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THE TRANSFER AND SYMPLECTIC COBORDISM

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Abstract. The main result of this paper is the nilpotency fomula $\phi_i^4 = 0$, $\forall i \geq 1$ for N. Ray classes ϕ_i in the torsion of the symplectic bordism ring MSp_*

Introduction

This paper is organised as follows. Section 1 is devoted to calculation of the transfer homomorphism in the symplectic cobordism theory [D], [BG]. In particular, using the results of [BM], [Fe], [Sn] we calculate the transfer homomorphism for projective bundles associated with universal $Spin(m)$ bundles, $m = 3, 4, 5$. This section includes the following corollary in the case $m = 3$:

Let N be the normalizer of the torus $U(1)$ in $Sp(1)$; $\zeta \to BSp(1)$ be the universal $Sp(1)$ bundle and Λ be the universal $Spin(3)$ bundle over $BSpin(3) = BSp(1)$. Then the bundle $p : BN \to BSp(1)$ is the projective bundle associated with Λ. Let

$$x = pf_1(\zeta);$$

$$y = pf_1(p^*(\zeta));$$

$$e = pf_3(p^*(\Lambda \otimes_R H))$$

be the Conner-Floyd symplectic Pontryagin classes and

$$\tau_p^* : MSp^*(BN) \to MSp^*(BSp(1))$$

be the transfer homomorphism. Then τ_p^* satisfies the relations

$$\tau_p^*(1) = 1; \tag{1}$$

$$\tau_p^*(e) = 0. \tag{2}$$

In Section 2 we establish a connection of the Euler class e with the classes ϕ_i defined as follows:

Recall from [R] the classes θ_i arising from the expansion

$$pf_1((\eta - R) \otimes_R (\zeta - H)) = s \sum_{i \geq 1} \theta_i pf_1^i(\zeta) = s \sum_{i \geq 1} \theta_i x^i$$

in $MSp^4(S^1 \wedge BSp(1))$, where s is the generator of $MSp^1(S^1)$, $\eta \to S^1$ is the nontrivial real line bundle and ζ is as above. Also recall the relabelling $\theta_{2i} = \phi_i$ in

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MSp_{8i-3}, and from [Ro] that $\theta_{2i-1} = 0$ for $i > 1$. As proved in [R], each ϕ_i is an indecomposable torsion element of order 2.

It is shown in [Na] that the homomorphism π^* induced by $\pi : BU(1) \to BSp(1)$ is not a monomorphism in the symplectic cobordism theory. In particular (see Section 2)

$$\pi^*(\theta_1 x + \sum_{i\geq 1} \phi_i x^{2i}) = 0.$$

Using this observation and the results of [G], [GR], we state that in $MSp^*(BN)$

$$e = \sum_{i\geq 1} \phi_i^4 y^{8i}(1 + \sum_{j\geq 1} \alpha_j y^j) \tag{3}$$

for some coefficients $\alpha_j \in MSp_*$.

Applying (1), (2), (3) we have

$$\tau_p^*(e) = 0$$

by (2),

$$= \tau_p^*(\sum_{i\geq 1} \phi_i^4 y^{8i}(1 + \sum_{j\geq 1} \alpha_j y^j))$$

by (3),

$$= \sum_{i\geq 1} \phi_i^4 x^{8i}(1 + \sum_{j\geq 1} \alpha_j x^j)\tau_p^*(1)$$

by the transfer property,

$$= \sum_{i\geq 1} \phi_i^4 x^{8i}(1 + \sum_{j\geq 1} \alpha_j x^j)$$

by (1).

Thus we obtain

$$\sum_{i\geq 1} \phi_i^4 x^{8i} = 0$$

in $MSp^*(BSp(1)) = MSp_*[[x]]$.

This proves

Theorem. *$\phi_i^4 = 0$, $\forall i \geq 1$.*

We cannot use a reasoning similar to that of Section 2 for the self-conjugate cobordism, since in this theory it is impossible to construct characteristic classes with the required properties. Namely, as proved in [BaNa], for arbitrary natural classes

$$P_i(\xi^n) \in SC^{2i}(X)$$

in the self-conjugate cobordism theory

$$P(\xi^n) = 1 + P_1(\xi^n) + ... + P_n(\xi^n),$$

where $\xi^n \to X$ is the SC-vector bundle, the following conditions are contradictory:

1. $P_n(\xi^n)$ is the Euler class (normalization);
2. $P(\xi^n + \xi^m) = P(\xi^n)P(\xi^m)$ (the Whitney formula).

That is why in Section 3 we calculate the transfer homomorphism for the bundle of flags of the bundle Λ. As a corollary we obtain a new proof of the nilpotency formula for the N. Ray classes in the self-conjugate cobordism, which was proved for the first time in [Na].

As is known from [Mo] and [V], various three-fold products of N. Ray's family are nontrivial. In Section 4 we shall prove

Proposition 4.1. *All four-fold products of the N. Ray classes are zero, and the images of double products of these classes in self-conjugate cobordism are zero.*

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1. Calculation with Transfer

The result of this section is

Proposition 1. *Let $G_m = Spin(m)$ and $\xi^m \to BG_m$ be the universal $Spin(m)$ bundle, $m = 3, 4, 5$. Let*

$$p_m : P(\xi^m) \to BG_m$$

be the associated projective bundle with fibre RP^{m-1}, and $\lambda_m \to P(\xi^m)$ be the canonical real line bundle. Then the transfer homomorphism

$$\tau_m^* : MSp^*(P(\xi^m)) \to MSp^*(BG_m)$$

satisfies the relations

$$\tau_m^*(c_m^n) = 0, \tag{1.1}$$

for all $n \geq 1$, where $c_m = pf_1(\lambda_m \otimes_R H)$ is the first Conner-Floyd symplectic Pontryagin class;

$$\tau_m^*(1) = \chi(RP^{m-1}), \tag{1.2}$$

where $\chi(RP^{m-1})$ is the Euler characteristic of RP^{m-1} and hence is equal to 1 if $m = 3, 5$, and to 0 if $m = 4$;

$$\tau_m^*(e_m) = 0, \tag{1.3}$$

where $e_m = e(p_m^*(\xi^m \otimes_R H))$ is the Euler class.

For the proof we need the following facts.

1.4. *Spin*(m) **bundles.** It is well known that the groups

$$Spin(2), Spin(3), Spin(4), Spin(5), Spin(6)$$

are isomorphic to

$$S^1 = U(1), S^3 = Sp(1) = SU(2), Sp(1)^2, Sp(2), SU(4).$$

The inclusions $Spin(i) \to Spin(i + 1)$ up to an isomorphism are described as follows:

$Spin(2) \to Spin(3)$ is the standard $U(1) \to Sp(1)$;

$Spin(3) \to Spin(4)$ is the diagonal homomorphism $Sp(1) \to Sp(1)^2$;

$Spin(4) \to Spin(5)$ is the embedding $Sp(1)^2 \to Sp(2)$ of diagonal matrices.

$Spin(5) \to Spin(6)$ is the embedding of matrices A for which $A^T JA = J$, where

$$J = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}.$$

Denote $Spin(m)$ by G_m and consider N_m, the normalizer of G_m in G_{m+1}. Then

N_2 consists of $U(1)$ and $jU(1)$, where j is the quaternionic unit;

N_3 consists of matrices $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ and $\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}$, a is the quaternion, $a\bar{a} = 1$;

N_4 consists of matrices $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ and $\begin{pmatrix} o & a \\ b & 0 \end{pmatrix}$, where a and b are quaternions, $a\bar{a} = b\bar{b} = 1$.

The universal $Spin(m)$ bundles ξ^m have the following description.

Case $m = 5$. $G_5 = Spin(5) = Sp(2)$ acts by conjugation on the 5-dimensional vector space of 2×2 quaternionic Hermitian matrices of zero trace. These matrices are of the form $\begin{pmatrix} a_0 & b \\ \bar{b} & -a_0 \end{pmatrix}$, where a_0 is real, b is a quaternion and $(b, a_0) \in R^5$.

Let $E \to BG_5$ be the principal $Spin(5)$ bundle. Then the above action of G_5 on R^5 defines the sphere bundle of ξ^5,

$$BG_4 = E \times_{G_5} S^4 \to BG_5;$$

and the projective bundle of ξ^5,

$$BN_4 = E \times_{G_5} RP^4 \to BG_5.$$

Case $m = 4$. The universal $Spin(4)$ bundle ξ^4 is

$$\zeta_1 \otimes_H \zeta_2^* \to BSp(1)^2,$$

where ζ_1, ζ_2 are the canonical symplectic line bundles, ζ_2^* is the symplectic conjugate of ζ_2 and $(q_1, q_2) \in Sp(1)^2 = G_4$ acts on $R^4 \cong H$ by $v \to q_1 v q_2^{-1}$.

This defines the sphere bundle and the projective bundle of ξ^4:

$$BG_3 = E \times_{G_4} S^3 \to BG_4,$$

$$BN_3 = E \times_{G_4} RP^3 \to BG_4.$$

Case $m = 3$. The universal $Spin(3)$ bundle ξ^3 is

$$\Lambda \to BSp(1),$$

where $1 + \Lambda = \zeta \otimes_H \zeta^*$. $G_3 = Sp(1)$ acts on R^3 as conjugation on the pure quaternion.

This defines the sphere bundle and the projective bundle of ξ^3:

$$BG_2 = E \times_{G_3} S^2 \to BG_3,$$

$$BN_2 = E \times_{G_3} RP^2 \to BG_3.$$

Consider now the standard inclusion $RP^3 \to RP^4$. This is G_4 equivariant, where G_4 acts on RP^3 as above and on RP^4 as a subgroup of G_5.

This defines the inclusion of the projective bundle $P(\xi^4)$ in $P(\xi^4 + 1)$:

$$l : BN_3 = P(\xi^4) = E \times_{G_4} RP^3 \to E \times_{G_4} RP^4 = P(\xi^4 + 1).$$

The inclusion $RP^2 \to RP^3$, induced by the embedding of the pure quaternions into $H \cong R^4$ is G_3-equivariant. Here G_3 acts on RP^2 as above and on RP^3 as a subgroup of G_4.

This defines the inclusion

$$m : BN_2 = P(\xi^3) = E \times_{G_3} RP^2 \to E \times_{G_3} RP^3 = P(1 + \xi^3).$$

Let

$$\lambda_3 \to P(\xi^3), \quad \lambda_4 \to P(\xi^4),$$

$$\tilde{\lambda_4} \to P(1+\xi^3), \quad \tilde{\lambda_5} \to P(\xi^4+1)$$

be the canonical real line bundles. Then it is easy to see

Lemma 1.5. $l^!(\tilde{\lambda_4}) = \lambda_3$, $m^!(\tilde{\lambda_5}) = \lambda_4$.

1.6. Double coset formula. Let G be a compact Lie group and H and K closed subgroups.

Recall that the bundle $\rho(H,G) : BH \to BG$ has the fibre G/H and structure group G. Consider the pullback of BH to BK,

$$\begin{array}{ccc} \Gamma = \times_K(G/K) & \longrightarrow & BH \\ \Big\downarrow{\scriptstyle \gamma} & & \Big\downarrow{\scriptstyle \rho(G,H)} \\ BK & \xrightarrow[\rho(K,G)]{} & BG \end{array}$$

$\gamma : \Gamma \to BK$ has the fibre G/H and structure group K.

Let $\tau^*(H,G)$ be the transfer homomorphsm associated to $\rho(H,G)$. Then there is a formula for calculation of $\rho^*(K,G)\tau^*(H,G)$.

Theorem [Fe]**.** *Let $\gamma : \Gamma \to BK$ be the fibre bundle with fibre $F = G/H$ and structure group K acting on the left on F. Let $\{M\}$ be the set of orbit-type manifold components of the orbit space $K|F$, and let q be any K-orbit in M. Let $\tilde{q}$ be the subbundle of γ corresponding to q. Let $k : \tilde{q} \to \gamma$ be the inclusion and $\chi^*(M) = \chi(\bar{M} - M)$. Then*

$$\tau_\gamma^* = \sum \chi^*(M)\tau_{\tilde{q}}^* k^*,$$

where the sum is over all the orbit-type manifold componemts $\{M\}$.

1.7. Calculations with transfer for sphere bundles. For the proof of Proposition 1 we need the following

Lemma 1.8. *Let $G_m = Spin(m)$ and $\rho(G_{m-1}, G_m) : BG_{m-1} \to BG_m$ be the sphere bundle of the universal $Spin(m)$ bundle ξ^m. Then*

$$\tau^*(G_{m-1}, G_m)(1) = \chi(S^{m-1}), \qquad m = 3, 4, 5,$$

in symplectic cobordism. Here $\chi(S^{m-1})$ is the Euler characterstic and because of this is equal to 2 if $m = 3, 5$, and 0 if $m = 4$.

Proof. Case $m = 4$. For the diagonal map $\rho = \rho(Sp(1), Sp(1)^2)$ we have $\rho(x_1) = \rho(x_2)$, where $x_i = cf_1(\zeta_i)$; ζ_1, ζ_2 are the canonical line symplectic bundles.

By the transfer property for $\tau^* = \tau^*(Sp(1), Sp(1)^2)$ we have

$$\tau^*(a)(x_1 - x_2) = \tau^*(\rho^*(x_1 - x_2)a) = 0$$

$\forall a \in MSp^*(BSp(1))$. Since $MSp^*(BSp(1)^2) = MSp^*[[x_1, x_2]]$, this proves that τ^* is the trivial homomorphism.

Case $m = 3$. Using the double coset formula for $\rho^*\tau^*$, we see that the double coset space $Sp(1)|Sp(1)^2|Sp(1)$ is the line segment, with isotropy group $Sp(1)$ at the endpoints and conjugate group of $U(1)$ in $Sp(1)$ in the interior. Taking into account the case $m = 4$, we have

$$0 = \tau^*(G_3, G_4)(1) = 2\tau^*(G_3, G_3)(1) - \tau^*(U(1), G_3)(1) = 2 - \tau^*(U(1), G_3)(1).$$

Since $\rho(U(1), G_3)$ is the sphere bundle of ξ^3, this proves the case $m = 3$.

Case $m = 5$. The sphere bundle of ξ^5 agrees with $\rho(Sp(1)^2, Sp(2))$. On the other hand this bundle is the quaternionic projective bundle associated to the universal symplectic plane bundle, and the statement is known from [D, p.235]. One may prove this case by the method we will use in the following section.

Proof of Propositions 1.1 and 1.2. Case $m = 5$**.** It is shown in [Sn, ch.1] that the following diagram of the stable maps is commutative (see also Remark 1.11):

$$\begin{array}{ccc} BG_4 = BSp(1)^2 & \longrightarrow & BSp(2) = BG_5 \\ \searrow & \swarrow \tau & \\ BZ_2 \wr Sp(1) = BN_4 & & \end{array}$$

where $BSp(1)^2 \to BSp(2)$ is induced by $\rho(Sp(1)^2, Sp(2))$ and $BSp(1)^2 \to BZ_2 \wr Sp(1)$ by $\rho(Sp(1)^2, Z_2 \wr Sp(1))$.

Since, as it is well known, $\rho^*(Sp(1)^2, Sp(2))$ is a monomorphism, this proves the case $m = 5$.

Proof of Propositions 1.1 and 1.2. Case $m = 4$**.** The following lemma immediately follows from the definitions of 1.4.

Lemma 1.9. *The double coset space* $G_4|G_5|N_4$ *is a line segment. One endpoint corresponds to an orbit consisting of one point* $(0, \pm 1) \in RP^4$*, where*

$$G_5|N_4 = RP^4 = \{\pm(v, w) | v \text{ is a quaternion, } w \text{ is a real, } vv^* + w^2 = 1\}.$$

The point $(0, \pm 1)$ *is a fixed point. The other endpoint corresponds to* RP^3*, consisting of points* $(\pm v, 0) \in RP^4$*. The isotropy groups for these points are conjugate groups of* N_3 *in* G_4*. The open interval corresponds to orbits* S^3 *consisting of points* $\pm(v, w)$*,* $0 < vv^* < 1$*. The isotropy groups for these points are conjugate groups of* G_3 *in* G_4*.*

Proof. For the point $(0, \pm 1) \in RP^4$ the isotropy group is obviously $Sp(1)^2$. For the points $(\pm v, 0) \in RP^4$, the isotropy group K_v for the given $(\pm v, 0)$ consists of elements (vqv^{-1}, q) and $(-vqv^{-1}, q)$ from the group $Sp(1)^2$. Hence

$$K_v = gN_3g^{-1}, \;\; g = (v, 1).$$

For the points $(\pm v, \pm w) \in RP^4$, $0 < vv^* < 1$, we have

$$(q_1, q_2)(\pm v, \pm w) = (\pm q_1 v q_2^{-1}, \pm w);$$

$$v = q_1 v q_2^{-1};$$

$$q_1 = v q_2 v^{-1}.$$

So for the given $(\pm v, \pm w)$ the isotropy group is the conjugate group of $Sp(1)$ in $Sp(1)^2$.

Combining Lemma 1.9 and the double coset formula for $\rho^*(G_4, G_5)T(N_4, G_5)$, we have

$$\rho^*(G_4, G_5)\tau^*(N_4, G_5)(1) = 1 - \tau^*(G_3, G_4)(1) + \tau^*(N_3, G_4)(1).$$

Since $\tau^*(N_4, G_5)(1) = 1$ and $\tau^*(G_3, G_4)(1) = 0$, this proves $\tau^*(N_3, G_4)(1) = 0$.

Consider now $\rho^*(G_4, G_5)\tau^*(N_4, G_5)(c_5^n)$. Again using the double coset formula above, this is decomposed into three summands. Of these, the two summands corresponding to the subbundles identity $BG_4 \to BG_4$ and $BG_3 \to BG_4$ are zero since there are no nontrivial real line bundles over BG_3 and BG_4. As for the third summand, it coincides with $\tau^*(N_3, G_4)(c_4^n)$ by Lemma 1.5.

Hence we have

$$\rho^*(G_4, G_5)\tau^*(N_4, G_5)(c_5^n) = 0$$

by the case $m = 5$,

$$= 0 - 0 + \tau^*(N_3, G_4)(c_4^n).$$

This proves the case $m = 4$.

Proof of Propositions 1.1, 1.2. Case $m = 3$**.** Consider now the double coset formula for $\rho^*(G_3, , G_4)\tau^*(N_3, G_4)$.

Recall, from 1.4, that the homogeneous space G_4/N_3 is the projective space

$$RP^3 = \{\pm h, hh^* = 1, h \in H\}.$$

It is easy to see the following.

Lemma 1.10. *The double coset space $G_3|G_4|N_3$ is a line segment. One endpoint corresponds to an orbit consisting of one point $(\pm 1) \in RP^3$. This point is fixed. The other endpoint corresponds to RP^2, consisting of points $\{\pm h, h$ pure quaternion, $hh* = 1\}$. The isotropy group for the given $(\pm h)$ is the conjugate group of N_2 in $Sp(1)$. The open interval corresponds to orbits S^2, consisting of points $(\pm h)$, whose real parts differ from 0 and ± 1. The isotropy groups for these points are the conjugate groups of $U(1)$ in $Sp(1)$.*

Using now the double coset formula, we obtain

$$0 = \rho * (G_3, G_4)\tau^*(N_3, G_4)(1)$$

by the case $m = 4$,

$$= 1 - \tau^*(U(1), Sp(1))(1) + \tau^*(N_2, Sp(1))(1)$$

by Lemma 1.10,

$$= 1 - 2 + \tau^*(N_2, Sp(1))(1)$$

by Lemma 1.8.

This proves that $\tau^*(N_2, Sp(1))(1) = 1$.

In the same spirit we obtain

$$0 = \rho * (G_3, G_4)\tau^*(N_3, G_4)(c_4^n)$$

by the case $m = 4$,

$$= 0 - 0 + \tau^*(N_2, Sp(1))(c_3^n)$$

by Lemma 1.10 and Lemma 1.5.

This proves the case $m = 3$.

Proof of Proposition 1.3. For $m = 3$ formula (1.3) coincides with (2) from the Introduction, which is the case we need to prove.

The projectivisation $p : BN \to BSp(1)$ of the bundle $\Lambda = \xi^3$ defines the canonical splitting over BN

$$p^*(\Lambda) = \mu + \lambda,$$

where μ and λ are a plane and a linear real bundle respectively.

Then we have the splitting

$$p^*(\Lambda \otimes_R H) = \mu \otimes_R H + \lambda \otimes_R H.$$

Apply now the Whitney formula to express the symplectic characteristic classes of the bundle $p^*(\Lambda \otimes_R H)$ in terms of the classes $\mu \otimes_R H$ and $\lambda \otimes_R H$. We obtain the equations

$$pf_1(p^*(\Lambda \otimes_R H)) = pf_1(\mu \otimes_R H) + pf_1(\lambda \otimes_R H);$$

$$pf_2(p^*(\Lambda \otimes_R H)) = pf_2(\mu \otimes_R H) + pf_1(\mu \otimes_R H)pf_1(\lambda \otimes_R H);$$

$$e = pf_3(p^*(\Lambda \otimes_R H) = pf_2(\mu \otimes_R H)pf_1(\lambda \otimes_R H).$$

Let $c = pf_1(\lambda \otimes_R H)$. Then the above equations give an exposition of e in terms of c and $pf_i(p^*(\Lambda \otimes_R H))$, $i = 1, 2$:

$$e = pf_2(p^*(\mu \otimes_R H))c$$

$$= [pf_2(p^*(\Lambda \otimes_R H)) - pf_1(\mu \otimes_R H)c]c$$

$$= pf_2(p^*(\Lambda \otimes_R H))c - [pf_1(p^*(\Lambda \otimes_R H)) - c]c^2$$

$$= pf_2(p^*(\Lambda \otimes_R H))c - pf_1(p^*(\Lambda \otimes_R H))c^2 + c^3.$$

Now apply the transfer homomorphism τ_p^* to this equation:

$$\tau_p^*(e) = \tau_p^*[p^*(pf_2(\Lambda \otimes_R H))c] - \tau_p^*[p^*(pf_1(\Lambda \otimes_R H))c^2] + \tau_p^*(c^3).$$

Taking into account the transfer property $\tau_p^*(p^*(t)) = t\tau_p^*(1)$, we obtain

$$\tau_p^*(e) = pf_2(\Lambda \otimes_R H)\tau_p^*(c) - pf_1(\Lambda \otimes_R H)\tau_p^*(c^2) + \tau_p^*(c^3).$$

But by virtue of Proposition 1.2 we have $\tau_p^*(c) = \tau_p^*(c^2) = \tau_p^*(c^3) = 0$. Therefore $\tau_p^*(e) = 0$.

The proofs of the cases $m = 4, 5$ are quite analogous. However the case $m = 4$ also follows from Proposition 1.1, namely, from the equality $\tau_4^*(1) = 0$:

$$\tau_4^*(e_4) = \tau_4^*(p_4^*(pf_4(\xi^4 \otimes_R H))) = pf_4(\xi^4 \otimes_R H)\tau_4^*(1) = pf_4(\xi^4 \otimes_R H) = 0.$$

Then, as proved in [GR], every $Spin(5)$ bundle and, in particular, ξ^4, is MSp-orientable and has zero Euler class. Thus $pf_5(\xi^5 \otimes_R H) = 0$, so we have nothing to prove in the case $m = 5$.

1.11. Remark on Propositions 1.1 and 1.2. Case $m = 5$. The commutativity of above diagram is stated by the method of equivariant vector fields on the homogeneus spaces [BM]. Namely there is [Sn, Example 1.13] an $Sp(1)^2$ equivariant vector field on $Sp(2)/Z_2 \wr Sp(1)$ with one singular point. Using this field we shall see here that in the case of the projective bundle $P(\xi^4 + 1)$ the transfer map is stably homotopic to the section of this bundle defined by the direct summand 1.

We need a simple particular case of [BM, Corollary 2.11]. Namely let $\pi : E \to B$ be the fiber bundle with fiber F. Suppose that F admits a G equivariant vector field with one singular point (fixed under the action G) and the Euler characteristic $\chi(F) = 1$. This fixed point obviously defines a section $i : B \to E$. Then i suspends to the transfer map $\tau(\pi)$, that is, $i^+ = \tau(\pi)$ in the track group $\{B^+, E^+\}$.

Taking into account Lemma 1.9, we see that the projective bundle $P(\xi^4 + 1)$, that is, the pullback of $BN_4 \to BG_5$ to BG_4, has section defined by the fixed point $(0, \pm 1) \in RP^4$ under the action of G_4. This section agrees with the section of $P(\xi^4 + 1)$ defined by the direct summand 1.

Lemma 1.12. *The above section of the projective bundle $P(\xi^4+1)$ suspends to the transfer map.*

Proof. Following [BM] we construct a $G_4 = Sp(1)^2$ equivariant vector field on $RP^4 = G_5/N_4$ with one zero point. It is easy to see that

$$G_5/N_4 = GL_2(H)/Z_2 \wr B(H)$$

where H is the quaternions, $GL_2(H)$ is the full linear group of 2×2 matrices, $B(H)$ are the all upper triangular matrices and the generator of Z_2 is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. This follows from the fact, that $GL_2(H)$ acts on $G_5/N_4 = S^4$, that is, on the manifold of flags $F_1 \subset F_2 = H^2$, with the isotropy group $B(H)$.

Now let v be a vector from the Lie algebra of $GL_2(H)$, for which

$$\omega = exp(v) = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$$

where x, y are real numbers and $x \neq y$.

Consider now the field φ_v on $GL_2(H)$ defined by the right translations:

$$\varphi_v(g) = dR_g(v), \qquad g \in GL_2(H).$$

This field induces the field ϱ_v on $GL_2(H)/Z_2 \wr B(H)$. The field ϱ_v is $Sp(1)^2$ equivariant, since $Sp(1)$ is a subgroup of the centralizer of ω . For the zero points of ϱ_v note that coset of g is the zero point if and only if $g^{-1}\omega g \in Z_2 \wr B(H)$, that is, $g \in Z_2 \wr B(H)$. Thus ϱ_v has one zero point. This proves Lemma 1.12.

The above lemma proves the analog of Proposition 1 for the projective bundle $P(\xi^4 + 1) \to BSp(1)^2$. But since this bundle is the pullback of $P(\xi^5) \to BG_5$ to BG_4 and the homomorphism induced by $BSp(1)^2 = BG_4 \to BG_5 = BSp(2)$ is a monomorphism, this proves the case $m = 5$.

2. Proof of (3) from the introduction

We need the following fact.

Propostion [Na]. *In $MSp^*(BU(1))$*

$$\theta_1 z + \sum_{k\geq 1} \phi_k z^{2k} = 0,$$

where $z = pf_1(\xi + \bar{\xi})$; ξ is the canonical complex line bundle; θ_1, ϕ_i are the Ray classes.

This follows immediately from the bundle relation

$$\eta \otimes_R (\xi + \bar{\xi}) = \xi + \bar{\xi}$$

in $KSp^0(S^1 \times BU(1))$ and from the definition of Ray classes.

Then, as it is known, any $Spin(4)$ bundle is MSp^* orientable. This follows from the isomorphism $KO^4 = KSp^0$: For the given KO orientation class of $Spin(4)$ bundle this isomorphism determines the symplectic bundle over the corresponding Thom space, and the first Conner-Floyd symplectic Pontryagin class of this symplectic bundle will be taken as the symplectic orientation class. So the $Spin(4)$ bundle $\zeta \otimes_H \zeta^* = 1 + \Lambda$, and because of this Λ is MSp^* orientable [RS].

By using these results and the fact that the bundle $BU(1) \to BSp(1)$ is the sphere bundle of Λ it is proved in [G] that the Thom class of the bundle Λ can be chosen in such a way that its restiction to the zero section $\tilde{e}(\Lambda)$ has the form

$$\tilde{e}(\Lambda) = \theta_1 x + \sum_{i\geq 1} \phi_i x^{2i},$$

where $x = pf_1(\zeta)$. For another proof, see [GR].

Since $2\theta_1 = 2\phi_i = 0$ [R] and $\theta_1^3 = 0$ [G], we obtain

$$\sum_{i\geq 1} \phi_i^4 x^{8i} = (\tilde{e}(\Lambda))^4 = \tilde{e}(\Lambda \otimes_R H)$$

But $\tilde{e}(\Lambda \otimes_R H)$ agrees with the ordinary Euler class $e(\Lambda \otimes_R H)$ up to multiplication by a unit of $MSp^*(BSp(1)_+)$, and we obtain

$$\sum_{i\geq 1} \phi_i^4 x^{8i} = e(\Lambda \otimes_R H)(1 + \sum_{j\geq 1} \alpha_j x^j)^{-1}$$

for some coefficients $\alpha_j \in MSp^*$. This proves (3).

3. Nilpotency formula in self-conjugate cobordism

Let $Q = \{\pm 1, \pm i, \pm j, \pm k\}$ be the quaternion group, N the normalizer of S^1 in S^3 as above and Z_4 a cyclic group generated by j.

Recall that $\rho(N, S^3)$ is the projective bundle of the universal $Spin(3)$ bundle $\Lambda \to BS^3$, and we have the canonical splitting

$$\rho^*(N, S^3)(\Lambda) = \mu + \lambda.$$

Here μ is a plane and λ is a line real bundle.

It is easy to see that the bundle $\rho(Q, N)$ is the projective bundle of μ and the bundle $\rho(Q, S^3)$ is the bundle of flags of the bundle Λ.

This defines the splittings

$$\rho^*(Q, S^3)(\Lambda) = \chi_1 + \chi_2 + \chi_3;$$

$$\rho^*(Q, N)(\mu) = \chi_2 + \chi_3;$$

$$\rho^*(Q, N)(\lambda) = \chi_1.$$

Here $\chi_3 = \chi_1 \otimes_R \chi_2$.

Proposition. *The transfer homomorphism $\tau^*(Q, S^3)$ satisfies the following relations:*

$$e(\Lambda \otimes_R C) = -\tau^*(Q, S^3)(e^2(\chi_i \otimes_R C)e(\chi_j \otimes_R C)) = 0 \tag{3.1}$$

in the self-conjugate cobordism theory and

$$e(\Lambda \otimes_R H) = -\tau^*(Q, S^3)(e^2(\chi_i \otimes_R H)e(\chi_j \otimes_R H)) = 0 \tag{3.2}$$

in the symplectic cobordism theory, where $i, j = 1, 2, 3$; $i \neq j$.

Proof of (3.1). The double coset space $N|S^3|N$ is a line segment. The isotropy groups are N and Q at the endpoints and Z_4 (generated by j) in the interior.

By the double coset theorem and Proposition 1, case $m = 3$, we have

$$0 = e(\lambda \otimes_R C) + \tau^*(Q, N)(e(\chi_2 \otimes_R C)) - \tau^*(Z_4, N)(\rho^*(Z_4, N)(e(\lambda \otimes_R C))).$$

But

$$e(\rho^*(N, S^3)(\Lambda \otimes_R C)) = e(\lambda \otimes_R C)e(\mu \otimes_R C)$$

and $\rho^*(Z_4, N)(\mu)$ has the section. Hence by the above splittings and transfer properties we obtain

$$e(\rho^*(N, S^3)(\Lambda \otimes_R C))$$

$$= -\tau^*(Q, N)(e(\chi_2 \otimes_R C))e(\mu \otimes_R C)$$

$$= -\tau^*(Q, N)(e^2(\chi_2 \otimes_R C)e(\chi_3 \otimes_R C)).$$

Since $\tau^*(N, S^3)(1) = 1$ by the analogue of Proposition 1 for the self-conjugate cobordism, this proves

$$e(\Lambda \otimes_R C) = -\tau^*(Q, S^3)(e^2(\chi_2 \otimes_R C)e(\chi_3 \otimes_R C)).$$

We may prove relations analogous to (3.1) by changing N to its conjugate subgroup in S^3, but this follows also by symmetry.

Now

$$\tau^*(Q, S^3)(e^2(\chi_1 \otimes_R C)e(\chi_2 \otimes_R C))$$

$$= \tau^*(N, S^3)(\tau^*(Q, N)(e^2(\chi_1 \otimes_R C)e(\chi_2 \otimes_R C)))$$

$$= \tau^*(N, S^3)(e^2(\lambda \otimes_R C)(-e(\lambda \otimes_R C) + \tau^*(Z_4, N)(\rho^*(Z_4, N)(e(\lambda \otimes_R C)))$$

$$= \tau^*(N, S^3)(-e^3(\lambda \otimes_R C)) + \tau^*(Z_4, S^3)(\rho^*(Z_4, N)(e^3(\lambda \otimes_R C))).$$

The first summand is zero by Proposition 1 (by its analogue). The second summand is also zero. This follows immediate from the following theorem

Theorem [Fe]**.** *Assume $N_G(H)/H$ is not discrete, where $N_G(H)$ is the normalizer of H in G. Then $\tau^*(H, G) = 0$.*

The proof of (3.2) is analogous.

Now since (see Section 2) the symplectic Euler class of $\Lambda \otimes_R H$ (the Euler class of $\Lambda \otimes_R C$ in SC^*) coincides with $\sum_{i\geq 1} \phi_i^4 x^{8i}$ (with the image of $\sum_{i\geq 1} \phi_i^2 x^{4i}$ in SC^* theory) up to multiplication by a unit of $MSp^0(BS^3_+)$ (by a unit of $SC^0(BS^3_+)$), this proves

Corollary 3.3. *$\phi_i^4 = 0$, and the images of ϕ_i^2 in self-conjugate cobordism are zero.*

Remark 3.4. It follows from the relation between the transfer and the umkehr map [BG], [BO] that Proposition 1 is true also for $m = 2$ and $m = 6$.

4. On four-fold products of Ray classes

Here we improve the above method and obtain

Proposition 4.1. *All four-fold products of Ray classes are zero, and the images of double products of these classes in self-conjugate cobordism are zero.*

The proof is organized as follows:

Let N be the normalizer of the torus $U(1)$ in $Sp(1)$ as above. Consider again the bundle

$$p : BN \to BSp(1)$$

and the map

$$f : BN \to BZ_2$$

induced by projection of N on the Weil group Z_2. Let τ_p be the transfer map for p.

We have the following relations.

Proposition 4.2. *In $MSp^*(BSp(1)^4) = MSp^*[[x_1, x_2, x_3, x_4]]$ we have*

$$\sum_{i,j,k,l\geq 1} \phi_i\phi_j\phi_k\phi_l x_1^{2i} x_2^{2j} x_3^{2k} x_4^{2l} = \sum_{m,n,p,q\geq 0} \tau_p^* f^*(\gamma_{mnpq}) x_1^m x_2^n x_3^p x_4^q,$$

where γ_{mnpq} are elements from $\tilde{MSp}^*(BZ_2)$.

Proposition 4.3. *In $SC^*(BSp(1)^2) = SC^*[[y_1, y_2]]$ we have*

$$\sum_{i,j\geq 1} \psi_i\psi_j y_1^{2i} y_2^{2j} = \sum_{m,n\geq 0} \tau_p^* f^*(\delta_{mn}) y_1^m y_2^n,$$

where ψ_i is the image of ϕ_i in self-conjugate cobordism and the δ_{mn} are elements from $\tilde{SC}^*(BZ_2)$.

We shall see later that the map $f\tau_p$ induces trivial homomorphism for any generalized cohomology theory h^* .

Proposition 4.4.

$$\tau_p^* f^*(a) = 0, \quad \forall a \in \tilde{h}^*(BZ_2);$$

$$\tau_p^*(1) = 1.$$

Thus the right sides of the relations from 4.2 and 4.3 are zero. This proves Proposition 4.1

Proofs of 4.2 *and* 4.3. We need a simple lemma about orientable bundles, whose proof follows from the fact that $KO^4(X) = KSp^0(X)$.

Let $\eta \to BZ_2$ be the universal $O(1)$ bundle and ζ, ζ^*, Λ the bundles from the introduction.

Lemma 4.5. *i) The bundle $\eta \otimes_R \zeta \otimes_H \zeta^* \to BZ_2 \times BSp(1)$ is MSp-orientable.*

ii) The bundle $\eta \otimes_R \sum_{i=1}^4 \Lambda_i \to BZ_2 \times BSp(1)^4$ is MSp-orientable.

iii) The bundle $\eta \otimes_R \sum_{i=1}^2 \Lambda_i \to BZ_2 \times BSp(1)^2$ is SC-orientable.

Proof. i) This bundle is a $Spin(4)$ bundle and so is MSp-orientable.

ii) Since $\zeta_i \otimes_H \zeta_i^* = \Lambda_i + R^1$, the bundle ii) is MSp-orientable as a difference of two MSp-orientable bundles

$$\eta \otimes_R \sum_{i=1}^{4} \zeta_i \otimes_H \zeta_i^* - \eta \otimes_R H.$$

iii) This bundle is a difference of SC-orientable bundles

$$\eta \otimes_R \sum_{i=1}^{2} \zeta_i \otimes_H \zeta_i^* - \eta \otimes_R C.$$

Recall from section 2 that

$$\tilde{e}(\Lambda) = \theta_1 + \sum_{i \geq 1} \phi_i x^{2i}, \qquad x = e(\zeta).$$

Any two orientation classes of the given orientable bundle agrees up to multiplication by an invertible element. So there is

$$\tilde{e} = \tilde{e}(\eta \otimes_R \sum_{i=1}^{4} \Lambda_i),$$

which as an element from

$$MSp^*(BZ_2 \times BSp(1)^4) = MSp^*(BZ_2)[[x_1, x_2, x_3, x_4]], \qquad x_i = e(\zeta_i),$$

has the form

$$\tilde{e} = \prod_{s=1}^{4} (\theta_1 + \sum_{r \geq 1} \phi_r x_s^{2r}) + \sum_{m,n,p,q \geq 0} \gamma_{mnpq} x_1^m x_2^n x_3^p x_4^q$$

$$= \sum_{i,j,k,l \geq 1} \phi_i \phi_j \phi_k \phi_l x_1^{2i} x_2^{2j} x_3^{2k} x_4^{2l} + \sum_{m,n,p,q \geq 0} \gamma_{mnpq} x_1^m x_2^n x_3^p x_4^q.$$

Here we take into account the relation $\theta_1 \phi_i \phi_j = 0$ from [G].

Consider now the map

$$g = (f, p) \times 1 : BN \times BSp(1)^3 \to BZ_2 \times BSp(1) \times BSp(1)^3.$$

Lemma 4.6. $g^*(\tilde{e}) = 0$.

Proof. Recall from Section 3 that $p^*(\Lambda) = \mu + \lambda$. But $f^*(\eta) = \lambda$ and $\lambda^2 = 1$. Thus the bundle

$$g^*(\eta \otimes_R \sum_{i=1}^{4} \Lambda_i) = \lambda(\mu + \lambda + \Lambda_2 + \Lambda_3 + \Lambda_4)$$

has the section. This proves the lemma.

We now have in $MSp^*(BN \times BSp(1)^3)$ the relation

$$\sum_{i,j,k,l \geq 1} \phi_i\phi_j\phi_k\phi_l p^*(x_1)^{2i} x_2^{2j} x_3^{2k} x_4^{2l} + \sum_{m,n,p,q \geq 0} f^*(\gamma_{mnpq}) p^*(x_1)^m x_2^n x_3^p x_4^q = 0.$$

After application of the transfer homomorphism for the bundle

$$p \times 1 : BN \times BSp(1)^3 \to BSp(1)^4$$

we get Proposition 4.2.

The proof of 4.3 is analogous.

Proof of 4.4. In fact this is a particular case of Proposition 1, although we should rewrite it as follows:

Proposition 4.7. *Let $G_m = Spin(m)$, and let $\xi^m \to BG_m$ be the universal $Spin(m)$ bundle, $m = 2, 3, 4, 5$. Let $p_m : P(\xi^m) \to BG_m$ be the projective bundle associated to ξ^m and let*

$$f_m : P(\xi^m) \to BZ_2$$

be the classifying map for the canonical real line bundle $\lambda_m \to P(\xi^m)$. Then $\tau_m^*(1)$ is equal to 0 if $m = 2, 4$ and equal to 1 if $m = 3, 5$;

$$\tau_m^*(a) = 0, \quad \forall a \in M\tilde{S}p^*(BZ_2).$$

The case $m = 3$ gives Proposition 4.4.

We also remark that using [Bu] and Proposition 1 one can obtain a new proof of the relation $\theta_1\theta_i\theta_j = 0$ proved in [GR]. Moreover, some relations between the θ_i's and the generators of the free part of the symplectic cobordism can be also derived. We plan to present the details in a future paper.

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