ONLY SINGLE TWISTS ON UNKNOTS CAN PRODUCE COMPOSITE KNOTS

CHUICHIRO HAYASHI AND KIMIHIKO MOTEGI

ABSTRACT. Let K be a knot in the 3-sphere S^3, and D a disc in S^3 meeting K transversely more than once in the interior.

For non-triviality we assume that $|K \cap D| \geq 2$ over all isotopy of K.

Let $K_n(\subset S^3)$ be a knot obtained from K by cutting and n-twisting along the disc D (or equivalently, performing $1/n$-Dehn surgery on ∂D).

Then we prove the following: (1) If K is a trivial knot and K_n is a composite knot, then $|n| \leq 1$; (2) if K is a composite knot without locally knotted arc in $S^3 - \partial D$ and K_n is also a composite knot, then $|n| \leq 2$.

We exhibit some examples which demonstrate that both results are sharp.

Independently Chaim Goodman-Strauss has obtained similar results in a quite different method.

1. INTRODUCTION

Let K be a knot in the 3-sphere S^3 and D an arbitrarily fixed disc intersecting K transversely in its interior. Suppose that $|K \cap D| \geq 2$ over all isotopies of K in $S^3 - \partial D$. Performing $1/n$-surgery along $C = \partial D$, we obtain a new knot K_n in S^3 as the image of K. We call K_n a *twisted knot* if the original knot K is unknotted in S^3. In the case where K intersects D in just two points, then a twisted knot K_n is always prime ([11], [14]), which is a generalization of "Unknotting number one knots are prime [10]". Mathieu asked in [5] whether twisted knots are always prime. Several families of twisted knots are known to be composite ([8], [12], [6]). All the examples are ± 1-twist.

In [7], it is shown that a twisted knot K_n can be composite only for $|n| \leq 5$. Recently Teragaito [13] has shown that 5 can be replaced by 2. The main result in this paper is

Theorem 1.1. *If a twisted knot K_n is composite, then $n = \pm 1$.*

We will prove also the following generalization. We say that K has *no locally knotted arc* in $S^3 - C$ if there is no 3-ball B^3 in $S^3 - C$ such that $(B^3, B^3 \cap K)$ is a knotted ball pair.

1-twist

$\cong$

FIGURE 1.1

(-2)-twist

$\cong$

FIGURE 1.2
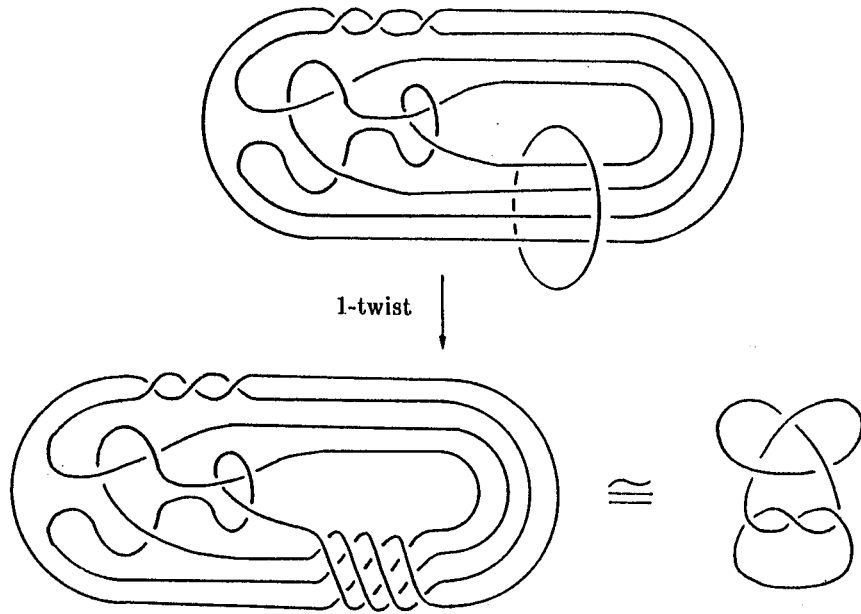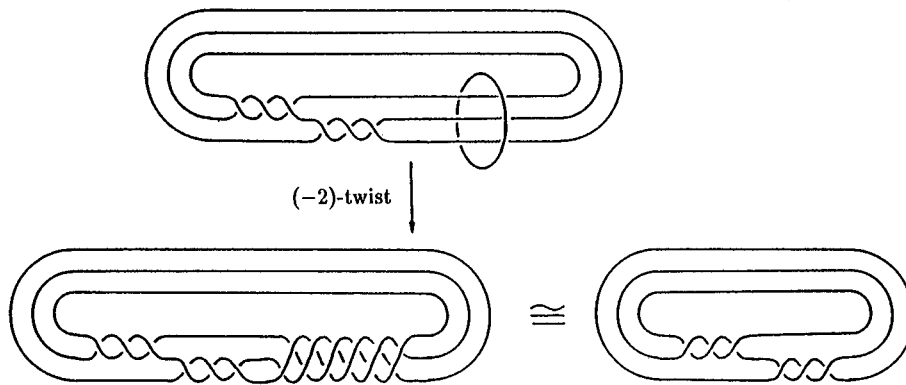
Theorem 1.2. *Suppose that K has no locally knotted arc in $S^3 - C$. (K is possibly knotted in S^3.) If K_m and K_n are composite knots, then $|m - n| \leq 2$.*

Theorem 1.2 is also sharp: there are examples such that K_m and K_n are both composite for m, n with $|m - n| = 1$ (Figure 1.1) and with $|m - n| = 2$ (Figure 1.2). The first example is given by Tetsuo Shibuya.

Independently, Chaim Goodman-Strauss has obtained Theorems 1.1 and 1.2 by a quite different method in his paper [2], in which he also shows that our example (Figure 1.2) is the only exceptional one for $|m - n| = 2$.

Acknowledgements

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2. Intersection of surfaces and associated graphs

To prove Theorems 1.1 and 1.2, it is sufficient to consider punctured annuli (derived from an essential annulus of composite knot exteriors) and punctured discs (derived from meridian discs of trivial knot exteriors).

However in this section we consider a more general situation as follows.

Let M be an orientable 3-manifold whose boundary ∂M contains a toral component T.

Let $(F_i, \partial F_i) \subset (M, \partial M)$ $(i = 1, 2)$ be an orientable surface, which is possibly compressible or ∂-compressible in M. Suppose that $\partial F_i \cap T \neq \emptyset$ and that the component of $\partial F_1 \cap T$ and that of $\partial F_2 \cap T$ have distinct slopes γ_1 and γ_2 on T. Then $\Delta = \Delta(\gamma_1, \gamma_2)$ will denote their minimal geometric intersection number. We assume that F_1 and F_2 intersect transversely and ∂F_1 and ∂F_2 intersect in the minimal number of points so that each component of ∂F_1 and that of ∂F_2 intersect just Δ times on T. We assign labels $1, \ldots, n_i$ to the components of $\partial F_i \cap T$ in the order which they appear on T. In the following we consider the case where $n_i \geq 2$ for $i = 1, 2$. Let $M(\gamma_i)$ denote the manifold obtained from M by γ_i-Dehn filling along T, that is, by attaching a solid torus V_i to M along T so that γ_i bounds a disc in V_i. We obtain a surface $F(\gamma_i)$ when we cap off all the components of $\partial F_i \cap T$ with discs. We can regard the discs as meridian discs of V_i and can suppose that $F(\gamma_i)$ is imbedded in $M(\gamma_i)$.

As in [3], we form the associated graph G_i on $F(\gamma_i)$. Put $\{i, j\} = \{1, 2\}$. Assign an arbitrary orientation to F_i and we refer to $+$ or $-$ boundary components of F_i, according to the direction of the induced orientation of a boundary component as it lies on T. Label the end points of arc components of $F_1 \cap F_2$ in F_i with the corresponding labels of components of $\partial F_j \cap T$. Thus around each component of $\partial F_i \cap T$ we see the labels $\{1, \ldots, n_j\}$ appearing sequentially (either clockwise or anticlockwise according to the sign of the boundary) Δ times. Capping off all the boundary components of $\partial F_i \cap T$ with discs, we regard these discs as forming the "fat" vertices of the graph G_i on the surface $F_i(\gamma_i)$. Every fat vertex v has the sign according to the sign of the component of $\partial F_i \cap T$. The edges of G_i are corresponding to the arcs of $F_1 \cap F_2$ in F_i. Note that we neglect circle components of $F_1 \cap F_2$. An edge e is an *interior edge* if it connects vertices (possibly the same vertex), and a *boundary edge* otherwise. If G_1 has k boundary edges, so does G_2, and vice versa. Let e be an interior edge. Then we call e a *sign-preserving edge* if it connects vertices which have the same sign, otherwise a *sign-reversing edge*. An edge is a *loop* if both its end points are in the same vertex. A loop is a sign-preserving edge.
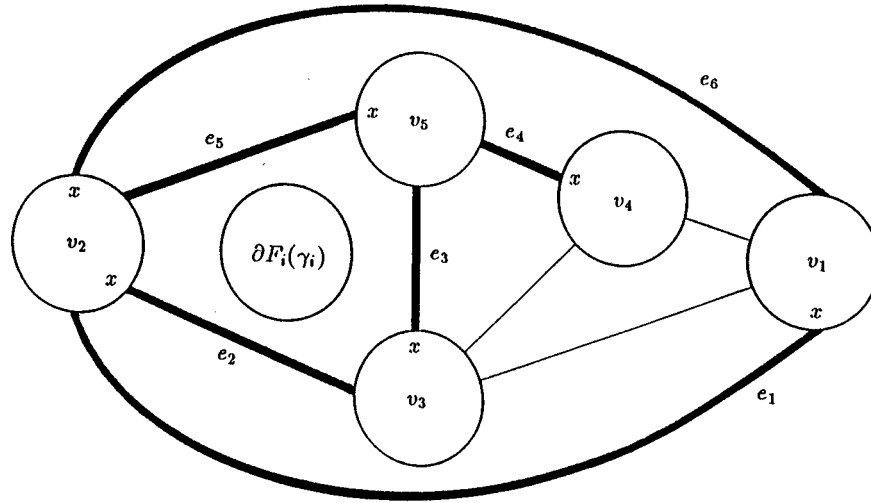
FIGURE 2.1

We thus obtain two labeled graphs, whose edges are in one to one correspondence. The orientability of F_i $(i = 1, 2)$ and M gives us the *parity rule* — an edge e in G_i is a sign-preserving edge if and only if the corresponding edge in G_j is a sign-reversing edge.

The edges and faces of G_i are open edges and faces, i.e., components of $G_i -$ {vertices} and $F(\gamma_i) - G_i$, respectively. An edge e is called a *double edge* of a face P if $e \subset \text{int}\overline{P}$, and a *single edge* otherwise. The *boundary* ∂P is a "subgraph" which consists of vertices and edges intersecting $\overline{P} - P$. A face P is called a *disc face* if P is an open disc. Let v be a fat vertex in ∂P and c a component of $\partial v -$ {end point of edges in ∂P} which intersects $\overline{P} - P$. Then the closure of c in ∂v is called a *corner* of ∂P.

Let x be a label of G_i $(i = 1, 2)$. An *x-edge* in G_i is an edge with label x at an end point.

A subgraph σ is an *x-edge cycle* if σ consists of sign-preserving x-edges and there is a disc face E of the subgraph σ such that $\sigma = \partial E$ (Figure 2.1). E is called a *face of the x-edge cycle*.

The length of an x-edge cycle is defined to be (the number of single edges of the cycle) $+ 2\times$(the number of double edges of the cycle). For example, the cycle in Figure 2.1 has length 7. An x-edge cycle σ is an *x-cycle* if σ contains no double edges and if each edge of the cycle has the label x at the head for some orientation of σ. If all the vertices in the face of an x-edge cycle (resp. x-cycle) have the same sign as those of the vertices of the cycle, then we call such an x-edge cycle (resp. x-cycle) a *great x-edge cycle* (resp. *great x-cycle*). A *Scharlemann cycle* is an x-cycle for some label x which bounds a disc face of G_i. Note that if a Scharlemann cycle is an x-cycle, then we can regard it as an $(x + 1)$-cycle or $(x - 1)$-cycle by reversing the orientation. A Scharlemann cycle is called a *trivial loop* if its length equals one, and a *non-trivial Scharlemann cycle* otherwise. Let t_i be the number of trivial loops in the graph G_i, and b_i the number of boundary edges each of which

has an end point in a fat vertex. Since $b_1 = b_2$, we denote this number by b. Then we have the following inequality.

Theorem 2.1. *If the graph G_i does not contain non-trivial Scharlemann cycles for $i = 1, 2$, then we have*

$$\Delta(\gamma_1, \gamma_2) \le 2 + (t_1 - \chi(F_1(\gamma_1)))/n_1 + (t_2 - \chi(F_2(\gamma_2)))/n_2 + b/n_1 n_2.$$

Before proving this theorem, as applications of this, we will prove Theorems 1.1 and 1.2 in the following two sections.

The proof of Theorem 2.1 will be given in Sect. 6.

3. Proof of Theorem 1.1

In this section we prove Theorem 1.1 by making use of Theorem 2.1.

Let us assume that a twisted knot K_n is composite. Put $M = S^3 - \text{int} N(K \cup C)$, $T = \partial N(C)$ and $T' = \partial N(K)$. Slopes on T, T' will be parametrized by $\mathbb{Q} \cup \{1/0\}$ in the usual way, using a preferred meridian-longitude pair ([9]). Since $K \cup C$ forms an unsplittable link in S^3, M is irreducible. Recall that $M(1/0)$ (resp. $M(1/n)$) is a 3-manifold obtained from M by attaching a solid torus $V_1 = N(C)$ (resp. V_2) to M along T so that $\gamma_1 = 1/0$ (resp. $\gamma_2 = 1/n$) bounds a disc in V_1 (resp. V_2). The 3-manifold $M(1/0)$ is a solid torus, because the original knot K is unknotted in S^3. We choose a meridian disc D of the solid torus $M(1/0) = S^3 - \text{int} N(K)$ so that $D \cap N(C)$ is a disjoint union of meridian discs of $N(C)$ and $n_1 = |D \cap N(C)|$ is minimal among all meridian discs of $M(1/0)$. Note that $n_1 \ge 2$, otherwise K_n would be also unknotted. Then $F_1 = D \cap M$ is an incompressible and ∂-incompressible planar surface in M because of the minimality of n_1. The surface F_1 has an *outer* boundary component in T' and n_1 *inner* boundary components each of which has the slope $\gamma_1 = 1/0$ on T.

Let S be a decomposing 2-sphere for the composite knot K_n. We can choose S so that $S \cap V_2$ is a disjoint union of meridian discs of V_2 and $n_2 = |S \cap V_2|$ is minimal. Note that n_2 is even since S is separating in S^3. Because the trivial knot K has no locally knotted arc in $S^3 - C$, $n_2 \ne 0$. Hence $n_2 \ge 2$. Then $F_2 = S \cap M$ is an incompressible, ∂-incompressible planar surface in M because of the incompressibility of the annulus $F_2(\gamma_2) \subset M(1/n)$ and of the minimality of n_2. F_2 has two outer boundary components lying in T' and n_2 inner boundary components lying in T. Each outer boundary component has the slope $1/0$ on T' and each inner boundary component has the slope $\gamma_2 = 1/n$ on T. By an isotopy of F_2, we may assume that F_1 and F_2 intersect transversely, and each outer boundary component of F_2 intersects the outer boundary component of F_1 exactly once, and each inner boundary component of F_2 intersects each inner boundary component of F_1 in $|n|$ points. By an innermost disc argument, we can assume that every circle component of $F_1 \cap F_2$ bounds a disc neither in F_1 nor in F_2, since F_1 and F_2 are incompressible in M.

As in Sect. 2, we form associated graphs G_1 and G_2 on the surfaces $F_1(\gamma_1)$ and $F_2(\gamma_2)$ respectively. They have at most two boundary edges. Note that neither G_1 nor G_2 contains a trivial loop by the ∂-incompressibility of F_1 and F_2 and the minimality of the number of intersection points of ∂F_1 and ∂F_2 on T.

Sublemma 3.1 ([1, Lemma 2.5.2], [11, Proposition 5.6]). *The graph G_i cannot contain a non-trivial Scharlemann cycle for $i = 1, 2$.*

Proof. Suppose that G_1 contains a non-trivial Scharlemann cycle σ_1 which is simultaneously an x-cycle and an $(x+1)$-cycle. Let P_1 be the face of σ_1. The disc $\overline{P_1}$ is imbedded in the 3-sphere $S_2^3 = M(1/n) \cup N(K_n)$. There is a 1-handle $H_2(\subset V_2)$ imbedded in S_2^3 such that $H_2 \cap S =$(the attaching discs Q_1 and Q_2 of H_2) and that Q_1 and Q_2 corresponding to the fat vertices of G_2 numbered x and $(x + 1)$. Let S' be the torus obtained by applying a surgery on S along H_2. The boundary loop $\partial \overline{P_1}$ is imbedded in S', and the open disc of P_1 does not intersect $S \cup H_2$. Since σ_1 is a non-trivial Scharlemann cycle, the algebraic intersection number of $\partial \overline{P_1}$ and the co-core of the 1-handle H_2 is greater than or equal to two. Let B^3 be the 3-ball bounded by S such that int$B^3 \cap H_2 = \emptyset$. Then $B^3 \cup H_2 \cup N(\overline{P_1})$ is a punctured lens space imbedded in S_2^3, a contradiction.

Suppose that G_2 contains a non-trivial Scharlemann cycle σ_2 which is simultaneously a y-cycle and $(y+1)$-cycle. Let P_2 be the face of σ_2. The disc $\overline{P_2}$ is imbedded in the 3-sphere $S^3 = M(1/0) \cup N(K)$. There is a 1-handle $H_1(\subset V_1)$ imbedded in S^3 such that $H_1 \cap D =$(the attaching discs R_1 and R_2 of H_1) and that R_1 and R_2 corresponding to the fat vertices of G_1 numbered y and $(y + 1)$. Thus we see that $N(D) \cup H_1 \cup N(\overline{P_2})$ is a punctured lens space, a contradiction. $\qquad\square$

Thus we can apply Theorem 2.1 and we have

$$|n| \leq 2 + (t_1 - \chi(F_1(1/0)))/n_1 + (t_2 - \chi(F_2(1/n)))/n_2 + b/(n_1 n_2).$$

Since $t_1 = t_2 = 0$, $b \leq 2$, $\chi(F_1(1/0)) = 1$ and $\chi(F_2(1/n)) = 0$, it follows that $|n| \leq 2 + (2 - n_2)/(n_1 n_2)$. Hence if $n_2 \geq 4$, we have $|n| \leq 1$ as desired.

In case $n_2 = 2$, Teragaito has proved in [13, Theorem 2] that $|n| \leq 1$. However for convenience of the readers we give a proof here following [13] with a slight modification. We will use Lemma 6.2 which is proved in Sect. 6 in this paper.

Since $S \cap V_2$ consists of two meridian discs of V_2, the graph G_2 has exactly two vertices X and Y whose signs are opposite.

If $b < 2$ then Sublemma 3.1 and Theorem 2.1 implies that $|n| \leq 1$.

Hence in the following we assume that $b = 2$. Note that the two boundary edges have end points in distinct boundary components of the annulus $\partial F_2(\gamma_2)$. If the two boundary edges are incident to the same vertex X, say, then there is a loop σ at Y since X and Y have the same valency. Since the two boundary edges form an essential arc on the annulus $F_2(\gamma_2)$, σ bounds a disc on $F_2(\gamma_2)$ which does not contain the vertex X. Hence there would be a trivial loop. This is a contradiction.

Thus for each vertex there is a single boundary edge incident to it in G_2. If there are more than $n_1 + t_1 - \chi(F_1(\gamma_1)) = n_1 - 1$ sign-reversing edges in G_2, then Lemma 6.2 (see Sect. 6) implies that G_1 contains a non-trivial Scharlemann cycle. Hence we can assume that there are at least $(((|n| \times n_1 - 1) \times 2)/2) - (n_1 - 1)$ sign-preserving edges in G_2, which are loops. Suppose that $|n| \geq 2$. Then there are more than or equal to $n_1/2$ loops at the vertex X, say. Each loop must be essential in the annulus $F_2(\gamma_2)$. Consider the edge e incident to X immediately to the right of the boundary edge. Then e must be a loop, for otherwise there would be a loop based at X which is inessential in $F_2(\gamma_2)$. Similarly, the edge e has the other end point immediately to the left of the boundary edge, and all the loops incident to X are parallel to e. Among the loops, if n_1 is even then there is an edge whose two end points have the same label, which contradicts the parity rule. If n_1 is odd then there are two edges which form a non-trivial Scharlemann cycle of length 2, which contradicts Sublemma 3.1. $\qquad\square$

4. PROOF OF THEOREM 1.2

We prove Theorem 1.2 as an application of Theorem 2.1.

Let M be the solid torus $S^3 - \text{int} N(C)$ and T the boundary $\partial M = \partial N(C)$. As in the proof of Theorem 1.1, we parametrize slopes on T by $\mathbb{Q} \cup \{1/0\}$ using preferred meridian-longitude pair. Recall that $M(1/m)$ (resp. $M(1/n)$) is a 3-sphere obtained from M by attaching a solid torus V_1 (resp. V_2) to M along T so that the slope $\gamma_1 = 1/m$ (resp. $\gamma_2 = 1/n$) bounds a disc in V_1 (resp. V_2).

Let S_1 (resp. S_2) be a decomposing 2-sphere for the composite knot K_m (resp. K_n), then $S_1 \cap N(K_m)$ (resp. $S_2 \cap N(K_n)$) consists of two meridian discs of $N(K_m)$ (resp. $N(K_n)$). We choose S_i so that $S_i \cap V_i$ is a disjoint union of meridian discs of V_i and that $n_i = |S_i \cap V_i|$ is minimal. Note that $n_i \geq 2$, because K_m and K_n have no locally knotted arcs in $S^3 - C$.

Put $X = M - \text{int} N(K) = S^3 - \text{int} N(K \cup C)$ and $T' = \partial N(K)$. Then $P_i = S_i \cap X$ is an incompressible, ∂-incompressible planar surface in X. We may assume that P_1 and P_2 intersect transversely, that $\partial P_1 \cap \partial P_2 \cap T' = \emptyset$, and that each component of $\partial P_1 \cap T$ intersects each component of $\partial P_2 \cap T$ in $|m - n|$ points on T. By an innermost disc argument, we can assume that every circle component of $P_1 \cap P_2$ bounds a disc neither in P_1 nor in P_2.

Let F_i be the surface $S_i \cap M$. As in Sect. 2, we form associated graphs G_i on $S_i = F_i(\gamma_i)$. The graphs G_1 and G_2 have no boundary edges. Furthermore since P_1 (resp. P_2) is ∂-incompressible in X and since ∂P_1 and ∂P_2 intersect in the minimal number of points on T, the face of a trivial loop of G_1 (resp. G_2) contains at least one of the two discs $S_1 \cap N(K_m)$ (resp. $S_2 \cap N(K_n)$). Hence G_i contains at most two trivial loops, i.e., $t_i \leq 2$.

Sublemma 4.1. *The graph G_i cannot contain a non-trivial Scharlemann cycle for $i = 1, 2$.*

Proof. Note that $M(1/m)$ and $M(1/n)$ are 3-spheres. Suppose that one of G_1 and G_2, say G_1, contains a non-trivial Scharlemann cycle σ which is simultaneously an x-cycle and an $(x + 1)$-cycle. Let P be the face of σ. The disc $\overline{P}$ is imbedded in the 3-sphere $M(1/n)$. There is a 1-handle $H(\subset V_2)$ imbedded in $M(1/n)$ such that $H \cap S_2 =$ (the attaching discs Q_1 and Q_2 of H) and that Q_1 and Q_2 are corresponding to the fat vertices of G_2 numbered with x and $(x + 1)$. Let S_2' be the torus obtained by applying a surgery on S_2 along H. The boundary of $\overline{P}$ is imbedded in S_2'. The open disc P may intersect S_2 in circles and does not intersect H. Each circle does not separate the two discs Q_1 and Q_2 on S_2 since there is an arc on $\partial \overline{P}$ connecting the two discs. Hence there is an innermost circle which on S_2 bounds a disc whose interior does not intersect $\overline{P}$. Let B_2^3 be the 3-ball bounded by the 2-sphere S_2 such that $\text{int} B_2^3 \cap H = \emptyset$. Then suitable surgery on $\overline{P}$ yields a new disc P' such that $P' \cap (B_2^3 \cup H) = P' \cap S_2' = \partial P' = \partial \overline{P}$. Since σ is a non-trivial Scharlemann cycle, $B_2^3 \cup H \cup N(P')$ is a punctured lens space in the 3-sphere $M(1/n)$, a contradiction. □

Since $\chi(F_1(1/m)) = 2$, $\chi(F_2(1/n)) = 2$, $t_1 \leq 2$, $t_2 \leq 2$ and $b = 0$, we have $|m - n| \leq 2$ by using Theorem 2.1. This completes the proof. □

5. Finding Scharlemann cycles

We work in the situation described in Sect. 2. Let G_i be a graph on $F_i(\gamma_i)$. Our purpose in this section is to give a new condition which implies existence of a Scharlemann cycle.

Proposition 5.1. *A face of a great x-edge cycle σ of G_i contains a face of a Scharlemann cycle.*

Remark. If the surface F_j is separating in the ambient 3-manifold M, then this result has been obtained by Gordon-Luecke [4]. However their proof does not work when F_j is non-separating in M.

Proof. Let P be the face of σ, and G_P the subgraph $(G_i \cap P) \cup \partial P$. We take the subgraph Λ of G_P such that { the vertices of Λ } = { the vertices of G_P} and that { the edges of Λ } = { the x-edges of G_P }. Since for each vertex v of Λ there is an x-edge incident to v in $\overline{P}$, there exists a disc face E of Λ in P. The boundary ∂E is an x-edge cycle and no corners of ∂E contain a label x in its interior.

Let G_E be the graph $(G_P \cap E) \cup \partial E$. Note that E contains no vertices of G_P. If E contains a loop of G_E, then E contains a face of a trivial Scharlemann cycle. Hence we assume that E contains no loops.

If ∂E is not a "circle" when we regard fat vertices as points, then we cut the graph G_E along double edges of ∂E and at fat vertices to which more than two edges of ∂E are incident so that ∂E deformed into a "circle". For example we cut the graph in Figure 2.1 into the graph in Figure 5.1. Let $\widehat{G_E}$ be the resulting graph, and $\widehat{E}$ the disc obtained by cutting E as above.

It is sufficient to find a Scharlemann cycle in $\widehat{G_E}$ such that its face is in $\widehat{E}$. We will show this by induction on the length ℓ of the x-edge cycle $\partial \widehat{E}$.
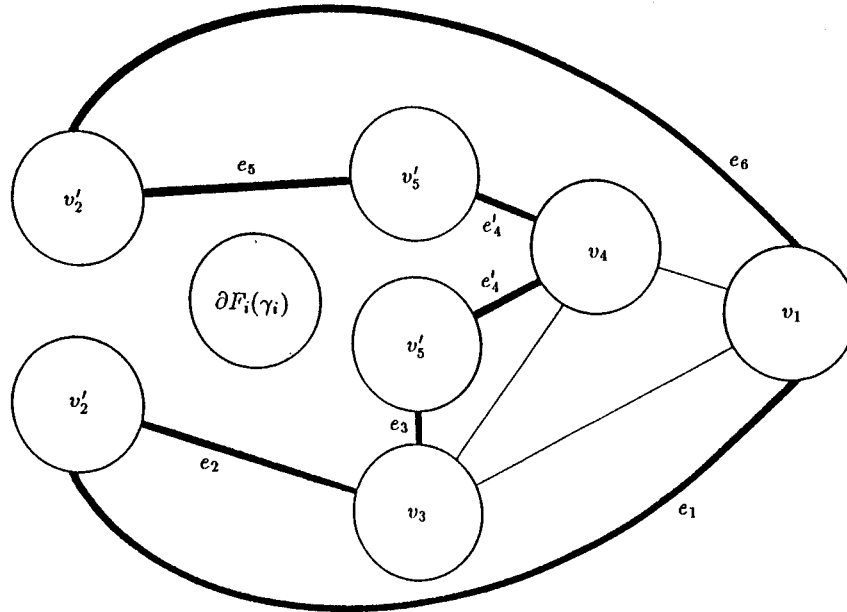

Figure 5.1

To study the graph $\widehat{G_E}$, we forget the geometric meaning and consider a more general situation, which is purely abstract in the sense that the parity rule does not hold. More precisely we consider the following graphs.

Let Γ be a graph on $F_i(\gamma_i)$ and σ an x-edge cycle in Γ. Let E be a disc face of σ. We assume that $\Gamma = (\Gamma \cap E) \cup \partial E$ (i.e., σ is the outermost cycle of Γ). Further Γ satisfies the following :

(1) σ is homeomorphic to a circle when we regard fat vertices as points,

(2) Γ has no vertices in E,

(3) each end of each edge is assigned a number $1, 2, \ldots$, or N ($N \geq 2$), and around each corner of σ the ends are labelled $i, i+1, \ldots, i+k$ (mod N) for some i and k in, say, anticlockwise order, where integers i and k may vary from corner to corner,

(4) there is no loop in E.

Note that all edges in Γ are sign-preserving edges and an x-edge cycle (resp. x-cycle) is automatically a great x-edge cycle (resp. great x-cycle) by the condition (2), and thus σ is a great x-edge cycle.

We call the following condition the *weak parity rule* — every sign-preserving edge has distinct labels at end points. (Clearly the parity rule implies the weak parity rule.) We call a sign-preserving edge which has the same label at both end points a *squashed Scharlemann cycle*, and continue to use the terminology established in Sect. 2 for the above graphs.

Lemma 5.2. *If the weak parity rule holds for edges in the open disc E, then Γ contains a non-squashed Scharlemann cycle (whose face is contained in E).*

In our original situation, regarding $\widehat{G_E}$ as Γ we find a Scharlemann cycle in $\widehat{G_E}$, hence in G_i, which establishes Proposition 5.1.

The remainder of this section is devoted to the proof of this lemma.

The proof will be done by induction on the length ℓ of σ. Considering the innermost x-edge cycle if necessary, we can assume that there are no x-edges in E.

Proof of Lemma 5.2. We start with the following sublemma.

Sublemma 5.3 (Special case of Lemma 5.2). *Suppose that $\sigma = \partial E$ is an x-cycle. If the weak parity rule holds for edges in E, then Γ contains a non-squashed Scharlemann cycle whose face is contained in E.*

Proof. The proof of Lemma 2.6.2 [1] still works for this sublemma.   □

If ∂E does not have a corner such that both its two end points (i.e., ∂(corner)) have the label x (we call such a corner an *x-x corner*), then ∂E is an x-cycle. Hence by Sublemma 5.3, E contains a non-squashed Scharlemann cycle.

We assume in the rest of the proof that there exists an x-x corner.

Sublemma 5.4. *Assume that the weak parity rule holds for edges in E. If there is a family of more than $N/2$ parallel edges in Γ, then we have:*

(1) there is a non-squashed Scharlemann cycle (of length two) in that family and its face is contained in E, or

(2) N is odd and the family consists of just $(N+1)/2$ parallel edges including a squashed Scharlemann cycle which is an edge on ∂E.

Proof. The sets of labels at the two ends of the parallel family are not disjoint. Hence there is a squashed Scharlemann cycle or an x-cycle (for some label x) in that family.

In the former case the squashed Scharlemann cycle is an edge of σ and N must be odd, otherwise E contains a squashed Scharlemann cycle, a contradiction. If there are more than $(N+1)/2$ parallel edges, then there is a non-squashed Scharlemann cycle of length two in that family.

In the latter case (i.e., there is an x-cycle), then Sublemma 5.3 implies existence of a non-squashed Scharlemann cycle. □

First we consider the case $\ell = 2$.

Since there is an x-x corner, say Q, Γ consists of $N + 1$ parallel sign-preserving edges. Then Sublemma 5.4 shows that Γ contains a non-squashed Scharlemann cycle.

Let us assume that $\ell \geq 3$.

An edge e is called a *diagonal edge* if e is incident to an x-x corner Q and has the other end point in a corner R which is not adjacent to Q on σ. When $\ell = 3$, there is no diagonal edge.

We divide into two cases depending upon whether a diagonal edge e exists or not.

Case 1: There is no diagonal edge on Γ.

Then for every x-x corner Q we can find more than $N/2$ parallel edges incident to Q since there are $N + 1$ edges incident to Q. If N is even, then we have a (non-squashed) Scharlemann cycle by Sublemma 5.4. If N is odd, then we have the following two possibilities by Sublemma 5.4.

(i) Γ contains a (non-squashed) Scharlemann cycle, or

(ii) the two edges which have the label x at the end points in Q have also the label x at the other end points, and the edges incident to Q are divided into two families of $(N+1)/2$ parallel edges which connect Q with two corners adjacent to Q on ∂E (see Figure 5.2).

We may assume that all the x-x corners satisfy the condition (ii), because otherwise we can find a non-squashed Scharlemann cycle. Under this assumption we show that every corner on ∂E is an x-x corner.

Let $Q_0, Q_1, \ldots, Q_m$ be the corners on ∂E appearing sequentially in this order. Let e_i be the edge on ∂E connecting Q_i and Q_{i+1}, where e_m connects Q_m and Q_0. The edge e_i has the tail in the corner Q_i and the head in the corner Q_{i+1}. We assume that Q_0 is an x-x corner and show that the next corner Q_1 is also an x-x corner. The label of the head of e_0 is x and the label at the tail of e_m is also x by the condition (ii). If the tail of e_1 has the label x, then we are done. Assume for a contradiction that the tail of e_1 does not have the label x. In general, suppose that the label at the tail of e_i is not x for some i $(1 \leq i \leq m - 1)$. Then the head of e_i has the label x. If the corner Q_{i+1} is an x-x corner, then by the condition (ii), the tail of e_i has the label x, a contradiction. Hence Q_{i+1} is not an x-x corner. Thus the tail of e_{i+1} does not have the label x. Inductively the assumption that the tail of e_1 does not have the label x implies that the tail of e_i does not have the label x for $i = 1, \ldots, m - 1$. Therefore particularly the head of e_{m-1} has the label x, and Q_m is an x-x corner (because the tail of e_m has the label x). Hence by the condition (ii), the tail of e_{m-1} has the label x, a contradiction. This proves Q_1 is also an x-x corner, and hence all the corners are x-x corners, all of which satisfy the
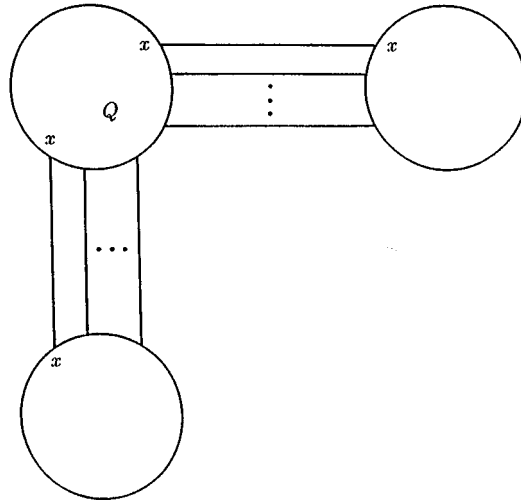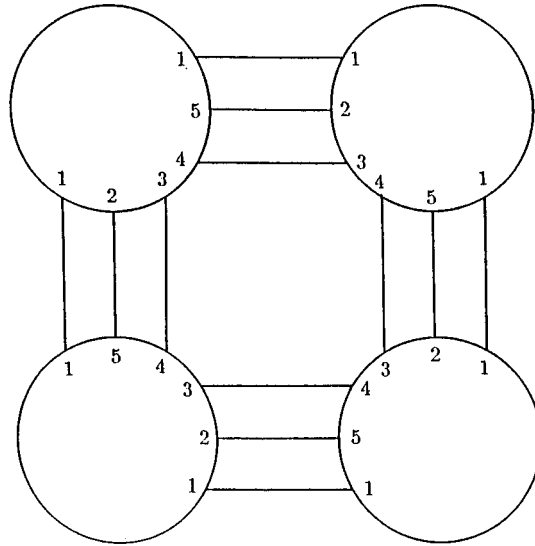
FIGURE 5.2

FIGURE 5.3

condition (ii). Then m families of $(N+1)/2$ parallel edges form $(N+1)/2$ m-gons (see Figure 5.3).

Hence the innermost m-gon is a non-squashed Scharlemann cycle.

Case 2 : There is a diagonal edge e on Γ, i.e., there is an edge e such that e is incident to an x-x corner Q and has the other end point in a corner R which is not adjacent to Q on the x-edge cycle σ.
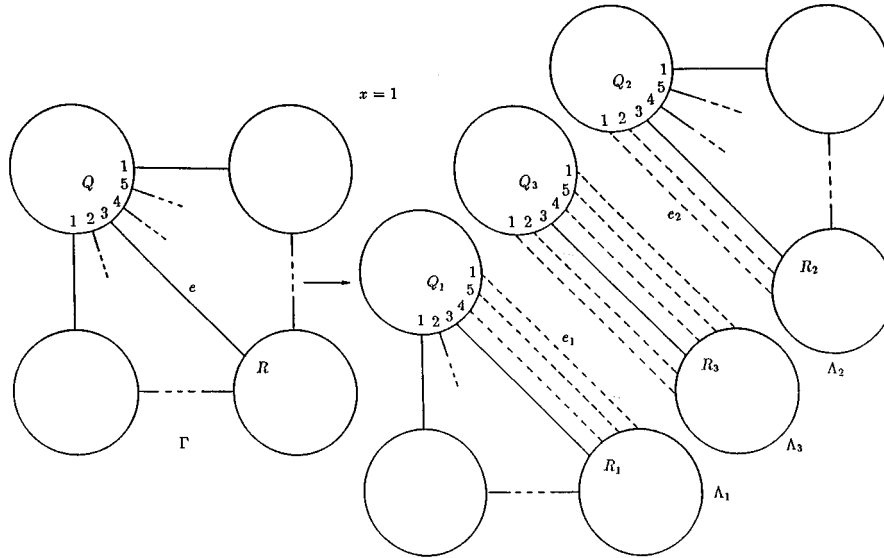
FIGURE 5.4

Let U (resp. V) be the vertex which contains Q (resp. R). Let k be the label at the end of e at Q. Now we formally construct three graphs. The diagonal edge e divides the disc E into two discs E_1 and E_2. Then $\overline{E_1} \cap Q$ contains labels $\{x, x+1, \ldots, k\}$ and $\overline{E_2} \cap Q$ contains labels $\{k, \ldots, x-1, x\}$. Let Λ'_i be the graph $(\Gamma \cap E_i) \cup \partial E_i$ for $i = 1, 2$. Let Λ_1 (resp. Λ_2) be the graph obtained from Λ'_1 (resp. Λ'_2) by adding edges which are parallel to e and incident to Q with labels $k+1, \ldots, x-1, x$ (resp. $x, x+1, \ldots, k-1$). Let $e_i \subset \Lambda_i$ be the outermost additional edge. Let D_i be the disc bounded by the two edges e and e_i. We construct the third graph Λ_3 as $((\Lambda_1 \cap D_1) \cup \partial D_1) \cup ((\Lambda_2 \cap D_2) \cup \partial D_2)$, and denote the disc $\mathrm{int}\overline{D_1 \cup D_2}$ by D_3. The graph Λ_3 consists of two vertices U, V and $N+1$ parallel edges connecting them. See Figure 5.4.

In the graph Λ_i let R_i be the corner of ∂D_i on ∂V which contains the end of e. We note that e does not have the label k at R_i because the weak parity rule holds in E. The graph Λ_i is bounded by an x-edge cycle σ_i, and these graphs satisfy the conditions (1)–(4). However these graphs may not satisfy the weak parity rule on $\Lambda_i - (\sigma_i \cup e)$. In the graph Λ_3, we can easily observe that

Sublemma 5.5. *(1) If the corner R_3 is also an x-x corner, then the graph Λ_3 has just three (possibly squashed) Scharlemann cycles and only one is in D_3. If N is even, then all the three Scharlemann cycles are squashed. If N is odd, then just one of them (which is in D_3) is non-squashed (see Figure 5.5).*

(2) If the corner R_3 is not an x-x corner, then let f be the edge which has the label x at R_3. The graph Λ_3 has just one (possibly squashed) Scharlemann cycle between the edges f and e_i for $i = 1$ and 2, respectively (see Figure 5.5).

Let us assume that R_3 is also an x-x corner. Recall that the diagonal edge e is not a squashed Scharlemann cycle. Hence by Sublemma 5.5, one of the two graphs Λ_1 and Λ_2, say Λ_1, contains just one squashed Scharlemann cycle e_1 and contains no more Scharlemann cycles in the disc D_1. Then in $\mathrm{int}(\overline{E_1} \cup \overline{D_1})$, the weak parity
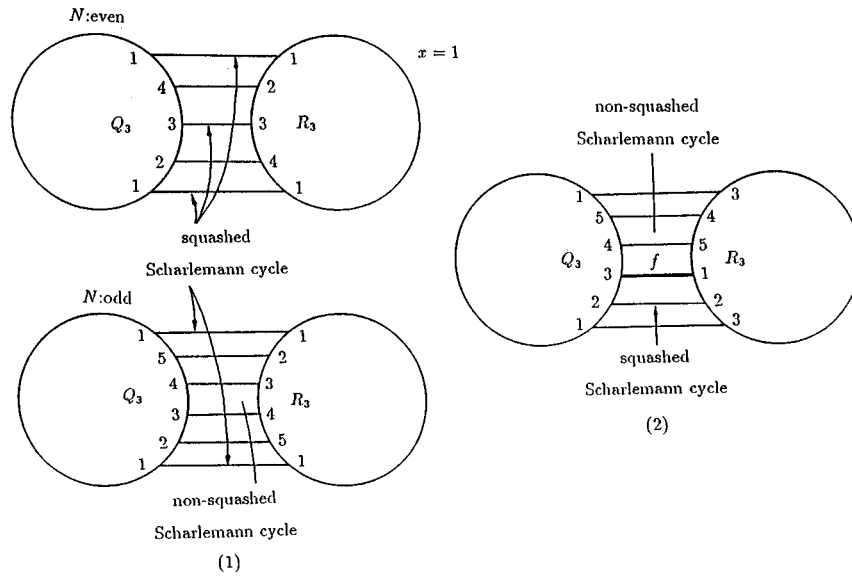
FIGURE 5.5

rule holds. Hence by the inductive hypothesis, there is a face of a non-squashed Scharlemann cycle in $E_1 \cup D_1$. Since there is no non-squashed Scharlemann cycle in $\overline{D_1}$, we can find a face of a non-squashed Scharlemann cycle in E_1.

Now let us assume that R_3 is not an x-x corner. Then we divide into two subcases according to the position of f and the diagonal edge e in Λ_3.

First suppose that one of the two graphs Λ_1 and Λ_2, say Λ_1, does not contain a (possibly squashed) Scharlemann cycle in $\overline{D_1}$. Then the graph Λ_1 is bounded by the x-edge cycle σ_1 and in $\text{int}(\overline{E_1} \cup \overline{D_1})$ the weak parity rule holds. Hence there is a face of a non-squashed Scharlemann cycle in E_1 by the inductive hypothesis.

Suppose that both graphs Λ_1 and Λ_2 contains just one (possibly squashed) Scharlemann cycle in $\overline{D_1}$ and $\overline{D_2}$ respectively. Then at least one of the two graphs Λ_1 and Λ_2, say Λ_1, has the edge f in $\overline{D_1}$. (f may be coincide with e.) Substituting the edge f for the edge e_1 in the x-edge cycle σ_1, we can obtain a new x-edge cycle ρ such that the weak parity rule holds in the face H of ρ. By the inductive hypothesis, there is a face of a non-squashed Scharlemann cycle. In any case, since $H \cap D_1$ does not contain a face of a Scharlemann cycle, the above one exists in E_1. This completes the proof of Lemma 5.2. □

The proof of Proposition 5.1 is now complete. □

6. Calculation of Euler characteristic — Proof of Theorem 2.1

The goal in this section is to prove Theorem 2.1.

Lemma 6.1. *Suppose that for some label x the graph G_i contains more than $(n_i + t_i - \chi(F_i(\gamma_i)))$ sign-preserving edges each of which has the label x at an end point. Then G_i contains a non-trivial Scharlemann cycle.*

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Proof. Let Λ be the subgraph of G_i consisting of the above sign-preserving edges and all the vertices of G_i. The graph Λ may have an isolated vertex. Let f_d denote the number of disc faces of Λ. Applying Euler's formula for the graph Λ on $F_i(\gamma_i)$ we have

$$n_i - (n_i + t_i - \chi(F_i(\gamma_i)) + 1) + \Sigma\chi(face) \geq \chi(F_i(\gamma_i)).$$

Hence $f_d \geq \Sigma\chi(face) \geq t_i + 1$. Thus there is a disc face D of Λ such that $\overline{D}$ does not contain a trivial loop of G_i. In other words there is a great x-edge cycle which is not a trivial loop and whose face does not contain a trivial loop of G_i. This implies that G_i contains a non-trivial Scharlemann cycle by Proposition 5.1. $\square$

The following is equivalent to Lemma 6.1.

Lemma 6.2. *Let v be a vertex of G_j. Suppose that there are more than $(n_i + t_i - \chi(F_i(\gamma_i)))$ sign-reversing edges incident to v. Then G_i contains a non-trivial Scharlemann cycle.*

The next lemma is a consequence of Lemma 6.1.

Lemma 6.3. *Suppose that the graph G_i contains more than $(n_i + t_i - \chi(F_i(\gamma_i))) \times n_j/2$ sign-preserving edges. Then G_i contains a non-trivial Scharlemann cycle.*

Proof. In the graph G_i there are n_j labels $1, \ldots, n_j$, and by the assumption there are more than $(n_i + t_i - \chi(F_i(\gamma_i))) \times n_j$ end points of sign-preserving edges. Hence for some label x there are more than $(n_i + t_i - \chi(F_i(\gamma_i)))$ sign-preserving edges each of which has the label x at an end point. Thus Lemma 6.1 implies that G_i contains a non-trivial Scharlemann cycle. $\square$

Now we can prove Theorem 2.1.

Proof of Theorem 2.1. Suppose that neither G_1 nor G_2 contains a non-trivial Scharlemann cycle. In the graph G_1 there are $\Delta n_1 n_2$ endpoints of the edges around the fat vertices. Lemma 6.2 shows that there are less than or equal to $(n_2 + t_2 - \chi(F_2(\gamma_2)))n_1$ end points of sign-reversing edges in G_1. Hence there are more than or equal to $\Delta n_1 n_2 - (n_2 + t_2 - \chi(F_2(\gamma_2)))n_1 - b$ end points of sign-preserving edges. By Lemma 6.3 this is less than or equal to $(n_1 + t_1 - \chi(F_1(\gamma_1)))n_2$. Thus we have

$$\Delta n_1 n_2 - (n_2 + t_2 - \chi(F_2(\gamma_2)))n_1 - b \leq (n_1 + t_1 - \chi(F_1(\gamma_1)))n_2.$$

Hence we have the desired inequality :

$$\Delta \leq 2 + (t_1 - \chi(F_1(\gamma_1)))/n_1 + (t_2 - \chi(F_2(\gamma_2)))/n_2 + b/(n_1 n_2).$$

$\square$

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Department of Mathematical Sciences, University of Tokyo, Hongo 7-3-1, Bunkyo-ku, Tokyo 113, Japan

Department of Mathematics, College of Humanities & Sciences, Nihon University Sakurajosui 3-25-40, Setagaya-ku, Tokyo 156, Japan
    E-mail address: motegi@math.chs.nihon-u.ac.jp