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SINGULAR SET OF SOME KÄHLER ORBIFOLDS

THALIA D. JEFFRES

Abstract. We consider some examples of orbifolds with positive first Chern class. Applying a result of Ding and Tian, we show that the singularities must be very mild if the orbifold admits a Kähler-Einstein metric.

1. Background

A complex manifold is Kähler-Einstein if it admits a metric whose Ricci and Kähler forms are related by $\rho = k\omega$ for a real number k. The first Chern class will be positive, negative, or zero according to the sign of k. This provides a necessary condition for the existence of a Kähler-Einstein metric. In fact, if $C_1(M) = 0$ or $C_1(M) < 0$, this condition is also sufficient, but for $C_1(M) > 0$ there are examples demonstrating its insufficiency. Additionally, for $C_1(M) > 0$, obstructions have been found by Matsushima, Futaki, and others. These are discussed in the book by Futaki ([F]). Since these automatically vanish when there are no nontrivial holomorphic vector fields, Calabi asked whether a complex manifold with $C_1 > 0$ and no nontrivial holomorphic vector fields must admit a Kähler-Einstein metric. Tian has given an affirmative answer for the surface case ([T]).

Kähler-Einstein metrics can also be defined on many singular varieties, such as Kähler orbifolds with normal singularities. These singularities are of particular interest because they appear in the Hausdorff limit of smooth varieties. For singular surfaces with $C_1 > 0$ new obstructions arise. Indeed, in contrast to the smooth case, there are two-dimensional orbifolds which possess no nontrivial holomorphic vector fields and yet do not admit a Kähler-Einstein metric. More specifically, Ding and Tian proved that a cubic surface in CP^3 has a Kähler-Einstein metric if and only if it is stable in the sense of Mumford. It seems to be that the correct condition for the existence of a Kähler-Einstein metric is a certain notion of stability in geometric invariant theory.

In this paper we examine a special case. Suppose that f and g are homogeneous quadratic polynomials whose zero locus in CP^4 is

$$\Sigma_{f,g} = \big\{[z_0, \cdots, z_4] | f = g = 0\big\}.$$

Suppose also that $\Sigma_{f,g}$ is an orbifold whose singularities are rational double points. Note that such an orbifold does indeed have positive first Chern class. It is a quartic del Pezzo orbifold. Then the main theorem is as follows:

Main Theorem. *If the orbifold described above admits a Kähler-Einstein metric, then it may have only the mildest singularities, those of type* A_1.

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Referring to the work of Mabuchi and Mukai ([MM]), this establishes semistability, giving the link indicated above. It should be noted in passing that in contrast to the work of Mabuchi and Mukai, we do not assume here that the orbifolds are limits of smooth surfaces.

2. Outline of the Method

The method may be outlined as follows. Assuming first that a more severe singularity occurs, we use the classification theory of singularities (see for example [BK] or [BPV]) to recover some information about the form of the original defining polynomials. Typically this orbifold will not admit a nontrivial holomorphic vector field. It may, however, be moved through a one-parameter family of automorphisms of the ambient space to a limit orbifold which does. Computation on this limit of a generalized Futaki invariant developed by Ding and Tian in which the structure of the singularities plays a role can yield information about the original orbifold. Namely, by choosing the automorphism carefully, negativity of this invariant when evaluated on the vector field generated by this one-parameter family implies that the original orbifold does not admit a Kähler-Einstein metric.

This conclusion follows from the main theorem shown by Ding and Tian, which is Theorem 0.2 of their paper ([DT]).

Theorem 1 (Ding and Tian). *Let X be a Kähler orbifold embedded in CP^N such that the hyperplane bundle restricts to a multiple of the anticanonical bundle K_X^{-1}, and σ_t a one-parameter subgroup in $Aut(CP^N) = SL(N+1, C)$. Let V be the induced holomorphic vector field. Assume that the $X_t = \sigma_t(X)$ converge to an irreducible, normal, and nondegenerate variety X_∞ and that X_∞ admits a Kähler-Einstein orbifold metric. Then the generalized Futaki invariant computed on X_∞, $F_\infty(V)$, is nonnegative.*

For the purposes of the computations in this paper, there is the following formula for the Futaki invariant, also established by Ding and Tian. Before stating this, we first establish some notations. Let $\Lambda = \Lambda_0 \cup \Lambda_1$ be the fixed point set of the one-parameter family of automorphisms $\{\sigma_t\}$, where Λ_i consists in the components with dimension i. Let N_λ denote the normal bundle of the component $Z_\lambda \subset \Lambda$ in σ. Define a map $L(V) : N_\lambda \longrightarrow N_\lambda$ by $L(V)Y = (D_V Y)^\perp$, where D denotes the connection. Suppose that $L(V)$ is nondegenerate, that is, that $det(L(V))$ is nonzero.

Theorem 2 (Ding and Tian). *For complex dimension two,*

$$F_X(V) = \sum_{\Lambda_0} \frac{1}{|\Gamma_x|} \frac{(divV)^3}{detL(V)} + \sum_{\Lambda_1} div_C(V) \left(2deg(C) + \frac{1}{|\Gamma|} \left(2 - 2g(C) - \sum_{x \in C} \frac{|\Gamma_x| - 1}{|\Gamma_x|} \right) \right)$$

Here, $g(C)$ denotes the genus of the one-dimensional component C. $|\Gamma_x|$ is the order of the uniformization group at the point x, and $|\Gamma|$ is the order of the local uniformization group at a generic point on C. $div_C(V)$ is the normal component of the divergence.

3. A Formula for A_n Singularities

A surface Σ with a rational double point may be expressed locally as a hypersurface in C^3. At the same time, the singularity is the quotient D^2/G, where G is a finite subgroup of $SU(2)$. Now if X is a vector field on all of C^3, it is useful to develop a formula which allows one to compute the contribution of this point to the Futaki invariant directly, without having first to parametrize the surface and compute the local lift X_0 of X to the disk D^2. The most commonly occurring singularities on these surfaces are the A_k singularities, and it is easy in this case.

Locally, such a singularity is described by $z_0z_1 + z_2^n$. There is a map from the disk to the singular surface,

$$(u, v) \to (u^n, v^n, \varepsilon uv),$$

where ε is any complex number with $\varepsilon^n = -1$. The vector field X is generated by a one-parameter family of automorphisms

$$\sigma(t) : (z_0, z_1, z_2) \to (t^{k_0}z_0, t^{k_1}z_1, t^{k_2}z_2),$$

so that

$$X = \sum_0^2 k_i z_i \frac{\partial}{\partial z_i}$$

If X_0 is a local lifting of X to the disk D^2, in other words, $\pi_* X_0 = X$, then one computes that

$$X_0 = \frac{1}{n}k_0 u\frac{\partial}{\partial u} + \frac{1}{n}k_1 v\frac{\partial}{\partial v},$$

so that

$$divX_0 = \frac{1}{n}(k_0 + k_1) = k_2.$$

$\frac{1}{n}(k_0 + k_1) = k_2$ holds because the one-parameter family of automorphisms leaves the surface invariant. Therefore the contribution to the Futaki invariant of this point is

$$\frac{(divX_0)^3}{|G|detX_0} = \frac{nk_2^3}{k_0k_1}$$

4. Proof of the Main Result

Since $\Sigma_{f,g}$ is not smooth, we may assume after a change of variables that a singularity occurs at the point $[1, 0, 0, 0, 0]$. Then after a further change of variables, we may assume that f and g are of the form

$$f = z_0z_1 + f_2(z_2, z_3, z_4),$$
$$g = g_2(z_1, z_2, z_3, z_4).$$

For the proof, we consider different cases, depending on the rank of f_2. The lowest rank case is the simplest, for one has:

Lemma 1. *If the rank of f_2 is one, then $\Sigma_{f,g}$ may have only singularities of type A_k.*

Proof of the lemma. Assume as before that the singularity occurs at $[1,0,0,0,0]$. Then if f_2 has rank one, f and g may be expressed as

$$f = z_0 z_1 + z_2^2,$$
$$g = z_1 \sum_1^4 a_i z_i + Q(z_2, z_3, z_4).$$

Showing that only A_k singularities occur is the same as showing that Q has rank at least two, because the singularity is described locally in U_0 by

$$G^* = a_1 z_2^4 - z_2^2 \sum_1^4 a_i z_i + Q(z_2, z_3, z_4),$$

and so

$$\frac{\partial^2 G^*}{\partial z_i \partial z_j}(0,0,0) = \frac{\partial^2 Q}{\partial z_i \partial z_j}(0,0,0).$$

If Q has rank less than two, it must have rank one, since g is irreducible, and so Q can be written as $Q(z_2, z_3, z_4) = (az_2 + bz_3 + cz_4)^2$. If b and c are both zero, then g is either reducible or degenerate, so we may assume this does not happen; therefore assume without loss of generality that it is b which is nonzero. Then after a change of variables, f and g may be written as

$$f = z_0 z_1 + z_2^2,$$
$$g = z_1 \sum_1^4 a_i z_i + z_3^2,$$

where a_4 cannot be zero or again $\Sigma_{f,g}$ will be degenerate. Then the further change of variables $w_4 = \sum_1^4 a_i z_i$ and $w_i = z_i$ for $i \neq 4$ expresses f and g as

$$\begin{aligned} f &= w_0 w_1 + w_2^2, \\ g &= w_1 w_4 + w_3^2, \end{aligned} \tag{1}$$

showing that this singularity is nonsimple. Therefore the rank of Q must be at least 2. □

We now proceed to the proof of the main theorem.

Case 1. The rank of f_2 is 1.

While the above shows that Q must have rank at least two, if we would like to assume $\Sigma_{f,g}$ has a singularity worse that A_1 at $[1,0,0,0,0]$ then the rank of Q cannot exceed two either. If it did, then

$$G^* = a_1 z_2^4 - z_2^2 \sum_2^4 a_i z_i + Q(z_2, z_3, z_4)$$

could be diagonalized as

$$G^* = w^2 + x^2 + z^2,$$

which describes an A_1 singularity. So we have

$$f = z_0 z_1 + z_2^2,$$
$$g = z_1 \sum_1^4 a_i z_i + Q(z_2, z_3, z_4),$$

with rank $Q = 2$.

After a linear change of variables, these may be rewritten as

$$f = z_0 z_1 + f_2(w, x, z),$$
$$g = a_1 z_1^2 + z_1 \ell(w, x, z) + w^2 + x^2,$$

where ℓ denotes a linear function. $f_2(w, x, z)$ still has rank 1, so

$$f_2(w, x, z) = (aw + bx + cz)^2.$$

One checks that c cannot be zero, or else $\Sigma_{f,g}$ has nonisolated singularities. This allows one to redefine z as $aw + bx + cz$ and rewrite f as $f = z_0 z_1 + z^2$. Completing the square on the w and x terms in g brings f and g to the form

$$f = z_0 z_1 + z^2,$$
$$g = a_1 z_1^2 + a_2 z_1 z + w^2 + x^2.$$

If $a_2 \neq 0$, then putting $\tilde{z} = a_1 z_1 + a_2 z$, redefining z_0 and absorbing the constants brings f and g to the form

$$f = z_0 z_1 + z^2,$$
$$g = z_1 z + w^2 + x^2.$$

If $a_2 = 0$, then f and g may be brought to the form

$$f = z_0 z_1 + z^2,$$
$$g = z_1^2 + w^2 + x^2.$$

In either case, $\Sigma_{f,g}$ already admits a nontrivial holomorphic vector field, and one computes $F(X) \neq 0$, showing that neither of these surfaces admits a Kähler-Einstein metric.

Case 2. The rank of f_2 is 2.

Now suppose the rank of f_2 is 2, so that f and g initially look like

$$f = z_0 z_1 + z_2^2 + z_3^2,$$
$$g = z_1 \sum_1^4 a_i z_i + Q(z_2, z_3, z_4).$$

First suppose that Q has rank 1. Then g may be written

$$g = z_1 \sum_1^4 a_i z_i + (a z_2 + b z_3 + c z_4)^2.$$

It is not necessary to consider the case in which $c = 0$. To see this, first ask whether it is possible to replace f by $f + \lambda g$ of smaller rank:

$$f + \lambda g = z_1\big(z_0 + \sum_1^4 \lambda a_i z_i\big) + z_2^2 + z_3^2 + \lambda(a z_2 + b z_3)^2.$$

Define

$$R_\lambda(z_2, z_3) = z_2^2 + z_3^2 + \lambda(az_2 + bz_3)^2.$$

Then

$$det(\lambda) := det\frac{\partial^2 R_\lambda}{\partial z_2 \partial z_3} = 4\big(1 + (a^2+b^2)\lambda\big),$$

so that if $a^2 + b^2 \neq 0$, then it is possible to solve $det(\lambda) = 0$, reducing the rank. We do not wish to study this, as it is included in Case 1, and so we may assume $a^2 + b^2 = 0$. Normalizing so that $a = 1$ and supposing without loss of generality that $b = +i$, we get

$$f = z_0 z_1 + (z_2 + iz_3)(z_2 - iz_3),$$
$$g = z_1 \sum_1^4 a_i z_i + (z_2 + iz_3)^2.$$

After the obvious change of variables, this becomes

$$f = z_0 z_1 + wx,$$
$$g = z_1(a_1 z_1 + a_2' w + a_3' x + a_4 z_4) + w^2.$$

a_4 must be nonzero or else $\Sigma_{f,g}$ is degenerate, so putting $z = a_1 z_1 + a_2' w + a_3' x + a_4 z_4$ gives

$$f = z_0 z_1 + wx,$$
$$g = z_1 z + w^2.$$

From this form it is easy to recognize that $\Sigma_{f,g}$ has nonisolated singularities. Therefore we may assume in the original expression for Q that $c \neq 0$. This allows us to simplify the form of f and g and finally write

$$f = z_0 z_1 + w^2 + x^2,$$
$$g = a_1 z_1^2 + z_1 \ell(w, x) + z^2.$$

Note that unlike the earlier example, here $\Sigma_{f,g}$ does not admit a nontrivial holomorphic vector field. We want to choose a 1-parameter family of automorphisims in such a way that

1. $\Sigma_{f,g} \xrightarrow{\sigma(t)} \Sigma_\infty$ with Σ_∞ irreducible and nondegenerate,
2. Σ_∞ is invariant under $\sigma(t)$, so that $\sigma'(1)$ is a nontrivial holomorphic vector field on Σ_∞, and
3. $F_\infty\big(\sigma'(1)\big) < 0$.

This will show, by the theorem of Ding and Tian, that the original $\Sigma_{f,g}$ cannot admit a Kähler-Einstein metric.

If we write

$$\sigma(t) : [z_0, \cdots, z_4] \mapsto [z_0, t^{\alpha_1} z_1, \cdots, t^{\alpha_4} z_4],$$

consideration of the three requirements above already determines the weights α_i. For instance, since g may have few terms, and we want Σ_∞ to be irreducible, try to choose $\sigma(t)$ so that f is invariant. Then $\sigma(t)$ must have the form

$$\sigma(t) : [z_0, z_1, w, x, z] \mapsto [z_0, t^{2\alpha} z_1, t^\alpha w, t^\alpha x, t^\beta z].$$

Polynomials f_t and g_t are determined by

$$\sigma(t)(\Sigma_{f,g}) = \Sigma_{f_t,g_t},$$

in other words, the image $\sigma(t)(\Sigma_{f,g})$ is the zero locus of f_t and g_t. One computes

$$g_t = a_1 t^{2\beta-4\alpha} z_1^2 + t^{2\beta-3\alpha} z_1 \ell(w,x) + z^2.$$

ℓ is not zero since the original surface was irreducible. So for g_t to converge, β must be $\beta = \frac{3}{2}\alpha$. Then $\Sigma_{f,g}$ converges to a limit orbifold Σ_∞ described by

$$f_\infty = z_0 z_1 + w^2 + x^2,$$
$$g_\infty = z_1 \ell(w,x) + z^2$$

via $\sigma(t) : [z_0, z_1, w, x, z] \mapsto [z_0, t^{2\alpha} z_1, t^\alpha w, t^\alpha x, t^{\frac{3}{2}\alpha} z]$.

Writing f_2 as $f_2 = w^2 + x^2$ was useful to fix the weights and determine $\sigma(t)$, but to compute the Futaki invariant it is easier now to write this as $f_2 = wx$ so that $f = z_0 z_1 + wx$; since w and x have the same weights, the form of $\sigma(t)$ is not affected.

The fixed point set of $\sigma(t)$ is

$$\Lambda(x) = \big\{[1,0,0,0,0], [0,1,0,0,0], [0,0,1,0,0], [0,0,0,0,1]\big\}.$$

The first fixed point is a D_4 singularity and the others are smooth. One computes

$$F_\infty\big(\sigma'(1)\big) = \frac{1}{4}\alpha - \frac{27}{4}\alpha + \frac{1}{4}\alpha + \frac{1}{4}\alpha < 0.$$

Now suppose that rank $Q = 2$. Then there is a linear change of variables so that

$$f = z_0 z_1 + f_2(w,x,z),$$
$$g = a_1 z_1^2 + z_1 \ell(w,x,z) + w^2 + x^2.$$

f_2 still has rank two, and it must contain z, for if it does not, it is possible to reduce the rank by replacing f with $f + \lambda g$.

Next, consider two cases, depending on the form of $\ell(w,x,z)$.

I. Suppose that $\ell(w,x,z) = aw + bx + cz$, with $c \neq 0$. Then after a change of variables, f and g may be written in the form

$$f = z_0 z_1 + f_2(w,x,z),$$
$$g = z_1 z + w^2 + x^2.$$

Rewriting f in more detail indicates how to choose $\sigma(t)$ and Σ_∞:

$$f = z_0 z_1 + Az^2 + z\ell(w,x) + Q(w,x),$$
$$g = z_1 z + w^2 + x^2.$$

Here, A or ℓ can be zero, but they cannot both vanish.

If $A \neq 0$, choose

$$\sigma(t) : [z_0, z_1, w, x, z] \mapsto [z_0, t^{\frac{4}{3}\alpha} z_1, t^\alpha w, t^\alpha x, t^{\frac{2}{3}\alpha} z].$$

The same computation as before shows that $F_\infty\big(\sigma'(1)\big) < 0$.

If $A = 0$, then choose

$$\sigma(t) : [z_0, z_1, w, x, z] \mapsto [z_0, t^{3\alpha} z_1, t^{2\alpha} w, t^{2\alpha} x, t^\alpha z]$$

and converge to

$$f_\infty = z_0 z_1 + z\ell(w,x),$$
$$g_\infty = z_1 z + w^2 + x^2.$$

Again, to compute $F_\infty\bigl(\sigma'(1)\bigr)$ it is more convenient to rewrite these as

$$f_\infty = z_0 z_1 + z\hat{\ell}(w,x),$$
$$g_\infty = z_1 z + wx,$$

and it is also important to notice that $\hat{\ell}$ must contain both w and x or else the original surface $\Sigma_{f,g}$ has nonisolated singularities. Then one computes $F_\infty\bigl(\sigma'(1)\bigr) < 0$.

II. Now suppose $\ell(w,x,z)$ does not contain z, in other words that

$$f = z_0 z_1 + f_2(w,x,z),$$
$$g = a_1 z_1^2 + z_1\ell(w,x) + w^2 + x^2.$$

By completing the square in g on w and x, and rescaling, one may assume

$$f = z_0 z_1 + f_2(w,x,z),$$
$$g = z_1^2 + w^2 + x^2.$$

Rewriting f in more detail as

$$f = z_0 z_1 + az^2 + z\ell(w,x) + Q(w,x),$$

one checks that if $A = 0$, then $\Sigma_{f,g}$ has nonisolated singularities.

Choose $\sigma(t) : [z_0, z_2, w, x, z] \mapsto [z_0, t^{2\alpha} z_1, t^{2\alpha} w, t^{2\alpha} x, t^\alpha z]$, $\Sigma_{f,g} \xrightarrow{\sigma(t)} \Sigma_\infty$, described by

$$f_\infty = z_0 z_1 + z^2$$
$$g_\infty = z_1^2 + w^2 + x^2,$$

or, equivalently,

$$f_\infty = z_0 z_1 + z^2,$$
$$g_\infty = z_1^2 + wx.$$

The fixed point set of $\sigma(t)$ is

$$\Lambda(X) = \bigl\{[1,0,0,0,0]\bigr\} \cup \bigl\{[0,z_1,w,x,0] | z_1^2 + wx = 0\bigr\}.$$

$[1,0,0,0,0]$ is an A_3 singularity, and contributes α to the Futaki invariant. The contribution of the 1-dimensional component, C, is

$$div_C x\left(2\cdot degC + \frac{1}{|\Gamma|}\bigl(2 - 2g(C) - \sum_{x\in C}\frac{|\Gamma_x|-1}{|\Gamma_x|})\right)$$
$$= -\alpha\left(2\cdot 2 + 1\bigl(2 - 0 - (\frac{1}{2} + \frac{1}{2}))\right),$$

so $F_\infty\bigl(\sigma'(1)\bigr) = \alpha - 15\alpha < 0$. This completes the second case, when f_2 has rank 2.

Case 3. f has full rank.

In this case, f and g may be written as

$$f = z_0 z_1 + z_2^2 + z_3^2 + z_4^2,$$
$$g = z_1 \sum_1^4 a_i z_i + Q(z_2, z_3, z_4).$$

We consider separately the cases rank $Q = 1$ and rank $Q = 2$, beginning with the former. In this instance, f and g may be written, after a change of variables, as

$$f = z_0 z_1 + f_2(w, x, z),$$
$$g = z_1 w + x^2.$$

$f_2(w, x, z)$ is still homogeneous, quadratic, and has rank 3. Furthermore, f_2 must contain the term z^2 or else the surface will have nonisolated singularities, so f and g are of the form

$$f = z_0 z_1 + Q(w, x) + z^2,$$
$$g = z_1 w + x^2.$$

To find $\sigma(t)$ and Σ_∞, we reason this way; we must have $g_\infty = g$, and we also want to keep z^2 in f_∞. This partially determines the weights;

$$\sigma(t) : [z_0, z_1, w, x, z] \mapsto [z_0, t^{2\alpha} z_1, t^{2\beta-2\alpha} w, t^\beta x, t^\alpha z]$$

with $2\alpha + \beta = 2\gamma$. Then compute that $\sigma(t)(\Sigma_{f,g})$ is the zero locus of g and of

$$f_t = z_0 z_1 + a t^{6\alpha-4\beta} w^2 + b t^{4\alpha-3\beta} wx + c t^{2\alpha-2\beta} x^2 + z^2.$$

In order for f_t to converge as $t \to \infty$, there are the following conditions:

$$a \neq 0 \quad \Rightarrow \quad \alpha \leq \frac{2}{3}\beta,$$
$$b \neq 0 \quad \Rightarrow \quad \alpha \leq \frac{3}{4}\beta,$$
$$c \neq 0 \quad \Rightarrow \quad \alpha \leq \beta.$$

So if a is nonzero, we must choose $\alpha = \frac{2}{3}\beta$. If $a = 0$, we must choose $\alpha = \frac{3}{4}\beta$, for in this case b cannot be zero.

In the former possibility, then, $\Sigma_{f,g}$ converges to a limit orbifold described by

$$f_\infty = z_0 z_1 + w^2 + z^2,$$
$$g_\infty = z_1 w + x^2$$

via $\sigma(t) : [z_0, z_1, w, x, z] \mapsto [z_0, t^{\frac{4}{3}\beta} z_1, t^{\frac{2}{3}\beta} w, t^\beta x, t^{\frac{2}{3}\beta} z]$. An earlier computation shows that

$$F_\infty\big(\sigma'(1)\big) < 0.$$

In the latter possibility, we have

$$f_\infty = z_0 z_1 + wx + z^2,$$
$$g_\infty = z_1 w + x^2,$$

via $\sigma(t) : [z_0, z_1, w, x, z] \mapsto [z_0, t^{6\beta} z_1, t^{2\beta} w, t^{4\beta} x, t^{3\beta} z]$. The fixed point set is

$$\Lambda(X) = \{[1, 0, 0, 0, 0], [0, 1, 0, 0, 0], [0, 0, 1, 0, 0]\}.$$

It is interesting that among the limit orbifolds which arise in this context, this one has the highest order singularity. $[1, 0, 0, 0, 0]$ is a D_5 singularity; the others are smooth points. One computes $F_\infty\big(\sigma'(1)\big) = \frac{\beta}{3} - \frac{5^3}{6}\beta + \frac{1}{2}\beta < 0$.

It remains now to examine what happens when rank $Q = 2$. A linear change of variables diagonalizes Q:

$$f = z_0 z_1 + f_2(w, x, z),$$
$$g = a_1 z_1^2 + z_1 \ell(w, x, z) + w^2 + x^2.$$

Write $f_2 = az^2 + z\ell(w, x) + Q(w, x)$. If $a \neq 0$, normalize so that $a = 1$ and complete the square so that f and g now have the form

$$f = z_0 z_1 + Q(w, x) + z^2,$$
$$g = a_1 z_1^2 + z_1 \ell(w, x, z) + w^2 + x^2.$$

To keep Σ_∞ as uncomplicated as possible and to avoid converging to something possibly irreducible or degenerate, try to keep w^2, x^2, and z^2; $\sigma(t)$ must then look like

$$\sigma(t) : [z_0, z_1, w, x, z] \mapsto [z_0, t^{2\alpha} z_1, t^\beta w, t^\beta x, t^\alpha z].$$

Then $\sigma(t)(\Sigma_{f,g})$ is the zero locus of

$$f_t = z_0 z_1 + t^{2\alpha - 2\beta} Q(w, x) + z^2,$$
$$g_t = a_1 t^{2\beta - 4\alpha} z_1^2 + t^{\beta - 2\alpha} z_1 \ell(w, x) + c t^{2\beta - 3\alpha} z_1 z + w^2 + x^2.$$

If $c \neq 0$, we must choose $\beta = \frac{3}{2}\alpha$, and then obtain

$$f_\infty = z_0 z_1 + z^2,$$
$$g_\infty = z_1 z + w^2 + x^2$$

via $\sigma(t) : [z_0, z_1, w, x, z] \mapsto [z_0, t^{2\alpha} z_1, t^{\frac{3}{2}\alpha} w, t^{\frac{3}{2}\alpha} x, t^\alpha z]$. If $c = 0$, choose $\beta = 2\alpha$ and then obtain, after completing the square,

$$f_\infty = z_0 z_1 + z^2,$$
$$g_\infty = z_1^2 + w^2 + x^2.$$

For both of these possibilities, we already computed $F_\infty\big(\sigma'(1)\big) < 0$.

We now must consider the case $a = 0$, i.e., f_2 does not contain z^2.

After a change of variables, f and g may be written as either

$$f = z_0 z_1 + z\ell(w, x) + Q(w, x),$$
$$g = z_1^2 + wx$$

or

$$g = z_1 z + wx.$$

But the first possibility for g may be eliminated because this surface has nonisolated singularities. In the remaining case, there are two further cases. If $\ell(w, x) = aw + bx$ with a and b both nonzero, then $\Sigma_{f,g}$ can be moved to Σ_∞ with

$$f_\infty = z_0 z_1 + z\ell(w, x),$$
$$g_\infty = z_1 z + wx$$

via $\sigma(t) : [z_0, z_1, w, x, z] \mapsto [z_0, t^{3\alpha} z_1, t^{2\alpha} w, t^{2\alpha} x, t^\alpha z]$, for which we already know $F_\infty\big(\sigma'(1)\big) < 0$ from before. If, however, one of a or b is zero, then this choice of

$\sigma(t)$ produces a limit Σ_∞ with nonisolated singularities. Suppose without loss of generality that $b = 0$, so that f and g look like

$$f = z_0 z_1 + zw + Q(w,x),$$
$$g = z_1 z + wx.$$

Since f_2 has rank 3, Q must contain x^2, so $f = z_0z_1 + zw + Aw^2 + Bwx + Cx^2$, $C \neq 0$. Try to converge to $f_\infty = z_0z_1 + zw + Cx^2$; indeed, this establishes the weights as

$$\sigma(t) : [z_0, z_1, w, x, z] \mapsto [z_0, t^{2\alpha} z_1, t^{\frac{3}{2}\alpha} w, t^\alpha x, t^{\frac{1}{2}\alpha} z],$$

and

$$f_\infty = z_0 z_1 + zw + Cx^2,$$
$$g_\infty = z_1 z + wx.$$

This surface has one singularity, of type A_4:

$$\Lambda(X) = \{[1,0,0,0,0], [0,1,0,0,0], [0,0,1,0,0], [0,0,0,0,1]\},$$

$$F_\infty\big(\sigma'(1)\big) = \frac{5}{12}\alpha - \frac{27}{4}\alpha + \frac{4}{3}\alpha + 0 < 0.$$

This completes the last case, and with it, the proof of the main result.

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Department of Mathematics, State University of New York, Stony Brook, New York 11794-3651

Current address: Department of Mathematics, University of California at Irvine, Irvine, California 92697-3875

E-mail address: `tjeffres@math.uci.edu`