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CONGRUENCES BETWEEN MODULAR FORMS, CYCLIC ISOGENIES OF MODULAR ELLIPTIC CURVES, AND INTEGRALITY OF p-ADIC L-FUNCTIONS

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ABSTRACT. Let Γ be a congruence subgroup of type (N_1, N_2) and of level N. We study congruences between weight 2 normalized newforms f and Eisenstein series E on Γ modulo a prime $\mathfrak{p}$ above a rational prime p. Assume that $p \nmid 6N$, E is a common eigenfunction for all Hecke operators and f is ordinary at $\mathfrak{p}$. We show that the abelian variety associated to f and the cuspidal subgroup associated to E intersect non-trivially in their p-torsion points. Let A be a modular elliptic curve over $\mathbb{Q}$ with good ordinary reduction at p. We apply the above result to show that an isogeny of degree divisible by p from the optimal curve A_1 in the $\mathbb{Q}$-isogeny class of elliptic curves containing A to A extends to an étale morphism of Néron models over $\mathbb{Z}_p$ if $p > 7$. We use this to show that p-adic distributions associated to the p-adic L-functions of A are $\mathbb{Z}_p$-valued.

INTRODUCTION

Let A be a modular elliptic curve of conductor N defined over $\mathbb{Q}$ and let $p > 2$ be a prime at which A has good ordinary reduction. Let Δ be a positive integer prime to p. In [9], Mazur and Swinnerton-Dyer construct, using modular symbols, an $H_1(A, \mathbb{Z}) \otimes \mathbb{Q}_p$-valued measure $\mu_{A,\Delta}$ on $\mathbb{Z}_{p,\Delta}^* = \varprojlim (\mathbb{Z}/p^n\Delta\mathbb{Z})^*$ (inverse limit being with respect to natural maps) associated to A, and define the p-adic L-function of A to be the p-adic Mellin transform of $\mu_{A,\Delta}$ which interpolates the values of the complex L-functions $L(A, \chi, z)$ at $z = 1$ for Dirichlet characters χ of conductor $p^n\Delta$, $n \geq 0$. In the light of Iwasawa theory, one would expect $\mu_{A,\Delta}$ to be $H_1(A, \mathbb{Z}) \otimes \mathbb{Z}_p$-valued. This is known when $\Delta = 1$.

Let $\pi : X_0(N) \to A$ be a modular parametrization (i.e. a non-constant $\mathbb{Q}$-morphism) which sends the cusp ∞ to the origin of A. Let ω_A be a Néron differential on A. Then $\pi^*\omega_A = c(\pi)f(q)dq/q$, where $c(\pi) \in \mathbb{Q}^*$ and $f(q)dq/q$ is a normalized newform on $\Gamma_0(N)$. $c(\pi)$ is called the Manin constant of π and is conjectured by Manin to be ± 1 when A is strong Weil [8]. Stevens in [23] studies parametrizations $\pi : X_1(N) \to A$ (which send the cusp 0 $(= \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right])$ to the origin) and refines Manin's conjecture as follows.

Conjecture 0.1. ([23, Conj. I]) *For every modular elliptic curve A over $\mathbb{Q}$, there is a modular parametrization $X_1(N) \to A$ whose Manin constant is ± 1.*

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In [12], p-adic distributions on $\mathbb{Z}_{p,\Delta}^*$ are constructed more generally for modular forms of weight ≥ 2. In the case of f arising from A as above, the construction yields an $\mathcal{L}(A) \otimes \mathbb{Q}_p$-valued measure $\nu_{A,\Delta}$ where $\mathcal{L}(A)$ is the period lattice of A with respect to ω_A (cf. [23, §4]). If $\Delta = 1$, $\mu_{A,\Delta}$ and $\nu_{A,\Delta}$ coincide (up to a p-adic unit) under the identification $H_1(A, \mathbb{Z}) \cong \mathcal{L}(A)$ but may differ in general (due to the p-adic "multiplier" [12, §14]). It is known ([23, Thms. 1.6, 4.6]) that $c(\pi) \in \mathbb{Z}$ and $c(\pi)\nu_{A,\Delta}$ is $\mathcal{L}(A) \otimes \mathbb{Z}_p$-valued for any modular parametrization $\pi : X_1(N) \to A$. Thus if Conjecture 0.1 is true, then $\nu_{A,\Delta}$ is $\mathcal{L}(A) \otimes \mathbb{Z}_p$-valued. We study in this paper the integrality of $\nu_{A,\Delta}$ and prove the following (cf. [23, §4, Conj. IV])

Theorem 0.2. *With the above notation and assumptions, $\nu_{A,\Delta}$ is $\mathcal{L}(A) \otimes \mathbb{Z}_p$-valued for $p > 7$.*

To prove Theorem 0.2, it suffices to show $p \nmid c(\pi)$ for some modular parametrization $\pi : X_1(N) \to A$. Let $\mathcal{A}$ be the $\mathbb{Q}$-isogeny class of elliptic curves over $\mathbb{Q}$ containing A. Then there are a unique curve (up to $\mathbb{Q}$-isomorphism) A_1 in $\mathcal{A}$ and a modular parametrization $\pi_1 : X_1(N) \to A_1$ such that if $\pi : X_1(N) \to A'$ is a modular parametrization of a curve $A' \in \mathcal{A}$, then there is a $\mathbb{Q}$-isogeny $\beta : A_1 \to A'$ such that $\pi = \beta \circ \pi_1$. We call A_1 the optimal curve in $\mathcal{A}$ and π_1 an optimal parametrization. By a result analogous to [11, Cor. 4.1], $c(\pi_1) \in \mathbb{Z}[1/2n]^*$ where n is the largest square dividing N. (Mazur proves this for $X_0(N)$-parametrizations of strong Weil curves, but his method works also for $X_1(N)$-parametrizations of optimal curves.) In particular, $p \nmid c(\pi_1)$. The next step is to look at $\mathbb{Q}$-isogenies between A_1 and A. In this direction, we prove

Theorem 0.3. *Suppose $p > 7$. Let $\beta : A_1 \to A$ be a cyclic $\mathbb{Q}$-isogeny of degree divisible by p. Then β is étale at p.*

Here we say that an isogeny $\alpha : E_1 \to E_2$ of elliptic curves over $\mathbb{Q}$ is étale at p if the morphism $\alpha_{/\mathbb{Z}_p} : E_{1/\mathbb{Z}_p} \to E_{2/\mathbb{Z}_p}$ of Néron models over $\mathbb{Z}_p$ is étale. The proof of Theorem 0.3 relies on Theorem 0.4 below, which reflects a general principle in the theory of modular curves that whenever there is a congruence between two modular forms, there should be a fusion module which explains the congruence. Let Γ be a congruence group of type (N_1, N_2) and of level N ([21, 1.1]). Let f (resp. E) be a weight two normalized newform (resp. an Eisenstein series) on Γ. Assume that E is a common eigenfunction for all Hecke operators and that there is a place $\mathfrak{P}$ of $\overline{\mathbb{Q}}$ such that the Fourier coefficients of f and E are congruent mod $\mathfrak{P}$. We say that $\mathfrak{P}$ is an Eisenstein prime for E and f. Let A_f be the abelian subvariety over $\mathbb{Q}$ of the Jacobian J_Γ of the modular curve associated to Γ ([18, Thm. 7.14]) and K_f the field generated over $\mathbb{Q}$ by the Fourier coefficients of f. There is an embedding of K_f into $\mathrm{End}(A_f) \otimes_{\mathbb{Z}} \mathbb{Q}$ (*loc. cit.*). Let C_E be the cuspidal subgroup of J_Γ associated to E ([21, 1.8]). Let $\mathfrak{p}$ be the prime of K_f below $\mathfrak{P}$. We say that f is ordinary at $\mathfrak{p}$ if the p-th Fourier coefficient of f is a unit mod $\mathfrak{p}$. By considering the q-expansions in characteristic p of $A_f[\mathfrak{p}]$ and $C_E[p]$, we prove

Theorem 0.4. *Let $\mathfrak{P}$ be an Eisenstein prime for f and E. Assume that $\mathfrak{P} \nmid 6N$ and f is ordinary at $\mathfrak{p}$. Then $A_f[\mathfrak{p}] \cap C_E \neq 0$.*

Returning to the proof of Theorem 0.3, we let $\beta : A_1 \to A$ be a cyclic isogeny of degree divisible by p. Assuming that the Galois character on the subgroup of order p of $\ker \beta$ is not the trivial character or the Teichmüller character, we show that there is an Eisenstein series E on $\Gamma_1(N)$ arising from the Galois representation on

the p-torsion points of A_1 whose Fourier coefficients are congruent mod $\mathfrak{P}$ to those of f (Prop. 2.6). Theorem 0.4 shows that A_1 and the cuspidal subgroup of the Jacobian of $X_1(N)$ associated to E intersect non-trivially in their p-torsion points. Using the classification theorem of rational cyclic isogenies of elliptic curves over $\mathbb{Q}$ ([11], [7]), we deduce that $p \leq 7$ (§2.3).

We prove Theorems 0.4, 0.3 and 0.2 in Sections 1, 2 and 3 respectively . For a field K, we write $\overline{K}$ for an algebraic closure of K, K_s for the separable closure of K in $\overline{K}$ and G_K for $\mathrm{Gal}(K_s/K)$. Throughout the paper, we fix a place $\mathfrak{P}$ of $\overline{\mathbb{Q}}$ above p and identify the residue field of the valuation ring of $\overline{\mathbb{Q}}$ at $\mathfrak{P}$ as an algebraic closure $\overline{\mathbb{F}}_p$ of $\mathbb{F}_p$. Also fix an embedding of $\overline{\mathbb{Q}}$ into $\mathbb{C}_p$ so that $\mathfrak{P}$ is the place induced by it. For a prime l, D_l will denote a decomposition group in $G_{\mathbb{Q}}$ for l and I_l its inertia subgroup. For a commutative group scheme G and α an endomorphism of G, $G[\alpha]$ will denote the kernel of α and $G[\alpha^\infty] = \bigcup_{n\geq 0} G[\alpha^n]$. If $\mathfrak{a}$ is a set of endomorphisms of G, then $G[\mathfrak{a}]$ will denote $\bigcap_{\alpha\in\mathfrak{a}} G[\alpha]$. When $G = \mathbb{G}_m$ is the multiplicative group, we write μ_n for $\mathbb{G}_m[n]$. We shall use the following notation for cusps: if Γ is a congruence subgroup of level N and $a,\ b \in \mathbb{Z}$ are such that $(a, b, N) = 1$, then $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] = \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]_\Gamma$ will denote the Γ-equivalence class of the cusp a/b.

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1. Fusion module of an Eisenstein prime

In this section, we prove Theorem 0.4. Let Γ be a congruence subgroup of type (N_1, N_2) and of level N. Let f (resp. E) be a weight two normalized newform (resp. an Eisenstein series) on Γ. Suppose that E is a common eigenfunction for all Hecke operators. Let a_n $(n \geq 0)$ (resp. b_n $(n \geq 0)$) be the Fourier coefficients of f (resp. E). (We take $a_0 = 0$.) Assume that $\mathfrak{P}$ is an Eisenstein prime associated to f and E, i.e. $a_n \equiv b_n \pmod{\mathfrak{P}}$ for all $n \geq 0$. We write $f \equiv E \pmod{\mathfrak{P}}$. Assume further that $\mathfrak{P} \nmid 6N$ and f is ordinary at $\mathfrak{p}$. We first review some properties of regular differentials on modular curves and their q-expansions (§1.2) and the q-expansions of the p-torsion points of the Jacobian of a curve over $\overline{\mathbb{F}}_p$ (§1.3). We show that the q-expansions in characteristic p of $A_f[\mathfrak{p}]$ and the cuspidal subgroup $C_E[p]$ coincide. This enables us to conclude that $A_f[\mathfrak{p}]$ and C_E intersect non-trivially.

1.1. **Modular curves and Hecke operators.** Let $\mathfrak{H}$ be the upper half plane and $\mathfrak{H}^* = \mathfrak{H} \cup \mathbb{P}^1(\mathbb{Q})$. The quotient $\mathfrak{H}/\Gamma$ is an open Riemann surface which can be compactified to a projective algebraic curve $X_{\Gamma/\mathbb{C}} = \mathfrak{H}^*/\Gamma$ over $\mathbb{C}$ by the addition of cusps. By Shimura [18, §6.7], $X_{\Gamma/\mathbb{C}}$ has a canonical model X_Γ over $\mathbb{Q}$. The moduli interpretation of X_Γ is that it is the coarse moduli scheme of the functor associating to each $\mathbb{Q}$-scheme S the S-isomorphism classes of generalized elliptic curves over S with an H-orbit of level N-structures where H is the image of Γ under the natural map $\Gamma \to \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$. Write $X(N)$, $X_0(N)$, $X_1(N)$ for $\Gamma = \Gamma(N)$, $\Gamma_0(N)$, $\Gamma_1(N)$ respectively.

Let T_l $(l \nmid N)$ and U_l $(l|N)$ be the usual Hecke correspondences on X_Γ (see for example [18, Chap. 7]). For $m \in (\mathbb{Z}/N\mathbb{Z})^*$, let $\sigma_m \in \mathrm{SL}_2(\mathbb{Z})$ be such that

$$\sigma_m \equiv \begin{pmatrix} * & 0 \\ 0 & m \end{pmatrix} \pmod{N}$$

and write $\langle m\rangle$ for the Hecke correspondence corresponding to σ_m. Let J_Γ be the Jacobian of X_Γ and $\mathbb{T} \subset \mathrm{End}(J_\Gamma) \otimes_{\mathbb{Z}} \mathbb{Q}$ the Hecke algebra generated by the images of T_l ($l \nmid N$), U_l ($l|N$) and $\langle m\rangle$, $m \in (\mathbb{Z}/N\mathbb{Z})^*$. Denote the images of T_l, U_l and $\langle m\rangle$ in $\mathbb{T}$ by the same symbols. For brevity, we shall sometimes write T_l for U_l when $l|N$ below.

1.2. **Regular differentials and their q-expansions.** Let ζ_N be a primitive N-th root of unity. Let $\mathcal{O}$ be the completion of $\mathbb{Z}[\zeta_N]$ at the prime below $\mathfrak{P}$, and K the field of fractions of $\mathcal{O}$. Fix an embedding of K in $\mathbb{C}$. Let $X_{\Gamma/\mathcal{O}}$ be the normalization of the j-line $\mathbb{P}^1_{/\mathcal{O}}$ in the function field of $Y_{\Gamma/K}$, where the morphism $Y_{\Gamma/K} \to \mathbb{P}^1_{/\mathcal{O}}$ is defined on points by sending an elliptic curve E with level H-structure to the j-invariant of E. Then $X_{\Gamma/\mathcal{O}}$ is smooth. For any ring R over $\mathcal{O}$, let $\Omega_{/R} = \Omega_{X_\Gamma/R}$ be the sheaf of regular differentials with respect to $X_{\Gamma/R} \to \mathrm{Spec}\, R$ ([5, I 2.1]) and $\Omega_{/R}(\mathrm{cusps}) = \Omega_{X_\Gamma/R}(\mathrm{cusps})$ the sheaf which, when restricted to the complement of the cuspidal sections, is the sheaf of regular differentials and whose sections in a neighborhood of the cuspidal sections are meromorphic differentials with at worst simple poles along those sections ([10, II 3]). We consider only rings R which are flat over $\mathcal{O}$ or $\mathcal{O}/p^n\mathcal{O}$ for some n.

Proposition 1.1. ([24, Prop. 6.1], [6, Prop. 5.1]) *Let $R \to R'$ be a morphism of rings which are flat over $\mathcal{O}$ or $\mathcal{O}/p^n\mathcal{O}$ for some n. Then*

$$\begin{aligned} H^0(X_{\Gamma/R}, \Omega_{/R}) \otimes_R R' &\cong H^0(X_{\Gamma/R'}, \Omega_{/R'}), \\ H^0(X_{\Gamma/R}, \Omega_{/R}(\mathrm{cusps})) \otimes_R R' &\cong H^0(X_{\Gamma/R'}, \Omega_{/R'}(\mathrm{cusps})). \end{aligned}$$

We consider the q-expansions of regular differentials. Since $H^0(X_{\Gamma/R}, \Omega_{/R}) = H^0(X(N)_{/R}, \Omega_{/R})^H$ by [5, VII 3.3], we can restrict ourselves to the case $\Gamma = \Gamma(N)$. Let $\mathrm{Tate}(q)$ be the Tate curve with N-sides over $\mathcal{O}[\![q^{\frac{1}{N}}]\!]$. $\mathrm{Tate}(q)$ with Drinfeld basis $(\zeta_N, q^{\frac{1}{N}})$ defines a point on $X_{\Gamma/\mathcal{O}}$, and the corresponding morphism

$$\tau_{/\mathcal{O}} : \mathrm{Spec}\, \mathcal{O}[\![q^{\frac{1}{N}}]\!] \to X_{\Gamma/\mathcal{O}}$$

can be identified with the formal completion of $X_{\Gamma/\mathcal{O}}$ along the section corresponding to the cusp ∞ ($= \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$) ([5, VII 2.4]). For any R as above, we then have a morphism

$$\tau_{/R} : \mathrm{Spec}\, R[\![q^{\frac{1}{N}}]\!] \to X_{\Gamma/R}.$$

For any $\omega \in H^0(X_{\Gamma/R}, \Omega_{/R})$, we define the q-expansion of ω at ∞ to be the element $\varphi_R(\omega) \in R[\![q^{\frac{1}{N}}]\!]$ such that

$$\tau^*_{/R}\omega = \varphi_R(\omega) dq^{\frac{1}{N}}/q^{\frac{1}{N}}.$$

This defines the *q-expansion morphism* $\varphi_R : H^0(X_{\Gamma/R}, \Omega_{/R}) \to R[\![q^{\frac{1}{N}}]\!]$. Similarly, there is a q-expansion map $\varphi_R : H^0(X_{\Gamma/R}, \Omega_{/R}(\mathrm{cusps})) \to R[\![q^{\frac{1}{N}}]\!]$. Let $B^0(\mathcal{O})$ (resp. $B(\mathcal{O})$) be the submodule of $\mathcal{O}[\![q^{\frac{1}{N}}]\!]$ consisting of the q-expansions at ∞ of cusp forms (resp. holomorphic modular forms) of weight 2 on Γ with coefficients in $\mathcal{O}$. For any R as above, let $B^0(R) = B^0(\mathcal{O}) \otimes R$ and $B(R) = B(\mathcal{O}) \otimes R$. One can show, using Prop. 1.1, that $\varphi_R(H^0(X_{\Gamma/R}, \Omega_{/R})) \subset B^0(R)$ ([24, Prop. 6.2]). Similarly, we have $\varphi_R(H^0(X_{\Gamma/R}, \Omega_{/R}(\mathrm{cusps}))) \subset B(R)$. So we have maps

$$\varphi_R : \quad H^0(X_{\Gamma/R}, \Omega_{/R}) \longrightarrow B^0(R), \tag{1.1}$$

$$\varphi_R : \quad H^0(X_{\Gamma/R}, \Omega_{/R}(\mathrm{cusps})) \longrightarrow B(R). \tag{1.2}$$

One can define an action of $\mathbb{T}$ on $H^0(X_{\Gamma/R}, \Omega_{/R}(\text{cusps}))$ using the definition of Hecke correspondences, and one on $B(R)$ by its action on the q-expansions of classical modular forms. With these actions, (1.1) and (1.2) are then $\mathbb{T}$-morphisms. Furthermore, if $g \in R[\![q^{\frac{1}{N}}]\!]$ with zero constant term is a common eigenvector for T_l $(l \nmid N)$ and U_l $(l|N)$ with eigenvalues c_l, then the usual recursive relations ([18, (3.5.12)]) show that g is determined by the c_l up to multiplication by a constant.

1.3. **q-expansions of p-torsion points.** Let X be a smooth projective curve defined over $\overline{\mathbb{F}}_p$ and let J be its Jacobian. Let Ω^1_X be the canonical sheaf of differentials on X and let $\mathcal{C}$ be the Cartier operator on $H^0(X, \Omega^1_X)$ ([3]). There is a canonical isomorphism (cf. [15, §11, Prop. 10])

$$\delta : J[p] \to H^0(X, \Omega^1_X)^{\mathcal{C}} \tag{1.3}$$

where $J[p] = \{x \in J(\overline{\mathbb{F}}_p) : px = 0\}$ and $H^0(X, \Omega^1_X)^{\mathcal{C}} = \{\omega \in H^0(X, \Omega^1_X) : \mathcal{C}\omega = \omega\}$. The definition of δ is as follows: if x in the domain is represented by a divisor D on $X_{/\overline{\mathbb{F}}_p}$ such that $pD = (g)$ where (g) is the divisor of g, then $\delta(x) = dg/g$.

Proposition 1.2. ([24, Prop. 6.5], [6, Prop. 5.2]) *Let J be the Jacobian of $X = X_{\Gamma/\overline{\mathbb{F}}_p}$ and let $\varphi_{\overline{\mathbb{F}}_p}$ be as in* (1.1). *Then $\varphi_{\overline{\mathbb{F}}_p} \circ (\delta \otimes 1)$ induces an injection $\varphi : J[p] \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p \hookrightarrow B^0(\overline{\mathbb{F}}_p)$ such that $\varphi \circ T_n^* = T_n \circ \varphi$ for all $n \geq 1$ where T_n is the n-th Hecke operator and T_n^* is the dual of the endomorphism of J which T_n induces by Pic functoriality.*

We call φ in Proposition 1.2 the q-expansion map of the p-torsion points of J. Let A_f be the abelian subvariety over $\mathbb{Q}$ of the Jacobian J_Γ of X_Γ associated to f ([18, Thm. 7.14]). Let K_f be the field generated over $\mathbb{Q}$ by the a_n and ι the embedding $K_f \hookrightarrow \text{End}(J_\Gamma) \otimes_{\mathbb{Z}} \mathbb{Q}$ from the construction of A_f (*loc. cit.*). Let $\mathfrak{p}$ be the prime of K_f below $\mathfrak{P}$.

Proposition 1.3. *Assume that $\mathfrak{p} \nmid N$ and f is ordinary at $\mathfrak{p}$. Then the image of $A_f[\mathfrak{p}]_{/\mathbb{F}_p}(\overline{\mathbb{F}}_p) \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p$ under φ is the $\overline{\mathbb{F}}_p$-module generated by the reduction $\overline{f}$ of $f(q)$ in $B^0(\overline{\mathbb{F}}_p)$.*

Proof. By [18, Thm. 7.14], A_f is stable under subrings of $\text{End}_{\mathbb{Q}}(J_\Gamma) = \text{End}(J_\Gamma) \otimes_{\mathbb{Z}} \mathbb{Q}$ induced by the Hecke correspondences T_n via Albanese functoriality and Pic functoriality which are related as follows. The endomorphism of J_Γ which T_n induces by Albanese functoriality is the endomorphism denoted ξ_n in [18, Chap. 7]. Its dual is the endomorphism of J_Γ which T_n induces by Pic functoriality. (For more details, see [14, p. 444].) By [18, Thm. 7.14(b)], $\xi_n x = \iota(a_n)x$ for all $x \in A_f$ and all n. Since J_Γ has good reduction at p, the morphism $A_{f/\mathbb{Z}_p} \to J_{\Gamma/\mathbb{Z}_p}$ is a closed immersion ([11, Prop. 1.2]), hence $A_f[p]_{/\mathbb{F}_p} \hookrightarrow J_\Gamma[p]_{/\mathbb{F}_p}$. Since f is ordinary at $\mathfrak{p}$, $A_f[\mathfrak{p}]_{/\mathbb{F}_p}(\overline{\mathbb{F}}_p)$ is a one-dimensional vector space over the residue field of $\mathfrak{p}$ (cf. [25, Thm. 2.2]). The action of $\iota(a_n)$ on $A_f[\mathfrak{p}]_{/\mathbb{F}_p}(\overline{\mathbb{F}}_p)$ is given by multiplication by a_n mod $\mathfrak{p}$. For any x in $A_f[\mathfrak{p}]_{/\mathbb{F}_p}(\overline{\mathbb{F}}_p)$ and $n \geq 1$, it follows from Prop. 1.2 and the above discussion that

$$T_n\varphi(x \otimes 1) = \varphi(T_n^* x \otimes 1) = \varphi(\iota(a_n)x \otimes 1) = a_n\varphi(x \otimes 1).$$

Now $\overline{f}$ is a common eigenvector for all T_n with eigenvalues a_n mod $\mathfrak{p}$. Thus $\varphi(x \otimes 1)$, $\overline{f} \in B^0(\overline{\mathbb{F}}_p)$ are two common eigenvectors of T_n for all n with the same eigenvalues. (Note that $B^0(\overline{\mathbb{F}}_p) \subset q\overline{\mathbb{F}}_p[\![q]\!]$.) So $\varphi(x \otimes 1) = c\overline{f}$ for some $c \in \overline{\mathbb{F}}_p^*$. This proves the proposition. □

Remark 1.4. Let $\sigma : K_f \hookrightarrow \mathbb{C}$ be an embedding. Let f^σ be the cusp form obtained by applying σ to the coefficients of f (cf. [18, Thm. 7.14]), A_f^σ the abelian subvariety of J_Γ associated to f^σ and $\mathfrak{p}^\sigma$ the σ-conjugate of $\mathfrak{p}$. Then a_p^σ is a unit mod $\mathfrak{p}^\sigma$, so f^σ is ordinary at $\mathfrak{p}^\sigma$. If σ extends to an element in the decomposition group $D_\mathfrak{P}$ for $\mathfrak{P}$ (we view K_f as a subfield of $\overline{\mathbb{Q}}$), then the same arguments as in Proposition 1.3 show that $\varphi\left(A_f^\sigma[\mathfrak{p}^\sigma]_{/\mathbb{F}_p}(\overline{\mathbb{F}}_p) \otimes \overline{\mathbb{F}}_p\right) = \overline{\mathbb{F}}_p \cdot \overline{f^\sigma}$.

1.4. **Cuspidal subgroups associated to Eisenstein series.** We recall some results in [21] and [22] which we need below (Props. 1.5, 1.6 and 1.7). Let $\mathcal{E}_\Gamma$ be the space of weight 2 Eisenstein series on Γ. For any $E \in \mathcal{E}_\Gamma$, let ω_E be the differential form on $\mathfrak{H}/\Gamma$ whose pull-back to $\mathfrak{H}$ is $E(z)dz$. Let $\mathcal{P}(E)$ be the image of

$$H_1(\mathfrak{H}/\Gamma, \mathbb{Z}) \to \mathbb{C}, \qquad \gamma \mapsto \int_\gamma \omega_E.$$

For any $\mathbb{Z}$-module $M \subset \mathbb{C}$, let $\mathcal{E}_\Gamma(M) = \{E \in \mathcal{E}_\Gamma : \mathcal{P}(E) \subset M\}$. We then have

Proposition 1.5. ([22, Prop. 1.1(a)]) *For any $\mathbb{Z}$-module $M \subset \mathbb{C}$, the natural map $\mathcal{E}_\Gamma(\mathbb{Z}) \otimes_\mathbb{Z} M \to \mathcal{E}_\Gamma(M)$ is an isomorphism.*

In [21, §1.8], Stevens showed how one can associate to an arbitrary $E \in \mathcal{E}_\Gamma$ a subgroup C_E of the cuspidal group C_Γ of J_Γ. The construction of C_E is as follows. Let $\mathbf{cusps} = \mathbf{cusps}(\Gamma)$ denote the set of cusps on X_Γ and $\mathrm{Div}^0(\mathbf{cusps})$ the group of divisors of degree zero supported on the cusps. For any $\mathbb{Z}$-module M, define $\mathrm{Div}^0(\mathbf{cusps}; M) = \mathrm{Div}^0(\mathbf{cusps}) \otimes_\mathbb{Z} M$. Let

$$\delta_\Gamma(E) = \sum_{x \in \mathbf{cusps}} r_x(E) \cdot x \quad \in \mathrm{Div}^0(\mathbf{cusps}; \mathbb{C})$$

where $r_x(E) = 2\pi i \,\mathrm{res}_x(\omega_E)$ and $\mathrm{res}_x(\omega_E)$ is the residue of ω_E at the cusp x. We note that by [22, Thm 1.3(a)],

$$r_{[\begin{smallmatrix} r \\ s \end{smallmatrix}]}(E) = e([\begin{smallmatrix} r \\ s \end{smallmatrix}]) \cdot a_0(E|\gamma_{[\begin{smallmatrix} r \\ s \end{smallmatrix}]}) \tag{1.4}$$

where $e([\begin{smallmatrix} r \\ s \end{smallmatrix}])$ is the ramification index of $[\begin{smallmatrix} r \\ s \end{smallmatrix}]$ over $X(1)$, $\gamma_{[\begin{smallmatrix} r \\ s \end{smallmatrix}]} \in \mathrm{SL}_2(\mathbb{Z})$ is such that $\gamma_{[\begin{smallmatrix} r \\ s \end{smallmatrix}]} \cdot i\infty$ represents $[\begin{smallmatrix} r \\ s \end{smallmatrix}]$ and $a_0(E|\gamma_{[\begin{smallmatrix} r \\ s \end{smallmatrix}]})$ is the constant term of the Fourier expansion of $E|\gamma_{[\begin{smallmatrix} r \\ s \end{smallmatrix}]}$. Let $\mathcal{R}(E)$ be the $\mathbb{Z}$-submodule of $\mathbb{C}$ generated by the coefficients of $\delta_\Gamma(E)$ and let

$$\mathcal{R}(E)^* = \{\eta \in \mathrm{Hom}_\mathbb{Q}(\mathcal{R}(E) \otimes_\mathbb{Z} \mathbb{Q}, \mathbb{Q}) : \eta(\mathcal{R}(E)) \subset \mathbb{Z}\}.$$

The subgroup C_E of C_Γ associated to E is by definition the image of the composition

$$\begin{array}{ccccc} \mathcal{R}(E)^* & \longrightarrow & \mathrm{Div}^0(\mathbf{cusps}) & \xrightarrow{\theta} & C_\Gamma, \\ \eta & \longmapsto & \eta(\delta_\Gamma(E)), & & \end{array} \tag{1.5}$$

where θ sends a divisor to its divisor class. Note that since $r_x(E)$ is the integral of ω_E along some cycle around x, $\mathcal{R}(E) \subset \mathcal{P}(E)$. Let $A_E = \mathcal{P}(E)/\mathcal{R}(E)$.

Proposition 1.6. ([22, Thm. 1.2(a)]) *For any $E \in \mathcal{E}_\Gamma$, there is a perfect duality $C_E \times A_E \to \mathbb{Q}/\mathbb{Z}$.*

Suppose $E \in \mathcal{E}_\Gamma$ is a common eigenfunction for all T_l and $\langle l \rangle$, $l \nmid N$. Then there are Dirichlet characters ϵ_1 and ϵ_2 modulo N such that $E|T_l = (\epsilon_1(l) + l\epsilon_2(l))E$ for each prime $l \nmid N$ ([21, 3.2.2, (3.2.3)]). We say that E has signature ϵ_1, ϵ_2. Let $\mathbb{Z}[\epsilon_1, \epsilon_2]$ (resp. $\mathbb{Q}[\epsilon_1, \epsilon_2]$) be the ring generated by the values of ϵ_1 and ϵ_2 over $\mathbb{Z}$ (resp. $\mathbb{Q}$). By [21, 3.2.1, 3.2.2], $\mathcal{P}(E)$ and $\mathcal{R}(E)$ are fractional ideals of

$\mathbb{Q}[\epsilon_1, \epsilon_2]$. Let $\mathfrak{o} = \mathbb{Z}[1/2, \epsilon_1, \epsilon_2]$, $\mathfrak{a} = \mathfrak{o} + \mathfrak{o}\mathcal{R}(E)$ and $\mathfrak{b}$ the $\mathfrak{o}$-module generated by $\{1, B_{1,\epsilon_1^{-1}}, B_{1,\epsilon_2}, S(\epsilon_1)B_{2,\epsilon_2}, S(\epsilon_2)B_{2,\epsilon_1^{-1}}\}$, where for $i = 1, 2$, $B_{i,-}$ are the generalized Bernoulli numbers and

$$S(\epsilon_i) = \begin{cases} \phi(N_i) & \text{if } \epsilon_i = 1, \\ 0 & \text{otherwise,} \end{cases}$$

and ϕ is the Euler function.

Proposition 1.7. ([21, Thm. 3.6.1]) *Let $E \in \mathcal{E}_\Gamma$ have signature ϵ_1, ϵ_2. Then $\mathfrak{a} \subset \mathfrak{o}\mathcal{P}(E) \subset \mathfrak{b}$.*

We next give a set of generators of the space over $\mathbb{Q}$ of weight 2 Eisenstein series E of level N with $\mathcal{R}(E) \subset \mathbb{Q}$. For fractional ideals $\mathfrak{a}_1$ and $\mathfrak{a}_2$ of $\mathbb{Q}$ and $a_1, a_2 \in \mathbb{Q}$, define

$$\begin{aligned} E(z, s) &= E(z, s; a_1, a_2; \mathfrak{a}_1, \mathfrak{a}_2) \\ &= -N(\mathfrak{a}_2)(2\pi)^{-2} \sum_{(m_1, m_2)} (m_1 z + m_2)^{-2} |m_1 z + m_2|^{-2s} \end{aligned} \tag{1.6}$$

for $z \in \mathfrak{H}$, $s \in \mathbb{C}$ with $\operatorname{Re} s > 2$, where the sum is over all pairs $(m_1, m_2) \in \mathbb{Q}^2 - (0, 0)$ such that $m_i \equiv a_i \pmod{\mathfrak{a}_i}$, $i = 1, 2$. For fixed z, $E(z, s)$ may be continued analytically to a meromorphic function in the s-plane which is holomorphic at $s = 0$ (cf. [19, §3]). Define

$$E(z) = E(z; a_1, a_2; \mathfrak{a}_1, \mathfrak{a}_2) = E(z, 0; a_1, a_2; \mathfrak{a}_1, \mathfrak{a}_2). \tag{1.7}$$

The Fourier expansion of $E(z)$ at ∞ is given by

$$-\delta(a_1, \mathfrak{a}_1)(2\pi)^{-2} N(\mathfrak{a}_2) \sum_{0 \neq d \equiv a_2 \pmod{\mathfrak{a}_2}} |d|^{-2} + \sum_{c \equiv a_1 \pmod{\mathfrak{a}_1}} \sum_{\substack{bc > 0 \\ b \in \mathfrak{a}_2^{-1}}} |b| e^{2\pi i(bcz + ba_2)} \tag{1.8}$$

where $\delta(a_1, \mathfrak{a}_1) = 1$ or 0 according as $a_1 \in \mathfrak{a}_1$ or not (cf. [19, (3.6)]). For any $(x, y) \in (N^{-1}\mathbb{Z}/\mathbb{Z})^2$, let

$$\phi_{(x,y)}(z) = N^{-2} \sum_{(a_1, a_2) \in (N^{-1}\mathbb{Z}/\mathbb{Z})^2} e^{2\pi i(a_2 x_1 - a_1 x_2)} E(z; a_1, a_2; \mathbb{Z}, \mathbb{Z}).$$

By Hecke (cf. [21, pp. 59-60]), $\{\phi_{(x,y)} : (x, y) \in (N^{-1}\mathbb{Z}/\mathbb{Z})^2 - (0, 0)\}$ spans the space of weight 2 Eisenstein series E of level N with $\mathcal{R}(E) \subset \mathbb{Q}$ over $\mathbb{Q}$. Another fact we need is that if U_Γ is the group of meromorphic functions on $X_{\Gamma/\mathbb{C}}$ whose divisors are supported on the cusps, then logarithmic differentiation gives an isomorphism

$$U_\Gamma/\mathbb{C}^* \xrightarrow{\sim} \mathcal{E}_\Gamma(\mathbb{Z}), \qquad g \mapsto \frac{1}{2\pi i} \frac{g'(z)}{g(z)}, \tag{1.9}$$

and if $E(z) = (2\pi i)^{-1} g'(z)/g(z)$, then $\delta_\Gamma(E) = (g)$ (cf. [22, §1]).

Lemma 1.8. *For any $E \in \mathcal{E}_\Gamma(\mathbb{Z})$, the Fourier coefficients of E at each cusp are in $(12N^2)^{-1}\mathbb{Z}$.*

Proof. By (1.9) and a theorem of Kubert (cf. [22, §3]), $2E$ is a $\mathbb{Z}$-linear combination of the $\phi_{(x,y)}$ with $(x, y) \in (N^{-1}\mathbb{Z}/\mathbb{Z})^2 - (0, 0)$. For any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, we have $\phi_{(x,y)}|\gamma = \phi_{(x,y)|\gamma}$, where $(x, y)|\gamma = (ax + cy, bx + dy)$ (cf. [21, 2.4.1(a)]). From [21, 2.4.2(a)], we see that the Fourier coefficients of $\phi_{(x,y)}$, $(x, y) \in (N^{-1}\mathbb{Z}/\mathbb{Z})^2 - (0, 0)$, at ∞ are in $(12N^2)^{-1}\mathbb{Z}$. □

Suppose now E is a common eigenfunction for T_l for all l and $\langle l\rangle$, $l \nmid N$, such that $\mathfrak{P}$ is an Eisenstein prime associated to f and E as at the beginning of §1.

Proposition 1.9. *Assume that $\mathfrak{p} \nmid 6N$ and f is ordinary at $\mathfrak{p}$. Then p divides the order of C_E.*

Proof. Let $\mathfrak{o}_1 = \mathbb{Z}[\epsilon_0', \epsilon_0\chi^{-1}]$ and $\mathfrak{o} = \mathfrak{o}_1[1/2]$. By Prop. 1.7, $N \cdot \mathcal{P}(E) \subset \mathfrak{o}$, so $E \in \mathcal{E}_\Gamma(N^{-1}\mathfrak{o})$. By Prop. 1.5, there exist $\lambda_i \in N^{-1}\mathfrak{o}$, $E_i \in \mathcal{E}_\Gamma(\mathbb{Z})$ such that $E = \sum_i \lambda_i E_i$. Since $\mathfrak{b} \subset N^{-1}\mathfrak{o}$, Prop. 1.7 shows that the non-integral part of $\mathfrak{o}\mathcal{P}(E)$ is prime to p. (By the integral (resp. non-integral) part of a fractional ideal, we mean the product of the factors in its prime decomposition which occur to a positive (resp. negative) exponent.) Since $\mathfrak{o}\mathcal{R}(E) \subset \mathfrak{o}\mathcal{P}(E)$, the non-integral part of $\mathfrak{o}\mathcal{R}(E)$ is prime to p also. We suppose $p \nmid \#C_E$ and derive a contradiction. By Prop. 1.6, $p \nmid \#(\mathcal{P}(E)/\mathcal{R}(E))$, so $p \nmid \#(\mathfrak{o}\mathcal{P}(E)/\mathfrak{o}\mathcal{R}(E))$. Since $\mathfrak{a} \subset \mathfrak{o}\mathcal{P}(E)$ by Prop. 1.7, $p \nmid \#(\mathfrak{o} + \mathfrak{o}\mathcal{R}(E))/\mathfrak{o}\mathcal{R}(E) = \#\mathfrak{o}/(\mathfrak{o} \cap \mathfrak{o}\mathcal{R}(E))$. This implies that the integral part of $\mathfrak{o}\mathcal{R}(E)$ is prime to p. Hence $\mathfrak{o}\mathcal{R}(E)$ is prime to p.

Let $\wp = \mathfrak{P} \cap \mathfrak{o}_1$ and let $\overline{\eta}_0$ be the composite homomorphism $\mathfrak{o}_1 \to \mathfrak{o}_1/\wp \xrightarrow{\mathrm{tr}} \mathbb{F}_p$ where tr is the trace map from $\mathbb{F} := \mathfrak{o}_1/\wp$ to $\mathbb{F}_p$. Since $\mathfrak{o}_1$ is projective over $\mathbb{Z}$, we can lift $\overline{\eta}_0$ to a surjective homomorphism $\eta_0 : \mathfrak{o}_1 \to \mathbb{Z}$. Extend η_0 $\mathbb{Q}$-linearly to a map $\mathcal{R}(E)\otimes\mathbb{Q} \to \mathbb{Q}$, denoted η_0 again. Since $\mathcal{R}(E)$ is prime to p, there exists $a \in \mathbb{Z}$, $(a, p) = 1$, such that $a\mathcal{R}(E) \subset \mathfrak{o}_1$. Since $a\mathcal{R}(E)$ is prime to p, $\eta_0(a\mathcal{R}(E)) = m\mathbb{Z}$ for some $m \in \mathbb{Z}$ prime to p. Let $\eta = \frac{a}{m}\eta_0$. Then $\eta \in \mathcal{R}(E)^*$ satisfies $\eta(\mathcal{R}(E)) = \mathbb{Z}$, $\eta(\mathfrak{o}) = \frac{a}{m}\mathbb{Z}[1/2]$ and $\eta(\wp) \subset \frac{p}{m}\mathbb{Z}$. So $\eta(N^{-1}\mathfrak{o}) = \frac{a}{mN}\mathbb{Z}[1/2]$, and we can choose $n \in \mathbb{Z}$, $p \nmid n$, such that $n_i = n\eta(\lambda_i) \in \mathbb{Z}$ for all i. Let

$$\eta(E) = \sum_i \eta(\lambda_i)E_i.$$

Then

$$\eta(E)(q) = \sum_i \eta(\lambda_i)\, E_i(q) = \eta(\sum_i \lambda_i\, E_i(q)) = \eta(E(q)),$$

where $\eta(E(q))$ is the q-series obtained by applying η to the coefficients of $E(q)$. Since $f \equiv E \pmod{\mathfrak{P}}$,

$$\eta(E)(q) \equiv \mathrm{Tr}_{\mathbb{F}/\mathbb{F}_p}(E(q)) \equiv \mathrm{Tr}_{\mathbb{F}/\mathbb{F}_p}(f(q)) \pmod{p\mathbb{Z}_{(p)}}, \tag{1.10}$$

where $\mathrm{Tr}_{\mathbb{F}/\mathbb{F}_p}(E(q))$ is the q-series obtained by reducing the coefficients of $E(q)$ mod $\wp$ and taking the trace from $\mathbb{F}$ to $\mathbb{F}_p$, and similarly for $\mathrm{Tr}_{\mathbb{F}/\mathbb{F}_p}(f(q))$, and $\mathbb{Z}_{(p)}$ is the localization of $\mathbb{Z}$ at p. By (1.9), there exist $g_i \in U_\Gamma$ such that $(2\pi i)^{-1}g_i'(z)/g_i(z) = E_i(z)$ for all i. Let $g = \prod_i g_i^{n_i}$ and $\omega = dg/g$. Then $n\eta(E)(q)$ is the q-expansion of ω at ∞ and $(g) = \delta_\Gamma(n\eta(E))$. In particular, g is non-constant. Since $\eta(E) = \sum_i \eta(\lambda_i)E_i$ with $E_i \in \mathcal{E}_\Gamma(\mathbb{Z})$ and $\eta(\lambda_i) \in \mathbb{Z}_{(p)}$ for all i, Lemma 1.8 gives that $\eta(E)$ has p-integral Fourier coefficients at each cusp for $p \nmid 6N$. It follows from [5, VII 3.9(ii)] that $\eta(E)$ is a modular form with coefficients in $\mathcal{O}$. (Recall $\mathcal{O}$ was the completion of $\mathbb{Z}[\zeta_N]$ at the prime below $\mathfrak{P}$.) Hence ω arises by extension of scalars to $\mathbb{C}$ from an element (denoted ω again) in $H^0(X_1(N)_{/\mathcal{O}}, \Omega_{/\mathcal{O}}(\mathrm{cusps}))$ with $\varphi_{\mathcal{O}}(\omega) = n\eta(E)(q)$.

Write X for X_Γ. Let $\mathcal{O}^*(\mathrm{cusps})$ denote the sheaf on $X_{/\mathcal{O}}$ which when restricted to the complement of the cuspidal divisors is the sheaf of invertible elements of $\mathcal{O}_X$ and whose sections in a neighborhood of the cuspidal divisors consist of functions with divisors supported on those divisors. We see from the exact sequence

$$0 \to \mathcal{O}^* \to H^0(X_{/\mathcal{O}}, \mathcal{O}^*(\mathrm{cusps})) \xrightarrow{d\log} H^0(X_{/\mathcal{O}}, \Omega_{/\mathcal{O}}(\mathrm{cusps}))$$

that g comes from a function (denoted g again) in $H^0(X_{/\mathcal{O}}, \mathcal{O}^*(\text{cusps}))$ up to an element in $\mathcal{O}^*$. Let g_0 be the function on $X_{/\overline{\mathbb{F}}_p}$ obtained from g by the base change $\text{Spec}\,(\overline{\mathbb{F}}_p) \to \text{Spec}\,(\mathcal{O})$. Then g_0 is a non-constant function. Since $X_{/\mathcal{O}}$ is smooth over $\text{Spec}\,(\mathcal{O})$, $(g_0) = \delta_\Gamma(n\eta(E))_{/\overline{\mathbb{F}}_p}$ by [17, Thm. 20], where $\delta_\Gamma(n\eta(E))_{/\overline{\mathbb{F}}_p}$ is the pull-back of $\delta_\Gamma(n\eta(E))$ to $X_{/\overline{\mathbb{F}}_p}$. (The cuspidal sections of $X_{/\mathbb{Z}[1/N,\zeta_N]}$ are all disjoint over $\text{Spec}\,\mathbb{Z}[1/N,\zeta_N]$, cf. [5, VII §2].) Let $\omega_{/\overline{\mathbb{F}}_p}$ be the image of ω in $H^0(X_{/\overline{\mathbb{F}}_p}, \Omega_{/\overline{\mathbb{F}}_p}(\text{cusps}))$. By (1.10), $\varphi_{\overline{\mathbb{F}}_p}(\omega_{/\overline{\mathbb{F}}_p}) = n \cdot \text{Tr}_{\mathbb{F}/\mathbb{F}_p}(f(q))$. For each $\tau \in \text{Gal}\,(\mathbb{F}/\mathbb{F}_p)$, fix a lift σ_τ of τ in the decomposition group $D_{\mathfrak{P}}$ of $G_{\mathbb{Q}}$ for $\mathfrak{P}$, and let $x_\tau \otimes c_\tau \in A_f^{\sigma_\tau}[\mathfrak{p}^{\sigma_\tau}]_{/\mathbb{F}_p}(\overline{\mathbb{F}}_p) \otimes \overline{\mathbb{F}}_p$ be such that $\varphi(x_\tau \otimes c_\tau) = \overline{f^{\sigma_\tau}}$ (cf. Prop. 1.3, Remark 1.4). Then

$$\varphi_{\overline{\mathbb{F}}_p}(\omega_{/\overline{\mathbb{F}}_p}) = n\sum_\tau \overline{f^{\sigma_\tau}} = n\cdot\varphi\left(\sum_\tau x_\tau \otimes c_\tau\right) \in \varphi\left(J_\Gamma[p]_{/\mathbb{F}_p}(\overline{\mathbb{F}}_p) \otimes \overline{\mathbb{F}}_p\right)$$

and so $\omega_{/\mathbb{F}_p} \in H^0(X_{/\mathbb{F}_p}, \Omega^1_X)^{\mathcal{C}} \otimes \overline{\mathbb{F}}_p$. Since $\omega_{/\overline{\mathbb{F}}_p}$ is of the form dg_0/g_0, $\omega_{/\overline{\mathbb{F}}_p} \in H^0(X_{/\overline{\mathbb{F}}_p}, \Omega^1_X)$ by [3, Thm. 2]. Hence there exists $x \in A_f[p]_{/\mathbb{F}_p}(\overline{\mathbb{F}}_p)$ such that $\delta(x) = \omega_{/\overline{\mathbb{F}}_p}$ by (1.3). If x is represented by a divisor D on $X_{/\overline{\mathbb{F}}_p}$, then $pnD \equiv (g_0)$ (mod $p\text{Div}^0(X_{/\overline{\mathbb{F}}_p})$), where $\text{Div}^0(X_{/\overline{\mathbb{F}}_p})$ is the group of divisors of degree 0 on $X_{/\overline{\mathbb{F}}_p}$. Since $p \nmid n$ and $\delta_\Gamma(n\eta(E)) = n\delta_\Gamma(\eta(E))$, $pD \equiv \delta_\Gamma(\eta(E))_{/\overline{\mathbb{F}}_p}$ (mod $p\text{Div}^0(X_{/\overline{\mathbb{F}}_p})$). Hence the coefficients of $\delta_\Gamma(\eta(E))_{/\overline{\mathbb{F}}_p}$ are all divisible by p. But this contradicts the fact that $\eta(\mathcal{R}(E)) = \mathbb{Z}$. Hence $p|\#C_E$. This proves Proposition 1.9. □

Let $J_{\Gamma/\mathbb{Z}}$ be the Néron model of $J_{\Gamma/\mathbb{Q}}$. Let $C_{\Gamma/\mathbb{Z}}$ (resp. $C_{E/\mathbb{Z}}$) be the scheme-theoretic closure of C_Γ (resp. C_E) in $J_{\Gamma/\mathbb{Z}}$.

Corollary 1.10. *With notation as above, $C_E[p]_{/\mathbb{F}_p}(\overline{\mathbb{F}}_p) \neq 0$.*

Proof. Since the cusps of $X_1(N)$ are rational over $\mathbb{Q}(\zeta_N)$ and $p \nmid N$, C_E is unramified at p. Thus $C_{E/\mathbb{Z}_p}$ is a Néron model of $C_{E/\mathbb{Q}_p}$ ([1, 7.1, Cor. 6]). By Prop. 1.9, there exists $x \in C_{E/\mathbb{Z}_p}(K)$ of exact order p, where K is the field of fractions of $\mathcal{O}$. Let $\tilde{x} \in C_{E/\mathbb{Z}_p}(\mathcal{O})$ be the $\mathcal{O}$-valued point corresponding to x. Since $C_E[p]_{/\mathbb{Z}_p}$ is finite flat, the specialization lemma in [11, §1] shows that the specialization of $\tilde{x}$ to the special fiber has order p, so $C_E[p]_{/\mathbb{F}_p}(\overline{\mathbb{F}}_p) \neq 0$. □

Next we determine the image of $C_E[p]_{/\mathbb{F}_p}(\overline{\mathbb{F}}_p)$ under φ.

Lemma 1.11. *For any prime l and any $E \in \mathcal{E}_\Gamma$, we have*

$$T_l^*(\delta_\Gamma(E)) = \delta_\Gamma(E|T_l)$$

where T_l^* acts on $\delta_\Gamma(E)$ via its action on the cusps.

Proof. Let $\pi : X(N) \to X_\Gamma$ be the natural projection. The induced map

$$\pi^* : \text{Div}^0(\mathbf{cusps}(\Gamma)) \longrightarrow \text{Div}^0(\mathbf{cusps}(\Gamma(N)))$$

is injective. It is easy to check that π^* commutes with the actions of T_l^*. Thus it is enough to prove the lemma with $\Gamma = \Gamma(N)$. As $\{\phi_{(x,y)} : (x,y) \in (N^{-1}\mathbb{Z}/\mathbb{Z})^2 - (0,0)\}$ spans the space of weight 2 Eisenstein series E of level N with $\mathcal{R}(E) \subset \mathbb{Q}$ over $\mathbb{Q}$, it suffices by Prop. 1.5 to prove the lemma for $E = \phi_{(x,y)}$, $(x,y) \in (N^{-1}\mathbb{Z}/\mathbb{Z})^2 - (0,0)$, and all primes l. For $l \nmid N$, the lemma follows from [21, 1.3.2, 2.4.7, 3.2.1].

Suppose now $l|N$. We have the double coset decomposition:

$$\Gamma \begin{pmatrix} l & 0 \\ 0 & 1 \end{pmatrix} \Gamma = \bigcup_{k=0}^{l-1} \begin{pmatrix} l & 0 \\ Nk & 1 \end{pmatrix} \Gamma.$$

So

$$\begin{aligned} T_l^* \delta_\Gamma(\phi_{(x,y)}) &= \sum_{\left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right] \in \mathbf{cusps}} r_{\left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right]}(\phi_{(x,y)}) \cdot \sum_{k=0}^{l-1} \left[\begin{matrix} lr \\ Nkr+s \end{matrix}\right] \\ &= \sum_{\left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right]} \Big(\sum r_{\left[\begin{smallmatrix} r' \\ s' \end{smallmatrix}\right]}(\phi_{(x,y)}) \Big) \left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right] \\ &= \sum_{\left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right]} \sum_{k=0}^{l-1} r_{\left[\begin{smallmatrix} r+Nks \\ ls \end{smallmatrix}\right]}(\phi_{(x,y)}) \left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right], \end{aligned} \tag{1.11}$$

where the unindexed sum is over all $\left[\begin{smallmatrix} r' \\ s' \end{smallmatrix}\right] \in \mathbf{cusps} = \mathbf{cusps}(\Gamma(N))$ such that $\left[\begin{smallmatrix} lr' \\ Nkr'+s' \end{smallmatrix}\right] = \left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right]$ for some $0 \le k \le l-1$. Let $S = \{\left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right] \in \mathbf{cusps} : l \nmid r\}$ and $S' = \mathbf{cusps} - S$. We split the sum over $\left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right]$ in (1.11) into two sums Σ_1 and Σ_2 over S and S' respectively. By [21, Props. 2.4.1(a), 2.4.2(a)], $a_0(\phi_{(x,y)}|\gamma_{\left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right]}) = \frac{1}{2} B_2(rx+sy)$, where $B_2(t)$ is the second Bernoulli function. We remark that $B_2(t)$ is periodic with period 1. So from (1.4),

$$r_{\left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right]}(\phi_{(x,y)}) = \frac{1}{2} B_2(rx+sy). \tag{1.12}$$

For $\left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right] \in S$ we have $\left[\begin{smallmatrix} r+Nks \\ ls \end{smallmatrix}\right] = \left[\begin{smallmatrix} r \\ ls \end{smallmatrix}\right]$, so

$$\begin{aligned} r_{\left[\begin{smallmatrix} r+Nks \\ ls \end{smallmatrix}\right]}(\phi_{(x,y)}) = r_{\left[\begin{smallmatrix} r \\ ls \end{smallmatrix}\right]}(\phi_{(x,y)}) &= \frac{1}{2} e(\left[\begin{smallmatrix} r \\ ls \end{smallmatrix}\right]) B_2(rx+lsy) = \frac{1}{l} e(\left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right]) a_0(\phi_{(x,ly)}|\gamma_{\left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right]}) \\ &= \frac{1}{l} r_{\left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right]}(\phi_{(x,ly)}) \end{aligned}$$

since $e(\left[\begin{smallmatrix} r \\ ls \end{smallmatrix}\right]) = e(\left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right])/l$ for $l|N$ and since $l \nmid r$. Hence

$$\Sigma_1 = \sum_{\left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right] \in S} r_{\left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right]}(\phi_{(x,ly)}) \left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right]. \tag{1.13}$$

For $\left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right] \in S'$ we have $\left[\begin{smallmatrix} r+Nks \\ ls \end{smallmatrix}\right] = \left[\begin{smallmatrix} r/l+Nks/l \\ s \end{smallmatrix}\right]$, so by (1.12)

$$\Sigma_2 = l \sum_{\left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right] \in S'} \sum_{k=0}^{l-1} e(\left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right]) \frac{1}{2} B_2(rx/l + Nxks/l + sy). \tag{1.14}$$

Write $x = a/N_2$ with $N_2|N$ and $(a, N_2) = 1$. We now divide into two subcases according as l divides N/N_2 or not. Suppose first $l \nmid (N/N_2)$. Then $l \nmid a$, so

$$\Sigma_2 = l \sum_{\left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right] \in S'} \sum_{k=0}^{l-1} e(\left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right]) \frac{1}{2} B_2(rx/l + ks/l + sy) \left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right]. \tag{1.15}$$

By [21, Prop. 2.4.7] and its proof, we have

$$\phi_{(x,y)}|T_l = \phi_{(x,ly)} = \sum_{k=0}^{l-1} \sum_{j=0}^{l-1} \phi_{((x+j)/l, y+k/l)}. \tag{1.16}$$

For $\left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right] \in S'$, $0 \le k \le l-1$ and $0 \le j \le l-1$, we have

$$\begin{aligned} r_{\left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right]}(\phi_{((x+j)/l, y+k/l)}) =& e(\left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right]) \frac{1}{2} B_2 \left(\frac{x+j}{l} r + \left(y + \frac{k}{l} \right) s \right) \\ =& e(\left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right]) \frac{1}{2} B_2 \left(\frac{xr}{l} + ys + \frac{ks}{l} \right). \end{aligned} \tag{1.17}$$

Putting (1.11), (1.13), (1.15)-(1.17) together, we have $T_l^* \delta_\Gamma(\phi_{(x,y)}) = \delta_\Gamma(\phi_{(x,y)}|T_l)$ for $l \nmid (N/N_2)$.

Suppose now $l|(N/N_2)$. The proof of [21, Prop. 2.4.7] shows that

$$\phi_{(x,y)}|T_l = l \sum_{k=0}^{l-1} \phi_{((x+k)/l, y)}. \tag{1.18}$$

By (1.13), [21, Prop. 2.4.2(b)] and (1.12), we have

$$\begin{aligned} \Sigma_1 =& \sum_{\left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right] \in S} \sum_{k=0}^{l-1} \sum_{j=0}^{l-1} r_{\left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right]}(\phi_{((x+k)/l, y+j/l)}) \left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right] \\ =& \sum_{\left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right] \in S} e(\left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right]) \sum_{k=0}^{l-1} \sum_{j=0}^{l-1} \frac{1}{2} B_2(rx/l + sy + kr/l + sj/l). \end{aligned} \tag{1.19}$$

As k and j run through $\{0, 1, \cdots, l-1\}$, $kr+sj \pmod l$ runs through $\{0, 1, \cdots, l-1\}$ l times. Thus (1.19) and (1.12) give

$$\begin{aligned} \Sigma_1 =& l \sum_{\left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right] \in S} e(\left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right]) \sum_{k=0}^{l-1} \frac{1}{2} B_2(rx/l + k/l + sy) \left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right] \\ =& l \sum_{\left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right] \in S} e(\left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right]) \sum_{k=0}^{l-1} \frac{1}{2} B_2(rx/l + kr/l + sy) \left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right] \\ =& l \sum_{\left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right] \in S} \sum_{k=0}^{l-1} r_{\left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right]}(\phi_{((x+k)/l, y)}) \left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right]. \end{aligned} \tag{1.20}$$

On the other hand, (1.14) and (1.12) give

$$\begin{aligned} \Sigma_2 =& l^2 \sum_{\left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right] \in S'} e(\left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right]) \frac{1}{2} B_2(rx/l + sy) \left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right] \\ =& l \sum_{k=0}^{l-1} \sum_{\left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right] \in S'} r_{\left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right]}(\phi_{((x+k)/l, y)}) \left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right]. \end{aligned} \tag{1.21}$$

Combining (1.18), (1.20) and (1.21) gives $T_l^* \delta_\Gamma(\phi_{(x,y)}) = \delta_\Gamma(\phi_{(x,y)}|T_l)$. This proves Lemma 1.11. □

Observe that since $\mathcal{P}(E)$ and $\mathcal{R}(E)$ are fractional ideals of $\mathbb{Q}[\epsilon_1, \epsilon_2]$, A_E and by duality C_E have natural $\mathbb{Z}[\epsilon_1, \epsilon_2]$-module structures. The composite map (1.5) is a $\mathbb{Z}[\epsilon_1, \epsilon_2]$-module map with respect to these structures. If $x \in C_E$ is represented by $\eta(\delta_\Gamma(E))$ with $\eta \in \mathcal{R}(E)^*$, then Lemma 1.11 shows that $T_l^* x$ is represented by $\eta(\delta_\Gamma(E|T_l)) = \eta(b_l \delta_\Gamma(E))$, where b_l is the eigenvalue of T_l on E. Hence T_l^* acts by multiplication by b_l on C_E.

Proposition 1.12. *Under the same assumptions as in Proposition 1.9, the image of* $C_E[p]_{/\mathbb{F}_p}(\overline{\mathbb{F}}_p) \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p$ *under* φ *is the* $\overline{\mathbb{F}}_p$*-module generated by* $\overline{f}$*.*

Proof. Let $\eta \in \mathcal{R}(E)^*$. Choose $r \in \mathbb{Z}$ such that $r(\overline{\theta(\eta(\delta_\Gamma(E)))})$ is of order dividing p. Here $\overline{\theta(\cdot)}$ means the specialization to the special fiber of the $\mathcal{O}$-valued point of $C_{E/\mathbb{Z}_p}$ corresponding to $\theta(\cdot)$. By Prop. 1.2 and Lemma 1.11, we have for each prime l,

$$\begin{aligned} \varphi(r\overline{\theta(\eta(\delta_\Gamma(E)))})|T_l &= \varphi(rT_l^*\overline{\theta(\eta(\delta_\Gamma(E)))}) \\ &= b_l\varphi(r\overline{\theta(\eta(\delta_\Gamma(E)))}) \\ &= a_l\varphi(r\overline{\theta(\eta(\delta_\Gamma(E)))}). \end{aligned}$$

It follows that $\varphi(r\overline{\theta(\eta(\delta_\Gamma(E)))}) = c \cdot \overline{f}$ for some $c \in \overline{\mathbb{F}}_p$. Since $C_E[p]_{/\mathbb{F}_p}(\overline{\mathbb{F}}_p) \neq 0$ by Cor. 1.10 and $\eta \in \mathcal{R}(E)^*$ was arbitrary, $\varphi\left(C_E[p]_{/\mathbb{F}_p}(\overline{\mathbb{F}}_p) \otimes \overline{\mathbb{F}}_p\right) = \overline{\mathbb{F}}_p \cdot \overline{f}$. □

We can now complete the proof of Theorem 0.4. By Propositions 1.3, 1.12 and the injectivity of φ, we have $A_f[\mathfrak{p}]_{/\mathbb{F}_p}(\overline{\mathbb{F}}_p) \otimes \overline{\mathbb{F}}_p = C_E[p]_{/\mathbb{F}_p}(\overline{\mathbb{F}}_p) \otimes \overline{\mathbb{F}}_p$, hence $A_f[\mathfrak{p}]_{/\mathbb{F}_p}(\overline{\mathbb{F}}_p) = C_E[p]_{/\mathbb{F}_p}(\overline{\mathbb{F}}_p)$. Since the special fiber of $(A_f[\mathfrak{p}] \cap C_E)_{/\mathbb{Z}_p}$ is $A_f[\mathfrak{p}]_{/\mathbb{F}_p} \cap C_{E/\mathbb{F}_p}$, it follows that $A_f[\mathfrak{p}] \cap C_E \neq 0$. This proves Theorem 0.4.

2. Cyclic isogenies of modular elliptic curves

In this section, we prove Theorem 0.3. Let A_1 be an optimal curve over $\mathbb{Q}$ of conductor N. Let $p > 2$ be a prime where A_1 has good ordinary reduction and let $\beta : A_1 \to A$ be a cyclic $\mathbb{Q}$-isogeny of degree divisible by p. Let $\epsilon : G_\mathbb{Q} \to \mathrm{Aut}(\ker\beta[p])$ be the character giving the action of $G_\mathbb{Q}$ on $\ker\beta[p]$ and χ the Teichmüller character giving the action of $G_\mathbb{Q}$ on μ_p. Consider the following three cases:

1. $\epsilon = 1$,
2. $\epsilon = \chi$,
3. $\epsilon \neq 1,\ \chi$.

We shall show that if $p > 7$, the first two cases do not occur and β is étale at p in the last case.

2.1. **Reduction to** $\epsilon \neq 1, \chi$**.** If $\epsilon = 1$, then $\ker\beta[p] \subset A(\mathbb{Q})_{\mathrm{tors}}$. By Mazur's classification theorem [11, Thm. 2], this implies $p \leq 7$. Thus if $p > 7$, this case cannot occur.

Suppose next $\epsilon = \chi$, so $\ker\beta[p] \cong \mu_p$. Let $A' = A_1/\ker\beta[p]$ and let $\beta' : A_1 \to A'$ be the natural isogeny. By [13, §15, Thm. 1], the kernel of the dual isogeny $\check{\beta}' : A' \to A_1$ is the Cartier dual of $\ker\beta[p]$ and so is isomorphic to $\mathbb{Z}/p\mathbb{Z}$ as $G_\mathbb{Q}$-module. This implies that $p \mid \#A'(\mathbb{Q})_{\mathrm{tor}}$. By Mazur's classification theorem again, this cannot happen if $p > 7$.

We assume, henceforth, $\epsilon \neq 1, \chi$. Let $\rho : G_\mathbb{Q} \to \mathrm{Aut}(A_1[p]) \cong \mathrm{GL}_2(\mathbb{F}_p)$ be the Galois representation on the p-torsion points of A_1. Then with respect to a suitable basis, we have

$$\rho \sim \begin{pmatrix} \epsilon & * \\ 0 & \epsilon' \end{pmatrix} \tag{2.1}$$

for some character $\epsilon' : G_\mathbb{Q} \to \mathbb{F}_p^*$. The Weil pairing shows that $\det\rho(\mathrm{Frob}_l) \equiv l \pmod{p}$ for any prime $l \nmid Np$, where Frob_l is a Frobenius element of $G_\mathbb{Q}$ for l. It

follows from our assumption $\epsilon \neq 1, \chi$ that $\epsilon' \neq 1$. Since A_1 is ordinary at p, there is an exact sequence of finite flat group schemes over $\mathbb{Z}_p$

$$0 \to A_1[p]^0_{/\mathbb{Z}_p} \to A_1[p]_{/\mathbb{Z}_p} \to A_1[p]^{\text{ét}}_{/\mathbb{Z}_p} \to 0,$$

where the flanking terms are each of order p such that the inertia group I_p acts via χ on the $G_{\mathbb{Q}_p}$-module associated to $A_1[p]^0_{/\mathbb{Z}_p}$ and acts trivially on that associated to $A_1[p]^{\text{ét}}_{/\mathbb{Z}_p}$. It follows that exactly one of ϵ and ϵ' is unramifed at p. The next two lemmas show that we may assume ϵ' is unramified at p.

Lemma 2.1. *Let $\beta : A \to A'$ be a cyclic $\mathbb{Q}$-isogeny of elliptic curves over $\mathbb{Q}$. Suppose A has good reduction at p and $\ker\beta[p^\infty](\overline{\mathbb{Q}})$ is unramified at p. Then β is étale at p.*

Proof. Let $\beta_p : A_{/\mathbb{Z}_p} \to A'_{/\mathbb{Z}_p}$ be the extension of β to Néron models over $\mathbb{Z}_p$. To show that β_p is étale, we have to show that it is flat and unramified. Note that β_p is quasi-finite and flat [1, 7.3, Lemmas 1, 2]. We check that it is unramified at points of residue characteristic p. The kernel $\ker\beta_p$ of β_p is a finite flat group scheme over $\mathbb{Z}_p$. Let K be the extension of $\mathbb{Q}_p$ cut out by $\ker\beta_p(\overline{\mathbb{Q}}_p)$ and let $\mathcal{O}$ denote its ring of integers. Since A has good reduction at p, $A[m]$ is unramified at p for all m prime to p by the Néron-Ogg-Shafarevich criterion. By the assumption on $\ker\beta[p^\infty](\overline{\mathbb{Q}})$, $\ker\beta(\overline{\mathbb{Q}}_p)$ is unramified as $G_{\mathbb{Q}_p}$-module. Thus $\ker\beta_p$ is a Néron model of $\ker\beta_{p/\mathbb{Q}_p}$ ([1, 7.1, Cor. 6]). Let $x \in \ker\beta_p(\mathcal{O})$ be the $\mathcal{O}$-valued point corresponding to a generator of $\ker\beta_p(K)$. By the specialization lemma in [11, §1], the order of the specialization of x to the residue field of $\mathcal{O}$ equals the order of x. So if $\overline{\beta}_p$ is the reduction of β_p mod p, then

$$\#\ker\overline{\beta}_p = \#\ker\beta = \deg\beta = \deg\overline{\beta}_p.$$

By [20, 4.10(a)], $\#\ker\overline{\beta}_p$ equals the degree of separability $\deg_s\overline{\beta}_p$ of $\overline{\beta}_p$. So $\deg_s\overline{\beta}_p = \deg\overline{\beta}_p$; hence $\overline{\beta}_p$ is separable and unramified by [20, 4.10(c)]. This proves that β_p is étale. □

Lemma 2.2. *Let $\beta : A \to A'$ be a cyclic $\mathbb{Q}$-isogeny of degree d divisible by p. Suppose A has good ordinary reduction at p and $\ker\beta[p]$ is unramified at p. Then $\ker\beta[p^\infty]$ is unramified at p.*

Proof. Let ϵ_1 be the Galois character on $\ker\beta[p^\infty]$. Write $\epsilon_1 \oplus \epsilon_2$ for the semi-simplification of the Galois representation $\rho : G_{\mathbb{Q}} \to \mathrm{Aut}(A[p^r])$, where ϵ_2 is some Galois character and $p^r \| d$. Since p is ordinary, one of the characters ϵ_1 and ϵ_2 is unramified at p and the other when restricted to I_p is the cyclotomic character on μ_{p^r}. By our assumption, $\epsilon_1 \pmod p$ is unramified at p. Since $p > 2$, ϵ_1 is unramified at p. This proves the lemma. □

If ϵ is unramified at p, then by Lemma 2.2, $\ker\beta[p^\infty]$ is unramified at p. Thus β is étale at p by Lemma 2.1, hence Theorem 0.3. So we assume, from now on,

$$\epsilon \text{ is ramified at } p. \tag{2.2}$$

Let $\pi : X_1(N) \to A_1$ be a modular parametrization and let f be the associated weight 2 normalized newform on $\Gamma_1(N)$. Write

$$f(z) = \sum_{n=1}^{\infty} a_n q^n, \quad q = e^{2\pi i z}, \quad z \in \mathfrak{H} \tag{2.3}$$

for the q-expansion of f at the cusp ∞. We show that there is a weight 2 Eisenstein series E on $\Gamma_1(N)$ associated to ρ such that $\mathfrak{P}$ is an Eisenstein prime for E and f (Prop. 2.6). Applying Theorem 0.4 and the classification theorem of rational cyclic isogenies of elliptic curves over $\mathbb{Q}$, we deduce that $p \leq 5$ (§2.3).

2.2. **Eisenstein series associated to ρ.** Let χ_1 and χ_2 be two Dirichlet characters, not necessarily primitive, modulo N_1 and N_2 respectively. Put

$$G(a_1, a_2; \mathfrak{a}_1, \mathfrak{a}_2) = \frac{1}{N_2} \sum_{t \in N_2^{-1}\mathbb{Z}/\mathbb{Z}} e^{2\pi i(-ta_2)} E(z; a_1, t; \mathfrak{a}_1, \mathfrak{a}_2^{-1})$$

and

$$E(\chi_1, \chi_2) = \frac{1}{2} \sum_{a_1=0}^{N_1-1} \sum_{a_2=0}^{N_2-1} \chi_1(a_1)\chi_2(a_2) G(a_1, a_2; N_1\mathbb{Z}, \mathbb{Z}) \tag{2.4}$$

where $E(z; a_1, a_2; \mathfrak{a}_1, \mathfrak{a}_2)$ is as in (1.7).

Proposition 2.3. *Notation being as above, assume that not both χ_1 and χ_2 are the trivial character of any conductor. Then*

(a) *$E(\chi_1, \chi_2)$ is a weight 2 Eisenstein series on $\Gamma_0(N_1N_2)$ of character $(\chi_1\chi_2)^{-1}$.*
(b) *The Dirichlet series $L(E, s) := \sum_{n=1}^{\infty} b_n n^{-s}$ of $E(\chi_1, \chi_2)$ is*

$$L(\chi_1, s)L(\chi_2, s-1),$$

where b_n is the n-th Fourier coefficient of $E(\chi_1, \chi_2)$ at ∞.
(c) *If $\chi_1 \neq 1$, then the constant term of the Fourier expansion of $E(\chi_1, \chi_2)$ at ∞ is 0.*

Proof. (a) and (b) follow from [19, Prop. 3.4] for Artin characters of degree 1 of totally real number fields of degree > 1. But, as remarked in [25, §1.5], the same result holds for $\mathbb{Q}$ if χ_1 and χ_2 are not both the trivial character of any conductor. (c) follows from the Fourier expansion of $E(z; a_1, a_2; N_1\mathbb{Z}, \mathbb{Z})$ in (1.8), the definition of $E(\chi_1, \chi_2)$ in (2.4), and the assumption that $\chi_1 \neq 1$. □

Remark 2.4. Using (1.8), we find that the Fourier expansion of $E(\chi_1, \chi_2)$ at ∞ is

$$E(\chi_1, \chi_2)(z) = \sum_{c=1}^{\infty} \sum_{b=1}^{\infty} \chi_1(c)\chi_2(b) b e^{2\pi i(bcz)}. \tag{2.5}$$

From this, we can deduce Prop. 2.3(b) in a similar fashion to [21, Prop. 3.4.2(b)].

We want to apply Prop. 2.3 to certain Dirichlet characters associated to ϵ and ϵ' in (2.1) to get an Eisenstein series with suitable properties. For this, we consider the ramification behavior of the primes dividing N in $\mathbb{Q}(A_1[p])/\mathbb{Q}$. For each prime q, let T_q be the q-adic Tate module of A_1 and $V_q = T_q \otimes_{\mathbb{Z}_q} \mathbb{Q}_q$. Let $H_q = H^1_{\text{ét}}(A_1 \times_{\mathbb{Q}} \overline{\mathbb{Q}}, \mathbb{Q}_q)$. Then $V_q = H_q \otimes_{\mathbb{Q}_q} \mathbb{Q}_q(1)$, and the collection $\{H_q\}_q$ forms a compatible system V of q-adic representations of $G_{\mathbb{Q}}$ whose L-function $L(V, s)$ is defined by an Euler product (for $\operatorname{Re} s > \frac{3}{2}$):

$$L(V, s) = \prod_l L_l(V, s) = \prod_l \det(1 - \operatorname{Frob}_l^{-1} l^{-s} | (H_q)^{I_l})^{-1},$$

where for each l, q is a prime $\neq l$ and $(H_q)^{I_l}$ is the maximal subspace of H_q on which I_l acts trivially.

Let l be a prime dividing N. Suppose $l \| N$. Then A_1 has multiplicative reduction at l. By [16, §1.12], there is an unramified extension $K_l/\mathbb{Q}_l$ of degree ≤ 2 such that A_1 is isomorphic over K_l to the Tate curve $E_q = \mathbb{G}_m/q^{\mathbb{Z}}$ for some $q \in l\mathbb{Z}_l$ determined by the j-invariant of A_1. We have an exact sequence of G_{K_l}-modules for each n:

$$0 \to \mu_{p^n} \to A_1[p^n] \to \mathbb{Z}/p^n\mathbb{Z} \to 0. \tag{2.6}$$

As ρ is given by (2.1), both ϵ and ϵ' are unramified at l. On taking inverse limit over n and then tensoring with $\mathbb{Q}_p$, (2.6) gives a (non-split) exact sequence of G_{K_l}-modules:

$$0 \to \mathbb{Q}_p(1) \to V_p \to \mathbb{Q}_p \to 0.$$

Thus $G_{\mathbb{Q}_l}$ acts via some character ψ_l of $\mathrm{Gal}(K_l/\mathbb{Q}_l)$ on $(V_p)_{I_l}$ (= the maximal quotient of V_p on which I_l acts trivially). From (2.1), $\psi_l \equiv \epsilon|_{G_{\mathbb{Q}_l}} \pmod p$ or $\psi_l \equiv \epsilon'|_{G_{\mathbb{Q}_l}} \pmod p$. Let $S_1 = \{l : l \| N,\ \psi_l \equiv \epsilon|_{G_{\mathbb{Q}_l}} \pmod p\}$ and $S_2 = \{l : l \| N,\ l \notin S_1\}$.

Suppose now $l^2|N$. Then A_1 has additive reduction at l. By [16, §5.6 Prop. 23b], the images of I_l under ϵ and ϵ' are cyclic of order 2, 3, 4 or 6. In particular, ϵ and ϵ' are ramified at l.

Let $f_{\epsilon_{0,\mathrm{prim}}}$ and $f_{\epsilon'_{0,\mathrm{prim}}}$ be the conductors of the primitive Dirichlet characters $\epsilon_{0,\mathrm{prim}}$ and $\epsilon'_{0,\mathrm{prim}}$ associated to ϵ and ϵ' respectively. Let $\epsilon_{0,\mathrm{prim}}\chi^{-1}$ be the primitive Dirichlet character such that $\epsilon_{0,\mathrm{prim}} = \epsilon_{0,\mathrm{prim}}\chi^{-1} \cdot \chi$. Here we view χ as both a character of $G_{\mathbb{Q}}$ and a Dirichlet character. Since ϵ' is unramified at p (cf. (2.2)) and $\epsilon(l)\epsilon'(l) \equiv l \pmod p$ for any $l \nmid Np$, the conductor $f_{\epsilon_{0,\mathrm{prim}}\chi^{-1}}$ of $\epsilon_{0,\mathrm{prim}}\chi^{-1}$ is prime to p. Now let $\epsilon_0\chi^{-1}$ (resp. ϵ_0, ϵ'_0) be the Dirichlet character modulo $f_{\epsilon_0\chi^{-1}} := f_{\epsilon_{0,\mathrm{prim}}\chi^{-1}} \cdot \prod_{l\in S_2} l$ (resp. $f_{\epsilon_0} := f_{\epsilon_{0,\mathrm{prim}}} \cdot \prod_{l\in S_2} l$, $f_{\epsilon'_0} := f_{\epsilon'_{0,\mathrm{prim}}} \cdot \prod_{l\in S_1} l$) whose primitive character is $\epsilon_{0,\mathrm{prim}}\chi^{-1}$ (resp. $\epsilon_{0,\mathrm{prim}}$, $\epsilon'_{0,\mathrm{prim}}$). (For a Dirichlet character ψ modulo m, we set $\psi(n) = 0$ if $(n, m) \neq 1$.)

Lemma 2.5. (a) *$f_{\epsilon_0\chi^{-1}} f_{\epsilon'_0}$ divides N.*
(b) *$f_{\epsilon_0\chi^{-1}} f_{\epsilon'_0}$ and N have the same prime divisors.*

Proof. (a) By a result of Carayol [2, 0.8], the level N of f is equal to the conductor of the p-adic representation V_p. (For the definition of the latter, see for example [4, §1.1].) Put $\Phi = \ker\beta[p]$ and $\Phi' = A_1[p]/\Phi$. For any prime $l \neq p$, we have

$$\dim_{\mathbb{F}_p} A_1[p]^{G_i} \leq \dim_{\mathbb{F}_p} \Phi^{G_i} + \dim_{\mathbb{F}_p} \Phi'^{G_i},$$

where $G_0 \supset G_1 \supset \cdots$ is the series of ramification groups in $\mathrm{Gal}(\mathbb{Q}_l(A_1[p])/\mathbb{Q}_l)$. Since $\sum_{i=0}^{\infty}[G_0 : G_i]^{-1} \dim_{\mathbb{F}_p}(\Phi/\Phi^{G_i})$ is the l-part of $f_{\epsilon_{0,\mathrm{prim}}}$ and similarly for Φ' and ϵ', the l-part of $f_{\epsilon_{0,\mathrm{prim}}} f_{\epsilon'_{0,\mathrm{prim}}}$ divides N. To see that the l-part of $f_{\epsilon_0} f_{\epsilon'_0}$ divides N, we need only consider the case $l \| N$. In this case $l \nmid f_{\epsilon_{0,\mathrm{prim}}} f_{\epsilon'_{0,\mathrm{prim}}}$, so by the definition of f_{ϵ_0} and $f_{\epsilon'_0}$ we have $l \| f_{\epsilon_0} f_{\epsilon'_0}$. Hence $f_{\epsilon_0\chi^{-1}} f_{\epsilon'_0}$ divides N. This proves (a).

(b) Let l be a prime. If $l \| N$, then $l|f_{\epsilon_0}$ or $l|f_{\epsilon'_0}$. If $l^2|N$, then ϵ and ϵ' are both ramified at l, so $l|f_{\epsilon_0}$ and $l|f_{\epsilon'_0}$. This proves (b). □

We now apply Prop. 2.3 to $\chi_1 = \epsilon'_0$ and $\chi_2 = \epsilon_0\chi^{-1}$ to get an Eisenstein series $E = E(\epsilon'_0, \epsilon_0\chi^{-1})$. (Recall that, under our assumption (2.2), ϵ'_0 and $\epsilon_0\chi^{-1}$ are both non-trivial.)

Proposition 2.6. *The Fourier coefficients of the q-expansions of E and f at ∞ are congruent mod $\mathfrak{P}$:*

$$E(q) \equiv f(q) \pmod{\mathfrak{P}}.$$

Proof. By Prop. 2.3(b), the Euler factor $L_l(E,s)$ of $L(E,s)$ at a prime l is $(1-\epsilon_0'(l)l^{-s})^{-1}(1-\epsilon_0\chi^{-1}(l)l^{1-s})^{-1}$. We have a formal Euler product for the Dirichlet series $L(f,s)$ of f:

$$L(f,s) = \prod_l L_l(f,s) = \prod_{l|N}(1-a_l l^{-s})^{-1} \prod_{l\nmid N}(1-a_l l^{-s}+l^{1-2s})^{-1},$$

where a_l is the l-th Fourier coefficient of f in (2.3). By [2], $L(f,s) = L(V,s)$.

For $l \nmid Np$, we have $l\epsilon_0\chi^{-1}(l)\cdot\epsilon_0'(l) = l$ and, by the Eichler-Shimura relations,

$$a_l \equiv \mathrm{tr}(\rho(\mathrm{Frob}_l)) = \epsilon(\mathrm{Frob}_l) + \epsilon'(\mathrm{Frob}_l) \equiv l\epsilon_0\chi^{-1}(l) + \epsilon_0'(l) \pmod{\mathfrak{P}},$$

so

$$(1-a_l l^{-s}+l^{1-2s})^{-1} \equiv (1-l\epsilon_0\chi^{-1}(l)l^{-s})^{-1}(1-\epsilon_0'(l)l^{-s})^{-1} = L_l(E,s) \pmod{\mathfrak{P}}.$$

Let $l|N$. Then $L_l(f,s) = (1-a_l l^{-s})^{-1}$. Suppose $l\|N$. By (2.1) and the definition of ϵ_0' and $\epsilon_0\chi^{-1}$, we have

$$(1-a_l l^{-s})^{-1} \equiv (1-\epsilon^*(l)l^{-s})^{-1} \equiv L_l(E,s) \pmod{\mathfrak{P}},$$

where $\epsilon^* = \epsilon_0$ or ϵ_0' according as $l \in S_1$ or $l \in S_2$. Next suppose $l^2|N$. Then $a_l = 0$ and $L_l(f,s) = 1$. Since $\epsilon_0'(l) = \epsilon_0\chi^{-1}(l) = 0$, $L_l(E,s) = 1$.

Finally, suppose $l = p$. Since A_1 is ordinary at p, a_p is congruent mod $\mathfrak{P}$ to the eigenvalue of Frobenuis on the p-adic Tate module of $A_{1/\mathbb{F}_p}$. So we have

$$(1-a_p p^{-s}+p\cdot p^{-2s})^{-1} \equiv (1-a_p p^{-s})^{-1} \equiv (1-\epsilon_0'(p)p^{-s})^{-1} \pmod{\mathfrak{P}}.$$

Hence $L_l(f,s)$ is congruent mod $\mathfrak{P}$ to $L_l(E,s)$ for each l. This shows that $a_n \equiv b_n \pmod{\mathfrak{P}}$ for each $n \geq 1$ and, together with Prop. 2.3(c), proves the proposition. □

We show that $E = E(\epsilon_0', \epsilon_0\chi^{-1})$ is a common eigenfunction for all T_l and $\langle l\rangle$.

Lemma 2.7. *For any prime l, we have*

(a) $E|\langle l\rangle = E$ *if* $l \nmid N$;
(b) $E|T_l = (\epsilon_0'(l) + l\epsilon_0\chi^{-1}(l))E$.

Proof. (a) Since $\epsilon_0'(l)\cdot\epsilon_0\chi^{-1}(l) = 1$ for all $l \nmid N$, E is modular on $\Gamma_0(N)$ by Prop. 2.3(a). Hence $E|\langle l\rangle = E|\sigma_l = E$.

(b) We use the q-expansion of E at ∞ given by (2.5):

$$\begin{aligned} E(z) &= \sum_{c=1}^{\infty}\sum_{b=1}^{\infty} \epsilon_0'(c)\epsilon_0\chi^{-1}(b)bq^{bc}, \quad q = e^{2\pi iz}, \\ &= \sum_{n=1}^{\infty} b_n q^n, \end{aligned}$$

where $b_n = \sum_{bc=n}\epsilon_0'(c)\epsilon_0\chi^{-1}(b)b$ for each $n \geq 1$.

By Prop. 2.3(a), the level of E divides $f_{\epsilon_0\chi^{-1}}f_{\epsilon_0'}$, which has the same prime divisors as N by Lemma 2.5. The action of T_l on E is then given by ([18, (3.5.12)]):

$$E|T_l = \sum_{n=1}^{\infty} b_{ln}q^n + l\sum_{n=1}^{\infty} b_n q^{ln}, \quad l\nmid N, \tag{2.7}$$

$$= \sum_{n=1}^{\infty} b_{ln}q^n, \quad l|N. \tag{2.8}$$

Suppose first $l\nmid N$. Then

$$\begin{aligned} b_{ln} &= \sum_{bc=ln} \epsilon_0'(c)\epsilon_0\chi^{-1}(b)b \\ &= \sum_{\substack{bc=ln\\ l\nmid b}} \epsilon_0'(c)\epsilon_0\chi^{-1}(b)b + \sum_{\substack{bc=ln\\ l|b}} \epsilon_0'(c)\epsilon_0\chi^{-1}(b)b \\ &= \epsilon_0'(l)\sum_{\substack{bc'=n\\ l\nmid b}} \epsilon_0'(c')\epsilon_0\chi^{-1}(b)b + l\epsilon_0\chi^{-1}(l)\sum_{b'c=n} \epsilon_0'(c)\epsilon_0\chi^{-1}(b')b'. \end{aligned} \tag{2.9}$$

Since $\epsilon_0'(l)\epsilon_0(l) = \chi(l)$ for all $l\nmid Np$ and $p\nmid f_{\epsilon_0'}f_{\epsilon_0\chi^{-1}}$, it follows that $\epsilon_0'(l) = \epsilon_0\chi^{-1}(l^{-1})$ for all $l\nmid N$. So

$$\begin{aligned} lb_n =& l\sum_{bc=n} \epsilon_0'(c)\epsilon_0\chi^{-1}(b)b \\ =& \epsilon_0\chi^{-1}(l^{-1})\sum_{bc=n} \epsilon_0'(c)\epsilon_0\chi^{-1}(lb)lb \\ =& \epsilon_0\chi^{-1}(l^{-1})\sum_{\substack{bc=ln\\ l|b}} \epsilon_0'(c)\epsilon_0\chi^{-1}(b)b \\ =& \epsilon_0'(l)\sum_{\substack{bc=ln\\ l|b}} \epsilon_0'(c)\epsilon_0\chi^{-1}(b)b. \end{aligned} \tag{2.10}$$

(b) follows from (2.7), (2.9) and (2.10) for $l\nmid N$. For $l|N$, we have

$$b_{ln} = \begin{cases} l\epsilon_0\chi^{-1}(l)b_n & \text{if } l|f_{\epsilon_0'} \text{ and } l\nmid f_{\epsilon_0\chi^{-1}},\\ \epsilon_0'(l)b_n & \text{if } l|f_{\epsilon_0\chi^{-1}} \text{ and } l\nmid f_{\epsilon_0'},\\ 0 & \text{otherwise,}\end{cases}$$

and (b) follows from (2.8). □

2.3. **Completion of proof of Theorem 0.3 in the case $\epsilon\neq 1$, χ.** We can now complete the proof of Theorem 0.3 in this case, assuming (2.2). Applying Theorem 0.4 to f and E, we have $A_1[p]\cap C_E\neq 0$. By [21, Thm. 3.2.4], C_E is stable under the action of $G_{\mathbb{Q}}$, which is given by ϵ'. Since ϵ (resp. ϵ') is ramified (resp. unramified) at p, it follows that $\ker\beta[p]$ and $A_1[p]\cap C_E$ are two independent cyclic subgroups of order p defined over $\mathbb{Q}$. Hence $A_1/\ker\beta[p]$ is an elliptic curve over $\mathbb{Q}$ which has a cyclic subgroup of order p^2 defined over $\mathbb{Q}$. Kenku [7, Thm. 1] has shown that the table in the introduction of [11] is a complete list of d for which there is a rational cyclic d-isogeny of elliptic curves over $\mathbb{Q}$. This implies that $p\leq 5$. Thus if $p>7$, ϵ is unramified at p. This completes the proof of Theorem 0.3.

3. Integrality of p-adic L-functions

We now apply Theorem 0.3 to establish Theorem 0.2. We shall use the following result of Stevens.

Theorem 3.1. ([23, Thm. 4.6]) *Suppose $p > 2$. Let $\pi : X_1(N) \to A$ be a modular parametrization and $c(\pi)$ the Manin constant of π. Then $c(\pi)\nu_{A,\Delta}$ takes values in $\mathcal{L}(A) \otimes \mathbb{Z}_p$.*

To prove Theorem 0.2, it suffices by Theorem 3.1 to show that there is a modular parametrization $\pi : X_1(N) \to A$ such that $c(\pi)$ is a p-unit. Let $\mathcal{A}$ be the $\mathbb{Q}$-isogeny class of elliptic curves over $\mathbb{Q}$ containing A. Let A_1 be the optimal curve in $\mathcal{A}$ and $\pi_1 : X_1(N) \to A_1$ an optimal parametrization (cf. the introduction). Let n be the largest square dividing N.

Proposition 3.2. *With notation as above, $c(\pi_1) \in \mathbb{Z}[1/2n]^*$.*

Proof. The analogous result for a strong parametrization $\pi_0 : X_0(N) \to A_0$ (which takes the cusp ∞ to the origin of A_0) of the strong Weil curve $A_0 \in \mathcal{A}$ has been proved in [11, Cor. 4.1] by showing that $\pi_0 : X_0(N)^{\mathrm{smooth}}_{/\mathbb{Z}[\frac{1}{2n}]} \to A_{0/\mathbb{Z}[\frac{1}{2n}]}$ is a formal immersion along the ∞-section, where $X_0(N)^{\mathrm{smooth}}_{/\mathbb{Z}[\frac{1}{2n}]}$ is the smooth locus of $X_0(N)_{/\mathbb{Z}[\frac{1}{2n}]} \to \operatorname{Spec}\mathbb{Z}$. An analysis of the arguments used there shows that for an optimal parametrization $\pi_1 : X_1(N) \to A_1$ which takes the cusp 0 to the origin of A_1, $\pi_1 : X_1(N)^{\mathrm{smooth}}_{/\mathbb{Z}[\frac{1}{2n}]} \to A_{1/\mathbb{Z}[\frac{1}{2n}]}$ is a formal immersion along the 0-section. Since $X_1(N)_{/\overline{\mathbb{F}}_l}$ is irreducible if $l \nmid N$ and since the Atkin-Lehner involution w_N interchanges the two irreducible components of $X_1(N)_{/\overline{\mathbb{F}}_l}$ if $l \| N$, we have $c(\pi_1) \in \mathbb{Z}[\frac{1}{2n}]^*$. □

Lemma 3.3. *Let $X \xrightarrow{f} Y$, $Y \xrightarrow{g} Z$ be morphisms of schemes. Suppose that f is smooth. Then there is an exact sequence of $\mathcal{O}_X$-modules*

$$0 \to f^*\Omega^1_{Y/Z} \to \Omega^1_{X/Z} \to \Omega^1_{X/Y} \to 0,$$

where $\Omega^1_{X/Y}$ is the sheaf of relative differentials of degree 1 of X over Y and similarly for $\Omega^1_{Y/Z}$ and $\Omega^1_{X/Z}$, and X is considered as a Z-scheme via $g \circ f$.

Proof. See [1, 2.2 Prop. 5(b)] and the remark after it. □

Corollary 3.4. *Let $\beta : A \to A'$ be a $\mathbb{Q}$-isogeny of elliptic curves over $\mathbb{Q}$ étale at p. Let $\Omega^1_{A/\mathbb{Z}_p}$ and $\Omega^1_{A'/\mathbb{Z}_p}$ be the sheaves of Néron differentials on $A_{/\mathbb{Z}_p}$ and $A'_{/\mathbb{Z}_p}$ respectively. Then β induces an isomorphism*

$$\beta^* : H^0(A'_{/\mathbb{Z}_p}, \Omega^1_{A'/\mathbb{Z}_p}) \xrightarrow{\sim} H^0(A_{/\mathbb{Z}_p}, \Omega^1_{A/\mathbb{Z}_p}).$$

Proof. Since β is étale at p, $\beta_{/\mathbb{Z}_p}$ is smooth and $\Omega^1_{A_{/\mathbb{Z}_p}/A'_{/\mathbb{Z}_p}} = 0$. So the exact sequence in Lemma 3.3 gives $\beta^*\Omega^1_{A'/\mathbb{Z}_p} \cong \Omega^1_{A/\mathbb{Z}_p}$. Hence $H^0(A'_{/\mathbb{Z}_p}, \Omega^1_{A'/\mathbb{Z}_p}) \cong H^0(A_{/\mathbb{Z}_p}, \beta^*\Omega^1_{A'/\mathbb{Z}_p}) \cong H^0(A_{/\mathbb{Z}_p}, \Omega^1_{A/\mathbb{Z}_p})$. □

We can now prove Theorem 0.2. As remarked above, it suffices to show that there is a modular parametrization $\pi : X_1(N) \to A$ such that $c(\pi)$ is a p-unit. We show that a modular parametrization $\pi : X_1(N) \to A$ of minimal degree meets this requirement. By the definition of optimality, there is a $\mathbb{Q}$-isogeny $\beta : A_1 \to A$

such that $\pi = \beta \circ \pi_1$. We have $\deg \pi = \deg \beta \deg \pi_1$. It follows that $\deg \beta$ is minimal among all isogenies from A_1 to A. Thus β must be cyclic. Let ω_{A_1} and ω_A be Néron differentials on A_1 and A respectively, and let $c(\beta) \in \mathbb{Z}$ be such that $\beta^* \omega_A = c(\beta)\omega_{A_1}$. If $p \nmid \deg \beta$, then $p \nmid c(\beta)$. If $p | \deg \beta$, then β is étale at p by Theorem 0.3, and so $p \nmid c(\beta)$ by Corollary 3.4. Since $c(\pi) = c(\beta)c(\pi_1)$ and $c(\pi_1)$ is a p-unit by Prop. 3.2, $c(\pi)$ is a p-unit. This completes the proof of Theorem 0.2.

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