j, k; a_j, \ldots, a_k]$. Then $\gamma[i, k; a_i, \ldots, a_k] = \gamma[i, j; a_i, \ldots, a_{j-1}, b_j]$.*

3.5. **Proposition.** *Suppose that i and j are two different positive integers. Let l denote $|i-j|-1$. Suppose that $a_1, \ldots, a_l$ are real numbers. Then $\omega[i, j; a_1, \ldots, a_l, t]$ is not well defined for finitely many real numbers t. Moreover, there are numbers A, B, C and D such that $\omega[i, j; a_1, \ldots, a_l, t] = (At+B)/(Ct+D)$ for each t for which $\omega[i, j; a_1, \ldots, a_l, t]$ is well defined. In particular, it follows that, if $\omega[i, j; a_1, \ldots, a_l, t']$ and $\omega[i, j; a_1, \ldots, a_l, t'']$ are well defined and equal, then $t' = t''$.*

3.6. **Proposition.** *Suppose that i and j are positive integers such that $i \leq j$. Let $a_i, \ldots, a_j$ be rational numbers such that $\omega[i, j; a_i, \ldots, a_j]$ is well defined. Then there are two non-zero polynomials $P_1(x_{i+1}, \ldots, x_j)$ and $P_2(x_{i+1}, \ldots, x_j)$ with rational coefficients such that*

$$\omega[i, j; a_i, \ldots, a_j] = w_i P_1(w_{i+1}, \ldots, w_j) / P_2(w_{i+1}, \ldots, w_j).$$

3.7. **Proposition.** *Let p and q be positive integers such that $p \leq q$. Suppose that $c_p, \ldots, c_q$ are rational numbers such that $c_q \neq 0$. Then $\omega[p, q; c_p, \ldots, c_q]$ is well defined.*

We will prove Proposition 3.7 together with the following one.

3.8. **Proposition.** *Let i, j, k and l be positive integers such that $i \leq j$ and $k \leq l$. Suppose that $a_i, \ldots, a_j$, $b_k, \ldots, b_l$ are rational numbers such that if $i < j$, then $a_j \neq 0$ and, if $k < l$, then $b_l \neq 0$. Suppose also that*

$$(*) \qquad \gamma[i, j; a_i, \ldots, a_j] = \gamma[k, l; b_k, \ldots, b_l].$$

Then either $i = j$, $k = l$ and $a_i = a_k$ or $i = k$, $j = l$ and $a_n = b_n$ for each $n = i, \ldots, j$.

Proof of 3.7 *and* 3.8. Let $m = \max(j-i, l-k, q-p)$. If $m = 0$, then both propositions are obvious. Suppose that the propositions are true, if m is less than some integer $m_0 \geq 1$. We will prove that they are true, if $m = m_0$.

Proposition 3.7 (for $m < m_0$) implies that both numbers $\omega[i, j-1; a_{i+1}, \ldots, a_j]$ and $\omega[k, l-1; b_{k+1}, \ldots, a_l]$ are well defined. Let $P_1(x_{i+1}, \ldots, x_j)$, $P_2(x_{i+1}, \ldots, x_j)$,

$Q_1(x_{k+1},\dots,x_l)$ and $Q_2(x_{k+1},\dots,x_l)$ be polynomials with rational coefficients such that

$$\omega[i,j-1;a_{i+1},\dots,a_j]=w_iP_1(w_{i+1},\dots,w_j)/P_2(w_{i+1},\dots,w_j)$$

and

$$\omega[k,l-1;b_{k+1},\dots,a_l]=w_kQ_1(w_{k+1},\dots,w_l)/Q_2(w_{k+1},\dots,w_l).$$

It follows from (*) that $(a_i-b_k)P_2Q_2+w_iP_1Q_2-w_kP_2Q_1=0$. Since the polynomial on the left side of the last equation must be trivial, we have that $i=k$, $P_1Q_2-P_2Q_1=0$ and $a_i=b_i$. The condition (*) implies that

$$\omega[i,j-1;a_{i+1},\dots,a_j]=\omega[i,l-1;b_{i+1},\dots,b_l].$$

From the last equality we infer that

$$\gamma[i+1,j;a_{i+1},\dots,a_j]=\gamma[i+1,l;b_{i+1},\dots,b_l]$$

and Proposition 3.8 for $m=m_0$ follows from the inductive assumption.

Observe that $\omega[p,q;c_p,\dots,c_q]=w_p/\gamma[p+1,q+1;c_p,\dots,c_q]$. To complete the proof, we have to establish that $\omega[p,q;c_p,\dots,c_q]$ is well defined. By the inductive assumption, it will be enough to notice that $\gamma[p+1,q+1;c_p,\dots,c_q]$ is not equal to 0. But, since $0=\gamma[1,1;0]$, this is a particular case of already proven portion of Proposition 3.8. So Propositions 3.7 and 3.8 follow by induction.

3.9. **Proposition.** *Suppose that i and j are positive integers such that $i\le j$. Suppose that $a_i,a_{i+1},\dots,a_j,a_{j+1}$ are real numbers such that the continued fraction $b=\gamma[i,j+1;a_i,\dots,a_{j+1}]$ is well defined.*

Then $\omega[j,i;-a_j,-a_{j-1},\dots,-a_{i+1},b-a_i]$ is well defined and is equal to a_{j+1}.

Proof. We will prove the proposition by induction with respect to $j-i$. If $j-i=0$, then $b=a_i+w_i/a_{i+1}$ and $\omega[i,i;b-a_i]=a_{i+1}$. Suppose that the proposition is true if $j-i$ is less than some m. We will prove it for $j-i=m$.

Let $b'=\gamma[i+1,j+1;a_{i+1},\dots,a_{j+1}]$. Since $b=a_i+w_i/b'$, b' is well defined, $b-a_i\neq 0$ and $b'=w_i/(b-a_i)$. By the inductive assumption, the continued fraction $\omega[j,i+1;-a_j,-a_{j-1},\dots,-a_{i+2},b'-a_{i+1}]$ is well defined and equals to a_{j+1}. Substituting $b'=w_i/(b-a_i)$ to $\omega[j,i+1;-a_j,-a_{j-1},\dots,-a_{i+2},b'-a_{i+1}]$ we get $\omega[j,i+1;-a_j,-a_{j-1},\dots,-a_{i+2},-a_{i+1}+w_i/(b-a_i)]$. The last continued fraction is equal to $\omega[j,i;-a_j,-a_{j-1},\dots,-a_{i+1},b-a_i]$, so the proposition follows by induction.

3.10. **Proposition.** *Suppose i and n are integers such that $1\le i\le n$. Let $a_i,\dots,a_n\in\mathcal{A}$. Then there is a positive number ϵ such that if $b_i,\dots,b_n\in\mathcal{A}$ such that $v(b_j)=v(a_j)$ for $j=i,\dots,n$, then $|\gamma[i,n;b_i,\dots,b_n]|>\epsilon$.*

Proof. If $i=n$, then the proposition is obvious. Using the induction, we can assume that for each integer k, such that $i<k\le n$, there is a positive number ϵ_k such that $|\gamma[k,n;b_k,\dots,b_n]|>\epsilon_k$ for each $b_k,\dots,b_n\in\mathcal{A}$ such that $v(b_j)=v(a_j)$ for $j=k,\dots,n$.

3.10.1. *Claim.* Let κ be a positive number and let k be an integer such that $i-1 \leq k < n$. Then there is a finite sequence $b^1_{k+1}, b^2_{k+1}, \ldots, b^m_{k+1}$ of elements of $\mathcal{A}$ such that $|\omega[k, n-1; b_{k+1}, \ldots, b_n]| < \kappa$ for each $b_{k+1}, \ldots, b_n \in \mathcal{A}$ such that $v(b_j) = v(a_j)$ for $j = k+1, \ldots, n$ and $b_{k+1} \neq b^l_{k+1}$ for $l = 1, \ldots, m$.

Proof of 3.10.1. If $k+1 = n$, then $\omega[k, k; b_{k+1}] = w_k / b_{k+1}$. Since $|w_k / b_{k+1}| \geq \kappa$ for finitely many values of b_{k+1}, the claim is true when $k+1 = n$.

If $k < n-1$, then

$$\omega[k, n-1; b_{k+1}, \ldots, b_n] = w_k / (b_{k+1} + w_{k+1} / \gamma[k+2, n; b_{k+2}, \ldots, b_n]).$$

Since $|w_{k+1} / \gamma[k+2, n; b_{k+2}, \ldots, b_n]| < |w_{k+1} / \epsilon_{k+2}|$, $|\omega[k, n-1; b_{k+1}, \ldots, b_n]| < \kappa$ when $|b_{k+1}| > |w_k| / \kappa + |w_{k+1} / \epsilon_{k+2}|$. So the claim is true.

3.10.2. *Claim.* Let l be an integer such that $i \leq l \leq n$. Let $c_i, \ldots, c_l \in \mathcal{A}$. Then there is a positive number $\kappa(c_i, \ldots, c_l)$ such that $|\gamma[i, n; c_i, \ldots, c_l, b_{l+1}, \ldots, b_n]| > \kappa(c_i, \ldots, c_l)$ for each $b_{l+1}, \ldots, b_n \in \mathcal{A}$ such that $v(b_j) = v(a_j)$ for $j = l+1, \ldots, n$.

Proof of 3.10.2. If $l = n$, then $\kappa(c_i, \ldots, c_n) = |\gamma[i, n; c_i, \ldots, c_n]| > 0$, by Proposition 3.8. Now, we suppose that the claim is true for each l greater than some integer k and we will prove it for $l = k$.

Let $\kappa_1 = |\gamma[i, n; c_i, \ldots, c_k]|$. By Proposition 3.8, $\kappa_1 > 0$. There is a positive number κ_2 such that $|\gamma[i, n; c_i, \ldots, c_{k-1}, c_k + t]| > \frac{1}{2}\kappa_1$ for each real number t such that $|t| < \kappa_2$. Let $b^1_{k+1}, b^2_{k+1}, \ldots, b^m_{k+1}$ be as in Claim 3.10.1 used with $\kappa = \kappa_2$. Let $\kappa(c_i, \ldots, c_k)$ be the least of the numbers $\frac{1}{2}\kappa_1$, $\kappa\left(c_i, \ldots, c_k, b^1_{k+1}\right)$, $\kappa\left(c_i, \ldots, c_k, b^2_{k+1}\right)$, $\ldots$, $\kappa\left(c_i, \ldots, c_k, b^m_{k+1}\right)$. Let $b_{k+1}, \ldots, b_n \in \mathcal{A}$ be arbitrary numbers such that $v(b_j) = v(a_j)$ for $j = k+1, \ldots, n$. If $b_{k+1} = b^j_{k+1}$ for some $j = 1, \ldots, m$, then $|\gamma[i, n; c_i, \ldots, c_k, b_{k+1}, \ldots, b_n]| > \kappa\left(c_i, \ldots, c_k, b^j_{k+1}\right) \geq \kappa(c_i, \ldots, c_k)$. So we can assume that $b_{k+1} \neq b^j_{k+1}$ for $j = 1, \ldots, m$. Since $|\omega[k, n-1; b_{k+1}, \ldots, b_n]| < \kappa_2$ and

$$\gamma[i, n; c_i, \ldots, c_k, b_{k+1}, \ldots, b_n] = \gamma[i, n; c_i, \ldots, c_k + \omega[k, n-1; b_{k+1}, \ldots, b_n]],$$

we have that $|\gamma[i, n; c_i, \ldots, c_k, b_{k+1}, \ldots, b_n]| > \frac{1}{2}\kappa_1 \geq \kappa(c_i, \ldots, c_k)$. Thus the claim is true.

Let $b^1_{k+1}, b^2_{k+1}, \ldots, b^m_{k+1}$ be as in Claim 3.10.1 used with $k = i-1$ and $\kappa = |w_{i-1}|$. Let $b_i, \ldots, b_n$ be arbitrary numbers such that $v(b_j) = v(a_j)$ for each $j = i, \ldots, n$. Observe that if $b_i \neq b^j_i$ for each $j = 1, \ldots, m$, then $|w_{i-1} / \gamma[i, n; b_i, \ldots, b_n]| = \omega[i-1, n-1; b_i, \ldots, b_n] < |w_{i-1}|$ and consequently $|\gamma[i, n; b_i, \ldots, b_n]| > 1$. Now, define ϵ to be the least of the numbers 1, $\kappa\left(b^1_i\right)$, $\kappa\left(b^2_i\right)$, $\ldots$, $\kappa\left(b^m_i\right)$. Observe that the proposition follows readily from Claim 3.10.2.

3.11. **Proposition.** *Let i and n be integers such that $1 \leq i \leq n$. Suppose that $a_i, \ldots, a_n \in \mathcal{A}$. Then, for each positive number ϵ, there is a positive number $\eta^n_i(\epsilon)$ such that if t is a real number and $b_i, \ldots, b_n \in \mathcal{A}$ are such that $|t| < \eta^n_i(\epsilon)$ and $v(b_j) = v(a_j)$ for $j = i, \ldots, n$, then*

$$|\gamma[i, n; b_i, \ldots, b_n] - \gamma[i, n; b_i, \ldots, b_{n-1}, b_n + t]| < \epsilon.$$

Proof. If $i = n$, then $\eta_n^n(\epsilon) = \epsilon$ satisfies the proposition. Suppose that $k < n$ is an integer such that the proposition is true for each integer i such that $k < i \leq n$. We will show that the proposition is true for $i = k$.

Let ϵ be an arbitrary number. By Proposition 3.10, there is a positive number ϵ_1 such that $|\gamma[k+1, n; b_{k+1}, \ldots, b_n]| > \epsilon_1$ for each $b_{k+1}, \ldots, b_n \in \mathcal{A}$ such that $v(b_j) = v(a_j)$ for $j = k+1, \ldots, n$. Let ϵ_2 be the least of the numbers $\epsilon_1/2$ and ${\epsilon_1}^2/(2|w_k|)$. By the inductive assumption, there is $\eta_{k+1}^n(\epsilon_2)$ satisfying the proposition for $i = k+1$. Let t be a real number such that $|t| < \eta_{k+1}^n(\epsilon_2)$. Let $b_k, \ldots, b_n \in \mathcal{A}$ be such that $v(b_j) = v(a_j)$ for $j = k, \ldots, n$. Denote $\gamma[k+1, n; b_{k+1}, \ldots, b_n]$ and $\gamma[k+1, n; b_{k+1}, \ldots, b_{n-1}, b_n + t]$ by c and c_t, respectively. Since $|c - c_t| < \epsilon_2 \leq \epsilon_1/2$ and $|c| > \epsilon_1$, we have that $|c_t| > \epsilon_1/2$. Denote $\gamma[k, n; b_k, \ldots, b_n]$ and $\gamma[k, n; b_k, \ldots, b_{n-1}, b_n + t]$ by d and d_t, respectively. Since $d = b_k + w_k/c$ and $d_t = b_k + w_k/c_t$, we have that $|d - d_t| = |w_k| |c - c_t| / |c| |c_t| < 2|w_k| \epsilon_2/{\epsilon_1}^2 \leq \epsilon$. So the proposition is true with $\eta_k^n(\epsilon) = \eta_{k+1}^n(\epsilon_2)$.

3.12. **Definition.** Let $\underline{a}$ be a finite sequence $a_1, \ldots, a_n$ of elements of $\mathcal{A}$ and let ϵ be a positive number. By Proposition 3.11, for each pair of integers i and j such that $1 \leq i \leq j \leq n$, there is a number $\eta_i^j(\epsilon)$ such that

$$|\gamma[i, j; b_i, \ldots, b_j] - \gamma[i, j; b_i, \ldots, b_{j-1}, b_j + t]| < \epsilon$$

for each real number t and each $b_i, \ldots, b_j \in \mathcal{A}$ such that $|t| < \eta_i^j(\epsilon)$ and $v(b_m) = v(a_m)$ for $m = i, \ldots, j$. Let $\eta(\underline{a}, \epsilon)$ be the least of the numbers $\eta_i^j(\epsilon)$, where $1 \leq i \leq j \leq n$.

3.13. **Definition.** For any real number x, let $\mathrm{E}(x)$ denote the greatest integer less than or equal to x. Let $\mu(x)$ denote the least of the numbers $x - \mathrm{E}(x)$ and $\mathrm{E}(x) + 1 - x$.

3.14. **Proposition.** *Suppose i and j are integers such that $1 \leq i \leq j$. Let $a_i, \ldots, a_j$ be numbers from $\mathcal{A}$. Then there is a positive number ϵ such that if $b_i, \ldots, b_j \in \mathcal{A}$ are such that $v(b_m) = v(a_m)$ for $m = i, \ldots, j$, then*

$$\mu(\gamma[i, j; b_i, \ldots, b_j]) > \epsilon.$$

Proof. Using Proposition 3.10 twice, for $a_i, \ldots, a_j$ and then for $a_i + 1, a_{i+1}, \ldots, a_j$, we get a positive number ϵ such that $|\gamma[i, j; c_i, \ldots, c_j]| > \epsilon$ for each $c_i, \ldots, c_j \in \mathcal{A}$ such that $v(c_m) = v(a_m)$ for $m = i+1, \ldots, j$ and either $v(c_i) = v(a_i)$ or $v(c_i) = v(a_i + 1)$.

Let $b_i, \ldots, b_j \in \mathcal{A}$ be arbitrary numbers such that $v(b_m) = v(a_m)$ for $m = i, \ldots, j$. Let k be an arbitrary integer. Observe that $v(b_i - k) = v(a_i)$ if k is even, and $v(b_i - k) = v(a_i + 1)$ if k is odd. So

$$|\gamma[i, j; b_i, \ldots, b_j] - k| = |\gamma[i, j; b_i - k, b_{i+1}, \ldots, b_j]| > \epsilon.$$

The proposition follows from the above inequality used for $k = \mathrm{E}(\gamma[i, j; b_i, \ldots, b_j])$ and $k = \mathrm{E}(\gamma[i, j; b_i, \ldots, b_j]) + 1$.

3.15. **Definition.** Let $\underline{a}$ be a finite sequence $a_1, \ldots, a_n$ of elements of $\mathcal{A}$. By Proposition 3.14, for each pair of integers i and j such that $1 \leq i \leq j \leq n$, there is a number ϵ_j^i such that, if $b_i, \ldots, b_j \in \mathcal{A}$ are such that $v(b_m) = v(a_m)$ for $m = i, \ldots, j$, then $\mu(\gamma[i, j; b_i, \ldots, b_j]) > \epsilon_j^i$. Let $\epsilon(\underline{a})$ be the least of the numbers ϵ_j^i, where $1 \leq i \leq j \leq n$.

4. Auxiliary construction

4.1. **Definition.** Let $T = \{\emptyset\} \cup \bigcup_{j=1}^{\infty} I^j$ and let $T_n = \{\emptyset\} \cup \bigcup_{j=1}^{n} I^j$ for each positive integer n. For a point $t = (t_1, t_2, \ldots, t_n) \in T$, we will adopt the following notation: $r_j(t) = t_j$, $s_j(t) = (t_1, t_2, \ldots, t_j)$ and $\operatorname{len}(t) = n$. Additionally, set $r_0(t) = \emptyset$ and $s_0(t) = \emptyset$. Also $r_j(\emptyset) = s_j(\emptyset) = \emptyset$ and $\operatorname{len}(\emptyset) = 0$.

4.2. **Definition.** For each positive integer n, let $C(n)$ be the set of points $c = (c_1, c_2, \ldots, c_k) \in T_n$ for which there is a sequence $a_1, a_2, \ldots, a_n$ of elements of $\mathcal{A}$ and there is an increasing sequence of integers $\alpha(1), \ldots, \alpha(k), \alpha(k+1)$ such that $\alpha(1) = 1$, $\alpha(k+1) = n+1$ and $c_j = \gamma\left[\alpha(j), \alpha(j+1)-1; a_{\alpha(j)}, \ldots, a_{\alpha(j+1)-1}\right]$ for each $j = 1, \ldots, k$. By Proposition 3.8, the choice of $a_1, a_2, \ldots, a_n$ and $\alpha(1), \ldots, \alpha(k), \alpha(k+1)$ is unique for each $c \in C(n)$. We will denote these numbers by $a_1(c), a_2(c), \ldots, a_n(c)$ and $\alpha(1,c), \ldots, \alpha(k,c), \alpha(k+1,c)$, respectively. Additionally, let $C(0) = \{\emptyset\}$ and let $\alpha(1, \emptyset) = 1$.

The next two propositions follow readily from the definition of $C(n)$.

4.3. **Proposition.** *Suppose $c \in C(n)$ and j is an integer such that $0 \le j \le \operatorname{len}(c)$. Then $s_j(c) \in C(\alpha(j+1,c)-1)$, $\alpha(i,c) = \alpha(i, s_j(c))$ for $i = 1, \ldots, j+1$, and $a_k(c) = a_k(s_j(c))$ for $k = 1, \ldots, \alpha(j+1,c)-1$.*

4.4. **Proposition.** *Suppose $(c_1, \ldots, c_k)$ belongs to $C(i)$ and c_{k+1} is a number from $\mathcal{A} \cap I$. Then $(c_1, \ldots, c_k, c_{k+1}) \in C(i+1)$.*

The next proposition follows from 3.8.

4.5. **Proposition.** *Suppose $c \in C(n)$. Then $c \notin C(i)$ for each positive integer $i \ne n$.*

4.6. **Definition.** Suppose $c \in C(n)$. Denote by $\underline{a}$ the sequence $a_1(c), \ldots, a_n(c)$. Let $\epsilon(c) = \epsilon(\underline{a})$. Let $\eta(c)$ be a positive real number which is less than $\eta\left(\underline{a}, \frac{1}{2}\epsilon(c)\right)$ and such that $u_c = w_n / \eta(c)$ is an even integer. We can assume that $\epsilon(c)$ and $\eta(c)$ are chosen so that $\epsilon(c) \le \epsilon(s_j(c))$ and $\eta(c) \le \eta(s_j(c))$ for each positive integer $j \le \operatorname{len}(c)$. Observe that $\epsilon(c) < \frac{1}{2}$ and $\eta(c) < \frac{1}{2}$. Let $M(c) = 2/\epsilon(c) + 1/\eta(c)$.

4.7. *Remark.* We may assume that if $b, c \in C(n)$ are such that $v(a_i(b)) = v(a_i(c))$ for each $i = 1, \ldots, n$, then $\epsilon(b) = \epsilon(c)$, $\eta(b) = \eta(c)$ and $M(b) = M(c)$.

4.8. **Proposition.** *Suppose $c \in C(n)$ and i and j are integers such that $1 < i \le j \le n$. Let $b_i, \ldots, b_j \in \mathcal{A}$ are such that $v(b_m) = v(a_m(c))$ for $m = i, \ldots, j$ and $|b_i| > 2/\epsilon(c)$. Then*

$$|w_{i-1} / \gamma[i,j; b_i, \ldots, b_{j-1}, b_j + t]| < 1 / (|b_i| - 2/\epsilon(c))$$

for each real number t such that $|t| < \eta(c)$. In particular, if $|b_i| > M(c)$, then

$$|w_{i-1} / \gamma[i,j; b_i, \ldots, b_{j-1}, b_j + t]| < \eta(c)$$

for each real number t such that $|t| < \eta(c)$.

Proof. Suppose $i = j$. Then $|\gamma[i,j; b_i, \ldots, b_{j-1}, b_j + t]| = |b_i + t| > |b_i| - \eta(c) > 1/\eta(c)$. Since $0 < w_{i-1} < 1$ and $\eta(c) < 1/2 < 2/\epsilon(c)$, the first part of the proposition is true in this case.

Now, suppose $i < j$. Let y denote $\gamma[i+1,j;b_{i+1},\dots,b_j]$ and let y_t denote $\gamma[i+1,j;b_{i+1},\dots,b_{j-1},b_j+t]$. Observe that since $|y_t - y| < \frac{1}{2}\epsilon(c)$ and $|y| > \epsilon(c)$, we have that $|y_t| > \frac{1}{2}\epsilon(c)$. Thus $|\gamma[i,j;b_i,\dots,b_{j-1},b_j+t]| = |b_i + w_i/y_t| > |b_i| - 2/\epsilon(c)$. So the first part of the proposition is true. The second part follows in both cases from the equality $M(c) = 2/\epsilon(c) + 1/\eta(c)$.

4.9. **Definition.** Suppose $c \in C(n)$ and i and j are integers such that $1 \le i \le j \le n$. For any real number t, let $\gamma_c[i,j](t)$ denote $\gamma[i,j;a_i(c),\dots,a_{j-1}(c),t]$. The number $\gamma[i,j;a_i(c),\dots,a_j(c)]$ will be denoted by $\gamma_c[i,j]$. It can be readily verified that $\gamma_c[i,j](a_j(c)) = \gamma_c[i,j]$. Additionally $\gamma_c[i,n](t)$ will be denoted by $\gamma_c^i(t)$.

The next proposition follows from 3.4.

4.10. **Proposition.** *Suppose $c \in C(n)$ and i, j and k are integers such that $1 \le i \le j \le k \le n$. Then $\gamma_c[i,k] = \gamma_c[i,j](\gamma_c[j,k])$. Suppose t is a real number such that $\gamma_c[i,k](t)$ is well defined. Then $\gamma_c[i,k](t) = \gamma_c[i,j](\gamma_c[j,k](t))$. In particular, if $\gamma_c^i(t)$ is well defined, then $\gamma_c^i(t) = \gamma_c[i,j]\left(\gamma_c^j(t)\right)$.*

4.11. **Proposition.** *Suppose $c \in C(n)$ and i and j are integers such that $1 \le i \le j \le n$. Suppose t is a real number such that $|t| < \eta(c)$. Then $\gamma_c[i,j](a_j(c)+t)$ is well defined and $\mathrm{E}(\gamma_c[i,j]) < \gamma_c[i,j](a_j(c)+t) < \mathrm{E}(\gamma_c[i,j])$.*

Proof. Clearly, $\gamma_c[j,j](a_j(c)+t)$ is well defined. Let k be an arbitrary integer such that $i \le k \le j$ and $\gamma_c[k,j](a_j(c)+t)$ is well defined. It follows from the definition of $\eta(c)$ that $|\gamma_c[k,j](a_j(c)+t) - \gamma_c[k,j]| < \epsilon(c)$. Since $\epsilon(c) < \mu(\gamma_c[k,j])$, we get the result that $\mathrm{E}(\gamma_c[k,j]) < \gamma_c[k,j](a_j(c)+t) < \mathrm{E}(\gamma_c[k,j])$. In particular, $\gamma_c[k,j](a_j(c)+t)$ is not an integer and therefore is not 0. Now, $\gamma_c[k_1,j](a_j(c)+t) = a_{k-1}(c) + w_{k-1}/\gamma_c[k,j](a_j(c)+t)$ is well defined and the proposition follows by induction.

4.12. **Proposition.** *Suppose $c \in C(n)$. Then $\left|a_{\alpha(m,c)}\right| < M(c)$ for each $m = 1,\dots,\mathrm{len}(c)$.*

Proof. Observe that $r_m(c) = \gamma_c[\alpha(m,c),\alpha(m+1,c)-1]$ is between 0 and 1. If $\alpha(m+1,c) = \alpha(m,c)+1$, then $r_n(c) = a_{\alpha(m,c)}$ and the proposition is true, since $M(c) > 1$. If $\alpha(m+1,c) > \alpha(m,c)+1$, then $M(c) > 1/\epsilon(c) + 1 > 1/\epsilon(c) + |r_n(c)| = 1/\epsilon(c) + \left|a_{\alpha(m,c)} + w_{\alpha(m,c)}/\gamma_c[\alpha(m,c)+1,\alpha(m+1,c)-1]\right| \ge 1/\epsilon(c) + \left|a_{\alpha(m,c)}\right| - \left|w_{\alpha(m,c)}/\gamma_c[\alpha(m,c)+1,\alpha(m+1,c)-1]\right|$. Since $w_{\alpha(m,c)}$ is between 0 and 1 and $|\gamma_c[\alpha(m,c)+1,\alpha(m+1,c)-1]| > \epsilon(c)$, we get the result that $\left|w_{\alpha(m,c)}/\gamma_c[\alpha(m,c)+1,\alpha(m+1,c)-1]\right| < 1/\epsilon(c)$. Thus $M(c) > 1/\epsilon(c) + \left|a_{\alpha(m,c)}\right| - 1/\epsilon(c) = \left|a_{\alpha(m,c)}\right|$.

4.13. **Proposition.** *Suppose $b = (b_1,\dots,b_k)$ and $c = (c_1,\dots,c_k)$ are two points such that, for each $m = 1,\dots,k$, either $b_m = g_2(c_m)$ or $c_m = g_2(b_m)$. Let i be a positive integer. Then $b \in C(i)$ if and only if $c \in C(i)$.*

Proof. Suppose $b \in C(i)$. To prove the proposition it is enough to show that $c \in C(i)$. To this end we have to define $\alpha(m,c)$ for $m = 1,\dots,k+1$ and $a_j(c)$ for $j = 1,\dots,i$. Let $\alpha(m,c) = \alpha(m,b)$ for each $m = 1,\dots,k+1$. Suppose m is an arbitrary integer such that $1 \le m \le k$. Let n denote $\alpha(m+1,c) - 1$. To complete the proof of the proposition we have to define numbers $a_m,\dots,a_n$ from $\mathcal{A}$ such that $c_m = \gamma[m,n;a_m,\dots,a_n]$. Since either $b_m = g_2(c_m)$ or $c_m = g_2(b_m)$, we have to consider the following four cases: $b_m = 2c_m$, $b_m = 2 - 2c_m$, $c_m = 2b_m$ and $c_m = 2 - 2b_m$. For each $j = m, \ldots, n$, we will define a_j in the following way.

Case $b_m = 2c_m$.

$$a_j = \begin{cases} \frac{1}{2} a_j(b), & \text{if } j - m \text{ is even}, \\ 2a_j(b), & \text{if } j - m \text{ is odd}. \end{cases}$$

Case $b_m = 2 - 2c_m$.

$$a_j = \begin{cases} 2 - \frac{1}{2} a_m(b), & \text{if } j = m, \\ -\frac{1}{2} a_j(b), & \text{if } j > m \text{ and } j - m \text{ is even}, \\ -2a_j(b), & \text{if } j - m \text{ is odd}. \end{cases}$$

Case $c_m = 2b_m$.

$$a_j = \begin{cases} 2a_j(b), & \text{if } j - m \text{ is even}, \\ \frac{1}{2} a_j(b), & \text{if } j - m \text{ is odd}. \end{cases}$$

Case: $c_m = 2 - 2b_m$.

$$a_j = \begin{cases} 2 - 2a_m(b), & \text{if } j = m, \\ -2a_j(b), & \text{if } j > m \text{ and } j - m \text{ is even}, \\ -\frac{1}{2} a_j(b), & \text{if } j - m \text{ is odd}. \end{cases}$$

It may be readily verified that in each of the cases $c_m = \gamma[m, n; a_m, \ldots, a_n]$.

4.14. **Proposition.** *Suppose* $b = (b_1, \ldots, b_k)$ *and* $c = (c_1, \ldots, c_k)$ *are two points of* $C(i)$ *such that* $l = \alpha(k, b) = \alpha(k, c)$ *and* $c_k = 2b_k$. *Suppose* t *is a real number such that* $|t| < \eta(b)$ *and* $|t| < \frac{1}{2}\eta(c)$. *Let* t_c *be* $2t$, *if* $i - l$ *is even, and let* t_c *be* $t/2$, *if* $i - l$ *is odd. Then* $\gamma_c^l(a_i(c) + t_c) = 2\gamma_b^l(a_i(b) + t)$.

Proof. Let $a_{l+j} = 2a_{l+j}(b)$, if j is even, and let $a_{l+j} = \frac{1}{2} a_{l+j}(b)$, if j is odd. Observe that $\gamma[l, i; a_l, \ldots, a_i] = 2b_k = c_k$. By Proposition 3.8, $a_{l+j}(c) = a_{l+j}$ for $j = 0, \ldots, i - l$. Observe that $\gamma[l, i; a_l, \ldots, a_{i-1}, a_i + t_c] = 2\gamma_b^l(a_i(b) + t)$. Since $\gamma_c^l(a_i(c) + t_c) = \gamma[l, i; a_l, \ldots, a_{i-1}, a_i + t_c]$, the proposition is true.

4.15. **Proposition.** *Suppose* $b = (b_1, \ldots, b_k)$ *and* $c = (c_1, \ldots, c_k)$ *are two points of* $C(i)$ *such that* $l = \alpha(k, b) = \alpha(k, c)$ *and* $c_k = 2 - 2b_k$. *Suppose* t *is a real number such that* $|t| < \eta(b)$ *and* $|t| < \frac{1}{2}\eta(c)$. *Let* t_c *be* $-2t$, *if* $i - l$ *is even, and let* t_c *be* $-t/2$, *if* $i - l$ *is odd. Then* $\gamma_c^l(a_i(c) + t_c) = 2 - 2\gamma_b^l(a_i(b) + t)$.

Proof. Let $a_l = 2 - a_l(b)$. For each $j = 1, \ldots, i - l$, let $a_{l+j} = -2a_{l+j}(b)$, if j is even, and let $a_{l+j} = -\frac{1}{2} a_{l+j}(b)$, if j is odd. Observe that $\gamma[l, i; a_l, \ldots, a_i] = 2 - 2b_k = c_k$. By Proposition 3.8, $a_{l+j}(c) = a_{l+j}$ for $j = 0, \ldots, i - l$. Observe that $\gamma[l, i; a_l, \ldots, a_{i-1}, a_i + t_c] = 2 - 2\gamma_b^l(a_i(b) + t)$. Since $\gamma_c^l(a_i(c) + t_c) = \gamma[l, i; a_l, \ldots, a_{i-1}, a_i + t_c]$, the proposition is true.

4.16. **Definition.** Suppose $c = (c_1, \ldots, c_k) \in C(i)$. Let $l = \alpha(k, c)$. By Propositions 4.11 and 3.5, γ_c^l is a homeomorphism of $[a_i(c) - \eta(c), a_i(c) + \eta(c)]$ onto some interval $[\xi_-(c), \xi_+(c)] \subset I$. The point $c_k = \gamma_c^l(a_i(c)) \in (\xi_-(c), \xi_+(c))$. We will define a map $\tilde{\omega}_c : [\xi_-(c), \xi_+(c)] \to [-\eta(c), \eta(c)]$. Suppose $t \in [\xi_-(c), \xi_+(c)]$. If $i = l$, let $\tilde{\omega}_c(t) = t - a_i$. If $i > l$, let

$$\tilde{\omega}_c(t) = -a_i(c) + \omega[i-1, l; -a_{i-1}(c), \ldots, -a_{l+1}(c), t - a_l(c)].$$

By Propositions 3.5 and 3.9, $\tilde{\omega}_c$ is a homeomorphism of $[\xi_-(c), \xi_+(c)]$ onto the interval $[-\eta(c), \eta(c)]$. Note that $\tilde{\omega}_c(c_k) = 0$. Let $\Phi'_c : [\xi_-(c), c_k) \cup (c_k, \xi_+(c)] \to (-\infty, -u_c] \cup [u_c, \infty)$ be defined by $\Phi'_c(t) = w_i / \tilde{\omega}_c(t)$. Let $\Phi_c(t)$ denote $|\Phi'_c(t)|$.

The next proposition follows readily from the definition.

4.17. **Proposition.** *Suppose $c = (c_1, \ldots, c_k) \in C(i)$. Then the function Φ'_c is a homeomorphism of $[\xi_-(c), c_k) \cup (c_k, \xi_+(c)]$ onto $(-\infty, -u_c] \cup [u_c, \infty)$. Let l denote $\alpha(k, c)$. If b is a real number such that $|b| < \eta(c)$, then $\Phi'_c\left(\gamma_c^l(a_i(c) + b)\right) = w_i/b$.*

4.18. **Proposition.** *Suppose $b = (b_1, \ldots, b_k)$ and $c = (c_1, \ldots, c_k)$ are two points of $C(i)$ such that $l = \alpha(k, b) = \alpha(k, c)$ and $c_k = g_2(b_k)$. There is a positive number κ such that*

(1) $\Phi_c(g_2(z)) = \frac{1}{2}\Phi_b(z)$, if $i - l$ is even, and
(2) $\Phi_c(g_2(z)) = 2\Phi_b(z)$, if $i - l$ is odd,

for each real number z such that $0 < |z - b_k| < \kappa$.

Proof. For each real number $z \in [\xi_-(b), \xi_+(b)]$, there is a real number $t(z)$ such that $|t(z)| \leq \eta(b)$ and $z = \gamma_b^l(a_i(b) + t(z))$. There is a positive number κ such that $\xi_-(b) < b_k - \kappa$, $b_k + \kappa < \xi_+(b)$ and $|t(z)| < \eta(c)$ for each z with $|z - b_k| < \kappa$. Let z be an arbitrary number such that $0 < |z - b_k| < \kappa$. Let t denote $t(z)$. By Proposition 4.17,

$$\Phi_b(z) = |w_i/t|. \tag{0}$$

Since $b_k \neq \frac{1}{2}$, we can assume that κ is so small that either

$$c_k = 2b_k \text{ and } g_2(z) = 2z \tag{*}$$

or

$$c_k = 2 - 2b_k \text{ and } g_2(z) = 2 - 2z. \tag{**}$$

In each of these cases $i - l$ may be either even or odd. So we have to consider four cases.

Case: (*) *and $i - l$ is even.* By Proposition 4.14, $g_2(z) = \gamma_c^l(a_i(c) + 2t)$. By Proposition 4.17, $\Phi_c(g_2(z)) = |w_i/2t|$, so (1) follows from (0).

Case: (*) *and $i - l$ is odd.* By Proposition 4.14, $g_2(z) = \gamma_c^l\left(a_i(c) + \frac{1}{2}t\right)$. By Proposition 4.17, $\Phi_c(g_2(z)) = 2|w_i/t|$, so (2) follows from (0).

Case: (**) *and $i - l$ is even.* By Proposition 4.15, $g_2(z) = \gamma_c^l(a_i(c) - 2t)$. By Proposition 4.17, $\Phi_c(g_2(z)) = |w_i/2t|$, so (1) follows from (0).

Case: (**) *and $i - l$ is odd.* By Proposition 4.15, $g_2(z) = \gamma_c^l\left(a_i(c) - \frac{1}{2}t\right)$. By Proposition 4.17, $\Phi_c(g_2(z)) = 2|w_i/t|$, so (2) follows from (0).

4.19. **Definition.** Suppose $c = (c_1, \ldots, c_k) \in C(i)$ and $t \in (\xi_-(c), \xi_+(c))$. Let $I(c,t) \subset I$ be the interval such that Φ'_c restricted to $I(c,t)$ is a homeomorphism onto the interval $[\mathrm{E}(\Phi'_c(t)), \mathrm{E}(\Phi'_c(t)) + 1]$. Suppose $c' = (c_1, \ldots, c_{k'}) \in C(i')$ is a point such that $k' > k$ and $c = s_k(c')$. Let y denote the only element of $I(c,t)$ such that $v \circ \Phi'_c(y) = c_{k+1}$. Let $\theta(c', k, t) = (c_1, \ldots, c_{k-1}, y, c_{k+2}, \ldots, c_{k'})$.

Since $v(-z) = v(z)$ for each real number z, the following proposition is true.

4.20. **Proposition.** *Suppose $c = (c_1, \ldots, c_k) \in C(i)$ and $t \in (\xi_-(c), \xi_+(c))$. Then there is an integer m such that either $v \circ \Phi_c(z) = 2m + \Phi'_c(z)$ for each $z \in I(c,t)$ or $v \circ \Phi_c(z) = 2m - \Phi'_c(z)$ for each $z \in I(c,t)$.*

4.21. **Proposition.** *Suppose $c' = (c_1, \ldots, c_{k+1}) \in C(i')$. Let $c = s_k(c')$. Suppose $t \in (\xi_-(c), \xi_+(c))$ and $c'' = \theta(c', k, t)$. Let l' denote $\alpha(k+1, c')$. Let m be an integer satisfying the conclusion of Proposition* 4.20. *Then*

(1) *$c'' \in C(i')$,*
(2) *$\alpha(j, c'') = \alpha(j, c')$ for $j = 1, \ldots, k$,*
(3) *$\alpha(k+1, c'') = i' + 1$,*
(4) *$a_j(c'') = a_j(c')$ for each $j = 1, \ldots, l' - 1$,*
(5) *if $v \circ \Phi_c(t) = 2m + \Phi'_c(t)$, then $a_{l'}(c'') = a_{l'}(c') - 2m$ and $a_j(c'') = a_j(c')$ for each $j = l'+1, \ldots, i'$, and*
(6) *if $v \circ \Phi_c(t) = 2m - \Phi'_c(t)$, then $a_{l'}(c'') = 2m - a_{l'}(c')$ and $a_j(c'') = -a_j(c')$ for each $j = l'+1, \ldots, i'$.*

Proof. Let l and i denote $\alpha(k, c')$ and $l' - 1$, respectively. Observe that $c \in C(i)$ and $l = \alpha(k, c)$. We will consider two cases:

$$(*) \qquad v \circ \Phi_c(t) = 2m + \Phi'_c(t)$$

and

$$(**) \qquad v \circ \Phi_c(t) = 2m - \Phi'_c(t).$$

Let x denote $c_{k+1} - 2m$ in case of (*) and let x denote $2m - c_{k+1}$ in case of (**). By Proposition 4.20, $x = \Phi'_c(r_k(c''))$. It follows that $|x| > u_c$ and consequently $|w_i/x| < \eta(c)$. We will define numbers $a_1, \ldots, a_{l'}$. For each $j = 1, \ldots, i$, let $a_j = a_j(c') = a_j(c)$. If (*), then let $a_{l'} = a_{l'}(c') - 2m$ and let $a_j = a_j(c')$ for each $j = l'+1, \ldots, i'$. If (**), then let $a_{l'} = 2m - a_{l'}(c')$ and let $a_j = -a_j(c')$ for each $j = l'+1, \ldots, i'$. Observe that $x = \gamma[l', i'; a_{l'}, \ldots, a_{i'}]$. Let y denote $\gamma[l, i'; a_l, \ldots, a_{i'}]$. Observe that $y = \gamma[l, i; a_l, \ldots, a_{i-1}, a_i + w_i/x]$. Since $a_j = a_j(c)$ for $j = 1, \ldots, i$, $y = \gamma^l_c(a_i(c) + w_i/x)$. By Proposition 4.17, $\Phi'_c(y) = x$. Since $x = \Phi'_c(r_k(c''))$, $r_k(c'') = y$. It follows readily that c'' satisfies the conclusion of the proposition with $a_j(c'') = a_j$ for $j = 1, \ldots, i'$.

4.22. **Definition.** For an arbitrary point $c \in C(i)$, let $\tilde{M}(c)$ be the set of all integers $j = 1, \ldots, i$ such that either $|a_j(c)| > M(c)$ or $j = \alpha(m, c)$ for some $m = 1, \ldots, \mathrm{len}(c)$. We will define an equivalence relation $\approx$ on $C(i)$. For any $b, c \in C(i)$, $b \approx c$ provided that $\tilde{M}(b) = \tilde{M}(c)$ and $v(a_j(b)) = v(a_j(c))$ for each $j = 1, \ldots, i$. Let $D(i)$ be the set of equivalence classes of $\approx$. Note that if $d \in D(i)$ and $b, c \in d$, then $\epsilon(b) = \epsilon(c)$, $\eta(b) = \eta(c)$ and $M(b) = M(c)$. So, for each $d \in D(i)$, by $\epsilon(d)$, $\eta(d)$, u_d and $M(d)$ we will understand $\epsilon(c)$, $\eta(c)$, u_c and $M(c)$, respectively, where c is a point of d.

4.23. **Proposition.** *Suppose $d \in D(i)$, l and n are positive integers such that $l \le n \le i$. Let $a_l, \dots, a_{n-1}$ be elements of $\mathcal{A}$ and let $(b_j)_{j=1}^{\infty}$ be a sequence of elements of d such that $\lim_{j\to\infty} |a_n(b_j)| = \infty$ and $a_m(b_j) = a_m$ for each positive integers j and m such that $l \le m < n$. Let k be an integer such that $n \le k \le i$. Then $\lim_{j\to\infty} \gamma_{b_j}[l,k] = \gamma[l, n-1; a_l, \dots, a_{n-1}]$.*

Proof. By Proposition 4.8, we have that $\left| w_{n-1} / \gamma_{b_j}[n,k] \right| < 1/(|a_n(b_j)| - 2\epsilon(d))$. Since $\gamma_{b_j}[n-1,k] = a_{n-1}(b_j) + w_{n-1}/\gamma_{b_j}[n,k]$, $\lim_{j\to\infty} \gamma_{b_j}[n-1,k] = a_{n-1}(b_j)$. By Proposition 4.10, $\gamma_{b_j}[l,k] = \gamma_{b_j}[l,n-1]\left(\gamma_{b_j}[n-1,k]\right)$. Since $\gamma_{b_j}[l,n-1](\cdot)$ is continuous, $\lim_{j\to\infty} \gamma_{b_j}[l,k] = \gamma_{b_j}[l,n-1](a_{n-1}(b_j)) = \gamma[l,n-1; a_l, \dots, a_{n-1}]$.

The next proposition follows from the definition of Φ_c.

4.24. **Proposition.** *Suppose $d \in D(i)$ and $b = (b_1, \dots, b_j)$ and $c = (c_1, \dots, c_k)$ are two elements of d such that $b_j = c_k$. Then $\xi_-(b) = \xi_-(c)$, $\xi_+(b) = \xi_+(c)$ and $\Phi_b = \Phi_c$.*

4.25. **Proposition.** *Suppose $c' = (c_1, \dots, c_{k'}) \in C(i')$, $k < k'$, $c = s_k(c') \in C(i)$ and $t \in (\xi_-(c), \xi_+(c))$. If $\Phi_c(t) > 2M(c') + 1$, then $\theta(c', k, t) \approx c'$.*

Proof. Let c'' denote $\theta(c', k, t)$ and let l' denote $\alpha(k+1, c')$. In view of Proposition 4.21, it is enough to prove that

$$(*) \qquad |a_{l'}(c'')| > M(c'').$$

Let m be an integer satisfying the conclusion of Proposition 4.20. Observe that $|2m| \ge \Phi_c(t) - v \circ \Phi_c(t) \ge \Phi_c(t) - 1 > 2M(c')$. By Proposition 4.21 (5) and (6), $|a_{l'}(c'')| > |2m| - |a_{l'}(c')|$. Since, by Proposition 4.12, $|a_{l'}(c')| < M(c')$, we have that $|a_{l'}(c'')| > M(c')$. By Remark 4.7, $M(c'') = M(c')$ and (*) is true.

4.26. **Proposition.** *For each $d \in D(i)$ there is $c \in d$ such that* $\operatorname{len}(c) = 1$.

Proof. Let c' be an element of d such that $\operatorname{len}(c')$ is the least possible. Suppose $\operatorname{len}(c') > 1$. Then applying Proposition 4.25, we can obtain $c'' \in d$ such that $\operatorname{len}(c'') = \operatorname{len}(c') - 1$, which contradicts the choice of c'.

4.27. **Proposition.** *Suppose $d \in D(i')$ and $c' = (c_1, \dots, c_{k+1}) \in d$. Let $c = s_k(c')$. Suppose $t \in (\xi_-(c), \xi_+(c))$ is such that $\Phi_c(t) > M(c) + 2M(d) + 1$. Let $\tilde{t}$ and c'' denote $v \circ \Phi_c(t)$ and $\theta(c', k, t)$, respectively. Then*
(1) *$c'' \in d$,*
(2) *$(\xi_-(c''), \xi_+(c'')) \subset I(c,t)$,*
(3) *$t \in (\xi_-(c''), \xi_+(c''))$ if and only if $\tilde{t} \in (\xi_-(c'), \xi_+(c'))$, and*
(4) *if $t \in (\xi_-(c''), \xi_+(c''))$, then $\Phi_{c'}(\tilde{t}) = \Phi_{c''}(t)$.*

Proof. Since $\Phi_c(t) > 2M(d) + 1$, it follows from Proposition 4.25 that $c'' \in d$. Let m be an integer satisfying the conclusion of Proposition 4.20. We will consider the cases

$$(*) \qquad \tilde{t} = 2m + \Phi'_c(t) \text{ and } c_{k+1} = 2m + \Phi'_c(r_k(c''))$$

and

$$(**) \qquad \tilde{t} = 2m - \Phi'_c(t) \text{ and } c_{k+1} = 2m - \Phi'_c(r_k(c'')).$$

Since $\tilde{t} \in I$,

$$\text{(i)} \qquad |2m| \geq \Phi_c(t) - 1.$$

Let l and i denote $\alpha(k, c')$ and $\alpha(k+1, c')$, respectively. The following claim follows readily from Proposition 4.21.

4.27.1. *Claim.* Let $b \in \mathbf{R}$ be such that $|b| < \eta(d)$. Then $\gamma^i_{c''}(a_{i'}(c'') + b) = \gamma^i_{c'}(a_{i'}(c') + b) - 2m$ if (*), and $\gamma^i_{c''}(a_{i'}(c'') + b) = 2m - \gamma^i_{c'}(a_{i'}(c') - b)$ if (**).

4.27.2. *Claim.* Suppose $|b| < \eta(d)$. Then $\left|w_{i-1} / \gamma^i_{c''}(a_{i'}(c'') + b)\right| < \eta(c)$.

Proof of 4.27.2. It follows from Proposition 4.21 that $|a_i(c'')| = |a_i(c') - 2m|$. Since $|a_i(c') - 2m| \geq |2m| - |a_i(c')|$, by Proposition 4.12 we get that $|a_i(c'')| = |2m| - M(d)$. By (i), $|a_i(c'')| > \Phi_c(t) - 1 - M(d)$. Since $\Phi_c(t) > M(c) + 2M(d) + 1$, $|a_i(c'')| > M(c) + M(d) > 1/\eta(c) + 2/\epsilon(d)$. By Proposition 4.8, $\left|w_{i-1} / \gamma^i_{c''}(a_{i'}(c'') + b)\right| < 1/(|a_i(c'')| - 2\epsilon(d)) < \eta(c)$.

Observe that $l = \alpha(k, c)$, $i = \alpha(k+1, c)$ and $c \in C(i-1)$.

4.27.3. *Claim.* Suppose $|b| < \eta(d)$. Then

$$\gamma^l_{c''}(a_{i'}(c'') + b) = \gamma^l_c\left(a_{i-1}(c) + w_{i-1} / \gamma^i_{c''}(a_{i'}(c'') + b)\right).$$

Proof of 4.27.3. Let x denote $a_{i'}(c'') + b$, and let y denote $w_{i-1} / \gamma^i_{c''}(x)$. Observe that $\gamma^l_{c''}(x) = \gamma_{c''}[l, i-1](a_{i-1}(c'') + y)$. Since by Proposition 4.21, $a_j(c'') = a_j(c)$ for $j = l, \ldots, i-1$, $\gamma_{c''}[l, i-1](a_{i-1}(c'') + y) = \gamma_c[l, i-1](a_{i-1}(c) + y) = \gamma^l_c(a_{i-1}(c) + y)$ and the claim follows.

Let z be an arbitrary point of $(\xi_-(c''), \xi_+(c''))$. Then there is a real number b_z such that $|b_z| < \eta(d)$ and $z = \gamma^l_{c''}(a_{i'}(c'') + b_z)$. It follows from 4.27.2 , 4.27.3 and Proposition 4.17 that $\Phi'_c(z) = \gamma^i_{c''}(a_{i'}(c'') + b_z)$ and $\Phi'_c(r_k(c'')) = \gamma^i_{c''}(a_{i'}(c''))$. Since, by Proposition 4.11, $\mathrm{E}\left(\gamma^i_{c''}(a_{i'}(c'') + b_z)\right) = \mathrm{E}\left(\gamma^i_{c''}(a_{i'}(c''))\right)$, we get the result that $\mathrm{E}(\Phi'_c(z)) = \mathrm{E}(\Phi'_c(r_k(c'')))$. Since $\mathrm{E}(\Phi'_c(r_k(c''))) = \mathrm{E}(\Phi'_c(t))$, $z \in I(c, t)$ and the conclusion (2) of the proposition is true.

4.27.4. *Claim.* If $\tilde{t} \in (\xi_-(c'), \xi_+(c'))$, then $t \in (\xi_-(c''), \xi_+(c''))$.

Proof of 4.27.4. There is a number b' such that $|b'| < \eta(d)$ and $\tilde{t} = \gamma^i_{c'}(a_{i'}(c') + b')$. Let b'' denote b' if (*), and $-b'$ if (**). We will observe that

$$\text{(ii)} \qquad \Phi'_c(t) = \gamma^i_{c''}(a_{i'}(c'') + b'').$$

We will prove (ii) only in the case of (**). The proof in the remaining case is similar. Suppose (**). By Claim 4.27.1, $\tilde{t} = \gamma^i_{c'}(a_{i'}(c') + b') = 2m - \gamma^i_{c''}(a_{i'}(c'') - b')$. Since in this case $-b' = b''$ and $\Phi'_c(t) = 2m - \tilde{t}$, (ii) follows.

Since, by Claim 4.27.2, $\left|w_{i-1} / \gamma^i_{c''}(a_{i'}(c'') + b'')\right| < \eta(d)$, we infer from 4.17 and 4.27.3 that $\Phi'_c\left(\gamma^l_{c''}(a_{i'}(c'') + b'')\right) = \gamma^i_{c''}(a_{i'}(c'') + b'')$. Since Φ'_c is a homeomorphism, $t = \gamma^l_{c''}(a_{i'}(c'') + b'')$ and consequently $t \in (\xi_-(c''), \xi_+(c''))$.

Now, suppose $t \in (\xi_-(c''), \xi_+(c''))$. To complete the proof we have to show that $\tilde{t} \in (\xi_-(c'), \xi_+(c'))$ and $\Phi_{c'}\left(\tilde{t}\right) = \Phi_{c''}(t)$. There is a real number b such that $|b| < \eta(d)$ and $t = \gamma^l_{c''}(a_{i'}(c'') + b)$. By Proposition 4.17, $\Phi'_{c''}(t) = w_{i'}/b$. Since, by Claim 4.27.2, $\left|w_{i-1}/\gamma^i_{c''}(a_{i'}(c'')+b)\right| < \eta(d)$, it follows from Proposition 4.17 and Claim 4.27.3 that $\Phi'_c(t) = \gamma^i_{c''}(a_{i'}(c'')+b)$. We will now consider the cases (*) and (**) separately.

Suppose (*). Then $\tilde{t} = 2m+\Phi'_c(t) = 2m+\gamma^i_{c''}(a_{i'}(c'')+b)$. So, by Claim 4.27.1, $\tilde{t} = \gamma^i_{c'}(a_{i'}(c')+b)$. It follows that $\tilde{t} \in (\xi_-(c'), \xi_+(c'))$ and, by Proposition 4.17, $\Phi'_{c'}\left(\tilde{t}\right) = w_{i'}/b = \Phi'_{c''}(t)$.

Suppose (**). Then $\tilde{t} = 2m - \Phi'_c(t) = 2m - \gamma^i_{c''}(a_{i'}(c'')+b)$. So, by Claim 4.27.1, $\tilde{t} = \gamma^i_{c'}(a_{i'}(c')-b)$. It follows that $\tilde{t} \in (\xi_-(c'), \xi_+(c'))$ and, by Proposition 4.17, $\Phi'_{c'}\left(\tilde{t}\right) = -w_{i'}/b = -\Phi'_{c''}(t)$ and the proof of the proposition is complete.

4.28. **Proposition.** *Suppose $d \in D(i)$, b and c are two points of d such that $\operatorname{len}(b) = \operatorname{len}(c) = k$, $s_{k-1}(b) = s_{k-1}(c)$. Suppose that j, l and m are integers such that $\alpha(k,b) = \alpha(k,c) \le m \le l \le j \le i$ and $a_l(b) \ne a_l(c)$. Finally, suppose t_b and t_c are real numbers such that $|t_b - a_j(b)| \le \eta(d)$ and $|t_c - a_j(c)| \le \eta(d)$. Then $\gamma_b[m,j](t_b) \ne \gamma_c[m,j](t_c)$.*

Proof. Without loss of generality we may assume that l is the greatest number $\le i$ such that $a_l(b) \ne a_l(c)$. We will start the proof with the following claim.

4.28.1. *Claim.* $\gamma_b[l,j](t_b) \ne \gamma_c[l,j](t_c)$.

Proof of 4.28.1. Since $\mu(a_l(b)) > \epsilon(b) = \epsilon(d)$ and $\mu(a_l(c)) > \epsilon(c) = \epsilon(d)$, we have that $|\gamma_b[l,j] - \gamma_c[l,j]| = |a_l(b) - a_l(c)| > 2\epsilon(d)$. But, on the other hand, $|\gamma_b[l,j](t_b) - \gamma_b[l,j]| = |\gamma_b[l,j](t_b) - \gamma_b[l,j](a_j(b))| < \epsilon(d)$ and by the same argument $|\gamma_c[l,j](t_c) - \gamma_c[l,j]| = |\gamma_c[l,j](t_c) - \gamma_c[l,j](a_j(c))| < \epsilon(d)$, so the claim is true.

Suppose that the proposition is false. Let $m < l$ be the greatest integer such that $\gamma_b[m,j](t_b) = \gamma_c[m,j](t_c)$. Since $\gamma_b[m+1,j](t_b) = w_m/(\gamma_b[m,j](t_b) - a_m(b))$ and $\gamma_c[m+1,j](t_c) = w_m/(\gamma_c[m,j](t_c) - a_m(c))$, by the choice of m, we get that $a_m(b) \ne a_m(c)$. Since $m < l$, $a_n(b) \ne a_n(c)$ for some integer $n > m$. Let n be the least integer with this property. Since $b \approx c$, $a_n(b) \ne a_n(c)$ and $n > \alpha(k,b) = \alpha(k,c)$, we have that $|a_n(b)| > M(d)$ and $|a_n(c)| > M(d)$. Since $\gamma_b[n-1,j](t_b) = a_{n-1}(b) + w_{n-1}/\gamma_b[n,j](t_b)$, by Proposition 4.8, we get that $|\gamma_b[n-1,j](t_b) - a_{n-1}(b)| < 1/(|a_n(b)| - 2/\epsilon(d)) < 1/(M(d) - 2/\epsilon(d)) = \eta(d)$. Since $\gamma_b[m,j](t_b) = \gamma_b[m,n-1](\gamma_b[n-1,j](t_b))$ and also $\gamma_b[m,n-1] = \gamma_b[m,n-1](a_{n-1}(b))$, by the choice of $\eta(d)$, we get

$$(*) \qquad |\gamma_b[m,j](t_b) - \gamma_b[m,n-1]| < \epsilon(d).$$

By a similar proof, we obtain

$$(**) \qquad |\gamma_c[m,j](t_c) - \gamma_c[m,n-1]| < \epsilon(d).$$

By the choice of n, $|\gamma_b[m,n-1] - \gamma_c[m,n-1]| = |a_m(b) - a_m(c)|$. Since $a_m(b) \ne a_m(c)$, $v(a_m(b)) = v(a_m(c))$, $\mu(a_m(b)) > \epsilon(d)$ and $\mu(a_m(c)) > \epsilon(d)$, we have that $|a_m(b) - a_m(c)| > 2\epsilon(d)$ and thus $|\gamma_b[m,n-1] - \gamma_c[m,n-1]| > 2\epsilon(d)$. Now, applying (*) and (**) we get the result that $\gamma_b[m,j](t_b) \ne \gamma_c[m,j](t_c)$, which contradicts the choice of m.

Proposition 4.28 yields the following corollary.

4.29. **Proposition.** *Suppose $d \in D(i)$, b and c are two points of d such that $b \neq c$, $\operatorname{len}(b) = \operatorname{len}(c) = k$, $s_{k-1}(b) = s_{k-1}(c)$. Then the intervals $[\xi_-(b), \xi_+(b)]$ and $[\xi_-(c), \xi_+(c)]$ are disjoint.*

4.30. **Proposition.** *Suppose $d \in D(i)$, $b = (b_1, \ldots, b_j)$ and $c = (c_1, \ldots, c_k)$ are two points of d such that $b \neq c$, $j \geq k$ and $s_{k-1}(b) = s_{k-1}(c)$. Then $b_k \notin (\xi_-(c), \xi_+(c))$.*

Proof. If $j = k$, then the proposition follows from 4.29. Let $j > k$ be the least integer for which the proposition is false, i.e. $b_k \in (\xi_-(c), \xi_+(c))$. By taking a number t close enough to b_k we obtain $b' = \theta(b, k, t)$ such that $b' \neq c$, $b' \in d$ and $r_k(b') \in (\xi_-(c), \xi_+(c))$. Since $\operatorname{len}(b') = j - 1$, we get a contradiction with the choice of j.

4.31. **Definition.** Suppose $x = (x_1, \ldots, x_j) \in T$, $c = (c_1, \ldots, c_k) \in C(i)$, $j \geq k$, $s_{k-1}(x) = s_{k-1}(c)$ and $x_k \in (\xi_-(c), \xi_+(c))$. Let $\delta'(c, x)$ denote the point $(c_1, \ldots, c_k, v \circ \Phi_c(x_k), x_{k+1}, \ldots, x_j)$.

4.32. **Proposition.** *Suppose $x = (x_1, \ldots, x_j) \in T$, $c = (c_1, \ldots, c_k) \in C(i)$, $j \geq k$, $s_{k-1}(x) = s_{k-1}(c)$ and $x_k \in (\xi_-(c), \xi_+(c))$. Let i' be an integer greater than i. Then $\delta'(c, x) \in C(i')$ if and only if $x \in C(i')$ and $a_m(x) = a_m(c)$ for each integer m such that $\alpha(k, c) \leq m < \alpha(k+1, c)$. Additionally, if $\delta'(c, x) \in C(i')$ and $\left|a_{\alpha(k+1,c)}(x)\right| > M(x)$, then $\delta'(c, x) \approx x$.*

Proof. Let x' denote $\delta'(c, x)$.

Suppose $x' \in C(i')$. Then $\alpha(k, c) = \alpha(k, x')$, $\alpha(k+1, c) = \alpha(k+1, x')$ and $a_m(x') = a_m(c)$ for each positive integer $m < \alpha(k+1, c)$. Observe that $x = \theta(x', k, x_k)$. It follows from Proposition 4.21 that $x \in C(i')$ and $a_m(x) = a_m(c)$ for each positive integer $m < \alpha(k+1, c)$. It follows from the definition of $\approx$ that $x' \approx x$, if $\left|a_{\alpha(k+1,c)}(x)\right| > M(x)$.

Suppose $x \in C(i')$ and $a_m(x) = a_m(c)$ for each m such that $\alpha(k, c) \leq m < \alpha(k+1, c)$. Let l denote $\alpha(k, x) = \alpha(k, c)$ and let l' be $\alpha(k+1, x) - 1$. To prove that $x' \in C(i')$, it is enough to show that there are numbers $a_{i+1}, \ldots, a_{l'}$ in $\mathcal{A}$ such that $r_{k+1}(x') = \gamma[i+1, l'; a_{i+1}, \ldots, a_{l'}]$. Observe that $x_k = \gamma_x[l, l'] = \gamma_x[l, i](\gamma_x[i, l']) = \gamma_c^l(\gamma_x[i, l']) = \gamma_c^l(a_i(c) + w_i / \gamma_x[i+1, l'])$. By Proposition 4.17, $\Phi_c'(x_k) = \gamma_x[i+1, l']$. There is an integer q such that either

$$(*) \qquad 2q + \Phi_c'(x_k) = v \circ \Phi_c(x_k)$$

or

$$(**) \qquad 2q - \Phi_c'(x_k) = v \circ \Phi_c(x_k).$$

If (*), then set $a_{i+1} = 2q + a_{i+1}(x)$ and $a_m = a_m(x)$ for $m = i+2, \ldots, l'$. If (**), then set $a_{i+1} = 2q - a_{i+1}(x)$ and $a_m = -a_m(x)$ for $m = i+2, \ldots, l'$. It can be easily verified that $r_{k+1}(x') = v \circ \Phi_c(x_k) \gamma[i+1, l'; a_{i+1}, \ldots, a_{l'}]$.

4.33. **Proposition.** *Suppose $x = (x_1, \ldots, x_k) \in T$ and $d \in D(i')$ is such that for each positive real number κ there is $c' \in d$ such that $\operatorname{len}(c') \geq k$, $s_{k-1}(x) = s_{k-1}(c')$ and $|x_k - r_k(c')| < \kappa$. Then there is $c \in d$ such that $s_k(c) = x$. Let i be such that $x \in C(i)$. There is a positive number κ_0 such that if $b \in d$, $s_{k-1}(b) = s_{k-1}(x)$ and*

$0 < |r_k(b) - x_k| < \kappa_0$, *then* $\alpha(k+1, b) > i+1$, $|a_{i+1}(b)| > M(d)$ *and* $a_m(b) = a_m(x)$ *for each integer* m *such that* $\alpha(k, x) \leq m \leq i$.

Proof. Let $(b_j)_{j=1}^{\infty}$ be a sequence of points from d such that $\lim_{j\to\infty} r_k(b_j) = x_k$ and $s_{k-1}(x) = s_{k-1}(b_j)$ for each positive integer j. Observe that the number $\alpha(k, b_j)$ does not depend on the choice of j. We will denote this number by l. For each $j = 1, 2, \ldots$, there is an integer $m_j \leq i'$ such that $r_k(b_j) = \gamma_{b_j}[l, m_j]$. By replacing the sequence $(b_j)_{j=1}^{\infty}$ by its subsequence, we may assume that there is an integer $m' \leq i'$ such that $r_k(b_j) = \gamma_{b_j}[l, m']$. By Proposition 4.12, the sequence $(a_l(b_j))_{j=1}^{\infty}$ is bounded (by $M(d)$). Let i be the greatest integer $\leq m'$ such that the sequence $(a_m(b_j))_{j=1}^{\infty}$ is bounded for each integer m such that $l \leq m \leq i$. Since the sequence $(v(a_l(b_j)))_{j=1}^{\infty}$ is constant, the bounded sequence $(a_l(b_j))_{j=1}^{\infty}$ has finitely many values. By replacing the sequence $(b_j)_{j=1}^{\infty}$ by its subsequence, we may assume that for each m between l and i there is a number a_m such that $a_m(b_j) = a_m$ for each positive integer j. By Proposition 4.23, $\lim_{j\to\infty} r_k(b_j) = \gamma[l, i; a_l, \ldots, a_i]$. Since $\lim_{j\to\infty} r_k(b_j) = x_k$, $x_k = \gamma[l, i; a_l, \ldots, a_i]$. Observe that $x \in C(i)$. If $i = m'$, then $s_k(b_j) = x$ for each j. If $i < m'$, then, for large enough j, $c = \delta'(x, b_j) \in d$ and $x = s_k(c)$.

Let $(c_j)_{j=1}^{\infty}$ be a sequence of points from d such that $\lim_{j\to\infty} r_k(c_j) = x_k$, $r_k(c_j) \neq x_k$ and $s_{k-1}(x) = s_{k-1}(c_j)$. By Proposition 3.8, the representation of x_k as $\gamma[l, i; a_l, \ldots, a_i]$ is unique. Thus, since we could have started with $(c_j)_{j=1}^{\infty}$ instead of $(b_j)_{j=1}^{\infty}$, for j large enough we must have that $\alpha(k+1, b_j) > i+1$, $|a_{i+1}(b_j)| > M(d)$ and $a_m(b_j) = a_m$ for each integer m such that $\alpha(k, x) \leq m \leq i$. So the proof of the proposition is complete.

4.34. **Proposition.** *Suppose* i *and* i' *are positive integers such that* $i < i'$. *Suppose* $c = (c_1, \ldots, c_k) \in C(i)$ *and* $d \in D(i')$. *Then there is a positive number* κ *such that if* $b \in d$, $s_{k-1}(b) = s_{k-1}(c)$ *and* $0 < |r_k(b) - c_k| < \kappa$, *then* $\delta'(c, b) \in d$.

Proof. Let l denote $\alpha(k, c)$. By Proposition 4.33, there is a positive number κ such that if $b \in d$, $s_{k-1}(b) = s_{k-1}(c)$ and $0 < |r_k(b) - c_k| < \kappa$, then $\alpha(k+1, b) > i$, $|a_{i+1}(b)| > M(d)$ and $a_m(b) = a_m(c)$ for each integer m such that $l \leq m \leq i$. Now, the proposition follows from Proposition 4.32.

4.35. **Definition.** Let $Q_i^i = C(i) \times I$ for each integer $i \geq 0$. In particular, $Q_0^0 = I$. Suppose j and i are integers such that $j \geq i \geq 0$. We will define by recursion a set Q_i^j. The set Q_i^i is already defined. Let $Q_i^j = Q_i^{j-1} \backslash C(j-1)$. Let $P_n = \bigcup_{i=0}^{n-1} Q_i^n$.

4.36. **Definition.** Suppose $x = (x_1, \ldots, x_j) \in P_n$, $i < n$, $c = (c_1, \ldots, c_k) \in C(i)$, $j \geq k$, $s_{k-1}(x) = s_{k-1}(c)$ and $x_k \in (\xi_-(c), \xi_+(c))$. Let $\delta_n(c, x)$ be $\delta'(c, x)$, if $\delta'(c, x) \in P_n$, and let $\delta_n(c, x)$ be $s_{k+1}(\delta'(c, x))$, if $\delta'(c, x) \notin P_n$.

4.37. **Proposition.** *Suppose* $x = (x_1, \ldots, x_j) \in P_n$, $i < n$, $c = (c_1, \ldots, c_k) \in C(i)$, $j \geq k$, $s_{k-1}(x) = s_{k-1}(c)$ *and* $x_k \in (\xi_-(c), \xi_+(c))$. *Then* $\delta_n(c, x) \in P_n$.

Proof. Suppose $\delta_n(c, x) \notin P_n$. Then $\delta'(c, x) \notin P_n$ and $\delta_n(c, x) = s_{k+1}(\delta'(c, x))$. Since $s_k(\delta_n(c, x)) = c$, $\delta_n(c, x) \in C(i')$ for some positive integer $i' < n$. It follows from Proposition 4.32 that $s_k(x) \in C(i')$ and $a_m(s_k(x)) = a_m(c)$ for each integer m such that $\alpha(k, c) \leq m < \alpha(k+1, c)$. Since $x \in P_n$, $x \notin C(m)$ for each positive integer $m < n$ and $s_{j-1}(x) \in C(l)$ for some integer $l < n$. It follows from Proposition 4.32 that $\delta'(c,x) \notin C(m)$ for each positive integer $m < n$ and $s_j(\delta'(c,x)) \in C(l)$. Thus $\delta'(c,x) \in P_n$ and consequently $\delta_n(c,x) \in P_n$.

4.38. **Definition.** Let $d \in D(i)$. Suppose $c \in d$ and $\operatorname{len}(c) = k$. Let $L(c)$ denote the set $\{r_k(b) \mid b \in d \text{ and } s_{k-1}(b) = s_{k-1}(c)\}$. Let $\tilde{d}(c)$ denote the set $\{b \in d \mid \operatorname{len}(b) = k \text{ and } s_{k-1}(b) = s_{k-1}(c)\}$. Observe that if $b \in \tilde{d}(c)$, then $c \in \tilde{d}(b)$. Let $\Lambda(d)$ denote the collection $\left\{\tilde{d}(c) \mid c \in d\right\}$.

The following proposition is a simple consequence of the definitions.

4.39. **Proposition.** *Suppose $d \in D(i)$ and $z \in \Lambda(d)$. Let $b, c \in z$. Then* $\operatorname{len}(b) = \operatorname{len}(c)$*,* $L(b) = L(c)$ *and* $\alpha(j,b) = \alpha(j,c)$ *for* $j = 1, \ldots, \operatorname{len}(b) + 1$.

4.40. **Definition.** If $z \in \Lambda(d)$, then by $\operatorname{len}(z)$ we will denote $\operatorname{len}(c)$ for some $c \in z$. By $L(z)$ we will denote $L(c)$ for some $c \in z$. If j is an integer such that $1 \le i \le \operatorname{len}(x) + 1$, then by $\alpha(j,z)$ we will denote $\alpha(j,c)$ for some $c \in z$.

4.41. **Definition.** Suppose $d \in D(i)$ and $z \in \Lambda(d)$. By Proposition 4.29, the intervals $[\xi_-(c'), \xi_+(c')]$ and $[\xi_-(c''), \xi_+(c'')]$ are disjoint for each pair of different points c' and c'' from z. Let $\widehat{\Phi}_z : I \backslash L(z) \to [0,\infty)$ be defined by

$$\widehat{\Phi}_z(t) = \begin{cases} \Phi_c(t), & \text{if } t \in [\xi_-(c), \xi_+(c)] \text{ for some } c \in z, \\ u_d, & \text{otherwise.} \end{cases}$$

4.42. **Proposition.** *Suppose $d \in D(i)$ and $z \in \Lambda(d)$. Then $\widehat{\Phi}_z$ is continuous on $I \backslash L(z)$.*

Proof. Let k denote $\operatorname{len}(z)$ and let t be an arbitrary point of $I \backslash L(z)$. The following claim follows from Proposition 4.33.

4.42.1. *Claim.* There is a positive number κ_0 such that $r_k(c) \notin [t - \kappa_0, t + \kappa_0]$ for each $c \in d$.

Before we prove the proposition we need another claim.

4.42.2. *Claim.* There is a positive number κ_1 such that $[t - \kappa_1, t + \kappa_1]$ intersects $[\xi_-(c), \xi_+(c)]$ for at most one $c \in z$.

Proof of 4.42.2. By Proposition 4.29, t belongs to $[\xi_-(c), \xi_+(c)]$ for at most one $c \in d$. Suppose that the claim is false. Then there are three points c_1, c_2 and c_3 in z such that $[\xi_-(c_j), \xi_+(c)] \cap [t - \kappa_0, t + \kappa_0] \neq \emptyset$ for $j = 1, 2, 3$. For each $j = 1, 2, 3$, let t_j be a point in $[\xi_-(c_j), \xi_+(c_j)] \cap [t - \kappa_0, t + \kappa_0]$. Without loss of generality we may assume that $t_1 < t_2 < t_3$. By Proposition 4.29, t_1 and t_3 do not belong to $[\xi_-(c_2), \xi_+(c_2)]$. So $t_1 < r_k(c_2) < t_3$ and consequently $r_k(c_2) \in [t - \kappa_0, t + \kappa_0]$, which contradicts Claim 4.42.1.

Suppose $t \notin [\xi_-(c), \xi_+(c)]$ for each $c \in z$. Then it follows from Claim 4.42.2 that there is a positive number κ such that $[t - \kappa, t + \kappa] \cap [\xi_-(c), \xi_+(c)] = \emptyset$ for each $c \in z$. If follows that $\widehat{\Phi}_z$ is constant on $[t - \kappa, t + \kappa]$.

Suppose there is a point $c \in z$ such that $t \in [\xi_-(c), \xi_+(c)]$. By Proposition 4.17, $\widehat{\Phi}_z$ is continuous on $[\xi_-(c), \xi_+(c)]$. Observe that $\widehat{\Phi}_z(x) = u_d$ for each $x \in [t - \kappa_1, t + \kappa_1] \backslash [\xi_-(c), \xi_+(c)]$. Since $\widehat{\Phi}_z(\xi_-(c)) = u_d = \widehat{\Phi}_z(\xi_+(c))$, $\widehat{\Phi}_z$ is continuous at t.

4.43. **Definition.** Suppose $d \in D(i)$ and $t \in T$. If there is an integer j such that $s_j(t) \in d$, then let $\lambda'(d,t) = j$. If $s_j(t) \notin d$ for each positive integer j, then set $\lambda'(d,t) = 0$.

4.44. **Definition.** Suppose $d \in D(n)$ and $t \in T$ are such that $\lambda'(d,t) = 0$. Let $d(t)$ be the set of such points $c \in d$ that if k denote $\operatorname{len}(c)$, then $s_{k-1}(c) = s_{k-1}(t)$. Observe that if $c \in d$ and $\operatorname{len}(c) = 1$, then $c \in d(t)$. So, by Proposition 4.26, $d(t)$ is not empty. Let b be a point of $d(t)$ such that $\operatorname{len}(b)$ is maximal. Let $\lambda(d,t) = \tilde{d}(b)$. If $c \in d(t)$ is another point such that $\operatorname{len}(c) = \operatorname{len}(b)$, then $\tilde{d}(b) = \tilde{d}(c)$. So the definition $\lambda(d,t)$ does not depend on the choice of b.

4.45. **Lemma.** *Let n, i and i' be positive integers such that $i < n$ and $i' < n$. Suppose $d \in D(i')$ and $c = (c_1, \ldots, c_k) \in C(i)$. Let $P_n(c)$ denote the set of points $x \in P_n$ such that $\operatorname{len}(x) \geq k$, $s_{k-1}(x) = s_{k-1}(c)$, $r_k(x) \neq c_k$ and $r_k(x) \in (\xi_-(c), \xi_+(c))$. For each x in $P_n(c)$, let $\tilde{x}$ denote $\delta_n(c,x)$. Then there is a positive number $\kappa(c,d)$ such that if $x \in P_n(c)$ and $|r_k(x) - c_k| < \kappa(c,d)$, then the following statements are true.*

1. *If $0 < \lambda'(d,\tilde{x}) < k$, then $\lambda'(d,x) = \lambda'(d,\tilde{x})$.*
2. *If $\lambda'(d,\tilde{x}) = k$, then $\lambda'(d,x) = 0$ and $c \in d$.*
3. *If $\lambda'(d,\tilde{x}) > k$, then $\delta'(c,x) \in P_n$ and $\lambda'(d,x) = \lambda'(d,\tilde{x}) - 1$.*

Suppose, additionally, that $\lambda'(d,\tilde{x}) = 0$. Then $\lambda'(d,x) = 0$. Let x', x'' and q denote $\lambda(d,\tilde{x})$, $\lambda(d,x)$ and $\operatorname{len}(x')$, respectively. Then the following two statements are true.

4. *If $q \leq k$, then $x'' = x'$.*
5. *If $q > k$, then $\operatorname{len}(x'') = q - 1$ and $\widehat{\Phi}_{x'}(r_q(\tilde{x})) = \widehat{\Phi}_{x''}(r_{q-1}(x))$.*

Proof. Since $s_{k-1}(x) = s_{k-1}(\tilde{x})$, (1) is obvious. Suppose $\lambda'(d,\tilde{x}) = k$. Since $s_k(\tilde{x}) = c$, $c \in d$, it follows from Proposition 4.30 that $\lambda'(d,x) = 0$. So, (2) is true. (3) follows from Proposition 4.32.

Let l denote $\alpha(k,c)$. Proposition 4.33 implies the following claim.

4.45.1. *Claim.* There is a positive number κ_0 such that if $b \in d$, $s_{k-1}(b) = s_{k-1}(c)$ and $0 < |r_k(b) - c_k| < \kappa_0$, then $\alpha(k+1,b) > i$, $|a_{i+1}(b)| > M(d)$ and $a_m(b) = a_m(c)$ for each integer m such that $l \leq m \leq i$.

We may assume that $\kappa(c,d)$ is less than κ from the conclusion of Proposition 4.34. Now, suppose that $\lambda'(d,\tilde{x}) = 0$. It follows from Proposition 4.34 that $\lambda'(d,x) = 0$. Before we prove (4) and (5) we need the following three claims.

4.45.2. *Claim.* Suppose $b \in d$, $\operatorname{len}(b) = k+1$, $s_k(b) = c$, $c'' \in d$, $\operatorname{len}(c'') = k$, $s_{k-1}(c'') = s_{k-1}(c)$ and $r_k(x) \in (\xi_-(c''), \xi_+(c''))$. Then $r_k(c'') \in (\xi_-(c), \xi_+(c))$ and $c'' = \theta(c', k, r_{k+1}(x))$, where c' denote $\delta'(c,c'')$.

Proof of 4.45.2. Let κ_0 be as in Claim 4.45.1. Let z_- and z_+ be numbers such that $c_k - \kappa_0 < z_- < c_k < z_+ < c_k + \kappa_0$, $\xi_+(c) < z_-$ and $z_+ < \xi_+(c)$. Let t_- and t_+ be numbers such that $z_- < t_- < c_k < t_+ < z_+$, $\Phi_c(t_-) > \Phi_c(z_-) + 1$, $\Phi_c(t_+) > \Phi_c(z_+) + 1$, $\Phi_c(t_-) > 2M(d) + 1$ and $\Phi_c(t_+) > 2M(d) + 1$. Let b_- and b_+ denote $\theta(b,k,t_-)$ and $\theta(b,k,t_+)$, respectively. By Proposition 4.25, b_- and b_+ belong to d. Observe that $z_- < r_k(b_-) < c_k < r_k(b_+) < z_+$. We may assume that $\kappa(c,d)$ is so small that $r_k(b_-) < r_k(x) < r_k(b_+)$. Since $r_k(x) \in (\xi_-(c''), \xi_+(c''))$, by Proposition 4.30, we get that $r_k(b_-) < r_k(c'') < r_k(b_+)$. Consequently, $z_- < r_k(c'') < z_+$, $|r_k(c'') - c_k| < \kappa_0$ and $r_k(c'') \in (\xi_-(c), \xi_+(c))$. It follows from Claim

4.45.1 and Proposition 4.32 that $c' = \delta'(c, c'')$ is in d. Clearly, $c'' = \theta(c', k, r_k(c''))$. By Proposition 4.27 (2), $(\xi_-(c''), \xi_+(c'')) \subset I(c, r_k(c''))$. So $r_k(x) \in I(c, r_k(c''))$ and $I(c, r_k(c'')) = I(c, r_k(x))$. It follows that $\theta(c', k, r_k(c'')) = \theta(c', k, r_k(x))$ and the proof of the claim is complete.

4.45.3. *Claim.* Suppose $c'' \in d$, $l = \operatorname{len}(c'') > k$ and $s_{l-1}(c'') = s_{l-1}(x)$. Let $c' = \delta'(c, c'')$. Then $c' \in d$, $s_l(c') = s_l(\tilde{x})$ and $c'' = \theta(c', k, r_k(x))$.

Proof of 4.45.3. Since $l-1 \geq k$, it follows from Proposition 4.34 that $c' \in d$. Clearly, $s_l(c') = s_l(\tilde{x})$. Since $r_k(x) = r_k(c'')$, $\theta(c', k, r_k(x)) = \theta(c', k, r_k(c'')) = c''$.

4.45.4. *Claim.* Suppose $c' \in d$, $l = \operatorname{len}(c') > k$ and $s_{l-1}(c') = s_{l-1}(\tilde{x})$. Let $c'' = \theta(c', k, r_k(x))$. Then
(A) $c'' \in d$,
(B) $s_{l-2}(c'') = s_{l-2}(x)$,
(C) $r_l(\tilde{x}) \in (\xi_-(c'), \xi_+(c'))$ if and only if $r_{l-1}(x) \in (\xi_-(c''), \xi_+(c''))$, and
(D) if $r_l(\tilde{x}) \in (\xi_-(c'), \xi_+(c'))$, then $\Phi_{c'}(r_l(\tilde{x})) = \Phi_{c''}(r_{l-1}(x))$.

Proof of 4.45.4. We may assume that $\kappa(c, d)$ is so small that $\Phi_c(r_k(x)) > M(c) + 2M(d) + 1$. Observe that $s_{k-1}(c'') = s_{k-1}(c) = s_{k-1}(x)$. If $l = k+1$, then the claim follows from Proposition 4.27. So we may assume $l > k+1$. Since in this case $r_{k+1}(c') = r_{k+1}(\tilde{x})$ and $v \circ \Phi_c(r_k(x)) = r_{k+1}(x)$, we get the result that $r_k(c'') = r_k(x)$. It follows that $s_{l-2}(c'') = s_{l-2}(x)$. Since in this case $r_l(\tilde{x}) = r_{l-1}(x)$ and $r_l(c') = r_{l-1}(c'')$, the remaining part of the claim follows from Proposition 4.24.

Now, suppose $q \leq k$. Then it follows from Claim 4.45.3 that $\operatorname{len}(x'') \leq k$. Since $s_{k-1}(x) = s_{k-1}(\tilde{x})$, $\operatorname{len}(x'') = q$ and $x' = x''$. So (4) is true.

Finally, suppose $q > k$. Then it follows from Claim 4.45.3 that $\operatorname{len}(x'') < q$. By Claim 4.45.4 (A) and (B), $\operatorname{len}(x'') \geq q-1$. So $\operatorname{len}(x'') = q-1$. It remains to prove that

$$(*) \qquad \widehat{\Phi}_{x'}(r_q(\tilde{x})) = \widehat{\Phi}_{x''}(r_{q-1}(x)).$$

In our proof we will consider the following two cases.
(i) There is $c' \in x'$ such that $r_q(\tilde{x}) \in (\xi_-(c'), \xi_-(c'))$.
(ii) There is $c'' \in x''$ such that $r_{q-1}(x) \in (\xi_-(c''), \xi_-(c''))$.
If neither (i) nor (ii) applies, then $\widehat{\Phi}_{x'}(r_q(\tilde{x})) = u_d = \widehat{\Phi}_{x''}(r_{q-1}(x))$.

Case (i). Suppose there is $c' \in x'$ such that $r_q(\tilde{x}) \in (\xi_-(c'), \xi_+(c'))$. Let $c'' = \theta(c', k, r_k(x))$. By 4.45.4 (A) and (B), $c'' \in x''$. By 4.45.4 (C), $r_{q-1}(x) \in (\xi_-(c''), \xi_+(c''))$. Now, (*) follows from 4.45.4 (D).

Case (ii). Suppose there is $c'' \in x''$ such that $r_{q-1}(x) \in (\xi_-(c''), \xi_+(c''))$. Let c' denote $\delta'(c, c'')$. It follows from 4.45.2 and 4.45.3 that $c'' = \theta(c', k, r_k(x))$. By 4.45.4 (C), $r_q(\tilde{x}) \in (\xi_-(c'), \xi_+(c'))$. Now, (*) follows from Case (i).

5. Main construction

5.1. **Definition.** Suppose $d \in D(i)$ and $z \in \Lambda(d)$. Let j denote $\alpha(\operatorname{len}(z), x)$. Let $\tilde{\Phi}_z : \widetilde{B}_j \backslash {\beta_j}^{-1}(L(z)) \to \widetilde{B}_{i+1}$ denote the map $\tilde{\varphi}_{i+1} \circ \widehat{\Phi}_z \circ \beta_j$.

5.2. **Definition.** Suppose $c = (c_1, \ldots, c_k) \in C(i)$. Let $\widetilde{Q}(c)$ denote the set ${\beta_{\alpha(1,c)}}^{-1}(c_1) \times {\beta_{\alpha(2,c)}}^{-1}(c_2) \times \cdots \times {\beta_{\alpha(k,c)}}^{-1}(c_k) \times \widetilde{B}_{i+1}$. In particular, $\widetilde{Q}(\emptyset) = \widetilde{B}_1$.

For each integer $i \geq 0$, let $\widetilde{Q}_i^i$ be the union of $\bigcup_{c \in C(i)} \widetilde{Q}(c)$. Let $\widetilde{Q}$ denote the union $\bigcup_{i=0}^{\infty} \widetilde{Q}_i^i$.

For each $x \in \widetilde{Q}$, there is exactly one point $c(x)$ in $\bigcup_{i=0}^{\infty} C(i)$ such that $x \in \widetilde{Q}(c(x))$. It follows from Proposition 4.5 that $c(x) \neq c(y)$, if $x \in \widetilde{Q}_i^i$ and $y \in \widetilde{Q}_j^j$, where $i \neq j$. In particular, the sets $\widetilde{Q}_0^0, \widetilde{Q}_1^1, \ldots$ are mutually disjoint.

Each point $x \in \widetilde{Q}$ can be expressed in the form $(b_1, \ldots, b_k)$, where k is a positive integer and, for each $m = 1, \ldots, k$, b_m belongs to $\widetilde{B}_{\alpha(m,c(x))}$. As before, let $\operatorname{len}(x) = k$, $r_i(x) = b_i$ and $s_i(x) = (b_1, b_2, \ldots, b_i)$.

5.3. **Definition.** We will define a function $\tau : \widetilde{Q} \to T$. Suppose $x = (b_1, b_2, \ldots, b_k)$ is an arbitrary point of $\widetilde{Q}$. Let $c(x) = (c_1, \ldots, c_{k-1}) \in \bigcup_{i=0}^{\infty} C(i)$ be such that $x \in \widetilde{Q}(c(x))$. Let $c_k = \beta_{\alpha(k,c(x))}(b_k)$. Let $\tau(x) = (c_1, \ldots, c_k)$. In particular, if $k = 1$, $\tau(x) = \beta_1(x)$.

Let $\widetilde{H}_n = \left\{ b \in \widetilde{Q} \mid \tau(b) \in C(n) \right\}$.

5.4. **Definition.** Suppose that $m \geq i$. We will define by induction a set $\widetilde{Q}_i^m$. The set $\widetilde{Q}_i^i$ has already been defined. Let $\widetilde{Q}_i^m = \widetilde{Q}_i^{m-1} \backslash \widetilde{H}_{m-1}$ for $m > i$. Let $\widetilde{P}_n = \bigcup_{j=0}^{n-1} \widetilde{Q}_j^{n-1}$. Note that $\widetilde{P}_{n+1} = \widetilde{Q}_n^n \cup \left(\widetilde{P}_n \backslash \widetilde{H}_n \right)$. Let $\sigma_n : \widetilde{P}_{n+1} \to \widetilde{P}_n$ be defined by $\sigma_n(x) = x$ if $x \in \widetilde{P}_n \backslash \widetilde{H}_n$, and $\sigma_n(x) = s_{\operatorname{len}(x)-1}(x)$, if $x \in \widetilde{Q}_n^n$.

The next three propositions follow readily from the definition.

5.5. **Proposition.** *Suppose* $x \in \widetilde{Q}$. *Then* $x \in \widetilde{P}_n$ *if and only if* $\tau(x) \in P_n$.

5.6. **Proposition.** *Suppose* $x \in \widetilde{P}_n$ *and* j *is a positive integer such that* $0 < j < \operatorname{len}(x)$. *Then* $s_j(x) \notin \widetilde{P}_n$.

5.7. **Proposition.** ${\sigma_n}^{-1}(x) = x$ *for each* $x \in \widetilde{P}_n \backslash \widetilde{H}_n$, *and* ${\sigma_n}^{-1}(x) = \{x\} \times \widetilde{B}_{n+1}$ *for each* $x \in \widetilde{H}_n$.

5.8. **Definition.** Set $D(0) = \{\emptyset\}$. Let $\Delta(n) = \bigcup_{i=0}^{n-1} D(i)$.

5.9. **Definition.** Let i and n be integers such that $0 \leq i < n$. Suppose $d \in D(i)$. We will define a function $\sigma_d^n : \widetilde{P}_n \to \widetilde{B}_{i+1}$. Let x be an arbitrary point in $\widetilde{P}_n$. If $d = \emptyset$, then $\sigma_d^n(x) = r_1(x)$. If $d \neq \emptyset$ and $j = \lambda'(d, \tau(x)) > 0$, then $\sigma_d^n(x) = r_{j+1}(x)$. Now, suppose $d \neq \emptyset$ and $\lambda'(d, \tau(x)) = 0$. Let $\sigma_d^n(x) = \tilde{\Phi}_y(r_m(x))$, where $y = \lambda(d, \tau(x))$ and $m = \operatorname{len}(y)$.

It can be easily verified that:

5.10. **Proposition.** *Let* n *be a positive integer. Suppose* $d \in \Delta(n)$. *Let* x' *and* x'' *be two points of* $\widetilde{P}_n$ *such that* $\tau(x') = \tau(x'')$. *Then either* $\sigma_d^n(x') = \sigma_d^n(x'')$ *or there is an integer* k *such that* $\sigma_d^n(x') = r_k(x')$ *and* $\sigma_d^n(x'') = r_k(x'')$.

5.11. **Definition.** Suppose $b = (b_1, \ldots, b_j) \in \widetilde{P}_n$, $1 \leq i < n$, $c = (c_1, \ldots, c_k) \in C(i)$, $s_{k-1}(\tau(b)) = s_{k-1}(c)$ and $r_k(\tau(b)) \in (\xi_-(c), \xi_+(c))$. Let m denote $\alpha(k, c)$. Observe that $b_k \in \widetilde{B}_m$. Let $b'_k \in \widetilde{B}_m$ be such that $\beta_m(b'_k) = c_k$ and b'_k is in the same component of ${\beta_m}^{-1}((0,1))$ as b_k (see Proposition 2.11). Let $b''_k = \tilde{\varphi}_{i+1} \circ$

$\Phi_c(r_k(\tau(b)))$. Let $\tilde{\delta}'(c,b)$ denote the point $(b_1,\ldots,b_{k-1},b'_k,b''_k,b_{k+1},\ldots,b_j)$. If $\tilde{\delta}'(c,b)\in\widetilde{P}_n$, let $\tilde{\delta}_n(c,b)$ be $\tilde{\delta}'(c,b)$. If $\tilde{\delta}'(c,b)\notin\widetilde{P}_n$, let $\tilde{\delta}_n(c,b)$ be $s_{k+1}\left(\tilde{\delta}'(c,b)\right)$.

The next proposition follows from the definitions and Proposition 4.37.

5.12. **Proposition.** *Suppose* $b=(b_1,\ldots,b_j)\in\widetilde{P}_n$, $1\le i<n$, $c=(c_1,\ldots,c_k)\in C(i)$, $s_{k-1}(\tau(b))=s_{k-1}(c)$ *and* $r_k(\tau(b))\in(\xi_-(c),\xi_+(c))$. *Then* $\tau\left(\tilde{\delta}_n(c,b)\right)=\delta_n(c,\tau(b))$ *and* $\tilde{\delta}_n(c,b)\in\widetilde{P}_n$.

5.13. **Lemma.** *Let* n *and* i *be positive integers such that* $i<n$. *Suppose* $d\in\Delta(n)$ *and* $c=(c_1,\ldots,c_k)\in C(i)$. *Let* $\widetilde{P}_n(c)$ *denote the set of points* $b\in\widetilde{P}_n$ *such that* $\operatorname{len}(b)\ge k$, $s_{k-1}(\tau(b))=s_{k-1}(c)$, $r_k(\tau(b))\ne c_k$ *and* $r_k(\tau(b))\in(\xi_-(c),\xi_+(c))$. *For each* b *in* $\widetilde{P}_n(c)$, *let* $\tilde{b}$ *denote* $\tilde{\delta}_n(c,b)$. *Then* $\left|\sigma^n_d(b)-\sigma^n_d(\tilde{b})\right|\to 0$ *as* $r_k(\tau(b))\to c_k$ *where* $b\in\widetilde{P}_n(c)$.

Proof. Let j, $b_1,\ldots,b_j$, b'_k and b''_k be the same as in the definition of $\tilde{\delta}_n(c,b)$. It follows from Proposition 2.11 that

$$(*)\qquad |b_k-b'_k|\to 0 \text{ as } r_k(\tau(b))\to c_k.$$

We will divide the proof of the lemma into several cases and consider them separately.

Case $d=\emptyset$ *and* $k=1$. In this case $\sigma^n_d(b)=b_1=b_k$, $\sigma^n_d(\tilde{b})=b'_k$ and the conclusion of the lemma follows from (*).

Case $d=\emptyset$ *and* $k>1$. In this case $\sigma^n_d(b)=b_1=\sigma^n_d(\tilde{b})$ and the conclusion of the lemma is obvious.

Now, we can assume that $d\ne\emptyset$. The remaining cases will follow from the conclusion of Lemma 4.45. Let l denote $\lambda'(d,\tau(\tilde{b}))$.

Case $0<l<k$. Then $\lambda'(d,\tau(b))=l$, $\sigma^n_d(b)=b_{l+1}$ and $\sigma^n_d(\tilde{b})=r_{l+1}(\tilde{b})$. If $l+1<k$, then $r_{l+1}(\tilde{b})=b_{l+1}$ and the conclusion of the lemma is obvious. If $l+1=k$, then $r_{l+1}(\tilde{b})=b'_k$ and the conclusion of the lemma follows from (*).

Case $l=k$. In this case $\sigma^n_d(\tilde{b})=b''_k$, $c\in d$ and $\sigma^n_d(b)=\tilde{\Phi}_y(b_k))$, where y is the element of $\Lambda(d)$ containing c. Since $\tilde{\Phi}_y(b_k))=b''_k$, $\sigma^n_d(\tilde{b})=\sigma^n_d(b)$ and the conclusion of the lemma is obvious.

Case $l>k$. In this case $\tilde{b}=(b_1,\ldots,b_{k-1},b'_k,b''_k,b_{k+1},\ldots,b_j)$ and $\sigma^n_d(\tilde{b})=b_l=\sigma^n_d(b)$ and again the conclusion of the lemma is obvious.

In the remaining cases we will assume that $l=\lambda'(d,\tau(\tilde{b}))=0$. Then, by Lemma 4.45, $\lambda'(d,\tau(b))=0$. Let x', x'' and q denote $\lambda(d,\tau(\tilde{b}))$, $\lambda(d,\tau(b))$ and $\operatorname{len}(x')$, respectively.

Case $q<k$. Then $x'=x''$. It follows that $\sigma^n_d(b)=\tilde{\Phi}_{x''}(b_q)=\tilde{\Phi}_{x'}(b_q)=\sigma^n_d(\tilde{b})$.

Case $q=k$. Let y denote $x'=x''$. Observe that $\sigma^n_d(b)=\tilde{\Phi}_y(b_k)=\tilde{\varphi}_{i'+1}\circ\widehat{\Phi}_y(r_k(\tau(b)))$ and $\sigma^n_d(\tilde{b})=\tilde{\Phi}_y(b'_k)=\tilde{\varphi}_{i'+1}\circ\widehat{\Phi}_y(c_k))$, where $d\in D(i')$. Since

$c_k \notin L(y)$, $\widehat{\Phi}_y$ is continuous on $I\backslash L(y)$ and $\tilde{\varphi}_{n+1}$ is continuous; the conclusion of the lemma follows in this case.

Case $q > k$. It follows that $\sigma_d^n(b) = \tilde{\Phi}_{x''}(b_{q-1}) = \tilde{\varphi}_{n+1} \circ \widehat{\Phi}_{x''}(r_{q-1}(\tau(b)))$ and $\sigma_d^n(\tilde{b}) = \tilde{\Phi}_{x'}(r_q(\tilde{b})) = \tilde{\varphi}_{n+1} \circ \widehat{\Phi}_{x'}(r_q(\tau(\tilde{b})))$. By Lemma 4.45 (5), we get the result that $\widehat{\Phi}_{x''}(r_{q-1}(\tau(b))) = \widehat{\Phi}_{x'}(r_q(\tau(\tilde{b})))$. So $\sigma_d^n(b) = \sigma_d^n(\tilde{b})$ and the proof of the lemma is complete.

5.14. **Lemma.** *Let n and i be positive integers such that $i < n$. Suppose $d \in \Delta(n)$ and $c = (c_1, \dots, c_k) \in C(i)$. Let $\widetilde{P}_n'(c)$ denote the set of points $b \in \widetilde{P}_n$ such that* $\operatorname{len}(b) > k$ *and $s_k(\tau(b)) = c$. Let $b_k' \in \widetilde{B}_{\alpha(k,c)}$. For each $b = (b_1, \dots, b_j)$ in $\widetilde{P}_n'(c)$, let $\tilde{b}'$ denote the point $(b_1, \dots, b_{k-1}, b_k', b_{k+1}, \dots, b_j)$. Then $\left|\sigma_d^n(b) - \sigma_d^n(\tilde{b}')\right| \to 0$ as $b_k \to b_k'$ where $b \in \widetilde{P}_n'(c)$.*

Proof. It can be readily verified that either $\sigma_d^n(b) = \sigma_d^n(\tilde{b}')$ or $\sigma_d^n(b) = b_k$ and $\sigma_d^n(\tilde{b}') = b_k'$. So the lemma is true.

5.15. **Lemma.** *Let n be a positive integer. Suppose $x = (x_1, \dots, x_k) \in \widetilde{P}_n$ and $d \in \Delta(n)$. Let $\widetilde{P}_n''(x)$ denote the set of points $b \in \widetilde{P}_n$ such that* $\operatorname{len}(b) \geq k$ *and $s_{k-1}(b) = s_{k-1}(x)$. Then $|\sigma_d^n(b) - \sigma_d^n(x)| \to 0$ as $r_k(b) \to x_k$ where $b \in \widetilde{P}_n''(x)$.*

Proof. Clearly, the lemma is true if $d = \emptyset$. So we may assume that $d \neq \emptyset$. Since $x \in \widetilde{P}_n$, $\tau(x) \in P_n$ and consequently $\tau(x) \notin C(i)$ for each $i < n$. By Proposition 4.33, we may assume that

$$(*) \qquad s_k(\tau(b)) \neq s_k(c) \text{ for each } c \in d.$$

It follows that $\lambda'(d, \tau(x)) = \lambda'(d, \tau(b))$. Denote this number by j. Clearly $j < k$. If $j > 0$, then $\sigma_d^n(b) = r_{j+1}(b)$, $\sigma_d^n(x) = r_{j+1}(x)$ and the conclusion of the lemma holds. Now, suppose $j = 0$. It follows from (*) that $y = \lambda(d, \tau(x)) = \lambda(d, \tau(b))$. Let m denote $\operatorname{len}(y)$. Clearly $m \leq k$. Let i be an integer such that $d \in D(i)$. Observe that $\sigma_d^n(b) = \tilde{\Phi}_y(r_m(b))$ and $\sigma_d^n(x) = \tilde{\Phi}_y(x_m)$. If $m < k$, then the conclusion of the lemma follows from the equality $r_m(b) = r_m(b)$. Suppose $m = k$.

Let q denote $\alpha(k, s_{k-1}(\tau(x)))$. Observe that $\tilde{\Phi}_y = \tilde{\varphi}_{i+1} \circ \widehat{\Phi}_y \circ \beta_q$. Since $\tilde{\varphi}_{i+1}$ and β_q are continuous, $\widehat{\Phi}_y$ is continuous on $I\backslash L(y)$ and $\beta_q(x_k) \notin L(y)$, the proof of the lemma is complete.

5.16. **Definition.** Suppose $x = (x_1, \dots, x_{k+1}) \in \widetilde{Q}$. Let c denote $\tau(s_k(x))$. For each $j = 1, \dots, k+1$, let y_j denote $\tilde{f}_{\alpha(j,c)}(x_j)$ if $\alpha(j,c)$ is odd, and let y_j be $\tilde{f}_{\alpha(j,c)}{}^{-1}(x_j)$ if $\alpha(j,c)$ is even. Let $h(x)$ denote the point $(y_1, \dots, y_{k+1})$.

5.17. **Proposition.** *h restricted to $\widetilde{P}_n$ is a one-to-one map of $\widetilde{P}_n$ onto itself.*

Proof. Let $x = (x_1, \dots, x_{k+1})$ be an arbitrary point of $\widetilde{Q}$. Let c denote $\tau(s_k(x))$. For each $j = 1, \dots, k+1$, let z_j denote $\tilde{f}_{\alpha(j,c)}(x_j)$ if $\alpha(j,c)$ is even, and let z_j be $\tilde{f}_{\alpha(j,c)}{}^{-1}(x_j)$ if $\alpha(j,c)$ is odd. Let $h^{-1}(x)$ denote the point $(z_1, \dots, z_{k+1})$. Observe that h^{-1} so defined is the inverse of h. It follows from Propositions 2.12 and 4.13 that $h(x)$ and $h^{-1}(x)$ are in $\widetilde{P}_n$, if $x \in \widetilde{P}_n$.

The next proposition follows readily from the definitions of h and σ_n.

5.18. **Proposition.** *$h \circ \sigma_n(x) = \sigma_n \circ h(x)$ for each $x \in \widetilde{P}_{n+1}$.*

5.19. **Proposition.** *Suppose $x = (x_1, \ldots, x_k) \in \widetilde{P}_n$. Then $h^j(x) \neq x$ for each positive integer $j \leq n$.*

Proof. There is an integer i such that $0 \leq i < n$ and $s_{k-1} \circ \tau(x) \in C(i)$. Observe that $x_k \in \widetilde{B}_{i+1}$. Suppose that there is a positive integer $j \leq n$ such that $h^j(x) = x$. Observe that

$$(*) \qquad \tilde{f}_{i+1}{}^j(x_k) = x_k.$$

It follows from 2.13 that $\beta_{i+1}(x_k) \in \mathcal{A}$. By Proposition 4.4, $\tau(x) \in C(i+1)$. Thus $i+1$ must be equal to n. Now, (*) contradicts Proposition 2.9.

5.20. **Lemma.** *Suppose i and n are integers such that $0 < i < n$. Let $d \in \Delta(n)$ and let $b = (b_2, \ldots, b_k) \in \widetilde{Q}$ be such that $\tau(b) \in C(i)$. Let $P_n(b)$ denote the set of points $x \in \widetilde{P}_n$ such that $s_{k-1}(x) = s_{k-1}(b)$ and $r_k(\tau(x)) \neq r_k(\tau(b))$. Then $\left|\sigma_d^n\left(h\left(\tilde{\delta}_n(\tau(b), x)\right)\right) - \sigma_d^n\left(\tilde{\delta}_n(\tau(h(b)), h(x))\right)\right| \to 0$ as $r_k(x) \to b_k$, where $x \in P_n(b)$.*

Proof. Let c denote $\tau(b)$ and let c' be $\tau(h(b))$. Let $x = (x_1, \ldots, x_l)$ be a point of $P_n(b)$ and let $y = (y_1, \ldots, y_l)$ denote $h(x)$. Let $m = \alpha(k, c)$. Observe that $\alpha(k, c') = \alpha(k, \tau \circ s_{l-1}(x)) = \alpha(k, \tau \circ s_{l-1}(y)) = m$. Let f' denote $\tilde{f}_m$ if m is odd, and let f' be $\tilde{f}_m{}^{-1}$ if m is even. Let f'' denote $\tilde{f}_{i+1}$ if i is even, and let f' be $\tilde{f}_{i+1}{}^{-1}$ if i is odd. Since $x_k \to b_k$, we can assume that $\beta_m(x_k) \in (\xi_-(c), \xi_+(c))$ and $\tilde{x} = \tilde{\delta}'(c, x)$ is defined. Let x'_k denote $r_k(\tilde{x})$ and let x''_k denote $r_k(\tilde{x}) = \tilde{\varphi}_{i+1} \circ \Phi_c \circ \beta_m(x_k)$. Observe that $r_j(h(\tilde{x})) = y_j$ for $j = 1, \ldots, k-1$, $r_k(h(\tilde{x})) = f'(x'_k)$, $r_{k+1}(h(\tilde{x})) = f''(x''_k)$ and $r_j(h(\tilde{x})) = y_{j-1}$ for $j = k+2, \ldots, l+1$.

Since f' and β_m are continuous and $x_k \to b_k$, we may assume that $\beta_m(y_k) \in (\xi_-(c'), \xi_+(c'))$. Let $\tilde{y}$ denote $\tilde{\delta}'(c', y)$. Observe that $r_j(\tilde{y}) = y_j$ for each $j = 1, \ldots, k-1$ and $r_j(\tilde{y}) = y_{j-1}$ for each $j = k+2, \ldots, l+1$. So

$$(*) \qquad r_j(\tilde{y}) = r_j(h(\tilde{x})) \text{ for } j = 1, \ldots, k-1, k+2, \ldots, l+1.$$

Let A denote the arc in $\beta_m{}^{-1}((0,1))$ between x_k and x'_k. Since $\beta_m(f'(b_k)) = r_k(c')$ is neither 0, nor $\frac{1}{2}$, nor 1 and f' and β_m are continuous, there is a neighborhood U of b_k in $\widetilde{B}_m$ such that either $\beta_m \circ f'(U) \subset \left(0, \frac{1}{2}\right)$ or $\beta_m \circ f'(U) \subset \left(\frac{1}{2}, 1\right)$. Since $x_k \to b_k$, we may assume that $A \subset U$ and consequently $f'(x'_k)$ and $y_k = f'(x_k)$ are in the same component of $\beta_m{}^{-1}((0,1))$. We will prove the following claim.

5.20.1. *Claim.* $\beta_m \circ f'(x'_k) = \beta_m \circ f'(b_k)$.

Proof of 5.20.1. Suppose m is odd. Then $f' = \tilde{f}_m$. By Proposition 2.12, $\beta_m \circ f'(b_k) = g_2 \circ \beta_m(b_k)$ and $\beta_m \circ f'(x'_k) = g_2 \circ \beta_m(x'_k)$. Since $\beta_m(b_k) = \beta_m(x'_k)$, the claim is true if m is odd. So we may assume that m is even and $f' = \tilde{f}_m{}^{-1}$. It follows from Proposition 2.12 that $g_2 \circ \beta_m \circ f'(b_k) = \beta_m(b_k)$ and $g_2 \circ \beta_m \circ f'(x'_k) = \beta_m(x'_k)$. Since the points $\beta_m \circ f'(b_k)$ and $\beta_m \circ f'(x'_k)$ either both belong to the interval $\left(0, \frac{1}{2}\right)$ or they both belong to $\left(\frac{1}{2}, 1\right)$, they must be equal. So the proof of the claim is complete.

It follows from the claim that $r_k(\tilde{y}) = f'(x'_k)$ and consequently

$$(**) \qquad r_k(\tilde{y}) = r_k(h(\tilde{x})).$$

5.20.2. *Claim.* $\beta_{i+1} \circ r_{k+1}(h(\tilde{x})) = \beta_{i+1} \circ r_{k+1}(\tilde{y})$ and $|r_{k+1}(h(\tilde{x})) - r_{k+1}(\tilde{y})| \to 0$ as $x_k \to b_k$.

Proof of 5.20.2. Observe that $r_{k+1}(h(\tilde{x})) = f'' \circ \tilde{\varphi}_{i+1} \circ \Phi_c \circ \beta_m(x_k)$ and $r_{k+1}(\tilde{y}) = \tilde{\varphi}_{i+1} \circ \Phi_{c'} \circ \beta_m \circ f'(x_k)$. We will consider separately the cases where m is even and odd. In both cases we will consider the cases where i is even and odd.

Case m is odd. Then $f' = \tilde{f}_m$. Let z denote $\beta_m(x_k)$. By Proposition 2.12, $\beta_m \circ f'(x_k) = g_2(z)$. Again by 2.12, $r_k(c') = \beta_m \circ f'(b_k) = g_2 \circ \beta_m(b_k) = g_2 \circ r_k(c)$.

Subcase: *i is even.* Then $i - m$ is odd and, by Proposition 4.18, $\Phi_{c'} \circ \beta_m \circ f'(x_k) = \Phi_{c'} \circ g_2(z) = 2\Phi_c(z) = 2\Phi_c \circ \beta_m(x_k)$. Let $t = \Phi_c \circ \beta_m(x_k)$. Then $\Phi_{c'} \circ \beta_m \circ f'(x_k) = 2t$. Since i is even, $f'' = \tilde{f}_{i+1}$. So, in this case $r_{k+1}(h(\tilde{x})) = \tilde{f}_{i+1} \circ \tilde{\varphi}_{i+1}(t)$ and $r_{k+1}(\tilde{y}) = \tilde{\varphi}_{i+1}(2t)$. Since $t \to \infty$ as $x_k \to b_k$, the claim follows in this case from Proposition 2.16.

Subcase: *i is odd.* Then $i - m$ is even and, by Proposition 4.18, $\Phi_{c'} \circ \beta_m \circ f'(x_k) = \Phi_{c'} \circ g_2(z) = \frac{1}{2}\Phi_c(z) = 2\Phi_c \circ \beta_m(x_k)$. Let $t = \Phi_{c'} \circ \beta_m \circ f'(x_k)$. Then $\Phi_c \circ \beta_m(x_k) = 2t$. Since i is odd, $f'' = \tilde{f}_{i+1}{}^{-1}$. So, in this case $r_{k+1}(h(\tilde{x})) = \tilde{f}_{i+1}{}^{-1} \circ \tilde{\varphi}_{i+1}(2t)$ and $r_{k+1}(\tilde{y}) = \tilde{\varphi}_{i+1}(t)$. Since $t \to \infty$ as $x_k \to b_k$, the claim follows in this case from Proposition 2.15.

Case m is even. Then $f' = \tilde{f}_m{}^{-1}$. Let z denote $\beta_m \circ f'(x_k)$. By Proposition 2.12, $\beta_m(x_k) = \beta_m \circ \tilde{f}_m \circ \tilde{f}_m{}^{-1}(x_k) = g_2 \circ \beta_m \circ f'(x_k) = g_2(z)$. Again by 2.12, $r_k(c) = \beta_m(b_k) = \beta_m \circ \tilde{f}_m \circ \tilde{f}_m{}^{-1}(b_k) = g_2 \circ \tilde{f}_m{}^{-1}(b_k) = g_2 \circ r_k(c')$.

Subcase: *i is even.* Then $i - m$ is even and, by Proposition 4.18, $\Phi_c \circ \beta_m(x_k) = \Phi_c \circ g_2(z) = \frac{1}{2}\Phi_{c'}(z) = \frac{1}{2}\Phi_{c'} \circ \beta_m \circ f'(x_k)$. Let $t = \Phi_c \circ \beta_m(x_k)$. Then $\Phi_{c'} \circ \beta_m \circ f'(x_k) = 2t$. Since i is even, $f'' = \tilde{f}_{i+1}$. So, in this case $r_{k+1}(h(\tilde{x})) = \tilde{f}_{i+1} \circ \tilde{\varphi}_{i+1}(t)$ and $r_{k+1}(\tilde{y}) = \tilde{\varphi}_{i+1}(2t)$. Since $t \to \infty$ as $x_k \to b_k$ the claim follows in this case from Proposition 2.16.

Subcase: *i is odd.* Then $i - m$ is odd and, by Proposition 4.18, $\Phi_c \circ \beta_m(x_k) = \Phi_c \circ g_2(z) = 2\Phi_{c'}(z) = 2\Phi_{c'} \circ \beta_m \circ f'(x_k)$. Let $t = \Phi_{c'} \circ \beta_m \circ f'(x_k)$. Then $\Phi_c \circ \beta_m(x_k) = 2t$. Since i is odd, $f'' = \tilde{f}_{i+1}{}^{-1}$. So, in this case $r_{k+1}(h(\tilde{x})) = \tilde{f}_{i+1}{}^{-1} \circ \tilde{\varphi}_{i+1}(2t)$ and $r_{k+1}(\tilde{y}) = \tilde{\varphi}_{i+1}(t)$. Since $t \to \infty$ as $x_k \to b_k$ the claim follows in this case from Proposition 2.15.

It follows from (*), (**) and the first part of Claim 5.20.2 that $\tau(h(\tilde{x})) = \tau(\tilde{y})$. By Propositions 4.13 and 5.5, either $\tilde{\delta}_n(c, x) = \tilde{x}$ and $\tilde{\delta}_n(c', y) = \tilde{y}$ or $\tilde{\delta}_n(c, x) = s_{k+1}(\tilde{x})$ and $\tilde{\delta}_n(c', y) = s_{k+1}(\tilde{y})$. Now, the lemma follows from Proposition 5.10, (*), (**) and the second part of Claim 5.20.2.

5.21. **Definition.** For each positive integer n, we will construct a continuum X_n by introducing a topology on the set $\widetilde{P}_n$. In other words, we will construct a one-to-one map π_n of $\widetilde{P}_n$ onto some continuum X_n. For each integer $i = 0, 1, \dots$ and for each $d \in D(i)$, let $B(d)$ be a copy of $\widetilde{B}_{i+1}$. The continuum X_n will be a subset of $\prod_{d\in\Delta(n)} B(d)$. Let $\tilde{\sigma}_d^n$ be the projection of $\prod_{d\in\Delta(n)} B(d)$ onto $B(d)$. If $x \in P_n$, let $\pi_n(x)$ be the point of $\prod_{d\in\Delta(n)} B(d)$ such that $\tilde{\sigma}_d^n(\pi_n(x)) = \sigma_d^n(x)$ for each $d \in \Delta(n)$.

Observe that X_{n+1} is a subset of $X_n \times \prod_{d\in D(n)} B(d)$. Let σ'_n denote the projection of $X_n \times \prod_{d\in D(n)} B(d)$ onto X_n, and let $\tilde{\sigma}_n$ be the restriction of σ'_n to X_{n+1}. Observe that $\pi_n \circ \sigma_n = \tilde{\sigma}_n \circ \pi_{n+1}$.

5.22. **Proposition.** *X_n is compact.*

Proof. Let $(x_j)_{j=1}^\infty$ be an arbitrary sequence of points in $\widetilde{P}_n$. Let k be an integer such that the sequence $(s_{k-1}(x_j))_{j=1}^\infty$ is constant. Note that $1 \leq k \leq n+1$. We will show that the sequence $(\pi_n(x_j))_{j=1}^\infty$ has a convergent subsequence. If $k = n+1$, then the sequence $(\pi_n(x_j))_{j=1}^\infty$ is constant. So, in our proof we may assume that if $(y_j)_{j=1}^\infty$ is a sequence of points in $\widetilde{P}_n$ such that $(s_k(y_j))_{j=1}^\infty$ is constant, then $(\pi_n(y_j))_{j=1}^\infty$ has a convergent subsequence. Let $(z_1, \ldots, z_{k-1})$ denote the constant $s_{k-1}(x_j)$. Let b_j denote $r_k(x_j)$. Observe that there is an integer l such that $b_j \in \widetilde{B}_l$ for each positive integer j. Since $\widetilde{B}_l$ is compact, the sequence $(b_j)_{j=1}^\infty$ has a convergent subsequence. Without loss of generality, we may assume that $(b_j)_{j=1}^\infty$ converges to some $b \in \widetilde{B}_l$. Let z denote the point $(z_1, \ldots, z_{k-1}, b)$ and let $c = (c_1, \ldots, c_k)$ denote $\tau(z)$.

Suppose $c \notin C(i)$ for each $i < n$. Then $c \in P_n$ and $z \in \widetilde{P}_n$. It follows from Lemma 5.15 that the sequence $(\pi_n(x_j))_{j=1}^\infty$ converges to $\pi_n(z)$. So we may assume that there is an integer $i < n$ such that $c \in C(i)$.

Suppose that $\beta_l(b_j) = c_k$ for infinitely many integers j. In this case we may assume that $\beta_l(b_j) = c_k$ for each positive integer j. Let $y_j \in \widetilde{P}_n$ be such that $\operatorname{len}(y_j) = \operatorname{len}(x_j)$, $r_k(y_j) = b$ and $r_q(y_j) = r_q(x_j)$ for each $q = 1, \ldots, \operatorname{len}(y_j)$ such that $q \neq k$. By the inductive assumption, the sequence $(\pi_n(y_j))_{j=1}^\infty$ has a convergent subsequence. So, without loss of generality, we may assume that $(\pi_n(y_j))_{j=1}^\infty$ converges in X_n. By Lemma 5.14, the sequence $(\pi_n(x_j))_{j=1}^\infty$ converges to the same limit. So, we may assume that $\beta_l(b_j) \neq c_k$ for each positive integer j. In this case, let $y_j = \tilde{\delta}_n(c, x_j)$. As before, we may assume that $(\pi_n(y_j))_{j=1}^\infty$ converges in X_n. By Lemma 5.2, the sequence $(\pi_n(x_j))_{j=1}^\infty$ converges to the same limit.

5.23. **Definition.** Suppose $d \in D(n)$. Let $X_n(d)$ denote the set of points x of X_n such that $\tau \circ {\pi_n}^{-1}(x) \in d$.

5.24. **Proposition.** *The set $X_n(d)$ is compact for each $d \in D(n)$.*

Proof. Suppose $d \in D(n)$. Let $z = (z_1, \ldots, z_m) \in \widetilde{P}_n$ be a point such that $\pi_n(z)$ belongs to the closure of $X_n(d)$ in X_n. By Proposition 5.22, it is enough to show that $\pi_n(z) \in X_n(d)$. Let k be the greatest integer for which there is a sequence of points $(x_j)_{j=1}^\infty$ of ${\pi_n}^{-1}(X_n(d))$ such that $\lim_{j\to\infty} \pi_n(x_j) = \pi_n(z)$ and $s_{k-1}(x_j) = s_{k-1}(z)$ for each positive integer j. Note that $1 \leq k \leq m+1$.

If $k = m+1$, then, by Proposition 5.6, $x_j = z$ and the proposition is true. So, we may assume that $k \leq m$. Let c' denote $\tau \circ s_{k-1}(z)$. There is $d' \in \Delta(n)$ such that $c' \in d'$. Since $\sigma_{d'}^n(z) = z_k$ and $\sigma_{d'}^n(x_j) = r_k(x_j)$, $\lim_{j\to\infty} r_k(x_j) = z_k$. Let $c = (c_1, \ldots, c_k)$ denote $\tau \circ s_k(z)$. Clearly, $\lim_{j\to\infty} r_k(\tau(x_j)) = c_k$. By Proposition 4.33, there is $c'' \in d$ such that $s_k(c'') = c$.

Suppose $k=m$. Since $z\in\widetilde{P}_n$, $c\notin C(i)$ for each positive integer $i<n$. It follows that $s_k(c'')=c''$, $c\in d$ and consequently $\pi_n(z)\in X_n(d)$. So, we can assume $k<m$ and $c\in C(i)$ for some positive integer $i<n$.

Suppose that $r_k\circ\tau(x_j)=c_k$ for infinitely many integers j. In this case we may assume that $r_k\circ\tau(x_j)=c_k$ for each positive integer j. Let $y_j\in\widetilde{P}_n$ be such that $\operatorname{len}(y_j)=\operatorname{len}(x_j)$, $r_k(y_j)=z_k$ and $r_q(y_j)=r_q(x_j)$ for each $q=1,\dots,\operatorname{len}(y_j)$ such that $q\neq k$. By Lemma 5.14, $\lim_{j\to\infty}|\pi_n(y_j)-\pi_n(x_j)|=0$ and thus $\lim_{j\to\infty}\pi_n(y_j)=\pi_n(z)$. Since $\pi_n(y_j)\in X_n(d)$ and $s_k(x_j)=s_k(z)$ for each positive integer j, we have a contradiction with the choice of k.

So, we may assume that $r_k\circ\tau(x_j)\neq c_k$ for each positive integer j. In this case, let $y_j=\widetilde{\delta}_n(c,x_j)$. By Lemma 5.13, $\lim_{j\to\infty}|\pi_n(y_j)-\pi_n(x_j)|=0$ and thus $\lim_{j\to\infty}\pi_n(y_j)=\pi_n(z)$. Since $\pi_n(y_j)\in X_n(d)$ and $s_k(x_j)=s_k(z)$ for each positive integer j, we again get a contradiction with the choice of k.

5.25. **Proposition.** *$X_n(d)$ is 0-dimensional for each $d\in D(n)$.*

Proof. Suppose $d\in D(n)$. For an arbitrary point $c=(c_1,\dots,c_k)\in d$, let P_c denote the Cartesian product ${\beta_{\alpha(1,c)}}^{-1}(c_1)\times{\beta_{\alpha(2,c)}}^{-1}(c_2)\times\cdots\times{\beta_{\alpha(k,c)}}^{-1}(c_k)$. By proposition 2.11, P_c is 0-dimensional. It follows from Proposition 5.10 that π_n restricted to P_c is continuous. Since π_n is a bijection and P_c is compact, $\pi_n(P_c)$ is compact and 0-dimensional. Since $X_n(d)=\bigcup_{c\in d}\pi_n(P_c)$ and d is countable, the proposition is true.

The following well-known proposition is a simple consequence of [5, Theorem 1] and [15, Theorem 11] (see also [4]).

5.26. **Proposition.** *Suppose that $\sigma: X\to Y$ is a continuous map of a one-dimensional continuum X onto a tree-like continuum Y such that $\sigma^{-1}(y)$ is a tree-like continuum for each $y\in Y$. Then X is tree-like.*

5.27. **Proposition.** *X_n is a tree-like continuum.*

Proof. We will prove the proposition by induction. Since $X_1=\widetilde{B}_1$, the proposition is true for $n=1$. Suppose that the proposition is true for some integer $n\geq 1$. We will prove that X_{n+1} is tree-like. Since $\pi_n\circ\sigma_n=\tilde{\sigma}_n\circ\pi_{n+1}$, it follows from Proposition 5.7 that ${\tilde{\sigma}_n}^{-1}(x)$ is a one-point set for each $x\in X_n\backslash\pi_n\left(\widetilde{H}_n\right)$ and ${\tilde{\sigma}_n}^{-1}(x)=\pi_{n+1}\left(\{x\}\times\widetilde{B}_{n+1}\right)$ for each $x\in\pi_n\left(\widetilde{H}_n\right)$.

5.27.1. *Claim.* Suppose that $x\in\pi_n\left(\widetilde{H}_n\right)$. Then π_{n+1} restricted to $\{x\}\times\widetilde{B}_{n+1}$ is an embedding.

Proof of 5.27.1. Since $x\in\widetilde{H}_n$, $\tau(x)\in C(n)$. Let d be the element of $D(n)$ containing $\tau(x)$. If $d'\in\Delta(n+1)$ and $d'\neq d$, then $\sigma^{n+1}_{d'}$ is constant on $\{x\}\times\widetilde{B}_{n+1}$. On the other hand, $\sigma^{n+1}_{d}(x,b)=b$ for each $b\in\widetilde{B}_{n+1}$. Now, the claim follows from the definition of π_{n+1}.

It follows from the claim that ${\tilde{\sigma}_n}^{-1}$ is tree-like for each $x\in X_n$. By Proposition 5.26, to complete the proof it is enough to prove that X_{n+1} is 1-dimensional.

Recall that X_{n+1} is a subset of $X_n\times\prod_{d\in D(n)}B(d)$, where $B(d)=\widetilde{B}_{n+1}$. Let $d_1,d_2,\dots$ be an enumeration of $D(n)$. For each positive integer j, let Y_j denote the set $X_n\times B(d_1)\times B(d_2)\times\cdots\times B(d_j)$. Additionally, let Y_0 denote X_n. For each positive integer j, let κ_j be the projection of Y_j onto Y_{j-1}. Let ν_j be the projection of $X_n \times \prod_{d\in D(n)} B(d)$ onto Y_j.

By a proof similar to that of Claim 5.27.1, we get the following claim.

5.27.2. *Claim.* Suppose that $x \in \widetilde{H}_n$ and $\tau(x) \in d_m$. Then

(1) $\nu_j \circ \pi_{n+1}$ is constant on $\{x\} \times \widetilde{B}_{n+1}$ for each $j = 0, \ldots, m-1$, and

(2) $\nu_j \circ \pi_{n+1}$ restricted to $\{x\} \times \widetilde{B}_{n+1}$ is an embedding for each integer $j \geq m$.

For each integer $j \geq 0$, let $\tilde{Y}_j$ denote $\nu_j(X_{n+1})$ and let $\tilde{\kappa}_j$ denote the restriction of κ_j to $\tilde{Y}_j$. Observe that X_{n+1} is homeomorphic to the inverse limit of the system $\left\{\tilde{Y}_j, \tilde{\kappa}_j\right\}_{j=0}^{\infty}$. In order to prove that X_{n+1} is one-dimensional, it is enough to prove that $\tilde{Y}_j$ is one-dimensional for each integer $j \geq 0$. We will prove this by induction. Since $\tilde{Y}_0 = X_n$, $\tilde{Y}_0$ is tree-like and thus one-dimensional. Suppose $\tilde{Y}_{j-1}$ is one-dimensional for some positive integer j. We will prove that $\tilde{Y}_j$ is one-dimensional. Let p denote the map $\tilde{\kappa}_1\circ\tilde{\kappa}_2\circ\cdots\circ\tilde{\kappa}_{j-1}$. Observe that if $x \in \tilde{Y}_{j-1}$ and $\tau\circ{\pi_n}^{-1}\circ p(x) \notin d_j$, then ${\tilde{\kappa}_j}^{-1}$ is a one-point set. Let W denote the set $p^{-1}(X_n(d_j))$. By Proposition 4.20, W is compact. It follows from Claim 5.27.2 that p restricted to W is an injection. Hence W is homeomorphic to $X_n(d_j)$. By Proposition 4.21, W is a zero-dimensional compactum. By Claim 5.27.2 (2), ${\tilde{\kappa}_j}^{-1}(x)$ is homeomorphic to $\widetilde{B}_{n+1}$ for each $x \in W$. Since $\widetilde{B}_{n+1}$ is one-dimensional, ${\tilde{\kappa}_j}^{-1}(W)$ is a one-dimensional compactum. It follows from Claim 5.27.2 that ${\tilde{\kappa}_j}^{-1}(x)$ is a one-point set for each $x \in \tilde{Y}_{j-1}\backslash W$. So the set ${\tilde{\kappa}_j}^{-1}\left(\tilde{Y}_{j-1}\backslash W\right) = \tilde{Y}_j\backslash{\tilde{\kappa}_j}^{-1}(W)$ is one-dimensional and open in $\tilde{Y}_j$. Hence $\tilde{Y}_j = \left(\tilde{Y}_j\backslash{\tilde{\kappa}_j}^{-1}(W)\right) \cup {\tilde{\kappa}_j}^{-1}(W)$ is one-dimensional and the proof of the proposition is complete.

5.28. **Definition.** Let $h_n = \pi_n \circ h \circ {\pi_n}^{-1}$.

5.29. **Proposition.** *h_n is a continuous map of X_n onto itself.*

Proof. In view of Proposition 5.17, we have to show only that h_n is continuous. So to complete the proof of the proposition, it is enough to prove the following claim.

5.29.1. *Claim.* Suppose $x \in \widetilde{P}_n$ and $\underline{x} = (x_j)_{j=1}^{\infty}$ is a sequence of points of $\widetilde{P}_n$ such that $\lim_{j\to\infty} \pi_n(x_j) = \pi_n(x)$. Then the sequence $(\pi_n(h(x_j)))_{j=1}^{\infty}$ has an (infinite) subsequence which converges to $\pi_n(h(x))$.

For any sequence $\underline{z} = (z_j)_{j=1}^{\infty}$ of points from $\widetilde{P}_n$, let $K(\underline{z})$ be the greatest integer $k \leq \operatorname{len}(x) + 1$ such that the sequence $(s_{k-1}(z_j))_{j=1}^{\infty}$ contains an (infinite) subsequence of terms equal to $s_{k-1}(x)$. Observe that $1 \leq K(\underline{z}) \leq \operatorname{len}(x) + 1$. Let $k = K(\underline{x})$. By replacing the sequence $\underline{x}$ by its subsequence, we may assume that $s_{k-1}(x_j) = s_{k-1}(x)$ for each positive integer j.

Suppose $k = \operatorname{len}(x) + 1$. Then $s_{k-1}(x_j) = x \in \widetilde{P}_n$. By Proposition 5.6, $x_j = x$ and the claim is obvious. So we may assume that $k \leq \operatorname{len}(x)$. We may also assume that if $\underline{y} = (y_j)_{j=1}^{\infty}$ is a sequence of points of $\widetilde{P}_n$ such that $\lim_{j\to\infty} \pi_n(y_j) = \pi_n(x)$ and $K(\underline{y}) > k$, then the sequence $(\pi_n(h(y_j)))_{j=1}^{\infty}$ has an (infinite) subsequence which converges to $\pi_n(h(x))$.

Let $c = \tau(s_{k-1}(x))$. Observe that there is an integer $i < n$ such that $c \in C(i)$. Let b and b_j denote $r_k(x)$ and $r_k(x_j)$, respectively. Observe that b and b_j

are elements of $\widetilde{B}_{i+1}$. Let d be the element of $D(i)$ containing c. Observe that $\sigma_d^n(x) = b$ and $\sigma_d^n(x_j) = b_j$. Since $\lim_{j\to\infty} x_j = x$, it follows that

$$(*)\qquad \lim_{j\to\infty} b_j = b.$$

Let t and t_j denote $h(x)$ and $h(x_j)$, respectively. Clearly, $\pi_n(t) = h_n \circ \pi_n(x)$ and $\pi_n(t_j) = h_n \circ \pi_n(x_j)$. It follows that $r_m(t_j) = r_m(t)$ for each positive integer j and $m = 1, \ldots, k-1$. Let f' denote $\tilde{f}_{i+1}$ if i is even, and let f' be $\tilde{f}_{i+1}{}^{-1}$ if i is odd. Observe that $r_k(t) = f'(b)$ and $r_k(t_j) = f'(b_j)$. Since f' is continuous, it follows from (*) that

$$(**)\qquad \lim_{j\to\infty} r_k(t_j) = r_k(t).$$

Suppose $\operatorname{len}(x) = k$. Then $r_k(t) = t \in \widetilde{P}_n$. It follows from (**) and Lemma 5.15 that $\lim_{j\to\infty} \pi_n(t_j) = \pi_n(t)$. So, we may assume that $\operatorname{len}(x) > k$. Then $r_k(x)$ and $r_k(t)$ are not in $\widetilde{P}_n$.

We will consider the following two cases:

(1) $\beta_{i+1}(b_j) = \beta_{i+1}(b)$ for infinitely many positive integers j, and

(2) $\beta_{i+1}(b_j) \neq \beta_{i+1}(b)$ for each positive integer j.

Case (1). Then, by replacing the sequence $\underline{x}$ by its subsequence, we may assume that $\beta_{i+1}(b_j) = \beta_{i+1}(b)$ for each positive integer j. Let $y_j \in \widetilde{P}_n$ be such that $\operatorname{len}(y_j) = \operatorname{len}(x_j)$, $r_k(y_j) = b$ and $r_m(y_j) = r_m(x_j)$ for $m = 1, \ldots, k-1, k+1, \ldots, \operatorname{len}(y_j)$. By (*) and Lemma 5.14, $\lim_{j\to\infty} \pi_n(y_j) = \pi_n(x)$. Since $K\left((y_j)_{j=1}^{\infty}\right) > k$, the sequence $(\pi_n(h(y_j)))_{j=1}^{\infty}$ has an (infinite) subsequence which converges to $\pi_n(t)$. By replacing the sequence $\underline{x}$ by its subsequence, we may assume that $\lim_{j\to\infty} \pi_n(h(y_j)) = \pi_n(t)$. Observe that $\operatorname{len}(y_j) = \operatorname{len}(h(y_j)) = \operatorname{len}(t_j)$, $r_k(h(y_j)) = r_k(t)$ and $r_m(h(y_j)) = r_m(t_j)$ for $m = 1, \ldots, k-1, k+1, \ldots, \operatorname{len}(y_j)$. It follows from (**) and Lemma 5.14 that $\lim_{j\to\infty} |\pi_n(h(y_j)) - \pi_n(t_j)| = 0$. Therefore, $\lim_{j\to\infty} \pi_n(t_j) = \pi_n(t)$.

Case (2). Let $y_j = \tilde{\delta}(\tau(s_k(x)), x_j)$. By (*) and Lemma 5.13 $\lim_{j\to\infty} \pi_n(y_j) = \pi_n(x)$. Since $K\left((y_j)_{j=1}^{\infty}\right) > k$, the sequence $(\pi_n(h(y_j)))_{j=1}^{\infty}$ has an (infinite) subsequence which converges to $\pi_n(t)$. By replacing the sequence $\underline{x}$ by its subsequence, we may assume that

$$\text{(A)}\qquad \lim_{j\to\infty} \pi_n(h(y_j)) = \pi_n(t).$$

Let $z_j = \tilde{\delta}(\tau(s_k(t)), t_j)$. It follows from (**) and Lemma 5.13 that

$$\text{(B)}\qquad \lim_{j\to\infty} |\pi_n(z_j) - \pi_n(t_j)| = 0.$$

By Lemma 5.20, we get the result that

$$\text{(C)}\qquad \lim_{j\to\infty} |\pi_n(z_j) - \pi_n(h(y_j))| = 0.$$

By combining (A), (B) and (C), we get $\lim_{j\to\infty} \pi_n(t_j) = \pi_n(t)$. So the proof of the claim and the proposition is complete.

The next proposition follows from 5.18.

5.30. **Proposition.** $h_n \circ \tilde{\sigma}_n = \tilde{\sigma}_n \circ h_{n+1}$.

The next proposition follows from the definition of h_n and 5.19.

5.31. **Proposition.** *h_n does not have periodic points of periods smaller than or equal to n.*

5.32. **Theorem.** *There is a tree-like continuum X and a periodic point free homeomorphism of X onto itself.*

Proof. Let X be the inverse limit of the system $\{X_n, \tilde{\sigma}_n\}_{n=1}^{\infty}$. By Proposition 5.27, X is a tree-like continuum. By Proposition 5.30, the sequence $(h_n)_{n=1}^{\infty}$ induces a map $\tilde{h} : X \to X$. By Proposition 5.29, $\tilde{h}$ is a homeomorphism of X onto itself. Now, suppose that there is a positive integer j and there is a point $x = (x_1, x_2, \dots) \in X$ such that $\tilde{h}^j(x) = x$. It follows that ${h_n}^j(x_n) = x_n$ for each positive integer n. In particular, ${h_j}^j(x_j) = x_j$ in contradiction with Proposition 5.31.

References

1. D. P. Bellamy, *A tree-like continuum without the fixed-point property*, Houston J. Math. **6** (1980), 1–13. MR **81h:**54039
2. R. H. Bing, *The elusive fixed point property*, Amer. Math. Monthly **76** (1969), 119–132. MR **38:**5201
3. K. Borsuk, *Sur un continu acyclique qui se laisse transformer topologiquement en lui même sans points invariants*, Fund. Math. **24** (1935), 51–58.
4. K. Borsuk, *Theory of shape*, Polish Scientific Publishers, Warsaw, Poland, 1975. MR **54:**6132
5. J. H. Case and R. E. Chamberlin, *Characterizations of tree-like continua*, Pacific J. Math. **10** (1960), 73–84. MR **22:**1868
6. H. Cook, W. T. Ingram and A. Lelek, *Eleven annotated problems about continua*, Open Problems in Topology, North–Holland, Amsterdam, 1990, pp. 295–302. MR **91:**03
7. H. Cook, W. T. Ingram and A. Lelek, *A list of problems known as Houston problem book*, Continua with the Houston problem book, Lecture Notes in Pure and Appl. Math., Vol. 170, Marcel Dekker, New Yor, 1995, pp. 365–398. CMP 95:11
8. J. B. Fugate and L. A. Mohler, *A note on fixed points in tree–like continua*, Topology Proc. **2** (1977), 457–460. MR **80k:**54062
9. C. L. Hagopian, *A fixed-point theorem for tree-like continua*, Topology Proc. **16** (1991), 57–62. MR **94a:**54106
10. W. Lewis, *Continuum theory problems*, Topology Proc. **8** (1983), 361–394. MR **86a:**54038
11. P. Minc, *A tree-like continuum admitting fixed point free maps with arbitrarily small trajectories*, Topology and its Appl. **46** (1992), 99–106. MR **94a:**54108
12. L. G. Oversteegen and J. T. Rogers, Jr., *Tree-like continua as limits of cyclic graphs*, Topology Proc. **4** (1979), 507–515. MR **82c:**54032
13. L. G. Oversteegen and J. T. Rogers, Jr., *An inverse limit description of an atriodic tree-like continuum and an induced map without a fixed point*, Houston J. Math. **6** (1980), 549–564. MR **82j:**54075
14. L. G. Oversteegen and J. T. Rogers, Jr., *Fixed-point-free maps on tree-like continua*, Topology and its Appl. **13** (1982), 85–95. MR **83b:**54044
15. R. B. Sher, *Realizing cell-like maps in Euclidean space*, General Topology and its Appl. **2** (1972), 75–89. MR **46:**2683

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