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THE NONEXISTENCE OF EXPANSIVE HOMEOMORPHISMS OF A CLASS OF CONTINUA WHICH CONTAINS ALL DECOMPOSABLE CIRCLE-LIKE CONTINUA

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Abstract. A homeomorphism $f : X \to X$ of a compactum X with metric d is expansive if there is $c > 0$ such that if $x, y \in X$ and $x \neq y$, then there is an integer $n \in \mathbf{Z}$ such that $d(f^n(x), f^n(y)) > c$. It is well-known that p-adic solenoids S_p ($p \geq 2$) admit expansive homeomorphisms, each S_p is an indecomposable continuum, and S_p cannot be embedded into the plane. In case of plane continua, the following interesting problem remains open: For each $1 \leq n \leq 3$, does there exist a plane continuum X so that X admits an expansive homeomorphism and X separates the plane into n components? For the case $n = 2$, the typical plane continua are circle-like continua, and every decomposable circle-like continuum can be embedded into the plane. Naturally, one may ask the following question: Does there exist a decomposable circle-like continuum admitting expansive homeomorphisms? In this paper, we prove that a class of continua, which contains all chainable continua, some continuous curves of pseudo-arcs constructed by W. Lewis and all decomposable circle-like continua, admits no expansive homeomorphisms. In particular, any decomposable circle-like continuum admits no expansive homeomorphism. Also, we show that if $f : X \to X$ is an expansive homeomorphism of a circle-like continuum X, then f is itself weakly chaotic in the sense of Devaney.

1. Introduction

All spaces considered in this paper are assumed to be separable metric spaces. *Maps* are continuous functions. By a *compactum* we mean a nonempty compact metric space. A *continuum* is a connected compactum. A homeomorphism $f : X \to X$ of a compactum X with metric d is called *expansive* ([20], [1] and [2]) if there is $c > 0$ such that for any $x, y \in X$ with $x \neq y$, there is an integer $n \in \mathbf{Z}$ such that

$$d(f^n(x), f^n(y)) > c.$$

A homeomorphism $f : X \to X$ of a compactum X is *continuum-wise expansive* [8] if there is $c > 0$ such that if A is a nondegenerate subcontinuum of X, then there is an integer $n \in \mathbf{Z}$ such that

$$\operatorname{diam} f^n(A) > c,$$

where $\operatorname{diam} B = \sup\{d(x, y) \mid x, y \in B\}$ for a set B. Such a positive number c is called an *expansive constant* for f. Note that each expansive homeomorphism

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is continuum-wise expansive, but the converse assertion is not true. There are many continuum-wise expansive homeomorphisms which are not expansive (e.g., see [8], [9] and [11]). In fact, there are many decomposable circle-like continua admitting continuum-wise expansive homeomorphisms. By the definitions, we see that expansiveness and continuum-wise expansiveness do not depend on the choice of the metric d of X. These notions have been extensively studied in the area of topological dynamics, ergodic theory and continuum theory (e.g., see [1], [2], [7]–[12], [20], and [21]).

Let $f : X \to X$ be a homeomorphism of a compactum X. A (nonempty) closed subset M of X is a *minimal set* of f if M is f-invariant, i.e., $f(M) = M$, and for any $x \in M$, the orbit $O(f) = \{f^n(x) | \, n \in \mathbf{Z}\}$ is dense in M. Note that every homeomorphism of a compactum has a minimal set. For a point $x \in X$, the *ω-limit set* $\omega f(x)$ of x is the set

$$\begin{aligned} \omega f(x)(= \omega(x)) = \{y \in X | \, & \text{there is a sequence}\, n_1 < n_2 < \dots \\ & \text{of natural numbers such that } \lim_{i\to\infty} f^{n_i}(x) = y\}. \end{aligned}$$

Similarly, the *α-limit set* $\alpha f(x)(= \alpha(x))$ of x is the set $\omega f^{-1}(x)$.

Let X be a compactum. Let 2^X be the set of all nonempty closed sets of X and $C(X)$ the set of all nonempty subcontinua of X. Suppose that $U_1, \dots , U_n$ are nonempty open sets of X. Put

$$\langle U_1, \dots , U_n \rangle = \{A \in 2^X | \, A \cap U_i \neq \phi, A \subset \bigcup_{i=1}^{n} U_i\}.$$

Then

$$\begin{aligned} \beta = \{ & \langle U_1, U_2, \dots , U_n \rangle | \, n \geq 1 \text{ and} \\ & U_i \, (i \leq n) \text{ are nonempty open sets of } X\} \end{aligned}$$

is a base of 2^X, and it is called the *Vietoris topology*. Then 2^X and $C(X)$ are compacta. The spaces 2^X and $C(X)$ are called the *hyperspaces* of X. For a map $f : X \to X$, we define a map $f_* : 2^X \to 2^X$ by $f_*(A) = f(A)(= \{f(a)| \, a \in A\})$ for $A \in 2^X$. Also, put $C(f) = f_*|C(X) : C(X) \to C(X)$. Then X is identified with the closed invariant subset of singletons, i.e., degenerate subcontinua.

For the map $C(f) : C(X) \to C(X)$, we shall deal with $\omega(E) = \omega C(f)(E)$ and $\alpha(E) = \omega C(f)^{-1}(E)$ for $E \in C(X)$.

For a homeomorphism $f : X \to X$, if $Z \subset X$ is a closed invariant subset for X, then Z is *isolated* if for some neighborhood U of Z in X any orbit lying entirely in U is in fact in Z, i.e., $Z = \bigcap_{-\infty}^{\infty} f^n(U)$. Then f is expansive (resp. continuum-wise expansive) if and only if X is isolated in 2^X for f_* (resp. in $C(X)$ for $C(f)$) (see [1]).

Let $\mathbf{A}$ and $\mathbf{B}$ be closed $C(f)$-invariant sets in $C(X)$. Then we define the orderings ${}_*<, <_*$, and ${}_*<_*$ as follows: Define $\mathbf{A} \,{}_*< \mathbf{B}$ (resp. $\mathbf{A} <_* \mathbf{B}$) iff for any $A \in \mathbf{A}$ there is $B \in \mathbf{B}$ (resp. for any $B \in \mathbf{B}$ there is $A \in \mathbf{A}$) such that $A \subset B$. Also, define $\mathbf{A} \,{}_*<_* \mathbf{B}$ iff $\mathbf{A} \,{}_*< \mathbf{B}$ and $\mathbf{A} <_* \mathbf{B}$. Example: for $E_0, E_1 \in C(X)$, $E_0 \subset E_1$ implies $\omega(E_0) \,{}_*<_* \omega(E_1)$ and $\alpha(E_0) \,{}_*<_* \alpha(E_1)$.

For a homeomorphism $f : X \to X$, we define sets of *stable* and *unstable* nondegenerate subcontinua of X as follows (see [9]):

$$\mathbf{V}^s = \{A|\, A \text{ is a nondegenerate subcontinuum of } X \text{ such that } \lim_{n\to\infty} \operatorname{diam} f^n(A) = 0\},$$
$$\mathbf{V}^u = \{A|\, A \text{ is a nondegenerate subcontinuum of } X \text{ such that } \lim_{n\to\infty} \operatorname{diam} f^{-n}(A) = 0\}.$$

For each $0 < \delta < \epsilon$, put

$$\mathbf{V}^s(\delta;\epsilon) = \{A \in C(X)|\operatorname{diam} A \geq \delta, \text{ and } \operatorname{diam} f^n(A) \leq \epsilon \text{ for each } n \geq 0\},$$
$$\mathbf{V}^u(\delta;\epsilon) = \{A \in C(X)|\operatorname{diam} A \geq \delta, \text{ and } \operatorname{diam} f^{-n}(A) \leq \epsilon \text{ for each } n \geq 0\}.$$

Then $\mathbf{V}^\sigma(\delta;\epsilon)$ $(\sigma = u, s)$ is closed in $C(X)$. Note that if $f : X \to X$ is a continuum-wise expansive homeomorphism with an expansive constant $c > 0$, then for each $0 < \delta < \epsilon < c$ we have $\mathbf{V}^\sigma(\delta;\epsilon) \subset \mathbf{V}^\sigma$, and $\mathbf{V}^\sigma$ is an F_σ-set in $C(X)$.

A *chain* $C = [C_1, C_2, \ldots, C_m]$ of X is a finite collection of open sets of X satisfying the following property:

$$\operatorname{Cl}(C_i) \cap \operatorname{Cl}(C_j) \neq \phi \text{ if and only if } |i - j| \leq 1.$$

Each C_i is called a *link* of the chain C. Moreover, if for each $i = 1, \ldots, m$, $\operatorname{diam}(C_i) < \epsilon$, i.e., $\operatorname{mesh}(C) < \epsilon$, then we say that the chain C is an ϵ-chain. For a chain $C = [C_1, C_2, \ldots, C_m]$ and two points $p, q \in X$, if $p \in C_1$ and $q \in C_m$, we say that $C = [C_1, C_2, \ldots, C_m]$ is *a chain from* p *to* q. A continuum X is *chainable* if for any $\epsilon > 0$ there is an ϵ-chain covering of X.

If n is a natural number, let $I(n) = \{1, 2, \ldots, n\}$. A surjective function $f : I(m) \to I(n)$ is called a *pattern* provided that $|f(i + 1) - f(i)| \leq 1$ for each $i = 1, \ldots, m - 1$. Let $C = [C_1, C_2, \ldots, C_n]$ and $D = [D_1, D_2, \ldots, D_m]$ be chain coverings of X and let $f : I(m) \to I(n)$ be a pattern. We say that D follows the pattern f in C provided that $D_i \subset C_{f(i)}$ for each $i \in I(m)$.

Let $\mathcal{P}$ be a family of compact polyhedra. A continuum X is called a *$\mathcal{P}$-like continuum* if for any $\epsilon > 0$ there is an onto map $g : X \to P$ such that $P \in \mathcal{P}$ and $\operatorname{diam} g^{-1}(y) < \epsilon$ for each $y \in P$. Note that X is chainable if and only if X is arc-like. A *circular chain* differs from a chain in that the first and last links intersect. Then a continuum X is circle-like if and only if for any $\epsilon > 0$, there is an ϵ-circular chain covering of X.

Concerning expansive homeomorphisms, we have the following general problem:

Problem 1.1. What kinds of (plane) continua admit expansive homeomorphisms?

Note that p-adic solenoids S_p $(p \geq 2)$ are indecomposable circle-like continua admitting expansive homeomorphisms (see [21]), and they cannot be embedded into the plane R^2. On the other hand, each decomposable circle-like continuum X can be embedded into R^2, and $R^2 - X$ has at most 2 components. It is known that for each $n \geq 4$ there is a plane continuum X which is called a *Lake of Wada*, such that X admits an expansive homeomorphism and $R^2 - X$ has n components. It is not known whether there exists a plane continuum X such that X admits an expansive homeomorphism and X separates the plane R^2 into n components $(n \leq 3)$, or not. For the case $n = 2$, the typical continua are circle-like continua.

In [7, 8], we proved that if X is a tree-like continuum admitting a continuum-wise expansive homeomorphism, it must contain an indecomposable subcontinuum.

Also, in [10], we proved that chainable continua admit no expansive homeomorphisms. Naturally, we are interested in the following problem:

Problem 1.2. Does there exsist a decomposable circle-like continuum admitting an expansive homeomorphism?

In this paper, we prove that some kinds of continua, including all chainable continua, some continuous curves of pseudo-arcs constructed by W. Lewis and all decomposable circle-like continua, admit no expansive homeomorphisms. In particular, any decomposable circle-like continuum admits no expansive homeomorphism. For example, we know that a solenoid of pseudo-arcs and the circle of pseudo-arcs admit no expansive homeomorphisms. Also, we show that if $f : X \to X$ is an expansive homeomorphism of a circle-like continuum X, then f is itself weakly chaotic in the sense of Devaney.

2. Preliminaries

A continuum X is *decomposable* if there are two proper subcontinua A and B of X such that $A \cup B = X$. A continuum X is *indecomposable* if it is not decomposable. A continuum X is *hereditarily indecomposable* if each subcontinuum of X is indecomposable. The *pseudo-arc* P is characterized [4] as a (nondegenerate) hereditarily indecomposable chainable continuum. The pseudo-arc has many remarkable properties in topology and chaotic dynamics (e.g., see [3]–[6] and [13]–[16]). For example, the pseudo-arc P is homogeneous [3], each onto map of the pseudo-arc P is a near homeomorphism [15], and the pseudo-arc P admits chaotic homeomorphisms in the sense of Devaney (see [13]). Also, there is an onto map from the pseudo-arc P to each chainable continuum (see [6] and [14]).

From the proof of [8, Proposition 2.3] we have

Lemma 2.1. *Let $f : X \to X$ be a continuum-wise expansive homeomorphism of a compactum X with an expansive constant $c > 0$, and let $0 < \epsilon < c/2$. Then there is a positive number $\delta < \epsilon$ such that if A is a subcontinuum of X with $\operatorname{diam} A \leq \delta$ and $\operatorname{diam} f^m(A) \geq \epsilon$ for some integer $m \geq 0$ (resp. $m < 0$), then for each $n \geq m$ and for each $x \in f^n(A)$, there is a subcontinuum B of A such that $x \in f^n(B), \operatorname{diam} f^j(B) \leq \epsilon$ for $0 \leq j \leq n$ and $\operatorname{diam} f^n(B) = \delta$ (resp. for each $n \geq -m$ and for each $x \in f^{-n}(A)$, there is a subcontinuum B of A such that $x \in f^{-n}(B), \operatorname{diam} f^{-j}(B) \leq \epsilon$ for $0 \leq j \leq n$, and $\operatorname{diam} f^{-n}(B) = \delta$).*

Corollary 2.2. *Let $f : X \to X, c, \epsilon, \delta$ be as in Lemma 2.1.*

(a) *For every nondegenerate subcontinuum A of X with $\operatorname{diam} A \leq \delta$, exactly one of the two following assertions holds:*

1. *For all $n \geq 0, \operatorname{diam} f^n(A) \leq \epsilon$, in which case $A \in \mathbf{V}^s$ and $\omega(A) \subset X \subset C(X)$.*
2. *For $n \geq 0$ sufficiently large, $\operatorname{diam} f^n(A) \geq \delta$.*

(b) *For every subcontinuum A, either $\omega(A) \subset X \subset C(X)$ or $\operatorname{diam} E \geq \delta$ for all $E \in \omega(A)$.*

For $n \leq 0, \mathbf{V}^u$ and $\alpha(A)$, the similar properties are satisfied.

Lemma 2.3 ([8, Corollary 2.4])**.** *Let $f : X \to X$ be a continuum-wise expansive homeomorphism of a compactum X with $\dim X > 0$. Then the following are true.*

1. *$\mathbf{V}^u \neq \phi$ or $\mathbf{V}^s \neq \phi$.*

2. *If $\delta > 0$ is as in the above lemma, then for each $\gamma > 0$ there is a natural number $N(\gamma)$ such that if A is a subcontinuum of X with* $\operatorname{diam} A \geq \gamma$*, then either* $\operatorname{diam} f^n(A) \geq \delta$ *for each* $n \geq N(\gamma)$ *or* $\operatorname{diam} f^{-n}(A) \geq \delta$ *for each* $n \geq N(\gamma)$.

From the above lemma, we see that $\mathbf{V}^s \cap \mathbf{V}^u = \phi$ and moreover if $A \in \mathbf{V}^u, B \in \mathbf{V}^s$, then $\dim (A \cap B) \leq 0$.

Lemma 2.4. *Under the same hypothesis as in Lemma 2.3, let E_0, E_1 be nondegenerate subcontinua of X with $E_1 \in \omega(E_0)$. Then one of the following holds:*

1. *Every nondegenerate subcontinuum A_0 of E_0 with* $\operatorname{diam} A_0 < \delta$ *lies in* $\mathbf{V}^s$.
2. *There is a subcontinuum A_1 of E_1 with* $\operatorname{diam} A_1 = \delta$ *lying in* $\mathbf{V}^u$*. Moreover, if $E_0 \in \mathbf{V}^u$, then for any $x \in E_1$ there is a subcontinuum A_1 of E_1 such that* $\operatorname{diam} A_1 = \delta$ *and* $x \in A_1 \in \mathbf{V}^u$.

Proof. If the first condition is not true, then there is a subcontinuum B of E_0 with $0 < \gamma = \operatorname{diam} B < \delta$ and a natural number n such that $\operatorname{diam} f^n(B) > \epsilon$. Choose a sequence $0 = n_0 < n_1 < \dots$, of natural numbers such that $n_{i+1} - n_i \geq N(\gamma)$ (see Lemma 2.3) and $\lim_{i\to\infty} f^{n_i}(E_0) = E_1$. By using Lemmas 2.1 inductively, we can construct a sequence $B_0, B_1, \dots$ of subcontinua with $B_0 = B$, $\operatorname{diam} B_0 = \gamma < \delta$, $B_{i+1} \subset f^{n_{i+1}-n_i}(B_i)$, $\operatorname{diam} B_i = \delta$ $(i \geq 1)$, and $\operatorname{diam} f^{-j}(B_i) \leq \epsilon$ for each $0 \leq j \leq n_i$. We may assume that $\lim_{i\to\infty} B_i = A_1$. Then $A_1 \in \mathbf{V}^u$ and $A_1 \subset E_1$.

Moreover, suppose that $E_0 \in \mathbf{V}^u$. For any $x \in E_1$, we choose a sequence $x_0, x_1, \dots$ of points such that $x_i \in f^{n_i}(E_0)$ and $\lim_{i\to\infty} x_i = x$. Choose a subcontinuum B of E_0 such that $x_0 \in B$ and $\operatorname{diam} B = \gamma < \delta$. By Lemma 2.1, we can choose a sequence $B_0, B_1, \dots$ satisfying the above conditions with $x_i \in B_i$ for each i. Then $x \in A_1 \in \mathbf{V}^u$. □

Corollary 2.5. *Under the same hypothesis as in Lemma 2.3, let $\mathbf{A}$ be a minimal set of $C(f)$. Assume that there is a nondegenerate subcontinuum $A \in \mathbf{A}$.*

(a) *For all $A \in \mathbf{A}$,* $\operatorname{diam} A \geq \delta$.

(b) *Exactly one of the three following conditions holds for $\mathbf{A}$:*
   1. *For all $A \in \mathbf{A}$ and all subcontinua B of A with* $\operatorname{diam} B < \delta$*, $B \in \mathbf{V}^s$.*
   2. *For all $A \in \mathbf{A}$ and all subcontinua B of A with* $\operatorname{diam} B < \delta$*, $B \in \mathbf{V}^u$.*
   3. *For all $A \in \mathbf{A}$ there are subcontinua B_0, B_1 of A with* $\operatorname{diam} B_0 = \operatorname{diam} B_1$ $= \delta$ *and* $B_0 \in \mathbf{V}^s, B_1 \in \mathbf{V}^u$.

(c) *If $A \in \mathbf{A}$ and B is a nondegenerate subcontinuum of A with $B \notin \mathbf{V}^s$, then* $\operatorname{diam} E \geq \delta$ *for each $E \in \omega(B)$, $\omega(B)\ {}_* {<}_* \ \mathbf{A}$, and if $\mathbf{A}_0$ is a minimal set in $\omega(B)$, then $\mathbf{A}_0\ {}_* {<}_* \ \mathbf{A}$ as well.*

The following propositions are used in the sequel.

Proposition 2.6. *Let $f : X \to X$ be a continuum-wise expansive homeomorphism of a compactum X with* $\dim X > 0$*. Suppose that $\mathbf{B}$ is a $C(f)$-invariant set such that some element of $\mathbf{B}$ is nondegenerate. Then there exists a minimal set $\mathbf{A}\ {}_* {<}\ \mathbf{B}$ of $C(f)$ such that each element of $\mathbf{A}$ is nondegenerate, and such that for each $A \in \mathbf{A}$ and each nondegenerate subcontinuum B of A either $B \in \mathbf{V}^s$ or $\mathbf{A} \subset \omega(B)$, and either $B \in \mathbf{V}^u$ or $\mathbf{A} \subset \alpha(B)$.*

Proof. For pairs $(A, \mathbf{A})$ such that $\mathbf{A}$ is minimal, $\mathbf{A}\ {}_* {<}\ \mathbf{B}$, $A \in \mathbf{A}$ and A is a nondegenerate subcontinuum, we consider the order by inclusion of the A's. By Corollary 2.2,(b), there exists such a pair. If $\{(A_\alpha, \mathbf{A}_\alpha)\}$ is a totally ordered family, then $B = \bigcap_\alpha A_\alpha$ is a nondegenerate subcontinuum and so either $\omega(B)$ or $\alpha(B)$ contains a minimal subset $\mathbf{A}$ such that its elements are nondegenerate subcontinua and $\mathbf{A} \;{}_*\!<_* \mathbf{A}_\alpha$ for all α. For each α choose $B_\alpha \subset A_\alpha$ with $B_\alpha \in \mathbf{A}$. Then any limit point A of the net $\{B_\alpha\}$ is an element of $\mathbf{A}$ contained in all the A_α's. So Zorn's lemma applies to the pairs. If $(\tilde{A}, \mathbf{A})$ is minimal with respect to this ordering, then $\mathbf{A}$ satisfies the conclusion. In fact, if $A \in \mathbf{A}$ and B is a nondegenerate subcontinuum of A not in $\mathbf{V}^s$, then $\omega(B) \;{}_*\!<_* \mathbf{A}$ and $\mathbf{A}_0 \;{}_*\!<_* \mathbf{A}$ for any minimal subset $\mathbf{A}_0$ of $\omega(B)$. Then there is $A_0 \in \mathbf{A}_0$ such that $A_0 \subset \tilde{A}$, and so by minimality we see that $A_0 = \tilde{A}$ and so $\mathbf{A} = \omega(A_0) = \mathbf{A}_0 \subset \omega(B)$.

This completes the proof.

Proposition 2.7. *Under the same assumption as in the above proposition, the minimal set* $\mathbf{A}$ *satisfies one of the following conditions:*

1. *If some* $A_0 \in \mathbf{A}$ *contains an element of* $\mathbf{V}^u$*, then for any* $x \in A \in \mathbf{A}$*, there is a nondegenerate subcontinuum* A_x *of* A *such that* $x \in A_x \in \mathbf{V}^u$*, and if* A' *is a nondegenerate subcontinuum of* $A \in \mathbf{A}$ *with* $A' \notin \mathbf{V}^s$*, then for each* $H \in \mathbf{A}$ *there is a sequence* $n_1 < n_2 < \dots$ *of natural numbers such that*
$$\lim_{i\to\infty} f^{n_i}(A) = \lim_{i\to\infty} f^{n_i}(A') = H.$$
2. *If some* $A_0 \in \mathbf{A}$ *contains an element of* $\mathbf{V}^s$*, then for any* $x \in A \in \mathbf{A}$*, for any* $x \in A \in \mathbf{A}$*, there is a nondegenerate subcontinuum* A_x *of* A *such that* $x \in A_x \in \mathbf{V}^s$*, and if* A' *is a any nondegenerate subcontinuum of* $A \in \mathbf{A}$ *with* $A' \notin \mathbf{V}^u$*, then for each* $H \in \mathbf{A}$ *there is a sequence* $n_1 < n_2 < \dots$ *of natural numbers such that*
$$\lim_{i\to\infty} f^{-n_i}(A) = \lim_{i\to\infty} f^{-n_i}(A') = H.$$

Proof. We shall show the first case. Let $B \in \mathbf{V}^u$ and $B \subset A_0 \in \mathbf{A}$. By the above proposition, we see that $\mathbf{A} \subset \omega(B)$. By Lemma 2.4, we see that for any $x \in A \in \mathbf{A}$, there is $A_x \in \mathbf{V}^u$ such that $x \in A_x \subset A$. Since $\mathbf{A}$ is closed in $C(X)$, $\mathbf{A}$ contains an maximal element in order by inclusion. In fact, for a Whitney map $\mu : C(X) \to [0,1]$ (see [18]), we can choose an element T of $\mathbf{A}$ such that $\mu(T) = \max\{\mu(E) |\, E \in \mathbf{A}\}$. Then T is a maximal element of $\mathbf{A}$. Suppose that A' is a nondegenerate subcontinuum of $A \in \mathbf{A}$ with $A' \notin \mathbf{V}^s$. Let $H \in \mathbf{A}$. Since $\omega(A') \supset \mathbf{A}$ (see Proposition 2.6), $T \in \omega(A')$. Hence there is a sequence $i_1 < i_2 < \dots$ of natural numbers such that $\lim_{k\to\infty} f^{i_k}(A') = T$. We may assume that $\{f^{i_k}(A)\}_{k=1}^\infty$ is convergent. Since T is maximal in $\mathbf{A}$, we see that $\lim_{k\to\infty} f^{i_k}(A) = T$. Since $\mathbf{A}$ is minimal, $H \in \omega(T)$. Then we can choose a sequence $n_1 < n_2 < \dots$ of natural numbers such that
$$\lim_{i\to\infty} f^{n_i}(A') = \lim_{i\to\infty} f^{n_i}(A) = H.$$

This completes the proof.

Let $f : X \to X$ be a continuum-wise expansive homeomorphism of a compactum X with $\dim X > 0$. Note that every minimal set of f is 0-dimensional (see [8, Theorem 5.2]). Consider the following sets (see [12]):

1. $\mathcal{I}(f) = \{A \in 2^X |\, A \text{ is } f\text{-invariant}\}$.
2. $\mathcal{I}^+(f) = \{A \in \mathcal{I}(f) |\, \dim A > 0\}$.
3. $\mathcal{D}(f)$ is the set of all minimal elements of $\mathcal{I}^+(f)$ in the order by inclusion.

Note that $\mathcal{D}(f) \neq \phi$ (see [12, Proposition 2.4]) and if $Y \in \mathcal{D}(f)$, then $f_Y = f|Y : Y \to Y$ is *weakly chaotic in the sense of Devaney*, i.e., f_Y has *sensitive dependence on initial conditions*, f_Y is *topologically transitive* and the union of all minimal sets of f_Y is dense in Y ([12, Theorem 2.7]), i.e., the *min-center* of f_Y is Y (see [1, p. 70]).

Proposition 2.8. *Let $f : X \to X$ be a continuum-wise expansive homeomorphism of a compactum X with $\dim X > 0$. If $Y \in \mathcal{D}(f)$, then there is a minimal set $\mathbf{A}$ of $C(f)$ satisfying one of the conditions (1) and (2) as in Proposition 2.7 and $\bigcup\{A|\, A \in \mathbf{A}\} = Y$.*

Proof. Consider the map $f|Y : Y \to Y$. Then there is a minimal set $\mathbf{A}$ of $C(f|Y)$ as in Proposition 2.7. Put $Y' = \bigcup\{A|\, A \in \mathbf{A}\}$. Then Y' is f-invariant and $\dim Y' > 0$. Hence $Y = Y'$.

The following lemma follows from [3, Theorem 6] (see also [15, Lemmas 2 and 1.1]).

Lemma 2.9. *Let P be the pseudo-arc. Let $C = [C_1, C_2, \ldots, C_n]$ be a chain covering of P and $f : I(m) \to I(n)$ a pattern with $f(1) = 1$. Let $p \in C_1$. Then there is a chain covering $D = [D_1, D_2, \ldots, D_m]$ such that D refines the chain C, $p \in D_1$ and D follows the pattern f in C.*

The following lemma is a simple modification of the uniformization theorem of Mioduszewski (see [17] and [19]). For completeness, we give the proof.

Lemma 2.10. *Let $I = [0, 1]$ be the unit interval. Suppose that $f, g : I \to I$ are piecewise linear onto maps. If $f(0) = g(0) = 0$, then there are onto maps $a, b : I \to I$ such that $f \cdot a = g \cdot b$ and $a(0) = b(0) = 0$.*

Proof. Let $\psi : I^2 \to R$ be the map defined by $\psi(x, y) = f(x) - g(y)$. Note that I^2 is *unicoherent* (i.e., if A and B are coutinua with $A \cup B = I^2$, then $A \cap B$ is connected). In [17], Mioduszewski proved that there is a component K of $\psi^{-1}(0)$ such that K meets all four sides of I^2 (see also [19]). Note that each component of $\psi^{-1}(0)$ is a polyhedron. Let L be a component of $\psi^{-1}(0)$ containing the point $p = (0, 0) \in I^2$. Suppose, on the contrary, that $L \cap (I \times \{1\} \cup \{1\} \times I) = \phi$. Then there is an arc $\alpha : I \to I^2$ such that $\alpha(0) = (x_1, 0) \in I \times \{0\}$, $\alpha(1) = (0, y_1) \in \{0\} \times I$, and $\psi^{-1}(0) \cap \alpha(I) = \phi$. Note that $g(0) = 0 < f(x_1)$ and $g(y_1) > f(0)$. Hence we can see that there is a point $q = (q_1, q_2) \in \alpha(I)$ such that $f(q_1) = g(q_2)$, which implies that $q \in \phi^{-1}(0)$. This is a contradiction. Hence K contains L. By using this fact, we can choose desired maps $a, b : I \to I$.

3. The nonexistence of expansive homeomorphisms of certain continua

The following is the main theorem in this paper.

Theorem 3.1. *Let $f : X \to X$ be a homeomorphism of a compactum X. If there is a minimal set $\mathbf{A}$ of $C(f)$ such that some element A of $\mathbf{A}$ is a (nondegenerate) chainable continuum, then f is not expansive.*

Proof. Suppose, on the contrary, that f is expansive. Replace $\mathbf{A}$ if necessary by an $\mathbf{A}_0 \;{}_*\!\!< \mathbf{A}$ which satisfies the condition (1) of Proposition 2.7. Since every subcontinuum of a chainable continuum is also chainable, we may assume that $\mathbf{A}$ satisfies the conditions of Proposition 2.7,(1).

Let $c > 0$ be an expansive constant for f and $c/2 > \epsilon > 0$. Now, we shall prove the following property

(3.1.1)

For any $0 < \tau < \epsilon$ there are two points x, y of X and a natural number $n(\tau)$

$$\text{such that } d(x,y) \leq \tau, d(f^{n(\tau)}(x), f^{n(\tau)}(y)) \leq \tau, \text{ and}$$
$$\epsilon \leq \sup\{d(f^j(x), f^j(y)) |\, 0 \leq j \leq n(\tau)\} \leq 2\epsilon.$$

Let $A \in \mathbf{A}$ be a chainable continuum. Since A is chainable, there is a $\tau/4$-chain $C = [C_1, C_2, \ldots, C_r]$ in X which is an open covering of A. We can choose a subcontinuum B_1 of A such that $B_1 \in \mathbf{V}^u(\tau; \epsilon)$ (see (1) of Proposition 2.7), and we may assume that $\operatorname{diam}(B_1) = \tau$. Since $B_1 \in \mathbf{V}^u$ and f is expansive, we can choose a natural number N_1 such that if $x, y \in B_1$ and $d(x, y) \geq \tau/4$, then

$$\sup\{d(f^i(x), f^i(y)) |\, 0 \leq i \leq N_1\} > 2\epsilon.$$

Choose a subcontinuum B_2 of B_1 such that $\operatorname{diam} B_2 = \tau/2$. By the assumption, there is a sequence $n_1 < n_2 < \ldots$ of natural numbers such that $\lim_{i\to\infty} f^{n_i}(B_2) = \lim_{i\to\infty} f^{n_i}(A) = A$ (see Proposition 2.7). Hence, we can choose a natural number $N > N_1$ such that $f^N(B_1), f^N(B_2) \in \langle C_1, \ldots, C_r\rangle$. Choose a point $e \in B_2$ such that $f^N(e) \in C_1$. Since B_1, B_2 are chainable, by [6] or [14] there are onto maps $\psi_k : P \to B_k$ $(k = 1, 2)$ from the pseudo-arc P onto B_k. Let $p \in P$. Since P is homogeneous [3], we may assume that $\psi_k(p) = e$ for each $k = 1, 2$. Choose a chain covering $D = [D_1, \ldots, D_s]$ of P such that its mesh is sufficiently small. We may assume that if $x, y \in D_i \cup D_{i+1}$, then

$$\sup\{d(f^j(\psi_k(x)), f^j(\psi_k(y))) |\, 0 \leq j \leq N\} < \epsilon/2 \tag{3.1.2}$$

for each $k = 1, 2$. We may assume that $p \in D_1$ (see the proof of [3, Theorem 13]). Also we may assume that D is a refinement of the chains $C^k = (f^N \cdot \psi_k)^{-1}(C)$ $(k = 1, 2)$. Let $f_k : I(s) \to I(r)$ $(k = 1, 2)$ be patterns such that D follows the patterns f_k in C^k $(k = 1, 2)$. Then the patterns f_k $(k = 1, 2)$ induce maps $f_k : N(D) = N(\{D_1, \ldots, D_s\}) \to N(C) = N(\{C_1, \ldots, C_r\})$ which are natural simplicial maps from the nerve $N(D)$ of D to $N(C)$ of C with $f_k(D_j) = C_{f_k(j)}$ for each j.

Since the above nerves are arcs, we can consider that f_k is a map from the unit interval I onto I such that $f_k(0) = 0$ $(k = 1, 2)$. By Lemma 2.10, there are onto maps $g_k : I \to I$ such that $f_1 \cdot g_1 = f_2 \cdot g_2$ and $g_k(0) = 0$.

By using g_k $(k = 1, 2)$, we obtain patterns $g_k : I(l) \to I(s)$ satisfying the inequality $|f_1 g_1(j) - f_2 g_2(j)| \leq 1$ for each $j = 1, 2, \ldots, l$. By Lemma 2.9, we can choose chain coverings $E = [E_1, E_2, \ldots, E_l]$ and $F = [F_1, F_2, \ldots, F_l]$ of P such that E follows the pattern g_1 in D and F follows the pattern g_2 in D. We may assume that $p \in E_1 \cap F_1$.

Choose points $a_1, \ldots, a_l, b_1, \ldots, b_l$ of P beginning with $p = a_1 = b_1$ and such that $a_j \in E_j, b_j \in F_j$. Note that

$$d(f^N(\psi_1(a_j)), f^N(\psi_2(b_j))) \leq \tau.$$

For each $i = 1, 2, \ldots, l$, put

$$r_i = \sup\{d(f^j(\psi_1(a_i)), f^j(\psi_2(b_i))) |\, 0 \leq j \leq N\}.$$

Since the chain cover D is sufficiently small (see (3.1.2)), we may assume that

$$|r_i - r_{i+1}| < \epsilon.$$

Note that $r_1 = 0$. Since ψ_1 is surjective, there is a point a_u $(u \leq l)$ such that $d(\psi_1(a_u), B_2) \geq \tau/4$, and hence $d(\psi_1(a_u), \psi_2(b_u)) \geq \tau/4$. Thus $r_u > 2\epsilon$. Then we can choose $i \leq u$ such that $\epsilon \leq r_i \leq 2\epsilon$. The two points a_i, b_i satisfy the condition (3.1.1).

Let $\{\epsilon_i\}_{i=1}^{\infty}$ be a sequence of positive numbers such that $\lim_{i\to\infty} \epsilon_i = 0$. By the condition (3.1.1), there are two points $x_i, y_i \in X$ and a natural number $n(i)$ such that

$$d(x_i, y_i) < \epsilon_i, \qquad d(f^{n(i)}(x_i), f^{n(i)}(y_i)) < \epsilon_i$$

and

$$\epsilon \leq \sup\{d(f^j(x_i), f^j(y_i)) \,|\, 0 \leq j \leq n(i)\} \leq 2\epsilon.$$

Choose $0 < m(i) < n(i)$ such that $d(f^{m(i)}(x_i), f^{m(i)}(y_i)) \geq \epsilon$. We may assume that $\{f^{m(i)}(x_i)\}$ and $\{f^{m(i)}(y_i)\}$ are convergent to x_0 and y_0, respectively. Note that

$$\lim_{i\to\infty}(n(i) - m(i)) = \infty = \lim_{i\to\infty} m(i).$$

Then $x_0 \neq y_0$ and $d(f^n(x_0), f^n(y_0)) \leq 2\epsilon < c$ for all $n \in \mathbf{Z}$. This is a contradiction.

Corollary 3.2. *If X is a decomposable circle-like continuum, then X admits no expansive homeomorphism.*

Proof. Suppose, on the contrary, that there is an expansive homeomorphism $f : X \to X$. Since X is decomposable, there are two proper nonempty subcontinua A, B of X such that $A \cup B = X$. Since X is circle-like, A and B are chainable. Note that $A \cap B$ has at most 2 components [5, Theorem 5]. By [11, Theorem 3.6], there is a σ-chaotic continuum C of f. We may assume that $\sigma = u$. Then C is indecomposable (see [11, Corollary 5.3]) and is a proper subcontinuum of X. Note that $f^n(C) \cap A$ and $f^n(C) \cap B$ have at most 2 components. Since $f^n(C)$ is indecomposable, we can easily see that for each $n = 0, 1, \dots,$ $f^n(C) \subset A$ or $f^n(C) \subset B$. Hence we see that there is a minimal set $\mathbf{A}$ of $C(f)$ satisfying the condition of Theorem 3.1. By Theorem 3.1, f is not expansive.

Corollary 3.3. *Let $f : X \to X$ be a homeomorphism of a compactum X. Suppose that there are maps $\psi : X \to Y$ and $g : Y \to Y$ such that $\psi \cdot f = g \cdot \psi$ and for each $y \in Y$ $\psi^{-1}(y)$ is a (nondegenerate) chainable continuum. Then f is not expansive.*

Proof. Let $y_0 \in Y$. By Corollary 2.2, we may assume each element of $\omega(\psi^{-1}(y_0))$ is nondegenerate. Since $\psi \cdot f = g \cdot \psi$ and the collection $\{\psi^{-1}(y) \,|\, y \in Y\}$ is an upper semi-continuous decomposition of X, each element of $\omega(\psi^{-1}(y_0))$ is contained in some $\psi^{-1}(y)$, and hence it is chainable.

Take a minimal set $\mathbf{A}$ of $\omega(\psi^{-1}(y_0))$. Then each element of $\mathbf{A}$ is a chainable continuum. By Theorem 3.1, f is not expansive.

Corollary 3.4. *Let $f : X \to X$ be an expansive homeomorphism of a circle-like continuum X, and let $\delta > 0$ be a positive number as in Lemma 2.1. Then one of the following conditions is satisfied:*

(i) *If $A \in C(X)$ and $0 < \operatorname{diam} A < \delta$, then $A \in \mathbf{V}^u$, and if B is a nondegenerate subcontinuum of X, then $X \in \omega(B)$.*

(ii) *If $A \in C(X)$ and $0 < \operatorname{diam} A < \delta$, then $A \in \mathbf{V}^s$, and if B is a nondegenerate subcontinuum of X, then $X \in \alpha(B)$.*

Proof. By Lemma 2.3, we may assume that $\mathbf{V}^u \neq \phi$. Let $A \in \mathbf{V}^u$. Suppose, on the contrary, that $\omega(A)$ does not contain X. Then we obtain a minimal set $\mathbf{A} \subset \omega(A)$ of $C(f)$ satisfying the condition of Theorem 3.1. Hence f is not expansive, which is a contradiction. Since $X \in \omega(A)$, by Lemma 2.4 we see that for each $x \in X$ there is $x \in A_x \in \mathbf{V}^u(\delta;\epsilon)$, where δ, ϵ are as in Lemma 2.1. Suppose, on the contrary, that $\mathbf{V}^s \neq \phi$. Then we see also that for each $x \in X$ there is $x \in B_x \in \mathbf{V}^s(\delta;\epsilon)$. Since f is expansive, we know that $A_x \cap B_x = \{x\}$. Choose A_x and two points $y, z \in A_x$ such that x, y and z are different. Then there are three subcontinua B_x, B_y, B_z such that their diameters are small and B_x, B_y and B_z are mutually disjoint. Then $T = A_x \cup B_x \cup B_y \cup B_z$ is a triod. Since X is atriodic, this is a contradiction. Hence $\mathbf{V}^s = \phi$. Let A be a nondegenerate subcontinuum of X with $\operatorname{diam} A = \gamma < \delta$. Suppose, on the contrary, that $\sup\{\operatorname{diam} f^{-n}(A) |\, n \geq 0\} \geq \epsilon$. By using Lemmas 2.1 and 2.3 inductively, we have a sequence $n_1 < n_2 < \dots$ of natural numbers and a sequence $B_1, B_2, \dots$ of subcontinua such that $B_i \subset f^{-n_i}(A), \operatorname{diam} B_i = \delta$ and $\operatorname{diam} f^j(B_i) \leq \epsilon$ for each $0 \leq j \leq n_i$. We may assume that $\lim_{i\to\infty} B_i = B \in \mathbf{V}^s$. This is a contradiction. Hence we see that $A \in \mathbf{V}^u$. Clearly, if B is a nondegenerate subcontinuum, then $X \in \omega(B)$, because $B \notin \mathbf{V}^s$.

Corollary 3.5. *If $f : X \to X$ is an expansive homeomorphism of a circle-like continuum X, then f is itself weakly chaotic in the sense of Devaney.*

Proof. Consider the set $\mathcal{D}(f) \neq \phi$. Let $Y \in \mathcal{D}(f)$. Since $\dim Y > 0$, Y contains a nondegenerate subcontinuum. By Corollary 3.4, we see that $Y = X$. Hence f is weakly chaotic in the sense of Devaney.

Remark. In [16], Lewis showed that, for every 1-dimensional continuum M there exists a 1-dimensional continuum $\hat{M}$ such that $\hat{M}$ has a continuous decomposition $\psi : \hat{M} \to M$ into pseudo-arcs such that the decomposition space is homeomorphic to M and the decomposition elements are all terminal continua in $\hat{M}$, i.e., every subcontinuum of $\hat{M}$ either is contained in a single decomposition element or is a union of decomposition elements. More generally, let $\tilde{N}$ be a compactum that has an upper semi-continuous decomposition φ into indecomposable chainable continua such that the decomposition elements are all terminal, and let N be the decomposition space. Moreover, if each proper subcontinuum of N is decomposable, then for any homeomorphism $\tilde{h} : \tilde{N} \to \tilde{N}$ there is a homeomorphism $h : N \to N$ such that $\varphi \cdot \tilde{h} = h \cdot \varphi$. By Corollary 3.3, $\tilde{N}$ admits no expansive homeomorphism. The typical continua are solenoids of pseudo-arcs and hence they admit no expansive homeomorphisms.

Problem 3.6. Does there exist an indecomposable plane circle-like continuum which admits an expansive homeomorphism? In particular, does the pseudo-circle admit an expansive homeomorphism?

Problem 3.7. Does there exist a hereditarily indecomposable continuum which admits an expansive homeomorphism?

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