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CONTRACTIONS ON A MANIFOLD POLARIZED BY AN AMPLE VECTOR BUNDLE

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Abstract. A complex manifold X of dimension n together with an ample vector bundle E on it will be called a generalized polarized variety. The adjoint bundle of the pair (X, E) is the line bundle $K_X + det(E)$. We study the positivity (the nefness or ampleness) of the adjoint bundle in the case $r := rank(E) = (n-2)$. If $r \geq (n-1)$ this was previously done in a series of papers by Ye and Zhang, by Fujita, and by Andreatta, Ballico and Wisniewski.

If $K_X + detE$ is nef then, by the Kawamata-Shokurov base point free theorem, it supports a contraction; i.e. a map $\pi : X \longrightarrow W$ from X onto a normal projective variety W with connected fiber and such that $K_X + det(E) = \pi^* H$, for some ample line bundle H on W. We describe those contractions for which $dimF \leq (r-1)$. We extend this result to the case in which X has log terminal singularities. In particular this gives Mukai's conjecture 1 for singular varieties. We consider also the case in which $dimF = r$ for every fiber and π is birational.

Introduction

An algebraic variety X of dimension n (over the complex field) together with an ample vector bundle E on it will be called a generalized polarized variety. The adjoint bundle of the pair (X, E) is the line bundle $K_X + det(E)$. Problems concerning adjoint bundles have drawn a lot of attention from algebraic geometers: the classical case is when E is a (direct sum of) line bundles (a polarized variety), while the generalized case was motivated by the solution of the Hartshorne-Frankel conjecture by Mori ([Mo]), and by consequent conjectures of Mukai ([Mu]).

A first point of view is to study the positivity (the nefness or ampleness) of the adjoint line bundle in the case when $r = rank(E)$ is about $n = dimX$. This was done in a sequel of papers for $r \geq n-1$ and for a smooth manifold X ([YZ], [Fu2], [ABW2]). In this paper we want to discuss the next case, namely when $rank(E) = n-2$, with X smooth; we obtain a complete answer which is described in the theorem (5.1). This is divided into three cases, namely when $K_X + det(E)$ is not nef, when it is nef and not big, and finally when it is nef and big but not ample. If $n = 3$ a complete picture is already contained in the famous paper of Mori ([Mo1]), while the particular case in which $E = \oplus^{n-2}(L)$ with L a line bundle was also studied ([Fu1], [So]; in the singular case see [An]). Part 1 of the theorem was proved (in a slightly weaker form) by Zhang ([Zh]) and, in the case E is spanned by global sections, by Wisniewski ([Wi2]).

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Another point of view can be the following: let (X, E) be a generalized polarized variety with X smooth and $rankE = r$. If $K_X + det(E)$ is nef, then by the Kawamata-Shokurov base point free theorem it supports a contraction (see Theorem 1.2); i.e. there exists a map $\pi : X \to W$ from X onto a normal projective variety W with connected fiber and such that $K_X + det(E) = \pi^* H$ for some ample line bundle H on W. It is not difficult to see that, for every fiber F of π, we have $dimF \geq (r-1)$; equality holds only if $dimX > dimW$. In the paper we study the "border" cases: we assume that $dimF = (r-1)$ for every fiber and we prove that X has a $\mathbf{P}^r$-bundle structure given by π (Theorem 3.2). We consider also the case in which $dimF = r$ for every fiber and π is birational, proving that W is smooth and that π is a blow-up of a smooth subvariety (Theorem 3.1). This point of view was discussed in the case $E = \oplus^r L$ in the paper [AW].

Finally in section 4 we extend the Theorem 3.2 to the singular case, namely for a projective variety X with log-terminal singularities. In particular this gives Mukai's conjecture 1 for singular varieties.

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1. Notations and generalities

(1.1) We use the standard notation from algebraic geometry. In particular it is compatible with that of [KMM], to which we refer constantly. We just explain some special definitions and propositions used frequently.

In particular, in this paper X will always stand for a smooth complex projective variety of dimension n. Let $Div(X)$ be the group of Cartier divisors on X; denote by K_X the canonical divisor of X, an element of $Div(X)$ such that $\mathcal{O}_X(K_X) = \Omega^n_X$. Let $N_1(X) = \frac{\{1-cycles\}}{\equiv} \otimes \mathbf{R}$, $N^1(X) = \frac{\{divisors\}}{\equiv} \otimes \mathbf{R}$ and $\overline{\langle NE(X)\rangle} = \overline{\{\text{effective 1-cycles}\}}$; the last is a closed cone in $N_1(X)$. Let $\rho(X) = dim_{\mathbf{R}} N^1(X)$.

Suppose that K_X is not nef; that is, there exists an effective curve C such that $K_X \cdot C < 0$.

Theorem 1.2. [KMM] *Let X be as above and H a nef Cartier divisor such that $F := H^\perp \cap \overline{\langle NE(X)\rangle} \setminus \{0\}$ is entirely contained in the set $\{Z \in N_1(X) : K_X \cdot Z < 0\}$, where $H^\perp = \{Z : H \cdot Z = 0\}$. Then there exists a projective morphism $\varphi : X \to W$ from X onto a normal variety W with the following properties:*

- i) *For an irreducible curve C in X, $\varphi(C)$ is a point if and only if $H \cdot C = 0$, if and only if $cl(C) \in F$.*
- ii) *φ has only connected fibers.*
- iii) *$H = \varphi^*(A)$ for some ample divisor A on W.*
- iv) *The image $\varphi^* : Pic(W) \to Pic(X)$ coincides with $\{D \in Pic(X) : D \cdot C = 0$ for all $C \in F\}$.*

Definition 1.3. ([KMM], definition 3-2-3). Using the notation of the above theorem, the map φ is called a contraction (or an extremal contraction); the set F is an extremal face, while the Cartier divisor H is a supporting divisor for the map φ (or the face F). If $dim_{\mathbf{R}} F = 1$ the face F is called an extremal ray, while φ is called an elementary contraction.

Remark 1.4. We have also ([Mo1]) that if X has an extremal ray R then there exists a rational curve C on X such that $0 < -K_X \cdot C \leq n+1$ and $R = R[C] := \{D \in \langle NE(X)\rangle : D \equiv \lambda C, \lambda \in \mathbf{R}^+\}$. Such a curve is called an extremal curve.

Remark 1.5. Let $\pi : X \to V$ denote a contraction of an extremal face F, supported by $H = \pi^* A$. Let R be an extremal ray in F and $\rho : X \to W$ the contraction of R. Then π factors trough ρ (this is because $\pi^* A \cdot R = 0$).

Definition 1.6. To an extremal ray R we can associate:

i) its length $l(R) := min\{-K_X \cdot C$; for C a rational curve and $C \in R\}$
ii) the locus $E(R) :=$ {the locus of the curves whose numerical classes are in $R\} \subset X$.

A rational curve C in R such that $-K_X \cdot C = l(R)$ will be called a minimal curve

It is usual to divide the elementary contractions associated to an extremal ray R into three types, according to the dimension of $E(R)$ as follows.

Definition 1.7. We say that φ is of fiber type, respectively divisorial type, resp. flipping type, if $dim E(R) = n$, resp. $n-1$, resp. $< n-1$. Moreover an extremal ray is said to be not nef if there exists an effective $D \in Div(X)$ such that $D \cdot C < 0$.

The following very useful inequality was proved in [Io] and [Wi3].

Proposition 1.8. *Let φ be the contraction of an extremal ray R, $E'(R)$ any irreducible component of the exceptional locus and d the dimension of a fiber of the contraction restricted to $E'(R)$. Then*

$$dim E'(R) + d \geq n + l(R) - 1.$$

(1.9) Actually it is very useful to understand when a contraction is elementary, or in other words when the loci of two distinct extremal rays are disjoint. For this we will use in this paper the following results.

Proposition 1.10. [BS, Corollary 0.6.1] *Let R_1 and R_2 two distinct not nef extremal rays such that $l(R_1) + l(R_2) > n$. Then $E(R_1)$ and $E(R_2)$ are disjoint.*

Something can be said also if $l(R_1) + l(R_2) = n$:

Proposition 1.11. [Fu3, Theorem 2.4] *Let $\pi : X \to V$ be a birational contraction of a face F; suppose $n \geq 4$ and $l(R_i) \geq n-2$, for R_i extremal rays in F. Then the exceptional loci corresponding to different extremal rays are disjoint.*

Proposition 1.12. [ABW1] *Let $\pi : X \to W$ be a contraction of a face such that $dim X > dim W$. Suppose that for every rational curve C in a general fiber of π we have $-K_X \cdot C \geq (n+1)/2$. Then π is an elementary contraction except if*

a) *$-K_X \cdot C = (n+2)/2$ for some rational curve C on X, W is a point, X is a Fano manifold of pseudoindex $(n+2)/2$ and $\rho(X) = 2$; and if*
b) *$-K_X \cdot C = (n+1)/2$ for some rational curve C, and $dim\, W \leq 1$.*

Finally, the following definitions are used in the main theorem in section 5:

Definition 1.13. Let L be an ample line bundle on X. The pair (X, L) is called a scroll (respectively a quadric fibration, respectively a del Pezzo fibration) over a normal variety Y of dimension m if there exists a surjective morphism with connected fibers $\phi : X \to Y$ such that

$$K_X + (n-m+1)L \approx p^* \mathcal{L}$$

(respectively $K_X+(n-m)L \approx p^*\mathcal{L}$, respectively $K_X+(n-m-1)L \approx p^*\mathcal{L}$) for some ample line bundle $\mathcal{L}$ on Y. X is called a classical scroll (respectively quadric bundle) over a projective variety Y of dimension r if there exists a surjective morphism $\phi : X \to Y$ such that every fiber is isomorphic to $\mathbf{P}^{n-r}$ (respectively to a quadric in $\mathbf{P}^{n-r+1}$) and if there exists a vector bundle E of rank $n-r+1$ (respectively of rank $n-r+2$) on Y such that $X \simeq \mathbf{P}(E)$ (respectively exists an embedding of X as a subvariety of $\mathbf{P}(E)$).

2. A technical construction

Let E be a vector bundle of rank r on X and assume that E is ample (in Hartshorne's sense).

Remark 2.1. Let $f : \mathbf{P}^1 \to X$ be a non-constant map, and $C = f(\mathbf{P}^1)$. Then $detE \cdot C \geq r$.

In particular, if there exists a curve C such that $(K_X+detE)\cdot C \leq 0$ (for instance if (K_X+detE) is not nef), then there exists an extremal ray R such that $l(R) \geq r$.

(2.2) Let $Y = \mathbf{P}(E)$ be the associated projective space bundle, $p : Y \to X$ the natural map onto X and ξ_E the tautological bundle of Y. Then we have the formula for the canonical bundle $K_Y = p^*(K_X+detE) - r\xi_E$. Note that p is an elementary contraction.

Assume that K_X+detE is nef but not ample, and that it is the supporting divisor of an elementary contraction $\pi : X \to W$; let R be the associated extremal ray. Then $\rho(Y/W) = 2$ and $-K_Y$ is $\pi \circ p$-ample. By the relative Mori theory over W we have that there exists a ray on $NE(Y/W)$, say R_1, of length $\geq r$, not contracted by p, and a relative elementary contraction $\varphi : Y \to V$. We have thus the following commutative diagram:

$$\begin{array}{ccc} \mathbf{P}(E) = Y & \xrightarrow{\varphi} & V \\ \downarrow p & & \downarrow \psi \\ X & \xrightarrow{\pi} & W \end{array} \tag{2.1}$$

where φ and ψ are elementary contractions. Let $w \in W$ and let $F(\pi)_w$ be an irreducible component of $\pi^{-1}(w)$; choose also v in $\psi^{-1}(w)$ and let $F(\varphi)_v$ be an irreducible component of $\varphi^{-1}(v)$ such that $p(F(\varphi)_v) \cap F(\pi)_w \neq \emptyset$; then, by the commutativity of the diagram, $p(F(\varphi)_v) \subset F(\pi)_w$. Since p and φ are elementary contractions of different extremal rays, we have that $dim(F(\varphi) \cap F(p)) = 0$; that is, a curve which is contracted by φ cannot be contracted by p.

In particular this implies that $dim(p(F(\varphi)_v)) = dimF(\varphi)_v$; therefore

$$dimF(\varphi)_v \leq dimF(\pi)_w.$$

Remark 2.3. If $dimF(\varphi)_v = dimF(\pi)_w$. Then $dimF(\psi)_w := dim(\psi^{-1}(w)) = r-1$; if this holds for every $w \in W$ then ψ is equidimensional.

Proof. Let Y_w be an irreducible component of $p^{-1}\pi^{-1}(w)$ such that $\varphi(Y_w) = F(\psi)_w$. Then $dimF(\psi)_w = dimY_w - dimF(\varphi)_v = dimY_w - dimF(\pi)_w = dimF(p) = r-1$. □

(2.4) **Slicing techniques**. Let $H = \varphi^*(A)$ be a supporting divisor for φ such that the linear system $|H|$ is base point free. We assume as in (2.2) that (K_X+detE)

is nef, and we refer to the diagram (2.1). The divisor $K_Y + r\xi_E = p^*(K_X + detE)$ is nef on Y, and therefore $m(K_Y + r\xi_E + aH)$, for $m \gg 0$, $a \in \mathbf{N}$, is also a good supporting divisor for φ. Let Z be a smooth n-fold obtained by intersecting $r-1$ general divisors from the linear system $|H|$, i.e. $Z = H_1 \cap \cdots \cap H_{r-1}$ (this is what we call a slicing); let $H_i = \varphi^{-1}A_i$.

Note that the map $\varphi' = \varphi_{|Z}$ is supported by $|m(K_Y + r\xi_E + a\varphi^*A)_{|Z}|$; hence, by adjunction, it is supported by $K_Z + rL$, where $L = \xi_{E|Z}$. Let $p' = p_{|Z}$; by construction p' is finite.

If T is (the normalization of) $\varphi(Z)$ and $\psi' : T \to W$ is the map obtained by restricting ψ, then we have from (2.1) the following diagram:

$$(2.2) \qquad \begin{array}{ccc} Z & \xrightarrow{\varphi'} & T \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle \psi'} \\ X & \xrightarrow{\pi} & W \end{array}$$

In general the map φ' is well understood (for instance, in the case $r = n-2$ see the results in [Fu1] or in [An]). The goal is to "transfer" the information that we have on φ' to the map π. The following Proposition is an example.

We refer to the diagrams and notations of the above sections; in particular $\pi : X \to W$ is the elementary contraction of the ray R supported by $K_X + detE$. Therefore $l(R) \geq r$, and by Proposition 1.8 we have

$$dimE'(R) + d \geq n + r - 1,$$

where $E'(R)$ is an irreducible component of the exceptional locus and $d = dimF(\pi)$.

Proposition 2.5. *Assume that for every non-trivial fiber we have $dimF(\varphi) = dimF(\pi) = k$. Assume also that $l(R) = r$ and that for all fibers of φ*

$$(F(\varphi), \xi_{E|F(\varphi)}) \simeq (\mathbf{P}^k, \mathcal{O}(1)).$$

Then W has the same singularities as T.

Remark 2.6. The above proposition was proved in the case in which φ is birational and $k = r$ in [ABW2].

Proof. Let $w \in W$; by hypothesis and by Remark 2.3 any irreducible component F_i of a fiber $F(\psi)_w$ is of dimension $r-1$. This implies also that $F_i = \varphi(F(p))$ for some fiber of p.

Lemma 2.7. *There exists a fiber $F(p)_x$ such that $\varphi_{|F(p)_x} : F(p)_x \to F_i$ is of degree 1; that is, $\varphi_{|F(p)_x}$ is set-theoretically birational.*

Proof. For every $v \in V$ we have that $\varphi^{-1}_{|F(p)_x}(v) = F(p)_x \cap F(\varphi)_v$; therefore the lemma follows if we can prove that, for a general $v \in V$, with $\psi(v) = w$, $p_{|F(\varphi)_v} : F(\varphi)_v \to F(\pi)_w$ is set-theoretically birational.

We will need the following claim.

Claim 2.8. Let l be a line in $F(\varphi) \simeq \mathbf{P}^k$; then $-p^*K_X \cdot l = r$.

Proof of the claim. Let C a minimal curve in the ray R (see Definition 1.6); let $\nu : \mathbf{P}^1 \to C$ be its normalization. Thus $\nu^*E_{|C} = \oplus^r\mathcal{O}(1)$, and therefore $Y_C = \mathbf{P}(\nu^*E_C) = \mathbf{P}^1 \times \mathbf{P}^{r-1}$. Let $\tilde{\nu} : Y_C \to Y$ be the map induced by ν and let $\tilde{l}$ be a section of $\sigma : Y_C \to \mathbf{P}^1$; note that $\nu\sigma : \tilde{l} \to C$ is birational. Note also that $\tilde{\nu}^*\xi_E$ is the tautological bundle for Y_C; thus $1 = \tilde{\nu}^*\xi_E \cdot \tilde{l} = \xi_E \cdot \tilde{\nu}_*\tilde{l}$, hence $\tilde{\nu}_*\tilde{l} = l$. Therefore $p_*l = C$ and $-p^*K_X \cdot l = -K_X \cdot p_*l = -K_X \cdot C = r$. $\square$

Let R be the ramification divisor of $p' : Z \to X$ defined by the formula

$$K_Z = p'^*K_X + R.$$

Let l be a line in $F(\varphi) = \mathbf{P}^k \subset Z$; on one side we have that $-K_Z \cdot l = r$; on the other, by the above claim, $p^*K_X \cdot l = r$. Therefore $R \cdot l = 0$. Thus either $F(\varphi) \subset R$ or $F(\varphi) \cap R = \emptyset$. We want to prove that the latter is the case.

Lemma 2.9. *For a general choice of Z the ramification divisor R does not contain $F(\varphi) = \mathbf{P}^k \subset Z$; therefore $F(\varphi) \cap R = \emptyset$.*

Proof. It is enough to prove that there exists an $x \in F(\pi)_w$ such that $p^{-1}(x) \cap Z$ consists of d distinct points, where $d = deg(p' : Z \to X)$. Observe that this is true for every $x_1 \in X$ outside the branch locus and $d = \varphi^*A^{r-1} \cdot F(p)_{x_1} = \varphi^*A^{r-1} \cdot F(p)_x$, where $Z = \varphi^*A_1 \cap \ldots \cap \varphi^*A_{r-1}$ and $A_i \in |A|$. Moreover $p^{-1}(x) \cap Z = \bigcup_i p^{-1}(x) \cap F(\varphi)_{v_i}$, where the union is taken over all $v_i \in T \cap F_i$. Since $\varphi_{|F(p)_x} : F(p)_x \to F_i$ is generically unramified, choosing generic sections $A_i \in |A|$ yields that $p^{-1}(x) \cap F(\varphi)_{v_i}$ is a reduced cycle of length d_i for any i and $\sum_i d_i = d$. Hence $F(\varphi) \cap R = \emptyset$. $\square$

The exact sequence

$$\mathcal{I}/\mathcal{I}^2 \to \Omega_{Z/X} \otimes \mathcal{O}_{F(\varphi)} \to \Omega_{F(\varphi)/X} \to 0$$

yields that also $p_{|F(\varphi)} : \mathbf{P}^k \to F(\pi)$ is unramified. Let $f : \tilde{F} \to F(\pi)$ be the normalization and $g : \mathbf{P}^k \to \tilde{F}$ the map induced by p; then g is unramified and $\tilde{F}$ is smooth by Zariski's Main Theorem. Therefore $\tilde{F} \simeq \mathbf{P}^k$ by Lazarsfeld's result and g is an isomorphism; thus $p_{|F(\varphi)}$ is of degree 1. $\square$

Let $\varphi_{|F(p)} : F(p) \to F_i$ be as in the lemma, that is, $\varphi_{|F(p)}$ is set-theoretically birational. Let us follow an argument as in [Fu1, Lemma 2.12]. We can assume that the divisor A is very ample. Using Bertini's theorem we choose $r-1$ divisors $A_i \in |A|$ as above such that, if $T = \bigcap_i A_i$, then $T \cap \psi^{-1}(w)_{red} = N$ is a reduced 0-cycle and $Z = H_1 \cap \cdots \cap H_{r-1}$ is a smooth n-fold, where $H_i = \varphi^{-1}A_i$. Moreover the number of points in N is given by $A^{r-1} \cdot \psi^{-1}(w)_{red} = \sum_i A^{r-1} \cdot F_i = \sum_i d_i$. Note that, by the projection formula, we have $A^{r-1} \cdot F_i = \varphi^*A^{r-1} \cdot F(p)$; here we use the fact that the map $\varphi_{|F(p)}$ is set-theoretically birational. Moreover, since p is a projective bundle, the last number is constant, i.e. $\varphi^*A^{r-1} \cdot F(p) = d$ for all fibers $F(p)$; that is, the d_i's are constant.

Using that $\psi' := \psi_{|T} : T \to W$ is proper and finite over w, we now take a small enough neighborhood U of w, in the metric topology, such that any connected component U_λ of $\psi^{-1}(U) \cap T$ meets $\psi^{-1}(w)$ in a single point. Let ψ_λ be the restriction of ψ to U_λ and m_λ its degree. Then $deg\psi' = \sum m_\lambda \geq \sum_i d_i = \sum_i d$, and equality holds if and only if ψ is not ramified at w (remember that $\sum_i d_i$ is the number of U_λ).

The generic $F(\psi)_w$ is irreducible and generically reduced. Note that we can choose $\tilde{w} \in W$ such that $\psi^{-1}(\tilde{w}) = \varphi(F(p))$ and $deg\psi' = A^{r-1} \cdot \psi^{-1}(\tilde{w})$; the latter is possible by the choice of generic sections of $|A|$. Hence, by the projection formula, $deg\psi' = A^{r-1} \cdot \psi^{-1}(\tilde{w}) = \varphi^*A^{r-1} \cdot F(p) = d$; that is, $m_\lambda = 1$, and the fibers are irreducible. Since W is normal we can conclude, by Zariski's Main Theorem, that W has the same singularities as T. $\square$

Corollary 2.10. *In the hypothesis of the above proposition assume also that either φ is birational and $k = r$, or that φ is of fiber type and $k = (r-1)$. Then W is smooth.*

Proof. [AW, Theorem 4.1] applies to the map φ and gives that T is smooth and φ satisfies the hypothesis of Proposition 2.5 (for the fiber type case it is actually a theorem in [Fu1]). Thus by Proposition 2.5 also W is smooth. □

3. Some general applications

As an application of the above construction we will prove the following proposition; the case $r = n - 1$ was proved in [ABW2].

Theorem 3.1. *Let X be a smooth projective complex variety and E be an ample vector bundle of rank r on X. Assume that $K_X + detE$ is nef and big but not ample, and let $\pi : X \to W$ be the contraction supported by $K_X + detE$. Assume also that π is a divisorial elementary contraction, with exceptional divisor D, and that $dimF \leq r$ for all fibers F. Then W is smooth, π is the blow-up of a smooth subvariety $B := \pi(D)$, and $E = \pi^*E' \otimes [-D]$, for some ample E' on W.*

Proof. In the previous section (2.10) we have proved that W is smooth. Therefore π is a birational morphism between smooth varieties with exceptional locus a prime divisor and with equidimensional non-trivial fibers; by [AW, Corollary 4.11] this implies that π is a blow-up of a smooth subvariety in W.

We want to show that $E = \pi^*E' \otimes [-D]$. Let D_1 be the exceptional divisor of φ; first we claim that $\xi_E + D_1$ is a good supporting divisor for φ. Let C_1 be a minimal curve in the ray R_1 (see Definition 1.6), contracted by φ; we have that $\xi_E \cdot C_1 = 1$. Observe that $(\xi_E + D_1) \cdot C_1 = 0$, while $(\xi_E + D_1) \cdot C > 0$ for any curve C with $\varphi(C) \neq pt$ (in fact, ξ_E is ample and $D_1 \cdot C \geq 0$ for such a curve). Thus $\xi_E + D_1 = \varphi^*A$ for some ample $A \in Pic(V)$; moreover, using the projection formula, $A \cdot l = 1$, for any line l in the fiber of ψ. Hence, by Grauert's theorem, $V = \mathbf{P}(E')$ for some ample vector bundle E' on W. This yields, by the commutativity of diagram (1), $E \otimes D = p_*(\xi_E + D_1) = p_*\varphi^*A = \pi^*\psi_*A = \pi^*E'$. □

Similarly, for the fiber type case, we have the following.

Theorem 3.2. *Let X be a smooth projective complex variety and E be an ample vector bundle of rank r on X. Assume that $K_X + detE$ is nef and let $\pi : X \to W$ be the contraction supported by $K_X + detE$. Assume that $r \geq (n+1)/2$ and $dimF \leq r - 1$ for any fiber F of π. Then π is a fiber type contraction, W is smooth, and for any fiber $F \simeq \mathbf{P}^{r-1}$ and $E_{|F} = \oplus^r \mathcal{O}(1)$.*

Proof. Note that, by Proposition 1.8, π is a contraction of fiber type and all the fibers have dimension $r-1$. Moreover the contraction is elementary, by Proposition 1.12.

By Corollary 2.10 W is smooth. We want to use an inductive argument to prove the theorem. If $dimW = 0$ then this is Mukai's conjecture 1, which was proved by Peternell, Kollár, and Ye and Zhang (see for instance [YZ]). Let the theorem be true for dimension $m - 1$. Note that the locus over which the fiber is not $\mathbf{P}^{r-1}$ is discrete. In fact take a general hyperplane section A of W, and $X' = \pi^{-1}(A)$; then $\pi_{|X'} : X' \to A$ is again a contraction supported by $K_{X'} + detE_{|X'}$, such that $r \geq ((n-1)+1)/2$. Thus by induction A is smooth and all fibers over A are $\mathbf{P}^{r-1}$.

Let U be an open disk in the complex topology such that $U \cap SingW = \{0\}$. Then by Lemma 3.3, below, we obtain locally, in the complex topology, a π-ample line bundle L that restricted to the general fiber is $\mathcal{O}(1)$. Thus, as in [Fu1, Prop. 2.12], we can prove that all the fibers are $\mathbf{P}^{r-1}$. □

Lemma 3.3. *Let X be a complex manifold and $(W,0)$ an analytic germ such that $W \setminus \{0\} \simeq \Delta^m \setminus \{0\}$. Assume we have a holomorphic map $\pi : X \to W$ with $-K_X$ π-ample; assume also that $F \simeq \mathbf{P}^r$ for all fibers of π, $F \neq F_0 = \pi^{-1}(0)$, and that $codim F_0 \geq 2$. Then there exists a line bundle L on X such that L is π-ample and $L_{|F} = \mathcal{O}(1)$.*

Proof. (see also [ABW2, pp. 338-339]) Let $W^* = W \setminus \{0\}$ and $X^* = X \setminus F_0$. By abuse of notation set $\pi = \pi_{|X^*} : X^* \to W^*$; it follows immediately that $R^1\pi_*\mathbf{Z}_{X^*} = 0$ and $R^2\pi_*\mathbf{Z}_{X^*} = \mathbf{Z}$.

Using the Leray spectral sequence, we have that

$$E_2^{0,2} = \mathbf{Z} \text{ and } E_2^{p,1} = 0 \text{ for any } p.$$

Therefore $d_2 : E_2^{0,2} \to E_2^{2,1}$ is the zero map, and moreover we have the following exact sequence:

$$0 \to E_\infty^{0,2} \to E_2^{0,2} \xrightarrow{d_3} E_2^{3,0},$$

since the only non-zero map from $E_2^{0,2}$ is d_3 and hence $E_\infty^{0,2} = ker d_3$. On the other hand we have also, in a natural way, a surjective map $H^2(X^*, \mathbf{Z}) \to E_\infty^{0,2} \to 0$. Thus we get the following exact sequence:

$$H^2(X^*, \mathbf{Z}) \xrightarrow{\alpha} E_2^{0,2} \to E_2^{3,0} = H^3(W^*, \mathbf{Z}).$$

We want to show that α is surjective. If $dimW := w \geq 3$ then $H^3(W^*, \mathbf{Z}) = 0$ and we are done. Suppose $w = 2$; then $H^3(W^*, \mathbf{Z}) = \mathbf{Z}$; note that the restriction of $-K_X$ gives a non-zero class (in fact it is $r+1$ times the generator) in $E_2^{0,2}$ and is mapped to zero in $E_2^{0,3}$; thus the mapping $E_2^{0,2} \to E_2^{3,0}$ is the zero map and α is surjective. Since F_0 is of codimension at least 2 in X, the restriction map $H^2(X, \mathbf{Z}) \to H^2(X^*, \mathbf{Z})$ is a bijection. By the vanishing of $R_i\pi_*\mathcal{O}_X$ we get $H^2(X, \mathcal{O}_X) = H^2(W, \mathcal{O}_W) = 0$; hence also $Pic(X) \to H^2(X, \mathbf{Z})$ is surjective. Let $L \in Pic(X)$ be a preimage of a generator of $E_2^{0,2}$. By construction L_t is $\mathcal{O}(1)$, for $t \in W^*$. Moreover $(r+1)L = -K_X$ on X^*; thus, again by the codimension of X^*, this is true on X and L is π-ample. □

4. An approach to the singular case

The following theorem arose during a discussion between us and J.A. Wiśniewski; we would like to thank him. The idea to investigate this argument originated with Zhang [Zh2]. For the definition of log-terminal singularity we refer to [KMM].

Theorem 4.1. *Let X be an n-dimensional log-terminal projective variety and E be an ample vector bundle of rank r. Assume that $K_X + detE$ is nef and let $\pi : X \to W$ be the contraction supported by $K_X + detE$. Assume also that for any fiber F of π $dimF \leq r-1$, and that $r \geq (n+1)/2$ and $codimSing(X) > dimW$. Then X and W are smooth and, for any fiber, $F \simeq \mathbf{P}^{r-1}$.*

Proof. We will prove that X is smooth. Then we can apply Theorem 3.2. We consider in this case the associated projective space bundle Y and the commutative diagram

$$\begin{array}{ccc} \mathbf{P}(E) = Y & \xrightarrow{\varphi} & V \\ \downarrow p & & \downarrow \psi \\ X & \xrightarrow{\pi} & W \end{array} \tag{4.1}$$

as in 2.1); it is immediate that Y is Gorenstein and log-terminal; in particular it has Cohen-Macaulay singularities. Moreover, as in (3.1) $dimF(\varphi) \le dimF(\pi)$ and the map φ is supported by $K_Y + rH$, where $H = \xi_E + A$, with ξ_E the tautological line bundle and A a pull-back of an ample line bundle from V. It is known that a contraction supported by $K_Y + rH$ on a log terminal variety has to have fibers of dimension $\ge (r-1)$ and of dimension $\ge r$ in the birational case ([AW, remark 3.1.2]). Thus φ is not birational and all fibers have dimension $r-1$; moreover, by the Kobayashi-Ochiai criterion the general fiber is $F \simeq \mathbf{P}^{r-1}$. Imitating the proof of [BS, Prop 1.4], we have only to show that there are no fibers of φ entirely contained in $Sing(Y)$. Note that, by construction, $Sing(Y) \subset p^{-1}(SingX)$. Hence no fibers F of φ can be contained in $Sing(Y)$, and therefore the same proof of [BS, Prop. 1.4] applies. It follows that V is nonsingular, and $\varphi : Y \to V$ is a classical scroll. In particular Y is nonsingular, and therefore also X is nonsingular. □

As a corollary we obtain Mukai's conjecture 1 in the log terminal case (see also [Zh2]).

Corollary 4.2. *Let X be an n-dimensional log-terminal projective variety and E an ample vector bundle of rank $n+1$, such that $c_1(E) = c_1(X)$. Then $(X, E) = (\mathbf{P}^n, \oplus^{n+1}\mathcal{O}_{\mathbf{P}^n}(1))$.*

5. Main theorem

This section is devoted to the proof of the following theorem.

Theorem 5.1. *Let X be a smooth projective variety over the complex field of dimension $n \ge 3$ and E an ample vector bundle on X of rank $r = n-2$. Then we have:*

1) *$K_X + det(E)$ is nef unless (X, E) is one of the following:*
   - i) *there exist a smooth n-fold, W, and a morphism $\phi : X \to W$ expressing X as a blow up of a finite set B of points and an ample vector bundle E' on W such that $E = \phi^* E' \otimes [-\phi^{-1}(B)]$.*

Assume from now on that (X, E) is not as in i) above (that is eventually consider the new pair (W, E') coming from i)).

   - ii) *$X = \mathbf{P}^n$ and $E = \oplus^{n-2}\mathcal{O}(1)$ or $\oplus^2\mathcal{O}(2) \oplus^{n-4} \mathcal{O}(1)$ or $\mathcal{O}(2) \oplus^{n-3} \mathcal{O}(1)$ or $\mathcal{O}(3) \oplus^{n-3} \mathcal{O}(1)$.*
   - iii) *$X = \mathbf{Q}^n$ and $E = \oplus^{n-2}\mathcal{O}(1)$ or $\mathcal{O}(2) \oplus^{n-3} \mathcal{O}(1)$ or $\mathbf{E}(2)$ with $\mathbf{E}$ a spinor bundle on $\mathbf{Q}^n$.*
   - iv) *$X = \mathbf{P}^2 \times \mathbf{P}^2$ and $E = \oplus^2\mathcal{O}(1,1)$.*
   - v) *X is a del Pezzo manifold with $b_2 = 1$, i.e. $Pic(X)$ is generated by an ample line bundle $\mathcal{O}(1)$ such that $\mathcal{O}(n-1) = \mathcal{O}(-K_X)$ and $E = \oplus^{n-1}\mathcal{O}(1)$.*
   - vi) *X is a classical scroll or a quadric bundle over a smooth curve Y.*

vii) *X is a classical scroll over a smooth surface Y.*

2) *If $K_X + det(E)$ is nef then it is big unless there exists a morphism $\phi : X \to W$ onto a normal variety W supported by (a large multiple of) $K_X + det(E)$ and $dim(W) \leq 3$; let F be a general fiber of ϕ and $E' = E_{|F}$. We have the following according to $s = dimW$:*
   i) *If $s = 0$ then X is a Fano manifold and $K_X + det(E) = 0$. If $n \geq 6$ then $b_2(X) = 1$ except if $X = \mathbf{P}^3 \times \mathbf{P}^3$ and $E = \oplus^4 \mathcal{O}(1, 1)$.*
   ii) *If $s = 1$ then W is a smooth curve and ϕ is a flat (equidimensional) map. Then (F, E') is one of the pair described in* [PSW]*; in particular, F is either $\mathbf{P}^{n-1}$ or a quadric or a del Pezzo variety. If $n \geq 6$ then π is an elementary contraction. If the general fiber is $\mathbf{P}^{n-1}$ then X is a classical scroll, while if the general fiber is $\mathbf{Q}^{n-1}$ then X is a quadric bundle.*
   iii) *If $s = 2$ and $n \geq 5$, then W is a smooth surface, ϕ is a flat map and (F, E') is one of the pair described in the Main Theorem of* [Fu2]*. If the general fiber is $\mathbf{P}^{n-2}$, all the fibers are $\mathbf{P}^{n-2}$.*
   iv) *If $s = 3$ and $n \geq 5$, then W is a 3-fold with at most isolated singularities and X has at most isolated fibers of dimension $n-2$; all fibers over smooth point are isomorphic to $\mathbf{P}^{n-3}$.*

3) *Assume finally that $K_X + det(E)$ is nef and big but not ample. Then a high multiple of $K_X+det(E)$ defines a birational map, $\varphi : X \to X'$, which contracts an "extremal face" (see section* 2*). Let R_i, for i in a finite set of indices, be the extremal rays spanning this face; call $\rho_i : X \to W$ the contraction associated to one of the R_i. Then each ρ_i is birational and divisorial; if D is one of the exceptional divisors (we drop the index) and $B = \rho(D)$ we have that $dim(B) \leq 1$ and the following possibilities occur:*
   i) *$dimB = 0$, $D = \mathbf{P}^{n-1}$ and $D_{|D} = \mathcal{O}(-2)$; moreover $E_{|D} \simeq \oplus^{n-2}\mathcal{O}(1)$.*
   ii) *$dimB = 0$, D is a (possibly singular) quadric, $\mathbf{Q}^{n-1}$, and $D_{|D} = \mathcal{O}(-1)$; moreover $E_{|D} = \oplus^{n-2}\mathcal{O}(1)$.*
   iii) *$dimB = 1$, W and Z are smooth projective varieties, and ρ is the blow-up of W along Z. Moreover $E_{|F} = \oplus^{n-2}\mathcal{O}(1)$.*

If $n > 3$ then φ is a composition of "disjoint" extremal contractions as in i), ii) or iii).

Proof of part 1) of Theorem 5.1. Let (X, E) be a generalized polarized variety and assume that $K_X + det(E)$ is not nef. Then there exist on X a finite number of extremal rays, $R_1, \ldots, R_s$, such that $(K_X + det(E)) \cdot R_i < 0$, and therefore, by Remark 2.1, $l(R_i) \geq n - 1$.

Consider one of this extremal rays, $R = R_i$, and let $\rho : X \to Y$ be its associated elementary contraction. Then $L := -(K_X + det(E))$ is ρ-ample and so is the vector bundle $E_1 := E \oplus L$; moreover $K_X + det(E_1) = \mathcal{O}_X$ relative to ρ. To proceed we need a relative version of the theorem in [ABW2] which studies the positivity of the adjoint bundle in the case of $rankE_1 = n-1$. More precisely, we assume not that E_1 is ample but that it is ρ-ample (or equivalently a local statement in a neighborhood of the exceptional locus of the extremal ray R). For this we notice that the theorem in [ABW2] is true also in the relative case and can be proved verbatim using the relative minimal model theory instead of the absolute (see [KMM]; see also section 2 of [AW] for a discussion of the local setup).

Assume first that ρ is birational; then $K_X + det(E_1)$ is ρ-nef and ρ-big; note also that, since $l(R_i) \geq n - 1$, ρ is divisorial. Therefore we are in the (relative) case C of the theorem in [ABW2] (see also Theorem 3.1 with $r = n-1$); this implies that Y is smooth and ρ is the blow-up of a point in Y. Since $l(R_i) \geq n-1$, the exceptional loci of the birational rays are pairwise disjoint by Proposition 1.10. This gives Theorem 5.1 (i): the birational extremal rays have disjoint exceptional loci which are divisors isomorphic to $\mathbf{P}^{n-1}$ and which contract simultaneously to smooth distinct points on a n-fold W. The description of E follows trivially (see also [ABW2]).

If ρ is not birational then we are in case B of the theorem in [ABW2]; from this we obtain similarly as above the other cases of Theorem 5.1, with some trivial computations needed to recover E from E_1. Note that in the case of fibration over a surface, since all fibers are $\mathbf{P}^{n-2}$, then $n-1 = l(R) > detE \cdot R_i \geq n-2$; thus $l(R) = n-1$ and $detE \cdot R_i = n-2$. Then $-(detE + K_X)$ is a tautological bundle for the fibration, and the fibration is a scroll. This part was also independently proved in [Ma] □

Proof of part 2) of Theorem 5.1. Let $K_X + detE$ be nef but not big; then it is the supporting divisor of a face $F = (K_X + detE)^{\perp}$. Using ([KMM]) we can say that there exists a map $\pi : X \to W$ which is given by a high multiple of $K_X + detE$ and which contracts the curves in the face. Since $K_X + detE$ is not big, we have that $dimW < dimX$. Moreover for every rational curve C in a general fiber of π we have $-K_X \cdot C \geq n-2$ by Remark 2.1. We apply Proposition 1.12, which, together with the above inequality on $-K_X \cdot C$, gives that π is an elementary contraction if $n \geq 5$ unless either $n = 6$, W is a point and X is a Fano manifold of pseudoindex 4 and $\rho(X) = 2$, or $n = 5$ and $dimW \leq 1$.

By Proposition 1.8 we have the inequality

$$n + d \geq n + n - 2 - 1,$$

where d is the dimension of a fiber; in particular it follows that $dimW \leq 3$.

Case 5.2 ($dimW = 0$). Then $K_X + detE = 0$, and therefore X is a Fano manifold. By what just said above we have that $b_2(X) = 1$ if $n \geq 6$, with an exception which is a particular case of the following lemma for $n = 6$.

Lemma 5.3. *Let X be an n-dimensional projective manifold, E an ample vector bundle on X of rank $r+1$ such that $K_X + detE = 0$, and $n = 2r$. Assume moreover that $b_2 \geq 2$. Then $X = \mathbf{P}^r \times \mathbf{P}^r$ and $E = \oplus^r \mathcal{O}(1,1)$.*

Proof. The lemma is a slight generalization of [Wi1, Prop. B]; the proof is similar and for more details we refer to that paper. In particular, as in [Wi1] we can see that X has two extremal rays whose contractions π_i, $i = 1, 2$, are of fiber type with equidimensional fibers onto r-folds W_i and with general fiber $F_i \simeq \mathbf{P}^r$. We claim that the W_i are smooth and thus $W_i \simeq \mathbf{P}^r$. The contractions π_i are supported by $K_X + detE_i'$, with E_i' an ample vector bundle ($E_i' = E \times \pi^* A_i$ with A_i ample on W_i). Therefore we are in the hypothesis of Proposition 3.2. Thus the W_i are smooth and all the fibers are $\mathbf{P}^r$.

Let $T = \bigcap_{i=1}^{r-1} H_i$, where H_i are general elements of $\pi_1^*(\mathcal{O}(1))$. We claim that $T \simeq \mathbf{P}^1 \times \mathbf{P}^r$. In fact T is smooth and $\pi_{1|T}$ makes T a projective bundle over a line (since $H^2(\mathbf{P}^1, \emptyset^*) = 0$), that is, $T = \mathbf{P}(\mathcal{F})$. Moreover $\pi_{2_{|T}}$ is onto $\mathbf{P}^r$; therefore the claim follows. Therefore we conclude that $\pi_i^* \mathcal{O}_{\mathbf{P}^r}(1)_{|F_i} \simeq \mathcal{O}_{\mathbf{P}^r}(1)$ for $i = 1, 2$. This implies, by Grauert's Theorem, that the two fibrations are classical scrolls, that is,

$X = \mathbf{P}(\mathcal{F}_i)$, for $i = 1, 2$; moreover, computing the canonical class of X, the $\mathcal{F}_i$ are ample and the lemma easily follows. □

Case 5.4 ($dimW = 1$). Then W is a smooth curve and π is a flat map. Let F be a general fiber; then F is a smooth Fano manifold and $E_{|F}$ is an ample vector bundle on F of rank $n-2 = dimF - 1$ such that $-K_F = det(E_{|F})$. These pairs $(F, E_{|F})$ are classified in the Main Theorem of [PSW]; in particular, if $dimF \geq 5$, F is either $\mathbf{P}^{n-1}$ or $\mathbf{Q}^{n-1}$ or a del Pezzo manifold with $b_2(F) = 1$. Moreover, if $n \geq 6$, then π is an elementary contraction by Proposition 1.12.

Claim 5.5. Let $n \geq 6$ and assume that the general fiber is $\mathbf{P}^{n-1}$. Then X is a classical scroll and $E_{|F}$ is the same for all F.

Proof. (See also [Fu2].) Let $S = W \setminus U$ be the locus of points over which the fiber is not $\mathbf{P}^{n-1}$. Over U we have a projective fiber bundle. Since $H^2(U, \mathcal{O}^*) = 0$ we can associate this $\mathbf{P}$-bundle to a vector bundle $\mathcal{F}$ over U. Let $Y = \mathbf{P}(\mathcal{F})$ and H the tautological bundle; by abuse of language let H be the extension of H to X. Since π is elementary, H is an ample line bundle on X. Therefore by semicontinuity $\Delta(F, H_F) \geq \Delta(G, H_G)$, for any fiber G, where $\Delta(X, L)$ is Fujita's delta-genus. In our case this yields $0 = \Delta(F, H_F) \geq \Delta(G, H_G) \geq 0$. Moreover by flatness $(H_G)^{n-1} = (H_F)^{n-1} = 1$; by the Fujita classification of the pairs of delta genus zero we conclude that all G are equal to $\mathbf{P}^{n-1}$. Using again the Main Theorem of [PSW], we see that $E_{|G}$ is decomposable, hence rigid; that is, the decomposition is the same along all fibers of π. This concludes the proof of the claim. □

Claim 5.6. Let $n \geq 6$ and assume that the general fiber is $\mathbf{Q}^{n-1}$. Then X is a quadric bundle.

Proof. As above, let $S = W \setminus U$ be the locus of points over which the fiber is not a smooth quadric. Let $X^* = \pi^{-1}(U)$; then we can embed X^* in a fiber bundle of projective spaces over U, since it is locally trivial. Associate this P-bundle over U to a projective bundle and argue as before. □

Case 5.7 ($dimW = 2$). Assume that $n \geq 5$; then π is an elementary contraction. This implies first, by [ABW2, Prop. 1.4.1], that W is smooth; secondly that π is equidimensional, hence flat and the general fiber is $\mathbf{P}^{n-2}$ or $\mathbf{Q}^{n-2}$, see [Fu2].

Claim 5.8. Let $n \geq 5$ and let the general fiber be $\mathbf{P}^{n-2}$; then for any fiber $F \simeq \mathbf{P}^{n-2}$ and $E_{|F}$ is the same for all F.

Proof. Let $S \subset W$ be the locus of fibers that are not $\mathbf{P}^{n-2}$; then $dimS \leq 0$ since W is smooth. In fact, over a generic hyperplane section on W all fibers are $\mathbf{P}^{n-2}$ by [ABW2]. Let $U \subset W$ be an open set, in the complex topology, with $U \cap S = \{0\}$, and let $V \subset X$ be such that $V = \pi^{-1}(U)$. We are in the hypothesis of Lemma 3.3; thus we get a "tautological" line bundle H on V, and we conclude by [Fu1, Prop. 2.12].

There are two possible restrictions of E to the fiber, namely, $E_{|F} \simeq \mathcal{O}(2) \oplus (\oplus^{n-1}\mathcal{O}(1))$ or $E_{|F}$ is the tangent bundle. As observed by Fujita in [Fu2, 3.8 and 3.11], these two restrictions have a different behavior in the diagram 2.1): in the former φ is birational and $dimF(\varphi) = n-2$, while in the latter it is of fiber type and $dimF(\varphi) = n-3$. Hence the restriction has to be constant along all the fibers. □

Case 5.9 ($dimW = 3$). The general fiber is $\mathbf{P}^{n-3}$ (see for instance [Fu2]). Assume that $n \geq 5$; therefore π is elementary.

Since π is elementary, any fiber G has $codG \geq 2$. Let $S \subset W$ be the locus of point over which the fiber is not $\mathbf{P}^{n-3}$; $dimS \leq 0$ since a generic linear space section cannot intersect S, as above. Let $(W, 0)$ be an analytic germ of a smooth point of W. Then we are in the hypothesis of Proposition 2.5 and can assume that the contraction is supported (locally) by $K_X + (n-2)L$. Therefore, since $n \geq 5$, by [AW, Th. 4.1] all the fibers have dimension $n-3$. We conclude that all fibers over $(W, 0)$ are $\mathbf{P}^{n-3}$. □

Proof of part 3) of the theorem. In the last part of the theorem we assume that $K_X + detE$ is nef and big but not ample. Then $K_X + detE$ is a supporting divisor of an extremal face, F; let R_i be the extremal rays spanning this face. Fix one of these rays, say $R = R_i$, and let $\pi : X \to W$ be the elementary contraction associated to R.

We have $l(R) \geq n-2$; this implies first that the exceptional loci are disjoint if $n > 3$, by Proposition 1.11. Secondly, by the inequality 1.8), we have

$$dimE(R) + dimF(\pi) \geq 2n - 3.$$

Therefore $dimE(R) = n-1$ and either $dimF(\pi) = n-1$ or $dimF(\pi) = n-2$; thus $n-1 \geq l(R) \geq n-2$. If $B := \rho(E)$ and $D = E(R)$, this implies that $dimB = 0$ or 1.

If $dimB = 1$ then $dimF(\pi) = n-2$ for all fibers (note that since the contraction π is elementary there cannot be a fiber of dimension $n-1$); thus we can apply Theorem 3.1 with $r = n-2$. This will give the case 3(iii) of the theorem.

Now let $dimB = 0$ and consider again the construction in section 2; in particular we refer to the diagram 2.1). Let S be the extremal ray contracted by φ; note that $l(S) \geq n-2$ and that the inequality 1.8) gives

$$dimE(S) + dimF(\varphi) \geq 3n - 6;$$

in particular, since $dimF(\varphi) \leq dimF(\pi)$, we have two cases, namely $dimE(S) = 2n-5$ and $dimF(\varphi) = n-1$, or $dimE(S) = 2n-4$ and $dimF(\varphi) = n-1$ or $n-2$.

The case in which $dimE(S) = 2n-5$ will not occur. In fact, after "slicing", (see (2.4)), we would obtain a map $\varphi' = \varphi_{|Z}$ which would be a small contraction supported by a divisor of the type $K_Z + (n-2)L$, but this is impossible by the classification of [Fu1, Th. 4] (see also [An]).

Hence $dimE(S) = 2n-4$; that is, also φ is divisorial and $E(S) \cdot l_p = 0$, where l_p is a line in $F(p)$. In particular, $E(S) = p^*D$.

Suppose that the general fiber of φ, $F(\varphi)$, has dimension $n-2$. After slicing we obtain a map $\varphi' = \varphi_{|Z} : Z \to T$ supported by $K_Z + (n-2)L$, where $L = \xi_{E|Z}$. This map contracts divisors $\overline{D}$ in Z to curves; by ([Fu1, Th. 4]) we know that every fiber F of this map is $\mathbf{P}^{n-2}$ and that $\overline{D}_{|F} = \mathcal{O}(-1)$ (actually this map is a blow-up of a smooth curve in a smooth variety). In particular there are curves in Y, call them l, such that $-E(S) \cdot l = 1$. We will discuss this case in a while.

Assume now that the general fiber and therefore all have dimension $n-1$.

Lemma 5.10. *Under these hypotheses,* $l(R) = n-2$.

Let C be a minimal curve in R (see 1.6)), $\nu : \mathbf{P}^1 \to C$ its normalization, $\tilde{\nu} : Y_C = \mathbf{P}(\nu^* E_{|C}) \to Y$ the induced morphism and ξ_C the tautological bundle of Y_C; note that $\tilde{\nu}^* \xi_E = \xi_C$.

Let $g : Y_C \to F(\psi)_w$ be the morphism induced by φ on Y_C and

$$Y_C \xrightarrow{\alpha} V_1 \xrightarrow{\beta} F(\psi)_w$$

its Stein factorization. Assume by contradiction that $l(R) = n-1$; then $\nu^*(E_{|C}) = \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus n-3}$, and so Y_C has two contractions, the scroll structure and a blow-down to $\mathbf{P}^{n-2}$.

Let $\tilde{l}$ be a line contracted by the blow-down; then $\tilde{l}$ is contracted by g. In fact, by the projection formula $\xi_E \cdot \tilde{\nu}_* \tilde{l} = \xi_C \cdot \tilde{l} = 1$. Thus by the commutativity of the diagram $\tilde{\nu}_* \tilde{l}$ is a minimal curve in S.

Since α cannot contract all Y_C, then α is the blow-down. Since $dimF(\pi) = dimF(\varphi)$ by hypothesis, then by Remark 2.3 all fibers $F(\psi)$ have dimension $n-3$. So we get the contradiction that $\beta : \mathbf{P}^{n-2} \to F(\psi)_w$ is a finite map between two varieties of different dimension.

Slicing, we obtain a map $\varphi' = \varphi_{|Z} : Z \to T$ supported by $K_Z + (n-2)L$, where $L = \xi_{E|Z}$. This map contracts divisors $\overline{D}$ in Z to points; by ([Fu1]) we know that these divisors are either $\mathbf{P}^{n-1}$ with normal bundle $\mathcal{O}(-2)$ or $\mathbf{Q}^{n-1} \subset \mathbf{P}^n$ with normal bundle $\mathcal{O}(-1)$. In the latter case we have as above that there are curves l in Y such that $-E(S) \cdot l = 1$.

In these cases observe that $E(S) = p^*(D)$ and $K_X + (n-2)(-D)$ is a supporting divisor for π. Then by [Fu1] we conclude that $(D, D_{|D})$ is one of the pair listed in the theorem, and the theory of uniform bundles makes it easy to recover $E_{|D}$ ([OSS]).

There remains the case in which $\varphi' = \varphi_{|Z} : Z \to T$ contracts divisors $\overline{D} = \mathbf{P}^{n-1}$ with normal bundle $\mathcal{O}(-2)$ to points. We can apply Proposition 2.5 and show that the singularities of W are the same as those of T. Then, as in ([Mo1]), this means that we can factor π with the blow-up of the singular point. Let $X' = Bl_w(W)$; then we have a birational map $g : X \to X'$. Note that X' is smooth and that g is finite. Actually it is an isomorphism outside D, and cannot contract any curve of D. Assume to the contrary that g contracts a curve $C' \subset D$; let $N \in Pic(X')$ be an ample divisor. Then we have $g^*N \cdot C' = 0$ while $g^*N \cdot C \neq 0$, contradiction. Thus by Zariski's main theorem g is an isomorphism. This gives the case in 3i). □

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