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INTEGER TRANSLATION OF MEROMORPHIC FUNCTIONS

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Abstract. Let G be a given open set in the complex plane. We prove that there is an entire function such that its integer translations forms a normal family in a neighborhood of z exactly for z in G if and only if G is periodic with period 1, i.e., $z \pm 1 \in G$ for all $z \in G$.

1. Introduction and preliminaries

Let $\mathcal{S}$ be a space of functions defined on an open set G. For each positive integer n, we let T_n be an operator from $\mathcal{S}$ into itself. Our concern is the normality of the family $\{T_n(f)\colon n = 0, 1, 2, \dots\}$ for a function $f \in \mathcal{S}$. In this paper, we consider the case $T_n(f(z)) = f(z+n)$, translation by n.

To discuss the normality of families of meromorphic functions, we need the following notion.

Definition 1.1. Let f be a meromorphic function in the complex plane $\mathbf{C}$. We define the *spherical derivative* of f at z by

$$\rho(f(z)) = \frac{|f'(z)|}{1+|f(z)|^2}.$$

Theorem (Marty [6]). *A family $\mathcal{F}$ of functions $f(z)$, meromorphic in a domain D, is normal in D if and only if the set $\{\rho(f(z))\colon f \in \mathcal{F}\}$ is uniformly bounded on every compact subset of D.*

Now we state our definition and an example.

Definition 1.2. For a meromorphic function f, we define the set $\mathbf{N}(f)$ to be the set of all $z_0 \in \mathbf{C}$ such that the family $\{f(z+n)\colon n = 0, 1, 2, \dots\}$ is normal in a neighborhood of z_0.

Example 1.3. Let $f(z) = z \sin \pi z$; then $\mathbf{N}(f) = \mathbf{C} \backslash \mathbf{Z}$ where $\mathbf{Z}$ is the set of all integers.

Proof. For all integers n,

$$f(z+n) = (z+n)\sin \pi(z+n) = (z+n)\sin \pi z;$$

thus we have

$$f(z+n) \begin{cases} = 0, & \text{if } z \in \mathbf{Z}, \\ \to \infty, & \text{if } z \notin \mathbf{Z} \text{ and } n \to \infty. \end{cases}$$

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Hence no subsequence of $\{f(z+n)\colon n = 0, 1, 2, \dots\}$ converges on any neighborhood of any integer. But the sequence $\{f(z+n)\colon n = 0, 1, 2, \dots\}$ converges uniformly to infinity on any compact subset which does not contain integers. These facts prove that $\mathbf{N}(f) = \mathbf{C}\backslash\mathbf{Z}$. □

2. Some properties of $\mathbf{N}(f)$

From the definition, the set $\mathbf{N}(f)$ is open in the complex plane $\mathbf{C}$ and is periodic with period 1.

The set $\mathbf{N}(f)$ may not be connected. For a certain meromorphic function f, the set $\mathbf{N}(f)$ has infinitely many components.

Example 2.1. Let $f(z) = \tan e^z$; then the set $\mathbf{N}(f)$ is not connected.

Proof. We let

$$I_c = \{z \in \mathbf{C}\colon \text{Im } z = k\pi, k = 0, \pm 1, \pm 2, \dots\},$$

and we shall show that $\mathbf{N}(f) = \mathbf{C}\backslash I_c$. For $z = x + k\pi i \in I_c$, we have

$$|f(z+n)|^2 = |\tan e^{x+n+k\pi i}|^2 = \tan^2 e^{x+n}$$

and

$$|f'(z+n)| = |e^{x+n+k\pi i} \sec^2 e^{x+n+k\pi i}| = e^{x+n} \sec^2 e^{x+n}.$$

So for every real number x with $e^{x+n} \neq k\pi + \frac{\pi}{2}$ $(k = 0, \pm 1, \pm 2, \dots)$,

$$\begin{aligned} \rho(f(z+n)) &= \frac{e^{x+n} \sec^2 e^{x+n}}{1 + \tan^2 e^{x+n}} \\ &= \frac{e^{x+n}}{\sin^2 e^{x+n} + \cos^2 e^{x+n}} \\ &= e^{x+n} \to \infty \end{aligned}$$

as $n \to \infty$. But for an integer l every neighborhood of $x + l\pi i$ contains a point $x' + l\pi i$ such that $e^{x'+n} \neq k\pi + \frac{\pi}{2}$. Hence by Marty's Theorem, $x + l\pi i \notin \mathbf{N}(f)$ for all real x.

Now for $z_0 \notin I_c$, we choose a positive number ε so that $B(z_0, \varepsilon) \cap I_c = \varnothing$. Let F be a compact subset in $B(z_0, \varepsilon)$; then for $z = x + iy \in F$, we have

$$\begin{aligned} \rho(f(z+n)) &= \frac{|(\tan e^{z+n})'|}{1 + |\tan e^{z+n}|^2} \\ &= \frac{|e^{z+n} \sec^2 e^{z+n}|}{1 + |\tan e^{z+n}|^2} \qquad (2.1) \\ &= \frac{|e^{z+n}|}{|\sin e^{z+n}|^2 + |\cos e^{z+n}|^2}. \end{aligned}$$

For $z = x + iy$, it is easy to show

$$|\sin z|^2 + |\cos z|^2 = \tfrac{1}{2}(e^{2y} + e^{-2y})$$

and since

$$e^{z+n} = e^{x+n} \cos y + i e^{x+n} \sin y,$$

we can write

$$|\sin e^{z+n}|^2 + |\cos e^{z+n}|^2 = \tfrac{1}{2}(e^{2e^{x+n} \sin y} + e^{-2e^{x+n} \sin y}).$$

Thus from (2.1),

$$\begin{aligned}\rho(f(z+n)) &= \frac{2e^{x+n}}{e^{2e^{x+n}\sin y} + e^{-2e^{x+n}\sin y}} \\ &\le \frac{2e^{x+n}}{e^{2|\sin y|e^{x+n}}}.\end{aligned} \tag{2.2}$$

We let $\delta = \min\{|\sin y| \colon z = x+iy \in F\}$; then $\delta > 0$. We define a function g on the real line $\mathbf{R}$ by

$$g(t) = \frac{2e^t}{e^{2\delta e^t}};$$

then we get

$$g'(t) = 2e^{t-2\delta e^t}(1 - 2\delta e^t).$$

So the positive continuous function g takes the maximum value $\frac{1}{\delta e}$ at $t = -\log 2\delta$. Hence from (2.2), we have

$$\rho(f(z+n)) \le g(x+n) \le \frac{1}{\delta e}$$

for all $z \in F$ and $n = 0, 1, 2, \ldots$. Therefore z_0 belongs to the set $\mathbf{N}(f)$. This completes the proof. □

Let $\mathcal{A}$ be the space of all entire functions equipped with the topology of uniform convergence on compact subset of the complex plane $\mathbf{C}$.

Theorem 2.2. *If $\mathcal{F}$ is a family of entire functions which is dense in $\mathcal{A}$, then the family $\mathcal{F}$ is not normal on any open set in $\mathbf{C}$.*

Proof. For a given open set Ω and $z_0 \in \Omega$, we choose a positive number ε so that $\overline{B}(z_0, \varepsilon) \subset \Omega$ is a compact subset. By the density of $\mathcal{F}$, there exists a sequence $\{g_n(z)\} \subset \mathcal{F}$, such that

$$|g_n(z) - n(z - z_0)| < \frac{1}{n}$$

for all $z \in \overline{B}(z_0, \varepsilon)$. So we have

$$|g_n(z_0)| < \frac{1}{n} \to 0$$

as $n \to \infty$.

On the other hand, for $z \in \overline{B}(z_0, \varepsilon) \setminus \{z_0\}$,

$$|g_n(z)| > n|z - z_0| - \frac{1}{n} \to \infty$$

as $n \to \infty$. Hence the family $\mathcal{F}$ is not normal on Ω. □

An entire function f whose translates are dense in $\mathcal{A}$ is called *a universal entire function*. In 1929, the existence of a universal entire function was proved by G. D. Birkhoff [1]. For a proof of the following theorem, see [5, pp. 60–61].

Theorem (Birkhoff [1])**.** *There is an entire function whose integer translations are dense in the set $\mathcal{A}$ of all entire functions.*

Birkhoff's Theorem and Theorem 2.2 have the following consequence.

Corollary 2.3. *There is an entire function f such that $\mathbf{N}(f) = \varnothing$.*

In Definition 1.2, we considered positive integer translations only. At this time we allow the negative integer translations also.

Definition 2.4. Let f be a meromorphic function in the complex plane $\mathbf{C}$. We define as set $\mathbf{N}_B(f)$ by the set of all $z_0 \in \mathbf{C}$ such that the family $\{f(z+n) \colon n = 0, \pm 1, \pm 2, \dots\}$ is normal at z_0.

By the same reasons as in $\mathbf{N}(f)$, the set $\mathbf{N}_B(f)$ is a periodic open set with period 1. And clearly $\mathbf{N}_B(f)$ is a subset of $\mathbf{N}(f)$. But $\mathbf{N}_B(f)$ is different from $\mathbf{N}(f)$ for a certain meromorphic function f.

Theorem 2.5. *There is an entire function f such that $\mathbf{N}_B(f) \neq \mathbf{N}(f)$.*

Proof. We shall show that the entire function

$$f(z) = \frac{e^{z^2}}{\Gamma(z)}$$

satisfies the property by proving $0 \in \mathbf{N}(f) \backslash \mathbf{N}_B(f)$.

(i) $0 \notin \mathbf{N}_B(f)$: Let $0 < \varepsilon < \frac{1}{10}$ and n be a positive integer. Since

$$z\Gamma(z) = \Gamma(z+1),$$

we have

$$\begin{aligned} \Gamma(\varepsilon - n) &= \frac{\Gamma(\varepsilon - n + 1)}{\varepsilon - n} \\ &= \frac{\Gamma(\varepsilon)}{(\varepsilon - n)(\varepsilon - n + 1) \cdots (\varepsilon - 1)} \end{aligned}$$

and

$$|f(\varepsilon - n)| = \left| \frac{e^{(\varepsilon - n)^2} (\varepsilon - n)(\varepsilon - n + 1) \cdots (\varepsilon - 1)}{\Gamma(\varepsilon)} \right|. \tag{2.3}$$

The right-hand side of (2.3) tends to infinity as $n \to \infty$. But $f(-n) = 0$ for all positive integers. Hence 0 does not belong to $\mathbf{N}_B(f)$.

(ii) $0 \in \mathbf{N}(f)$: Since

$$\frac{\Gamma'(z)}{\Gamma(z)} = -\gamma + \sum_{m=1}^{\infty} \frac{z-1}{m(z+m-1)}$$

where

$$\begin{aligned} \gamma &= \lim_{n \to \infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n \right) \\ &= 0.5772157 \cdots, \end{aligned}$$

we get

$$\begin{aligned} f'(z) &= \frac{e^{z^2}}{\Gamma(z)} \left(2z - \frac{\Gamma'(z)}{\Gamma(z)} \right) \\ &= \frac{e^{z^2}}{\Gamma(z)} \left(2z + \gamma - \sum_{m=1}^{\infty} \frac{z-1}{m(z+m-1)} \right). \end{aligned}$$

Let K be a compact subset of $B(0, \frac{1}{10})$. Then for $z \in K$ and positive integer n,

$$\begin{aligned}|f'(z+n)| &= \left|\frac{e^{(z+n)^2}}{\Gamma(z+n)}\left(2(z+n)+\gamma-\sum_{m=1}^{\infty}\frac{z+n-1}{m(z+n+m-1)}\right)\right| \\ &\leq \left|\frac{e^{(z+n)^2}}{\Gamma(z+n)}\right|\left(2(n+1)+\gamma+n\sum_{m=1}^{\infty}\frac{1}{m^2}\right) \\ &= \left|\frac{e^{(z+n)^2}}{\Gamma(z+n)}\right|\left(2(n+1)+\gamma+\frac{\pi^2}{6}n\right).\end{aligned}$$

So we can write

$$\begin{aligned}\rho(f(z+n)) &\leq \frac{\left|\frac{e^{(z+n)^2}}{\Gamma(z+n)}\right|\left(2(n+1)+\gamma+\frac{\pi^2}{6}n\right)}{1+\left|\frac{e(z+n)^2}{\Gamma(z+n)}\right|^2} \\ &\leq \left|\frac{\Gamma(z+n)}{e^{(z+n)^2}}\right|\left(2(n+1)+\gamma+\frac{\pi^2}{6}n\right).\end{aligned} \tag{2.4}$$

By Stirling's formula,

$$\left|\frac{\Gamma(z+n)}{e^{(z+n)^2}}\right| \sim \frac{|\sqrt{2\pi}(z+n)^{z+n-1/2}e^{-(z+n)}|}{|e^{(z+n)^2}|}. \tag{2.5}$$

But for all $z \in K \subset B(0, \frac{1}{10})$, we have

$$\mathrm{Re}((z+n)^2+(z+n)) \geq n^2+\frac{n}{2}$$

and

$$|z+n|^{z+n-1/2} < e^{n\log(n+1)}.$$

Thus we have

$$\begin{aligned}&\frac{1}{n}-\frac{|(z+n)^{z+n-1/2}e^{-(z+n)}|}{|e^{(z+n)^2}|} \\ &\qquad > \frac{1}{n}-\frac{e^{n\log(n+1)}}{e^{n^2+\frac{n}{2}}} = \frac{e^{n^2+\frac{n}{2}}-ne^{n\log(n+1)}}{ne^{n^2+\frac{n}{2}}} \\ &\qquad > \frac{e^{n^2}-e^{n\log(n+1)}}{e^{n^2+\frac{n}{2}}} > 0.\end{aligned}$$

With (2.5), for sufficiently large n we obtain an inequality,

$$\left|\frac{\Gamma(z+n)}{e^{(z+n)^2}}\right| < \frac{\sqrt{2\pi}}{n}. \tag{2.6}$$

So we can write (2.4) as

$$\rho(f(z+n)) \leq \frac{\sqrt{2\pi}}{n}\left(2(n+1)+\gamma+\frac{\pi^2}{6}n\right) < 16$$

for all $z \in K$ and every positive integer n. Hence by Marty's theorem $0 \in \mathbf{N}(f)$. This completes the proof of the theorem. □

3. Main results

Now we shall prove our main result. We know that, for every meromorphic function f, the set $\mathbf{N}(f)$ is a periodic open set with period 1. To prove the existence part we shall construct an entire function with the help of the Weierstrass factorization theorem.

Theorem 3.1. *Let G be an open set in the complex plane $\mathbf{C}$. Then there exists an entire function f such that $\mathbf{N}(f) = G$ if and only if G is periodic with period* 1.

Theorem 3.1 is also true for the set $\mathbf{N}_B(f)$. We can prove it by a slight modification of the proof of Theorem 3.1.

Theorem 3.2. *Let G be an open set in the complex plane $\mathbf{C}$. Then there exists an entire function f such that $\mathbf{N}_B(f) = G$ if and only if G is periodic with period* 1.

Proof of Theorem 3.1. If $G = \mathbf{C}$, then we take a constant function for f. We assume G is a proper subset of $\mathbf{C}$. Since G is a periodic open set with period 1, it is enough to check for the set $\{z \in G\colon 0 \le \operatorname{Re}\ z < 1\}$ to show $\mathbf{N}(f) = G$. Because of its length we divide the proof into three steps and put each step as a section.

3.1. **Construction of an entire function.** Let G be a periodic open set in the complex plane $\mathbf{C}$ with period 1. For each positive integer i, we let

$$W_i = \{w_{ij}\colon j \ge i\}$$

be a countable dense subset of $R_i = \{z \in G^c\colon 0 \le \operatorname{Re}\ z < 1, i - 1 \le |\operatorname{Im}\ z| < i\}$ (for some i, the set W_i can be empty or finite). We let $W = \bigcup_{i=1}^{\infty} W_i$ and

$$G_0^c = \{z \in G^c\colon 0 \le \operatorname{Re}\ z < 1\}. \tag{3.1}$$

Then W is a countable dense subset of G_0^c.

From the set W, we form a sequence $Z = \{z_{ij}\}$ in G^c as in Table 3.1 so that it has no finite limit point. In the construction of the sequence Z, for some positive

Table 3.1. Sequence $Z = \{z_{ij}\}$

C_1	C_2	C_3	$\ldots$
$z_{11} = w_{11} + 2$	$z_{12} = w_{11} + (2^3 + 1)!$	$z_{13} = w_{11} + (3^3 + 1)!$	$\ldots$
	$z_{22} = w_{12} + (2^3 + 2)!$	$z_{23} = w_{12} + (3^3 + 2)!$	
	$z_{32} = w_{22} + (2^3 + 3)!$	$z_{33}+ = w_{22} + (3^3 + 3)!$	
		$z_{43} = w_{13} + (3^3 + 4)!$	
		$z_{53} = w_{23} + (3^3 + 5)!$	
		$z_{63} = w_{33} + (3^3 + 6)!$	

integer i if the set W_i contains only finitely many elements, for instance, suppose that $W_i = \{w_{ij}\colon i \le j \le k\}$, and if z_{st} is a translation of w_{il} where $l > k$, then we skip the term z_{st}. So if we let $\|C_j\|$ be the number of the elements in the jth column of the Table 3.1, then we have

$$\|C_j\| \le \frac{j(j+1)}{2}. \tag{3.2}$$

And for each $z_{ij} \in C_j$, we have

$$\text{Re } z_{ij} < 1 + \left(j^3 + \frac{j(j+1)}{2}\right)! \tag{3.3}$$

and

$$(j^3+1)! \le |z_{ij}| < j + 1 + \left(j^3 + \frac{j(j+1)}{2}\right)!. \tag{3.4}$$

We define a function g on the complex plane $\mathbf{C}$ by

$$g(z) = \prod_{j=1}^{\infty} \prod_{z_{ij} \in C_j} \left(1 - \frac{z}{z_{ij}}\right). \tag{3.5}$$

From (3.2) and (3.4) we have

$$\begin{aligned} \sum_{j=1}^{\infty} \sum_{z_{ij} \in C_j} \frac{1}{|z_{ij}|} &< \sum_{j=1}^{\infty} \frac{j(j+1)}{2} \frac{1}{(j^3+1)!} \\ &< e. \end{aligned}$$

Therefore the function g is entire and its only zeros are at the points $z_{ij} \in Z$.

Finally, we define a function f on the complex plane $\mathbf{C}$ by

$$f(z) = e^z g(z). \tag{3.6}$$

Then f is an entire function whose zero set is equal to that of g.

We shall show that f is a suitable function.

3.2. **For all** $z \in G$**, we have** $z \in \mathbf{N}(f)$**.** For $z_0 \in G$ (we assume $0 \le \text{Re}\, z_0 < 1$), $0 < \varepsilon < \frac{1}{10}$, let $F = \overline{B}(z_0, \varepsilon)$ be a compact subset of G. Since G^c is a closed set, there is a positive number ε_0 such that

$$d(F, G^c) = \varepsilon_0. \tag{3.7}$$

Let n be a sufficiently large positive integer and suppose that k is the largest integer such that

$$1 + \left(k^3 + \frac{k(k+1)}{2}\right)! < n. \tag{3.8}$$

We fix n for a moment and estimate the value of $|g(z+n)|$, for $z \in F$, and then we show $|f(n+z)| \to \infty$ uniformly on the compact set F as $n \to \infty$.

To estimate the value $|g(z+n)|$, for $z \in F$, we consider the following four cases. Throughout this paper C_j denotes the jth column of Table 3.1.

Case 1. $j < k$

For each $z_{ij} \in C_j$ and $z \in F$, we have

$$\begin{aligned} |z_{ij} - (z+n)| &> \text{Re}(z + n - z_{ij}) \\ &> n - 2 - \left(j^3 + \frac{j(j+1)}{2}\right)! \end{aligned}$$

by (3.3). Thus from the inequalities (3.4) and (3.8), we obtain

$$\begin{aligned}\left|1-\frac{z+n}{z_{ij}}\right| &= \frac{1}{|z_{ij}|}|z_{ij}-(z+n)| \\ &> \frac{n-2-(j^3+\frac{j(j+1)}{2})!}{j+1+(j^3+\frac{j(j+1)}{2})!} \\ &> \frac{(k^3+\frac{k(k+1)}{2})!-1-(j^3+\frac{j(j+1)}{2})!}{j+1+(j^3+\frac{j(j+1)}{2})!} \\ &> 1.\end{aligned}$$

Hence we can conclude

$$\prod_{j=1}^{k-1}\prod_{z_{ij}\in C_j}\left|1-\frac{z+n}{z_{ij}}\right| > 1. \tag{3.9}$$

Case 2. $j=k$

If $z_{ik}=w_{kk}+(k^3+\frac{k(k+1)}{2})!\in C_k$, then for all $z\in F$ we have

$$|z_{ik}-(z+n)|\geq \varepsilon_0$$

by (3.7). Thus by (3.4), we get

$$\left|1-\frac{z+n}{z_{ik}}\right| \geq \frac{\varepsilon_0}{k+1+(k^3+\frac{k(k+1)}{2})!}. \tag{3.10}$$

Now we suppose $z_{ik}\neq w_{kk}+(k^3+\frac{k(k+1)}{2})!$. Then from Table 3.1, we get

$$\text{Re }\ z_{ik} < 1+\left(k^3+\frac{k(k+1)}{2}-1\right)!$$

and

$$|z_{ik}| < k+1+\left(k^3+\frac{k(k+1)}{2}-1\right)!.$$

Hence for each $z_{ik}\in C_k$ ($z_{ik}\neq w_{kk}+(k^3+\frac{k(k+1)}{2})!$), we have

$$\begin{aligned}\left|1-\frac{z+n}{z_{ik}}\right| &= \frac{1}{|z_{ik}|}|z_{ik}-(z+n)| \\ &\geq \frac{n-\text{Re }\ z_{ik}-1}{|z_{ik}|} \\ &\geq \frac{n-2-(k^3+\frac{k(k+1)}{2}-1)!}{k+1+(k^3+\frac{k(k+1)}{2}-1)!}.\end{aligned}$$

But we defined $k\geq 2$ in (3.8) so that $1+(k^3+\frac{k(k+1)}{2})!<n$. Thus

$$\begin{aligned}\left|1-\frac{z+n}{z_{ik}}\right| &> \frac{(k^3+\frac{k(k+1)}{2})!-(k^3+\frac{k(k+1)}{2}-1)!-1}{k+1+(k^3+\frac{k(k+1)}{2}-1)!} \\ &> 1.\end{aligned} \tag{3.11}$$

From (3.10) and (3.11), we conclude

$$\prod_{z_{ik}\in C_k}\left|1-\frac{z+n}{z_{ik}}\right| > \frac{\varepsilon_0}{k+1+(k^3+\frac{k(k+1)}{2})!} \tag{3.12}$$

for all $z\in F$.

Case 3. $j=k+1$

We shall show that

$$\prod_{x_{ij}\in C_{k+1}}\left|1-\frac{z+n}{z_{ij}}\right| > \varepsilon_0^2\frac{[((k+1)^3+2)!-((k+1)^3+1)!]^{\frac{(k+1)(k+2)}{2}-2}}{\left[k+2+\left((k+1)^3+\frac{(k+1)(k+2)}{2}\right)!\right]^{\frac{(k+1)(k+2)}{2}}} \tag{3.13}$$

for all $z\in F$.

Let $z_{\alpha j}$, $z_{\beta_j}\in C_j$ be the first and second nearest points to the compact set $F+n$; then from (3.7) and Table 3.1, we have

$$|z_{ij}-(z+n)|\geq\begin{cases}\varepsilon_0, & \text{if } z_{ij}=z_{\alpha j} \text{ or } z_{\beta_j},\\ (j^3+2)!-(j^3+1)!, & \text{if } z_{ij}\neq z_{\alpha j} \text{ and } z_{\beta_j}.\end{cases}$$

So by (3.2) and (3.4), we have

$$\begin{aligned}\prod_{z_{ij}\in C_j}\left|1-\frac{z+n}{z_{ij}}\right| &\geq \frac{\prod_{z_{ij}\in C_j}|z_{ij}-(z+n)|}{\left(j+1+(j^3+\frac{j(j+1)}{2})!\right)^{\frac{j(j+1)}{2}}}\\ &> \varepsilon_0^2\frac{((j^3+2)!-(j^3+1)!)^{\frac{j(j+1)}{2}-2}}{\left(j+1+\left(j^3+\frac{j(j+1)}{2}\right)!\right)^{\frac{j(j+1)}{2}}}.\end{aligned} \tag{3.14}$$

Put $j=k+1$ into (3.14); then it becomes (3.13).

Case 4. $j>k+1$

We define a sequence $\{\alpha_k\}$ by

$$a_k=\min_{\substack{z\in F\\ n\in N(k)}}\left|\prod_{j=k+2}^{\infty}\prod_{z_{ij}\in C_j}\left(1-\frac{z+n}{z_{ij}}\right)\right| \tag{3.15}$$

where

$$N(k)=\left\{n\colon 1+\left(k^3+\frac{k(k+1)}{2}\right)!<n\leq 1+\left((k+1)^3+\frac{(k+1)(k+2)}{2}\right)!\right\}.$$

Since $F\subset G$, by the periodicity of G, we have $F+n\subset G$ for all positive integers n. Hence $|g(z+n)|\neq 0$ for all $z\in F$. Thus $a_k>0$ for all positive integers k.

We shall show that $\{a_k\}$ is eventually an increasing sequence. For convenience, we write $\min_{z,n_1}$ and $\min_{z',n_2}$ instead of $\min_{\substack{z\in F\\ n_1\in N(k)}}$ and $\min_{\substack{z'\in F\\ n_2\in N(k+1)}}$ (and likewise for max). To prove that a_k is increasing, we examine the ratio:

$$
\begin{aligned}
\frac{a_{k+1}}{a_k} &\geq \frac{\min_{z',n_2}\left|\prod_{j=k+3}^{\infty}\prod_{z'_{ij}\in C_j}(1-\frac{z'+n_2}{z'_{ij}}\right|}{\max_{z',n_1}\left|\prod_{j=k+2}^{\infty}\prod_{z_{ij}\in C_j}(1-\frac{z+n_1}{z'_{ij}}\right|}\\
&\geq \min_{z,n_1}\prod_{z_{ij}\in C_{k+2}}\left|\frac{z_{ij}}{z_{ij}-(z+n_1)}\right| \times \frac{\min_{z',n_2}\prod_{z'_{ij}\in C_{k+3}}\left|\frac{z'_{ij}-(z'+n_2)}{z'_{ij}}\right|}{\max_{z,n_1}\prod_{z_{ij}\in C_{k+3}}\left|\frac{z_{ij}-(z+n_1)}{z_{ij}}\right|}\\
&\quad\times \frac{\min_{z',n_2}\prod_{z'_{ij}\in C_{k+4}}\left|\frac{z'_{ij}-(z'+n_2)}{x'_{ij}}\right|}{\max_{z,n_1}\prod_{z_{ij}\in C_{k+4}}\left|\frac{z_{ij}-(z+n_1)}{z_{ij}}\right|}\times\cdots\\
&\geq \min_{z,n_1}\prod_{z_{ij}\in C_{k+2}}\left|\frac{z_{ij}}{z_{ij}-(z+n_1)}\right| \times \frac{\min_{z',n_2}\prod_{z'_{ij}\in C_{k+3}}|z'_{ij}-(z'+n_2)|}{\max_{z,n_1}\prod_{z_{ij}\in C_{k+3}}|z_{ij}-(z+n_1)|}\\
&\quad\times \frac{\min_{z',n_2}\prod_{z'_{ij}\in C_{k+4}}|z'_{ij}-(z'+n_2)|}{\max_{z,n_1}\prod_{z_{ij}\in C_{k+4}}|z_{ij}-(z+n_1)|}\times\cdots\\
&= \min_{z,n_1}\prod_{z_{ij}\in C_{k+2}}\left|\frac{z_{ij}}{z_{ij}-(z+n_1)}\right| \times \min_{z',n_2}\prod_{z'_{ij}\in C_{k+3}}|z'_{ij}-(z'+n_2)|\\
&\quad\times \frac{\min_{z',n_2}\prod_{z'_{ij}\in C_{k+4}}|z'_{ij}-(z'+n_2)|}{\max_{z,n_1}\prod_{z_{ij}\in C_{k+3}}|z_{ij}-(z+n_1)|}\\
&\quad\times \frac{\min_{z',n_2}\prod_{z'_{ij}\in C_{k+5}}|z'_{ij}-(z'+n_2)|}{\max_{z,n_1}\prod_{z_{ij}\in C_{k+4}}|z_{ij}-(z+n_1)|}\times\cdots.
\end{aligned}
\tag{3.16}
$$

We can easily check that the first and second terms of the last equation of (3.16) are greater than 1. From (3.4) and Table 3.1, for all $z \in F$, $n_1 \in N(k)$ and $z_{ij} \in C_{k+l-1}$ $(l \geq 4)$, we have

$$
|z_{ij}-(z+n_1)| \leq |z_{ij}| \leq k+l+\left((k+l-1)^3+\frac{(k+l-1)(k+l)}{2}\right)!
$$

and for $z' \in F$, $n_2 \in N(k+1)$ and $z'_{ij} \in C_{k+l}$ $(l \geq 4)$, we have

$$
\begin{aligned}
|z'_{ij}-(z+n_2)| &\geq \operatorname{Re}|z'_{ij}-(z+n_2)|\\
&\geq ((k+l)^3+1)!-2-\left[\left((k+2)^3+\frac{(k+2)(k+3)}{2}\right)!\right]\\
&> ((k+l)^3+1)!-(k+3)^3!.
\end{aligned}
$$

So we have an inequality,

$$
\begin{aligned}
&|z'_{ij}-(z'+n_2)|-|z_{ij}-(z+n_1)| \\
&\qquad > ((k+l)^3+1)!-(k+3)^3! \\
&\qquad\quad -\left[k+l+\left((k+l-1)^3+\frac{(k+l-1)(k+l)}{2}\right)!\right] \\
&\qquad = (k+3)^3!\left[(k+l)^3+1)\cdots((k+3)^3+1)-1\right. \\
&\qquad\quad \left.-\left(k+l+(k+l-1)^3+\frac{(k+l-1)(k+l)}{2}\right)\cdots((k+3)^3+1)\right].
\end{aligned}
\tag{3.17}
$$

It is easy to check that

$$(k+l)^3+1>k+l+(k+l-1)^3+\frac{(k+l-1)(k+l)}{2}$$

for $l \geq 4$. Hence the right-hand side of (3.17) is greater than 0 and we have

$$\left|\frac{z'_{ij}-(z'+n_2)}{z_{ij}-(z+n_1)}\right|>1$$

for all $z, z' \in F$, $n_1 \in N(k)$, $n_2 \in N(k+1)$, $z_{ij} \in C_{k+l-1}$ and $z'_{ij} \in C_{k+l}$. Since $\|C_{k+l}\| \geq \|C_{k+l-1}\|$, we have

$$\frac{\min_{z',n_2}\prod_{z'_{ij}\in C_{k+l}}|z'_{ij}-(z'+n_2)|}{\max_{z,n_1}\prod_{z_{ij}\in C_{k+l-1}}|z_{ij}-(z+n_1)|}>1$$

for all $l \geq 4$.

Now we can conclude that $\frac{a_{k+1}}{a_k} > 1$. Therefore $\{a_k\}$ is a positive and eventually increasing sequence.

From (3.9), (3.12), (3.13) and (3.15), for all $z \in f$ and $n \in N(k)$, we can write

$$
\begin{aligned}
|f(z+n)| &= |e^{z+n}g(z+n)| \\
&= |e^{z+n}| \times \prod_{j=1}^{k-1}\prod_{z_{ij}\in C_j}\left|1-\frac{z+n}{z_{ij}}\right| \times \prod_{z_{ij}\in C_k}\left|1-\frac{z+n}{z_{ij}}\right| \\
&\quad \times \prod_{z_{ij}\in C_{k+1}}\left|1-\frac{z+n}{z_{ij}}\right| \times \left|\prod_{j=k+2}^{\infty}\prod_{z_{ij}\in C_j}\left(1-\frac{z+n}{z_{ij}}\right)\right| \\
&> a_k e^n \frac{\varepsilon_0}{k+1+(k^3+\frac{k(k+1))}{2})!} \\
&\quad \times \varepsilon_0^2 \frac{[((k+1)^3+2)!-((k+1)^3+1)!]^{\frac{(k+1)(k+2)}{2}-2}}{\left[k+2+\left((k+1)^3+\frac{(k+1)(k+2)}{2}\right)!\right]^{\frac{(k+1)(k+2)}{2}}} \\
&> \varepsilon_0^3 a_k e^n \frac{[((k+1)^3+2)!-((k+1)^3+1)!]^{\frac{(k+1)(k+2)}{2}-2}}{\left[k+2+\left((k+1)^3+\frac{(k+1)(k+2)}{2}\right)!\right]^{\frac{(k+1)(k+2)}{2}+1}}.
\end{aligned}
\tag{3.18}
$$

Here we will show that

$$\log e^n-\left(\frac{(k+1)(k+2)}{2}+1\right)\log\left[k+2+\left((k+1)^3+\frac{(k+1)(k+2)}{2}\right)!\right]>0. \tag{3.19}$$

Since $n \in N(k)$, $1+(k^3+\frac{k(k+1)}{2})! < n$ by the definition of $N(k)$. We apply Stirling's formula to the left of (3.19); then we have

$$\begin{aligned}
&\log e^n - \left(\frac{(k+1)(k+2)}{2}+1\right)\log\left[k+2+\left((k+1)^3+\frac{(k+1)(k+2)}{2}\right)!\right] \\
&\geq \left(k^3+\frac{k(k+1)}{2}\right)! - \frac{1}{2}(k+2)^2\log(k+2)^3! \\
&\sim \left(k^3+\frac{k(k+1)}{2}\right)! - \frac{1}{2}(k+2)^2\log\sqrt{2\pi(k+2)^3}e^{-(k+2)^3}(k+2)^{3(k+2)^3} \\
&= \left(k^3+\frac{k(k+1)}{2}\right)! + \frac{1}{2}(k+2)^5 \\
&\quad - \frac{1}{2}(k+2)^2(3(k+2)^3\log(k+2)+\log\sqrt{2\pi(k+2)^3}) \\
&> \left(k^3+\frac{k(k+1)}{2}\right)! - \frac{3}{2}(k+2)^5\log(k+2) \\
&> 0
\end{aligned}$$

for large enough k. This implies

$$|f(z+n)| > \varepsilon_0^3 a_k[((k+1)^3+2)! - ((k+1)^3+1)!]^{\frac{(k+1)(k+2)}{2}-2} \tag{3.20}$$

and the right-hand side of (3.20) tends to infinity as $k \to \infty$. From (3.8), $k \to \infty$ as $n \to \infty$, so $|f(z+n)| \to \infty$ uniformly on the compact set F as $n \to \infty$. Therefore $z_0 \in \mathbf{N}(f)$.

3.3. **If** $z \in G^c$**, then** $z \notin \mathbf{N}(f)$**.** Since W is a countable subset of G_0^c, for each $w_{st} \in W$ and $\delta > 0$, we can choose a number δ_0, $0 < \delta_0 < \delta$, so that $w_{st}+\delta_0 \neq w_{kl}$ (and $\neq w_{kl}+1$) for all $w_{kl} \in W$. From the construction of the sequence Z, t is the smallest integer such that C_t contains the translation of w_{st}. And the translations of w_{st} have the form

$$z_{\gamma n} = w_{st} + (n^3+\gamma)! \tag{3.21}$$

where $n \geq t$ and $\gamma = \frac{t(t-1)}{2}+s$. We fix w_{st} and show $|f(z_{\gamma n}+\delta_0)| \to \infty$ as $n \to \infty$.

For each positive integer n, we let

$$I_n = \prod_{j=1}^{n-1}\prod_{z_{ij}\in C_j}\left(1-\frac{z_{\gamma n}+\delta_0}{z_{ij}}\right), \tag{3.22.1}$$

$$J_n = \prod_{z_{ij}\in C_n}\left(1-\frac{z_{\gamma n}+\delta_0}{z_{ij}}\right), \tag{3.22.2}$$

and

$$K_n = \prod_{j=n+1}^{\infty}\prod_{z_{ij}\in C_j}\left(1-\frac{z_{\gamma n}+\delta_0}{z_{ij}}\right). \tag{3.22.3}$$

We fix n for a moment and write

$$\begin{aligned} g(z_{\gamma n}+\delta_0) &= \prod_{j=1}^{\infty}\prod_{z_{ij}\in C_j}\left(1-\frac{z_{\gamma n}+\delta_0}{z_{ij}}\right) \\ &= I_n \times J_n \times K_n. \end{aligned} \tag{3.23}$$

We examine the moduli of I_n, J_n, and K_n.

(i) *Estimation of* $|I_n|$: Assume $\delta_0 < 1$; then by (3.4), for each $z_{ij} \in C_j$ where $1 \le j < n$,

$$\begin{aligned} \left|\frac{z_{\gamma n}+\delta_0}{z_{ij}}\right| &\ge \frac{n^3!}{j+1+(j^3+\frac{j(j+1)}{2})!} \\ &\ge \frac{n^3!}{n+((n-1)^3+\frac{n(n-1)}{2})!} \\ &> 2. \end{aligned}$$

So we obtain

$$\begin{aligned} \left|1-\frac{z_{\gamma n}+\delta_0}{z_{ij}}\right| &\ge \left|1-\left|\frac{z_{\gamma n}+\delta_0}{z_{ij}}\right|\right| \\ &> 1 \end{aligned}$$

and

$$|I_n| = \prod_{j=1}^{n-1}\prod_{z_{ij}\in C_j}\left|1-\frac{z_{\gamma n}+\delta_0}{z_{ij}}\right| > 1. \tag{3.24}$$

(ii) *Estimation of* $|J_n|$: Since

$$\begin{aligned} &\min\{|z_{\gamma n}-z_{in}|\colon z_{in}\in C_n, z_{in}\ne z_{\gamma n}\} \\ &\quad > |z_{1n}-z_{2n}| \ge \mathrm{Re}(z_{2n}-z_{1n}) \\ &\quad > (n^3+2)! - ((n^3+1)!+1) \\ &\quad > (n^3+1)! \end{aligned}$$

and

$$\begin{aligned} |J_n| &= \prod_{z_{in}\in C_n}\left|1-\frac{z_{\gamma n}+\delta_0}{z_{in}}\right| \\ &= \left|1-\frac{z_{\gamma n}+\delta_0}{z_{\gamma n}}\right| \times \prod_{\substack{z_{in}\in C_n \\ z_{in}\ne z_{\gamma n}}}\left|1-\frac{z_{\gamma n}+\delta_0}{z_{in}}\right| \\ &\ge \frac{\delta_0}{|z_{\gamma n}|} \times \prod_{\substack{z_{in}\in C_n \\ z_{in}\ne z_{\gamma n}}}\left|\frac{z_{in}-(z_{\gamma n}+\delta_0)}{z_{in}}\right|, \end{aligned}$$

with (3.2) and (3.4), we obtain the following inequality

$$|J_n| > \delta_0 \frac{((n^3+1)!)^{\frac{n(n+1)}{2}-1}}{\left(n+1+(n^3+\frac{n(n+1)}{2})!\right)^{\frac{n(n+1)}{2}}}. \tag{3.25}$$

(iii) $\{|K_n|\}$ *is an eventually increasing sequence*: We consider the ratio (for convenience, we let $z_{\gamma\hat{n}} = z_{\gamma(n+1)}$) :

$$
\begin{aligned}
\frac{k_{n+1}}{K_n} &= \prod_{j=n+1}^{\infty} \frac{\prod_{z'_{ij}\in C_{j+1}}\left(1-\frac{z_{\gamma\hat{n}}+\delta_0}{z'_{ij}}\right)}{\prod_{z_{ij}} \in C_j\left(1-\frac{z_{\gamma n}+\delta_0}{z_{ij}}\right)} \\
&= \prod_{z_{ij}\in C_{n+1}} \frac{z_{ij}}{z_{ij}-(z_{\gamma n}+\delta_0)} \times \frac{\prod_{z'_{ij}\in C_{n+2}}\left(1-\frac{z_{\gamma\hat{n}}+\delta_0}{z'_{ij}}\right)}{\prod_{z_{ij}\in C_{n+2}}\left(1-\frac{z_{\gamma n}+\delta_0}{z_{ij}}\right)} \\
&\quad \times \frac{\prod_{z'_{ij}\in C_{n+3}}\left(1-\frac{z_{\gamma\hat{n}}+\delta_0}{z'_{ij}}\right)}{\prod_{z_{ij}\in C_{n+3}}\left(1-\frac{z_{\gamma n}+\delta_0}{z_{ij}}\right)} \times \cdots \\
&= \prod_{z_{ij}\in C_{n+1}} \frac{z_{ij}}{z_{ij}-(z_{\gamma n}+\delta_0)} \times \frac{\prod_{z'_{ij}\in C_{n+2}}(z'_{ij}-(z_{\gamma\hat{n}}+\delta_0))}{\prod_{z_{ij}\in C_{n+2}}(z_{ij}-(z_{\gamma n}+\delta_0))} \qquad (3.26) \\
&\quad \times \frac{\prod_{z'_{ij}\in C_{n+3}}(z'_{ij}-(z_{\gamma\hat{n}}+\delta_0))}{\prod_{z_{ij}\in C_{n+3}}(z_{ij}-(z_{\gamma n}+\delta_0))} \times \cdots \\
&= \prod_{z_{ij}\in C_{n+1}} \frac{z_{ij}}{z_{ij}-(z_{\gamma n}+\delta_0)} \times \prod_{z'_{ij}\in C_{n+2}} (z'_{ij}-(z_{\gamma\hat{n}}+\delta_0)) \\
&\quad \times \frac{\prod_{z'_{ij}\in C_{n+3}}(z'_{ij}-(z_{\gamma\hat{n}}+\delta_0))}{\prod_{z_{ij}\in C_{n+2}}(z_{ij}=(z_{\gamma n}+\delta_0))} \times \frac{\prod_{z'_{ij}\in C_{n+4}}(z'_{ij}-(z_{\gamma\hat{n}}+\delta_0))}{\prod_{z_{ij}\in C_{n+3}}(z_{ij}-(z_{\gamma n}+\delta_0))} \\
&\quad \times \cdots .
\end{aligned}
$$

The moduli of the first and second parts of the last equation of (3.26) are greater than 1. And for $z'_{ij} \in C_{n+l+1}$ and $z_{ij} \in C_{n+l}$, $l \geq 2$, we have

$$
\begin{aligned}
\frac{|z'_{ij}-(z_{\gamma\hat{n}}+\delta_0)|}{|z_{ij}-(z_{\gamma n}+\delta_0)|} &> \frac{|z'_{ij}|-|z_{\gamma\hat{n}}+\delta_0|}{|z_{ij}|} \\
&> \frac{((n+l+1)^3+1)! - \left[(n+1)+1+\left((n+1)^3+\frac{(n+1)(n+2)}{2}\right)!\right]}{n+l+1+\left((n+l)^3+\frac{(n+l)(n+l+1)}{2}\right)!} \\
&> \frac{((n+l+1)^3+1)! - (n+2)^3!}{((n+l)^3+(n+l)^2)!} \\
&> 1
\end{aligned}
$$

for sufficiently large n. Since $\|C_{n+l+1}\| \geq \|C_{n+l}\|$,

$$
\frac{\prod_{z'_{ij}\in C_{n+l+1}} |z'_{ij}-(z_{\gamma\hat{n}}+\delta_0)|}{\prod_{z_{ij}\in C_{n+l}} |z_{ij}-(z_{ij}+\delta_0)|} > 1
$$

for all $l \geq 2$. Hence we can conclude that $|\frac{K_{n+1}}{K_n}| > 1$ for sufficiently large n.

Now we can estimate the value $|f(z_{\gamma n}+\delta_0)|$. From (3.23), (3.24) and (3.25), we obtain

$$\begin{aligned}|f(z_{\gamma n}+\delta_0)| &= |e^{z_{\gamma n}+\delta_0} g(z_{\gamma n}+\delta_0)| \\ &= |e^{z_{\gamma n}+\delta_0}|\cdot|I_n|\cdot|J_n|\cdot|K_n| \\ &\geq e^{(n^3+\gamma)!-1}\delta_0 \frac{((n^3+1)!)^{\frac{n(n+1)}{2}-1}}{\left(n+1+(n^3+\frac{n(n+1)}{2})!\right)^{\frac{n(n+1)}{2}}}|K_n|.\end{aligned} \tag{3.27}$$

Here we let

$$\alpha_n = \frac{e^{(n^3+\gamma)!-1}}{\left(n+1+\left(n^3+\frac{n(n+1)}{2}\right)!\right)^{n(n+1)}}.$$

Then for large enough n, by Stirling's formula we get

$$\begin{aligned}\log\alpha_n &\geq (n^3+\gamma)! - \frac{n(n+1)}{2}\log(n^3+n^2)! \\ &\sim (n^3+\gamma)! - \frac{n(n+1)}{2}\log\sqrt{2\pi(n^3+n^2)}e^{-(n^3+n^2)}(n^3+n^2)^{n^3+n^2} \\ &= (n^3+\gamma)! + \frac{n(n+1)}{2}(n^3+n^2) \\ &\qquad - \frac{n(n+1)}{2}[(n^3+n^2)\log(n^3+n^2)+\log\sqrt{2\pi(n^3+n^2)}] \\ &> (n^3+\gamma)! - \frac{n^3(n+1)^2}{2}\log(n^3+n^2) \\ &> 0.\end{aligned}$$

So $\alpha_n > 1$ and the inequality (3.27) becomes

$$|f(z_{\gamma n}+\delta_0)| > \delta_0\cdot|K_n|\cdot((n^3+1)!)^{\frac{n(n+1)}{2}-1}.$$

Since $\{|K_n|\}$ is an increasing sequence, $|f(z_{\gamma n}+\delta_0)|\to\infty$ as $n\to\infty$.

But for each positive integer n,

$$\begin{aligned}f(z_{\gamma n}) &= f(w_{st}+(n^3+\gamma)!) \\ &= 0\end{aligned}$$

by the definition of the function f. So w_{st} does not belong to $\mathbf{N}(f)$. We choose w_{st} arbitrarily from the set W and W is a dense subset of G_0^c. Hence no point in G_0^c can be in $\mathbf{N}(f)$. □

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