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HARISH-CHANDRA'S PLANCHEREL THEOREM FOR $\mathfrak{p}$-ADIC GROUPS

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ABSTRACT. Let G be a reductive $\mathfrak{p}$-adic group. In his paper, "The Plancherel Formula for Reductive $\mathfrak{p}$-adic Groups", Harish-Chandra summarized the theory underlying the Plancherel formula for G and sketched a proof of the Plancherel theorem for G. One step in the proof, stated as Theorem 11 in Harish-Chandra's paper, has seemed an elusively difficult step for the reader to supply. In this paper we prove the Plancherel theorem, essentially, by proving a special case of Theorem 11. We close by deriving a version of Theorem 11 from the Plancherel theorem.

1. INTRODUCTION

Let Ω be a commutative p-field. We shall employ the convention of referring to linear algebraic groups defined over Ω by bold Roman letters and the corresponding groups of points rational over Ω by the same Roman letter in ordinary type. Thus, in particular, the pair $(\mathbf{G}, \mathbf{Z})$ will denote a connected, reductive algebraic group $\mathbf{G}$ defined over Ω with $\mathbf{Z}$ the split component of the center of $\mathbf{G}$; the pair (G, Z) denotes the respective groups of Ω points of $\mathbf{G}$ and $\mathbf{Z}$. We call G a "reductive $\mathfrak{p}$–adic group" and note that G is a totally disconnected, separable unimodular group.

In [6] Harish-Chandra has summarized the theory underlying the Plancherel formula for G and sketched a proof of the Plancherel theorem. To complete this sketch it seems to this writer that details need to be supplied justifying only one assertion of [6], namely Theorem 11. Every other assertion in this paper can be readily proved either by using prior published work of Harish-Chandra or the present author's notes on Harish-Chandra's lectures. In particular, [8, §5.5] presents Harish-Chandra's theory of wave packets, proves the Plancherel and inversion formulas for the various wave packet algebras, and proves the orthogonality of the wave packet algebras associated to inequivalent tempered series of representations. Only the completeness theorem is lacking, i.e. the proof that the wave packet algebras span the Schwartz space.

In this paper, after briefly reviewing the Plancherel theorem, we prove our key result, which substitutes for Harish-Chandra's Theorem 11 in the proof of the Plancherel theorem. We close the paper with a version of Theorem 11 as a consequence of the Plancherel theorem.

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2. The Plancherel Theorem

We fix a unitary character χ of Z. We shall discuss the Plancherel theorem only for functions which satisfy

$$f(zg) = \chi(z)f(g) \qquad (z \in Z, g \in G).$$

To pass between a Plancherel theorem or formula for $L^2(G/Z, \chi)$ and a Plancherel theorem or formula for $L^2(G)$ is an exercise in abelian harmonic analysis which we shall not mention again.

We fix a maximal Ω split torus $\mathbf{A}_0$ of $\mathbf{G}$. Then $\mathbf{A}_0$ is unique up to conjugacy, so the integer $\dim(\mathbf{A}_0/\mathbf{Z}) = \ell = \mathrm{rank}_\Omega(\mathbf{G}) \geq 0$, the "semi-simple Ω rank" of $\mathbf{G}$, is an invariant of $\mathbf{G}$. We intend to prove the Plancherel theorem via an induction on $\mathrm{rank}_\Omega(\mathbf{G})$.

1 Standard Parabolic Subgroups and Levi Factors.
We also fix a minimal Ω parabolic subgroup $\mathbf{P}_0$ of $\mathbf{G}$ such that $\mathbf{A}_0 \subset \mathbf{P}_0$. Then we have a lattice of 2^ℓ "standard p-pairs" $(\mathbf{P}, \mathbf{A})$, such that $\mathbf{P}_0 \subseteq \mathbf{P} \subseteq \mathbf{G}$ and $\mathbf{Z} \subseteq \mathbf{A} \subseteq \mathbf{A}_0$, where $\mathbf{A} = \mathbf{A_P}$ is a maximal split torus in the radical of $\mathbf{P}$. Each parabolic subgroup has a "Levi decomposition", which is a semi-direct product of algebraic groups

$$\mathbf{P} = \mathbf{M} \ltimes \mathbf{N} = \mathbf{MN},$$

where $\mathbf{M} = \mathbf{M_P} = \mathbf{M_{A_P}}$ is the centralizer of $\mathbf{A_P}$ and the normal subgroup $\mathbf{N} = \mathbf{N_P}$ is the unipotent radical of $\mathbf{P}$; thus $\mathbf{M} = \mathbf{M_A}$ is uniquely determined by $\mathbf{A} = \mathbf{A_M}$ and vice versa. The group $\mathbf{M}$ is, like $\mathbf{G}$, a connected reductive Ω group which, if $\mathbf{A} \neq \mathbf{Z}$, is of semi-simple Ω rank strictly less than ℓ. In fact,

$$\mathrm{rank}_\Omega(\mathbf{M}) = \ell - \dim(\mathbf{A_M}/\mathbf{Z}) = \ell - \text{p-rank}(\mathbf{M}).$$

Every parabolic subgroup of $\mathbf{G}$ is conjugate to exactly one standard parabolic subgroup. We fix a subset $\mathcal{M}$ of the set of Levi factors of the 2^ℓ standard parabolic subgroups such that every Levi factor of a parabolic subgroup of $\mathbf{G}$ is conjugate to exactly one element of $\mathcal{M}$. For every $\mathbf{M} \in \mathcal{M}$ the pair $(\mathbf{M}, \mathbf{A_M})$ satisfies $\mathbf{M}_0 = \mathbf{M}_{\mathbf{A}_0} \subseteq \mathbf{M} \subseteq \mathbf{G}$ and $\mathbf{Z} \subseteq \mathbf{A_M} \subseteq \mathbf{A}_0$. Of course, $\mathbf{M}_0 \in \mathcal{M}$ and $\mathbf{G} \in \mathcal{M}$.

2 Good Maximal Compact Subgroups.
We fix an "A_0 good" maximal compact subgroup K of G in the sense of [8, §0.6].

Thus K is an open compact group which contains a set of representatives for the relative Weyl group $\mathrm{Norm}_{\mathbf{G}}(\mathbf{A}_0)/\mathrm{Cent}_{\mathbf{G}}(\mathbf{A}_0) = W(G/A_0)$. More generally, for any $\mathbf{M} \in \mathcal{M}$ and $\mathbf{A} = \mathbf{A_M}$ the group $W(G/A) = \mathrm{Norm}_{\mathbf{G}}(\mathbf{A})/\mathrm{Cent}_{\mathbf{G}}(\mathbf{A})$ is a subquotient of $W(G/A_0)$, so representatives $y(s)$ for $s \in W(G/A)$ may also be chosen in this case such that $y(s) \in K$.

The group K also contains a fundamental sequence $\{K_j\}_{j\geq 1}$ of neighborhoods of the identity of G such that, for each j, K_j is an open normal subgroup of K. Moreover, for each j and each standard p-pair (P, A) $(P = MN)$, with "opposite" $(\bar{P}, A)$ $(\bar{P} = M\bar{N})$ satisfying $N \cap \bar{N} = \{\mathrm{Id}\}$, the group $K_j = K_j^G$ has the "Iwahori factorization" $K_j = \bar{N}_j K_j^M N_j$, where $\bar{N}_j = K_j \cap \bar{N}$, $K_j^M = K_j \cap M$, and $N_j = K_j \cap N$ and each of the groups $\bar{N}_j$, N_j is a normal subgroup of K_j. The group $K^M = K \cap M$ is an A_0 good maximal compact subgroup of M and the sequence $\{K_j^M\}$ has the same properties relative to (M, A_0), K^M, and the standard parabolic subgroups of M as $\{K_j\}$ has relative to (G, A_0), K, and the standard parabolic subgroups of G.

Let $\Sigma^0(P_0, A_0)$ denote the set consisting of the ℓ simple roots with respect to (P_0, A_0) (cf. [8, §0.6]). Let $H : M_0 \to \mathfrak{a}_0$ be the "log" mapping to the "real Lie algebra" $\mathfrak{a}_0$ of A_0, let $C \subset \mathfrak{a}_0$ be the positive chamber relative to $\Sigma^0(P_0, A_0)$, and set $M_0^+ = H^{-1}(C) \subset M_0$. Then we have the Cartan decomposition $G = KM_0^+K$. Writing ${}^0M_0 = \ker(H) = M_0 \cap K$, we choose a set of representatives F for $M_0^+/{}^0M_0Z$. Then, $G = KFKZ$ and each element of F, except for a finite subset, is positive on at least one element of $\Sigma^0(P_0, A_0)$.

We also have the Iwasawa decomposition $G = KP_0 = P_0K$. Since all minimal Ω parabolic subgroups are conjugate in G, we have similar decompositions of G with respect to any Ω parabolic subgroup of G. For any such P let δ_P denote the modular function of P; it is defined such that if $d_\ell p$ is a left Haar measure on P, then $\delta_P(p)d_\ell p = d_r p$ is a right Haar measure on P. Using the Iwasawa decomposition, we extend δ_P to a function on G by setting $\delta_P(x) = \delta_P(p)$ when $x = kp$ with $k \in K$ and $p \in P$.

3 Orbits of Discrete Series Representations.

An irreducible admissible representation σ of $M \in \mathcal{M}$ is said to belong to the *discrete series* if for all $f \in \mathcal{A}(\sigma)$, the space spanned by the matrix coefficients of σ, we have

$$\int_{M/A_M} |f(m)|^2 dm^* < \infty,$$

where dm^* denotes a Haar measure on the quotient group. Every irreducible admissible representation σ of M has a central character, to be denoted η_σ, such that $\sigma(a)$ is the scalar operator $\eta_\sigma(a)I$ for all $a \in A_M$. If σ is a discrete series representation, then σ is a unitary representation, so η_σ is a unitary character.

For $M \in \mathcal{M}$ we let $\mathcal{E}_2(M, \chi)$ denote the set of classes of discrete series representations ω of M with central character η_ω satisfying $\eta_\omega|Z = \chi$.

We write $\mathcal{U}(M, 1_Z)$ for the group of all unramified unitary characters, χ_ν, of M with trivial restriction to Z. This group has the structure of a finite union of compact (real) tori and is thus a union of real manifolds. Each manifold has dimension equal the p-rank of $\mathbf{M}$.

The group $\mathcal{U}(M, 1_Z)$ acts on $\mathcal{E}_2(M, \chi)$ such that if $\sigma \in \omega \in \mathcal{E}_2(M, \chi)$ and $\chi_\nu \in \mathcal{U}(M, 1_Z)$, then

$$\chi_\nu \sigma = \chi_\nu \otimes \sigma \in \chi_\nu \cdot \omega = \omega_\nu.$$

We write $\mathcal{O}(\omega, \chi)$ for the orbit of ω with respect to this action. Each orbit is a union of finitely many connected real manifolds of dimension equal to the p-rank of M. [1]

We write $\mathcal{S}(M, M, \chi)$ for the set of orbits of discrete series representation classes ω of M with central character η_ω satisfying $\eta_\omega|Z = \chi$.

Recalling the sequence $\{K_j^M\}$ with respect to $M \in \mathcal{M}$ with maximal compact subgroup K^M, we also write $\mathcal{S}(M, M, \chi, j)$ for the set of all discrete series orbits for M such that if $\sigma \in \omega \in \mathcal{O}$, then $1_{K_j^M} \subset \sigma|K_j^M$. It is clear that this property is an orbit invariant.

[1]See [6, §2] or [8, §§5.3.1, 5.4.5] for more concerning the manifold structure on an orbit of discrete series classes. It is possible for an orbit to have fewer connected components than does $\mathcal{U}(M, 1_Z)$.

The group $W(G/A)$ acts on $\mathcal{S}(M,M,\chi,j)$. For $\sigma \in \omega \in \mathcal{S}(M,M,\chi,j)$ and $y = y(s) \in K$ a representative for $s \in W(G/A)$, we set $\sigma^y(m) = \sigma(y^{-1}my)$. The class of σ^y depends only on s; we denote it ω^s. Clearly, $\omega \in \mathcal{E}_2(M,\chi)$ if and only if $\omega^s \in \mathcal{E}_2(M,\chi)$. Moreover, since K_j is normal in K and since y normalizes M, y normalizes K_j^M; thus $1_{K_j^M} \subset \sigma|K_j^M$ if and only if $1_{K_j^M} \subset \sigma^y|K_j^M$, so $\omega \in \mathcal{S}(M,M,\chi,j)$ if and only if $\omega^s \in \mathcal{S}(M,M,\chi,j)$.

We write $W(\mathcal{O}) = W(G/A,\mathcal{O})$ for the stabilizer of $\mathcal{O} \in \mathcal{S}(M,M,\chi,j)$. We write $\mathcal{S}(G,M,\chi,j)$ for a fundamental set in $\mathcal{S}(M,M,\chi,j)$ for the action of $W(G/A)$. Thus, $\mathcal{S}(G,M,\chi,j)$ is a minimal subset of $\mathcal{S}(M,M,\chi,j)$ with the property that its $W(G/A)$ orbit equals $\mathcal{S}(M,M,\chi,j)$. Assuming that a choice of fundamental set $\mathcal{S}(G,M,\chi,j)$ has been made inductively such that $\mathcal{S}(G,M,\chi,j) \subset \mathcal{S}(G,M,\chi,j')$ whenever $j \le j'$, we set $\mathcal{S}(G,M,\chi) = \bigcup_j \mathcal{S}(G,M,\chi,j)$.

4 Manifolds of Tempered Representations of G.

We want to use the manifold structure on $\mathcal{E}_2(M,\chi)$ to parameterize the set of tempered representation classes of G. For this we have to recall some facts from the theory of induced representations of G.

Let $\mathbf{P} = \mathbf{MN}$ be a parabolic subgroup of G having $\mathbf{M}$ as a Levi factor. We consider unitary induction:

$$I(P,\sigma) = \mathrm{Ind}_P^G \delta_P^{1/2}\sigma,$$

where $\sigma \in \omega \in \mathcal{E}_2(M)$ and σ is inflated to P. Generically, the mapping $\omega_\nu \mapsto I(P,\omega_\nu)$, considered as a mapping from $\mathcal{O}(\omega,\chi)$ to the set of tempered representations of G with central character χ, sends each discrete series representation to an irreducible representation. In any case, the induced class $I(P,\omega_\nu)$ consists of finitely many direct summands and is independent of the choice of $\mathbf{P}$ having $\mathbf{M}$ as Levi factor and of $(\omega_\nu)^s$ for $s \in W(G/A)$. Thus, for $M \in \mathcal{M}$ and any orbit $\mathcal{O} \subset \mathcal{E}_2(M)$, we have a correspondence

$$\omega_\nu \mapsto I(G,M,\omega_\nu) = I(P,\omega_\nu)$$

which is generically a smooth covering map; on a lower dimensional set the image of ω_ν consists of a finite set of direct summands. Since every irreducible tempered representation class of G with central character χ is a subrepresentation class of $I(P,\omega)$ with $\omega \in \mathcal{E}_2(M,\chi)$ for some $M \in \mathcal{M}$, we have a surjective correspondence from

$$\bigcup_{M\in\mathcal{M}} \bigcup_{\mathcal{O}\in\mathcal{S}(G,M,\chi)} \mathcal{O}$$

to the set of all irreducible tempered representation classes of G. We let $I(G,M,\mathcal{O})$ denote the image of the set $\mathcal{O} \in \mathcal{S}(G,M,\chi)$ in the set of tempered representations of G under the correspondence $\omega \mapsto I(G,M,\omega)$; generically each point in the image has a preimage of cardinality equal to the order of $W(\mathcal{O})$.

We know that

$$I(G,M,\mathcal{O}) \cap I(G,M',\mathcal{O}') = \emptyset$$

if $M \ne M'$ or $\mathcal{O}' \notin W(G/A)\cdot\mathcal{O}$. Part of the content of the Plancherel theorem, implicit in this paper, is the assertion that the set

$$\bigcup_{M\in\mathcal{M}} \bigcup_{\mathcal{O}\in\mathcal{S}(G,M,\chi)} I(G,M,\mathcal{O})$$

is the support of the Plancherel measure.

We want to consider representations with K_j fixed vectors.

Lemma 1. *Let $M \in \mathcal{M}$ and let $P = MN$ be the standard parabolic subgroup of G having M as Levi factor. Let σ be any irreducible admissible representation of M. Then $1_{K_j} \subset I(P,\sigma)|K_j$ if and only if $1_{K_j^M} \subset \sigma|K_j^M$.*

Proof. Assume $1_{K_j^M} \subset \sigma|K_j^M$. Then $1_{K_j^M N}$ is a subrepresentation of the restriction to $K_j^M N$ of the inflation of σ to P. Let v in the representation space of σ be fixed by $K_j^M N$. Define a function h on G with values in the representation space of σ by setting

$$h(x) = \begin{cases} \sigma(p)v, & \text{for } x = p\bar{n} \in P\bar{N}_j; \\ 0, & \text{for } x \notin P\bar{N}_j. \end{cases}$$

Then $h(x)$ is a smooth function which lies in the representation space of $I(P,\sigma)$, and it is clear that h is fixed (under right translation) by K_j. The converse is just as easy a consequence of the definition of induced representation, and we omit its proof. □

Of course it follows from Lemma 1 that

$$\bigcup_{M\in\mathcal{M}} \bigcup_{\mathcal{O}\in\mathcal{S}(G,M,\chi,j)} I(G,M,\mathcal{O})$$

is the set of all tempered representation classes of G which have K_j fixed vectors.

5 The Schwartz Algebra.

A systematic introduction to the theory of the Schwartz space and tempered representations for $\mathfrak{p}$–adic groups is given in [8, Ch. 4]. Here we shall recall only enough of the Schwartz space theory so that we may explain the proof of the Plancherel theorem.

Let

$$\rho : g \mapsto \rho(g)_{ij} \qquad (1 \le i,j \le \deg(\rho))$$

be a rational representation of G, let

$$\|g\| = \max_{ij}(|\rho(g)_{ij}|, |\rho(g^{-1})_{ij}|),$$

and set

$$\sigma(g) = \log(\|g\|) \quad \text{and} \quad \sigma_*(g) = \inf_{z\in Z} \sigma(gz).$$

Define the spherical function

$$\Xi(g) = \int_K \delta_{P_0}(gk)^{-1} dk.$$

Let $C_c(G,\chi,j)$ denote the space of compactly supported mod Z, K_j bi-invariant functions $f : G \to \mathbb{C}$ such that $f(gz) = \chi(z)f(g)$ for all $g \in G, z \in Z$. Then $C_c(G,\chi,j)$ is a convolution algebra on G/Z with an identity element $E(\chi,j)$. We also write

$$C_c^\infty(G,\chi) = \bigcup_j C_c(G,\chi,j).$$

For any positive integer j we write $\mathcal{C}(G,\chi,j)=\mathcal{C}(G//K_j,\chi)$ for the space consisting of all functions $f : G \to \mathbb{C}$ which are K_j bi-invariant, satisfy $f(gz) = \chi(z)f(g)$ for all $g \in G, z \in Z$, and also satisfy

$$|f(g)| \leq \nu_r(f)\Xi(g)(1+\sigma_*(g))^{-r}$$

for some $\nu_r(f) > 0$ for all $r > 0$. This space is also a convolution algebra on G/Z with $E(\chi, j)$ as its identity element ([8, Theorem 4.4.2]). We set

$$\mathcal{C}(G,\chi) = \bigcup_j \mathcal{C}(G,\chi,j).$$

We call these spaces the *Schwartz spaces* or *Schwartz algebras* on G.

We know that

$$C_c^\infty(G,\chi) \subset \mathcal{C}(G,\chi) \subset L^2(G/Z)$$

([5, §14, Corollary 1], [8, Theorem 4.4.1]). For any $f \in \mathcal{C}(G,\chi)$ and any parabolic subgroup $P = MN$ of G Harish-Chandra sets

$$f^{(P)}(m) = \delta_P^{1/2}(m)\int_N f(mn)dn,$$

the integral being absolutely convergent for all $m \in M$ ([5, Lemma 12], [8, Theorem 4.4.3]).

A function $f \in \mathcal{C}(G,\chi)$ is called a *cusp form* if $f^{(P)} = 0$ for every parabolic subgroup $P \neq G$. We write ${}^0\mathcal{C}(G,\chi,j)$ for the subspace consisting of all cusp forms in $\mathcal{C}(G,\chi,j)$. Then ${}^0\mathcal{C}(G,\chi,j)$, being the intersection of the kernels of algebra homomorphisms of $\mathcal{C}(G,\chi,j)$, is a two-sided ideal of $\mathcal{C}(G,\chi,j)$.

6 The Orthogonal Decomposition of $\mathcal{C}(G,\chi)$ and the Plancherel Theorem.

It follows from [5, Theorem 26] or [8, Corollary 4.4.7 or Theorem 4.5.10] that for any discrete series representation π of G with central character χ the space $\mathcal{A}(\pi,\chi,j)$ spanned by the left and right K_j-invariant matrix coefficients of π is a (finite-dimensional) subspace of ${}^0\mathcal{C}(G,\chi,j)$. The space $\mathcal{A}(\pi,\chi,j)$ is also an ideal of $\mathcal{C}(G,\chi,j)$.

Expressed very tersely, the Plancherel theorem is the assertion:

Theorem 1. *For all positive integers j*

$${}^0\mathcal{C}(G,\chi,j) = \bigoplus_i \mathcal{A}(\pi_i,\chi,j),$$

the direct sum being over finitely many discrete series representations π_i of G with central character χ.

Theorem 1 means that every element of ${}^0\mathcal{C}(G,\chi)$ is a linear combination of matrix coefficients of discrete series representations. Moreover, it is asserted also that for any positive integer j the number of inequivalent discrete series representations π such that $1_{K_j} \subset \pi|K_j$ is finite. In other words, (cf. [6, Lemma 3, also Theorems 8, 9, and 10]):

Theorem 2. [2]

$$\dim {}^0\mathcal{C}(G,\chi,j) < \infty.$$

Let us show that Theorem 1 and Theorem 2 are equivalent. Since every discrete series representation is admissible, it is obvious that Theorem 1 implies Theorem 2. Conversely, Theorem 2 implies that ${}^0\mathcal{C}(G,\chi)$ is the representation space of an admissible representation of $G \times G$. Since the elements of ${}^0\mathcal{C}(G,\chi)$ are square integrable, it is unitary and hence a direct sum of irreducible unitary representations of $G \times G$ ([8, Lemma 1.6.5]). Since, clearly, any irreducible $G \times G$ submodule of ${}^0\mathcal{C}(G,\chi)$ is spanned by the set of matrix coefficients for an irreducible admissible representation of G, hence of a discrete series representation of G, Theorem 1 follows.

Theorem 2 is derived in [6, §14] from [6, Theorem 11].

Let $\mathcal{C}(\mathcal{O},\chi,j)$ be the ideal in $\mathcal{C}(G,\chi,j)$ consisting of the image of the wave packet map with respect to the tempered series $I(G,M,\mathcal{O})$ for $\mathcal{O} \in \mathcal{S}(G,M,\chi,j)$ ([8, §5.5]). From Theorems 1 and 2 and the orthogonality results proved in [8, §5.5] we obtain the following orthogonal decomposition of the Schwartz algebra:[3]

Theorem 3. *For any positive integer j*

$$\mathcal{C}(G,\chi,j) = \bigoplus_{M\in\mathcal{M}} \bigoplus_{\mathcal{O}\in\mathcal{S}(G,M,\chi,j)} \mathcal{C}(\mathcal{O},\chi,j),$$

the direct sum being finite and the ideals being pairwise orthogonal.

In [8, §5.5.2] an inversion formula is proved for ideals of the form $\mathcal{C}(\mathcal{O},\chi,j)$. Putting this inversion formula together with the decomposition of Theorem 3 gives the Fourier inversion formula for G which is summarized in [6].

Let $E(\mathcal{O},\chi,j) \in \mathcal{C}(\mathcal{O},\chi,j)$ denote the identity element of the wave packet algebra $\mathcal{C}(\mathcal{O},\chi,j)$ (cf. [8, Theorem 5.5.4.2]). We close with the idempotent formulation of Theorem 3:

Theorem 4. *For any positive integer j*

$$E(\chi,j) = \sum_{M\in\mathcal{M}} \sum_{\mathcal{O}\in\mathcal{S}(G,M,\chi,j)} E(\mathcal{O},\chi,j),$$

this sum of orthogonal idempotents being finite.

3. The Proof of the Plancherel Theorem

Let the split rank of the algebraic group $\mathbf{G}$ be $l > 0$ and assume that Theorems 1 and 2 are true for groups of split rank less than l. The case $l = 0$ is trivial, since, in this case, $G = M_0$, a compact mod center group; it follows that every continuous representation of G with unitary central character χ is a finite-dimensional, unitary,

[2]See [3] for an interesting proof that $\dim({}^0C_c(G,\chi,j)) < \infty$, a proof based on the theorem of [7]. Bushnell's argument is sufficient to prove that every irreducible unitary representation of G contains a dense admissible subrepresentation. The argument of Bushnell does not appear applicable to our case because tempered representations are probably not inductively ordered in general, i.e. probably not every tempered representation has an irreducible tempered quotient representation.

[3]Compare Theorems 3 and 4 to analogous results for $C_c(G,\chi,j)$ implied by the Bernstein center theory; cf. [1], [2]. The Bernstein center theory, by the way, also implies that every irreducible unitary representation has a dense admissible subrepresentation.

and discrete series representation. Furthermore, it is clear that, for every choice of χ, there are only finitely many representations with a given open kernel. [4]

By the induction hypothesis we have for any j the decomposition of $\mathcal{C}(G,\chi,j)$ given by Theorems 3 and 4 as a sum over finitely many orbits of tempered series for all $M \in \mathcal{M}$ such that $M \neq G$. It is sufficient to prove Theorem 2, and we do this by proving that the idempotent

$$
{}^0E(\chi,j) = E(\chi,j) - \sum_{M\in\mathcal{M}, M\neq G} \sum_{\mathcal{O}\in\mathcal{S}(G,M,\chi,j)} E(\mathcal{O},\chi,j) \tag{3.1}
$$

is of trace class. It follows from the induction hypothesis and the theory of wave packets developed in [8, §5.5] that ${}^0E(\chi,j)$ is the identity element of the ideal ${}^0\mathcal{C}(G,\chi,j)$ in $\mathcal{C}(G,\chi,j)$. Obviously, the convolution operator ${}^0E(\chi,j)$ projects $\mathcal{C}(G,\chi,j)$ on ${}^0\mathcal{C}(G,\chi,j)$. Its eigenvalues are either zero or one, so it has a finite trace if and only if Theorem 2 is true.

To begin the proof of Theorem 2, normalize the Haar measure on K_jZ/Z. For any $f \in {}^0\mathcal{C}(G,\chi,j)$ define

$$
T_f v(g) = \int_{K_j}\int_{G/Z} f(gkx^{-1})v(x)dx^*dk \tag{3.2}
$$

for $v \in \mathcal{C}(K_jZ\backslash G,\chi)$. Then T_f maps the dense subspace $\mathcal{C}(K_jZ\backslash G,\chi)$ of the Hilbert space $L^2(K_jZ\backslash G,\chi)$ into the smaller subspace $\mathcal{C}(G,\chi,j)$, in fact into the ideal ${}^0\mathcal{C}(G,\chi,j)$. We shall prove that the operator T_f on $L^2(K_jZ\backslash G,\chi)$ has finite trace and therefore also finite rank.

To choose an orthonormal basis of $L^2(K_jZ\backslash G,\chi)$ consisting of functions in the space $C_c(K_jZ\backslash G,\chi)$ let Y be a set of representatives for $K_jZ\backslash G$ and, for each $y\in Y$, set

$$
v_y(g) = \begin{cases} \chi(z), & \text{for } g = zky \text{ with } z\in Z \text{ and } k\in K_j,\\ 0, & \text{for } g\notin K_jZy. \end{cases} \tag{3.3}
$$

Clearly, the set of functions $\{v_y\}_{y\in Y}$ is an orthonormal basis of $L^2(K_jZ\backslash G,\chi)$.

For $w,y\in Y$ we consider the matrix coefficient [5]

$$
\begin{aligned}
(v_w, T_f v_y) &= \int_{G/Z}\int_{K_j}\int_{G/Z} \overline{v_w(g)} f(gkx^{-1})v_y(x)dx^*dkdg^*\\
&= \int_{K_jZw/Z}\int_{K_j}\int_{K_jZy/Z} \overline{v_w(g)} f(gkx^{-1})v_y(x)dx^*dkdg^*\\
&= \int_{K_j} f(wky^{-1})dk.
\end{aligned} \tag{3.4}
$$

To obtain a formula for trace(T_f) we have only to take $w=y$ and to sum over $y\in Y$. We obtain

$$
\operatorname{trace}(T_f) = \int_{G/Z}\int_{K_j} f(yky^{-1})dk\,dy^*, \tag{3.5}
$$

provided this integral is finite.

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[4] The Plancherel theorem for the case $l=1$ is proved in [8, §5.5.4] by a method which gives both the direct sum decomposition of $C_c^\infty(G,\chi)$ implied by the Bernstein center theory and an operator-valued Paley-Wiener theorem.

[5] The "overline" in the integrals denotes complex conjugation.

Theorem 5. *For any* $f \in {}^0\mathcal{C}(G, \chi, j)$

$$\int_{G/Z} \left| \int_{K_j} f(yky^{-1})dk \right| dy^* < \infty. \tag{3.6}$$

Proof. Using the set of representatives F chosen in §2.2 for $M_0^+/{}^0M_0Z$, we may write

$$\int_{G/Z} \left| \int_{K_j} f(yky^{-1})dk \right| dy^* \tag{3.7}$$

$$\approx \sum_{m \in F} \int_{K \times K} \left| \int_{K_j} \delta_{P_0}(m) f(k_1 m^{-1} k_2^{-1} k k_2 m k_1^{-1})\, dk \right| dk_1\, dk_2,$$

assuming all measures normalized. Since K normalizes K_j, we may omit the integration with respect to k_2. Similarly, we may integrate out K on the left or simply show that the integral is finite for each of finitely many representatives of K/K_j. In either case, it is enough to prove that

$$\sum_{m \in F} \left| \int_{K_j} \delta_{P_0}(m) f(m^{-1}km)dk \right| < \infty. \tag{3.8}$$

Since $H(M_0^+)$ is non-negative on $\Sigma^0(P_0, A_0)$ under the standard pairing ([8, Ch. 0]), we may partition F into subsets $S(P)$, (P, A) standard, such that $m \in S(P)$ if and only if $\langle \alpha, H(m) \rangle > 0$ for all $\alpha \in \Sigma^0(P, A)$, the set of simple roots which restrict non-trivially to A. For $m \in S(P)$ we have

$$H(m) = \sum_{\alpha \in \Sigma^0(P,A)} n_\alpha \lambda_\alpha, \tag{3.9}$$

where λ_α denotes the weight dual to $\alpha \in \Sigma^0(P, A)$ and the coefficients $n_\alpha = n_\alpha(m)$ are positive integers. Thus, excluding the finite subset of F which is orthogonal to every positive root, we have for each representative $m \in F$ a unique proper, standard parabolic subgroup $P(m) = P = MN$ and its opposite $P = M\bar{N}$ such that

(3.10)
$$\int_{K_j} f(m^{-1}km)dk = \int_{N_j} \int_{K_j^M} \int_{\bar{N}_j} f(m^{-1}n_j m \cdot m^{-1}m_j m \cdot m^{-1}\bar{n}_j m) d\bar{n}_j dm_j dn_j,$$

where $m^{-1}N_j m \subset K_j$, $m^{-1}K_j^M m$ lies in a compact subset of M which is independent of $m \in S(P)$, and

$$\int_{\bar{N}_j} f(m^{-1}m_j m \cdot m^{-1}\bar{n}_j m) d\bar{n}_j = \delta_{P_0}(m)^{-1} \int_{m^{-1}\bar{N}_j m} f(m^{-1}m_j m \cdot \bar{n}) d\bar{n} \tag{3.11}$$

with respect to a normalized measure on $\bar{N}$. Thus proving (3.8) may be reduced to checking that, for each standard pair (P, A),

$$\sum_{m \in S(P)} \left| \int_{K_j^M} \int_{m^{-1}\bar{N}_j m} f(m^{-1}m_j m \cdot \bar{n}) d\bar{n} dm_j \right| < \infty. \tag{3.12}$$

To prove (3.12) we first consider each integral separately. We use the hypothesis that $f \in {}^0\mathcal{C}(G)$, which implies that

$$\int_{\bar{N}} f(m^{-1}m_j m\bar{n})d\bar{n} = 0, \tag{3.13}$$

to make the substitution

$$\int_{m^{-1}\bar{N}_j m} f(m^{-1}m_j m \cdot \bar{n})d\bar{n} = -\int_{\bar{N}-m^{-1}\bar{N}_j m} f(m^{-1}m_j m \cdot \bar{n})d\bar{n}. \tag{3.14}$$

Since $m^{-1}K_j^M m$ lies in a fixed compact set independent of $m \in S(P)$ and f is K_j–invariant, we know that the absolute convergence of $\int_{\bar{N}} f(m\bar{n})d\bar{n}$ implied by [8, Theorem 4.4.3] is uniform with respect to the compact set containing $m^{-1}m_j m$. Thus we may omit $m^{-1}m_j m$ from the integral; for any $f \in \mathcal{C}(G)$, it is enough to estimate

$$\int_{\bar{N}-m^{-1}\bar{N}_j m} |f(\bar{n})|d\bar{n}. \tag{3.15}$$

We want to express this estimate in such a form that we may show that the sum over $m \in S(P)$ converges.

By [8, Theorem 4.2.1] and the Theorem of Tits [8, p.160], we have

$$\Xi(\bar{n}) \prec \delta_{P_0}(\bar{n})^{-1/2}(1+\sigma(\bar{n}))^{r_0} \tag{3.16}$$

for some $r_0 > 0$. If $f \in \mathcal{C}(G)$, then

$$|f(\bar{n})| \prec \Xi(\bar{n})(1+\sigma(\bar{n}))^{-r} \tag{3.17}$$

for any $r > 0$. Thus,

$$|f(\bar{n})| \prec \delta_{P_0}(\bar{n})^{-1/2}(1+\sigma(\bar{n}))^{-r} \tag{3.18}$$

for arbitrarily large positive r. On the other hand, [8, Theorem 4.3.7] implies that

$$\int_{\bar{N}} \delta_{P_0}(\bar{n})^{-1/2}(1+\sigma(\bar{n}))^{-r}d\bar{n} < \infty \tag{3.19}$$

for sufficiently large r. Therefore, certainly

$$\int_{\bar{N}-m^{-1}\bar{N}_j m} |f(\bar{n})|d\bar{n} < \infty \tag{3.20}$$

for any standard (P, A) with opposite $(\bar{P}, A)$ $(\bar{P} = M\bar{N})$ and for any $m \in S(P)$.

To show that

$$\sum_{m \in S(P)} \int_{\bar{N}-m^{-1}\bar{N}_j m} |f(\bar{n})|d\bar{n} < \infty \tag{3.21}$$

it is sufficient to show that for a sufficiently large choice of $r > 0$

$$\sum_{m \in S(P)} \int_{\bar{N}-m^{-1}\bar{N}_j m} \delta_{P_0}(\bar{n})^{-1/2}(1+\sigma(\bar{n}))^{-r}d\bar{n} < \infty. \tag{3.22}$$

It is obvious that the open compact set $m^{-1}\bar{N}_j m$, which may be viewed as a product of fractional ideals in some cartesian product of Ω's, expands as $m \in S(P)$ varies toward infinity in the chamber, and that

$$\bar{N} \subset \bigcup_{m \in S(P)} m^{-1}\bar{N}_j m. \tag{3.23}$$

We must show that the set $m^{-1}\bar{N}_j m$ expands rapidly enough to allow convergence of the sum.

Let

$$\bar{n} = \kappa(\bar{n})\mu_0(\bar{n})\eta_0(\bar{n}), \tag{3.24}$$

where $\kappa(\bar{n}) \in K$, $\mu_0(\bar{n}) \in M_0$, and $\eta_0(\bar{n}) \in N_0$. By [8, Corollary 4.3.2] we may without loss of generality assume that

$$\sigma(\bar{n}) = \langle \rho_P, H(\mu_0(\bar{n}))\rangle; \tag{3.25}$$

it follows from remarks on [8, p. 161] that $\langle \rho_P, H(\mu_0(\bar{n}))\rangle$ is a non-negative integer. For $m \in S(P)$ the integer $\langle \rho_P, H(m)\rangle$ is positive. Thus, by [8, Lemma 4.3.4],

$$\langle \rho_P, H(\mu_0(m^{-1}\bar{n}m))\rangle > \langle \rho_P, H(\mu_0(\bar{n}))\rangle \tag{3.26}$$

for all $\bar{n}$ outside a neighborhood of the identity in $\bar{N}$, a neighborhood which depends upon $m \in S(P)$. To complete the proof that the sum converges we need the following simple observation:

Lemma 2. *For $n > 0$ let $T(n)$ be the set of p–tuples of positive integers $(i_1, \cdots, i_p)$ such that $i_1 + \cdots + i_p = n$. Then $|T(n)| = O(n^{p-1})$.*

Proof. This is of course well known and easy to prove by an induction which we leave to the reader. □

Write $p = \text{p-rank}(M)$. Certainly,

$$\sum_{m\in S(P)} \int_{\bar{N}-m^{-1}\bar{N}_j m} \delta_{P_0}(\bar{n})^{-1/2}(1+\sigma(\bar{n}))^{-r} d\bar{n} \tag{3.27}$$

$$= \int_{\bar{N}} \delta_{P_0}(\bar{n})^{-1/2} C(\bar{n})(1+\sigma(\bar{n}))^{-r} d\bar{n},$$

where $C(\bar{n})$ is the cardinality of the set

$$\{m \in S(P) \ : \ \bar{n} \notin m^{-1}\bar{N}_j m\}. \tag{3.28}$$

It follows from Lemma 2 and (3.9) that

$$C(\bar{n}) = O(\sigma(\bar{n})^p). \tag{3.29}$$

Therefore,

$$\int_{\bar{N}} \delta_{P_0}(\bar{n})^{-1/2} C(\bar{n})(1+\sigma(\bar{n}))^{-r} d\bar{n} < \infty, \tag{3.30}$$

provided that r is large enough that

$$\int_{\bar{N}} \delta_{P_0}(\bar{n})^{-1/2}(1+\sigma(\bar{n}))^{-r+p} d\bar{n} < \infty. \tag{3.31}$$

This implies (3.22) and Theorem 5. □

4. Harish-Chandra's Theorem 11 and the Plancherel Theorem

In this section we show that Theorems 1 and 2 imply a version of [6, Theorem 11]. The version we prove is ostensibly weaker than the statement of Harish-Chandra's paper, but it is actually the form of the theorem that Harish-Chandra applies in [6, §14]. This author has found it technically cumbersome to derive Harish-Chandra's Theorem 11 from our Theorem 6; it is an exercise which requires no significant new ideas, so we include only a proof of Theorem 6. Finally we should remark that our proof of Theorem 6 merely adapts the proof of [4, Theorem 9].

Theorem 6. *Fix $\theta \in {}^0\mathcal{C}(G,\bar{\chi})$. Then*

$$\int_{G/Z}\left|\int_{G/Z} f(x)\theta(yxy^{-1})dx^*\right| dy^* < \infty$$

for $f \in \mathcal{C}(G,\chi)$. Put

$$\Theta(f) = \int_{G/Z}\int_{G/Z} f(x)\theta(yxy^{-1})dx^* dy^* \qquad (f \in \mathcal{C}(G,\chi)).$$

Then Θ is a continuous linear functional on $\mathcal{C}(G,\chi)$.

Proof. The integral under the absolute value sign exists for any $y \in G$ because $\mathcal{C}(G,\chi) \subset L^2(G/Z)$. Theorem 1 implies that θ is a finite linear combination of matrix coefficients of discrete series representations of G which have $\bar{\chi}$ as central character. Let (π, V) be any discrete series representation of G with $\bar{\chi}$ as central character; it is enough to check the convergence of the integral for

$$\theta(x) = (\pi(x)\phi, \psi),$$

where $(\ ,\ )$ denotes the inner product on V and $\phi, \psi \in V$. In this case,

$$\begin{aligned}\int_{G/Z} f(x)\theta(yxy^{-1})dx^* &= \int_{G/Z}(f(x)\pi(yxy^{-1})\phi,\psi)dx^* \\ &= \int_{G/Z}(f(x)\pi(x)\pi(y^{-1})\phi, \pi(y^{-1})\psi)dx^* \\ &= (\pi(f)\pi(y^{-1})\phi, \pi(y^{-1})\psi),\end{aligned}$$

where $\pi(f) = \int_{G/Z} f(x)\pi(x)dx^*$ is the Fourier transform with respect to π of the function $f \in \mathcal{C}(G,\chi)$. Since $f(kgk') = f(g)$ for all $g \in G$ and all $k, k' \in K_j$ for some j and since π is admissible, there is a finite orthonormal basis $\{\phi_s\}_{s=1}^l$ for the subspace $V^{K_j} \subset V$ on which $\pi|K_j = 1_{K_j}$; therefore,

$$\pi(f)\phi_s = \sum_{r=1}^{l} q_{rs}\phi_r \qquad (1 \leq s \leq l).$$

Therefore,

$$\begin{aligned}(\pi(f)\pi(y^{-1})\phi, \pi(y^{-1})\psi) &= \sum_{r,s}(\pi(f)(\pi(y^{-1})\phi,\phi_s)\phi_s, (\pi(y^{-1})\psi,\phi_r)\phi_r) \\ &= \sum_{r,s} q_{rs}(\phi,\pi(y)\phi_s)\overline{(\psi,\pi(y)\phi_r)}.\end{aligned}$$

This is a finite sum and the matrix coefficients of π belong to $L^2(G/Z)$, so

$$\begin{aligned}
&\int_{G/Z}\left|\int_{G/Z} f(x)\theta(yxy^{-1})dx^*\right| dy^* \\
&= \int_{G/Z}\left|\sum_{r,s} q_{rs}(\phi,\pi(y)\phi_s)\overline{(\psi,\pi(y)\phi_r)}\right| dy^* \\
&\leq \sum_{r,s}|q_{rs}|\int_{G/Z}|(\phi,\pi(y)\phi_s)\overline{(\psi,\pi(y)\phi_r)}|dy^* \\
&\leq \sum_{r,s}|q_{rs}|\left(\int_{G/Z}|(\phi,\pi(y)\phi_s)|^2dy^*\right)^{\frac{1}{2}}\left(\int_{G/Z}|(\psi,\pi(y)\phi_r)|^2dy^*\right)^{\frac{1}{2}} \\
&\leq d(\pi)^{-1}\|\phi\|\,\|\psi\|\sum_{r,s}|q_{rs}| \\
&< \infty,
\end{aligned}$$

where $d(\pi)$ is the formal degree of π with respect to the measure dy^*. As in [4, Theorem 9], using the defining relations for the formal degree, we also obtain an explicit formula:

$$\begin{aligned}
\Theta(f) &= \int_{G/Z}\int_{G/Z} f(x)\theta(yxy^{-1})dx^*dy^* \\
&= \int_{G/Z}\sum_{r,s} q_{rs}(\phi,\pi(y)\phi_s)\overline{(\psi,\pi(y)\phi_r)}dy^* \\
&= d(\pi)^{-1}(\phi,\psi)\sum_{r=1}^{l} q_{rr} \\
&= d(\pi)^{-1}(\phi,\psi)\mathrm{trace}(\pi(f)).
\end{aligned}$$

Since the character of a discrete series representation is a tempered distribution, it follows that the functional Θ is continuous with respect to the topology of $\mathcal{C}(G,\chi)$. □

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