SELF-DUALITY OPERATORS ON
ODD DIMENSIONAL MANIFOLDS

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ABSTRACT. In this paper we construct a new elliptic operator associated to any nowhere zero vector field on an odd-dimensional manifold and study its index theory. It turns out this operator has several geometric applications to conformal vector fields, self-dual vector fields, locally free S^1-actions and transversal hypersurfaces of these vector fields in an odd-dimensional manifold. In particular, we reveal a non-stable phenomena about the existence of conformal vector fields and self-dual vector fields in odd dimensions above 3. This is in sharp contrast to the stable phenomena about the existence of nowhere zero vector fields in odd dimensions. Besides these applications, the index formula of this new operator also gives the formulas for the dimensions of self-duality cohomology groups and for the virtual dimensions of the moduli spaces of anti-self-dual connections on 5-cobordisms, which are introduced in author's previous papers.

INTRODUCTION

It is well-known that topology of smooth 4-manifolds is more complicated than topology of higher dimensional smooth manifolds. However the topology of a smooth 4-manifold often can be captured by the topology of a line field on a much simpler smooth 5-manifold. For example, any exotic 4-space can be naturally represented by a smooth line field on the standard Euclidean 5-space $\mathbb{R}^5$ (Arnold [1]). (An exotic 4-space is a smooth 4-manifold which is homeomorphic but not diffeomorphic to the standard Euclidean 4-space $\mathbb{R}^4$.) This idea can also be applied to closed smooth 4-manifolds. In [12] we observe that such a representation of an exotic 4-space by a smooth line field is "faithful", which means that if there is a diffeomorphism of $\mathbb{R}^5$ sending a representative line field of one exotic 4-space to a representative line field of another exotic 4-space, then two exotic 4-spaces must be diffeomorphic. This observation motivates us to make the following definition. A *smooth 4-space* is a pair (M, L) where L is a smooth line field on a smooth 5-manifold M. Two smooth 4-spaces (M, L) and (M', L') are diffeomorphic if there is a diffeomorphism from M onto M' taking L to L'. By this notion it is natural to ask how to extend theories on smooth 4-manifolds to smooth 4-spaces. This suggests many interesting research topics. In addition it is perhaps more appropriate

to study deformations of smooth structures of smooth 4-spaces rather than those of smooth 4-manifolds alone.

The gauge theory on smooth 4-manifolds is one of the most profound theories. How can the gauge theory on smooth 4-manifolds be extended to smooth 4-spaces ? In [11] and [12] we proposed the self-dual Yang-Mills-type gauge theory and its dual Seiberg-Witten-type theory on smooth 5-manifolds with a smooth line field, that is, smooth 4-spaces. The linear models of these two non-linear theories are essentially a new operator which we are going to construct and study in this paper. We call this operator the *self-duality* operator, for it can be regarded as an analogue of the self-duality operator in the gauge theory on smooth 4-manifolds. Since we find our construction of the self-duality operator works quite well in any odd dimension ≥ 3, we are not going to restrict our studies only to dimension 5 in this paper.

To make the reading of this paper more pleasant, in Section 1 we discuss many applications of the self-duality operators to self-dual vector fields, nearly-self-dual vector fields and conformal vector fields. We first recall the notion of a self-dual vector field from [10] and generalize this to the notion of a nearly-self-dual vector field (see Definitions 1.1 and 5.1). Any conformal vector field on an odd-dimensional manifold is self-dual. Then we state a main theorem about the *non-stable* phenomena for the existence of *nearly-self-dual* vector fields in odd dimensions above 3, which is in sharp contrast to the stable phenomena for the existence of nowhere zero vector fields in odd dimensions (see Theorems 1.5 and 1.7 and Proposition 1.6). This result follows from the necessary conditions we find for the existence of nearly-self-dual vector fields on an odd-dimensional manifold (see Theorems 1.10, 1.11 and 1.15). In addition we obtain several topological inequalities about transversal hypersurfaces of nearly-self-dual vector fields (see Theorems 1.16, 1.17 and 5.7). The proofs of all these results are presented in the final section.

In Section 2 we construct the self-duality operator associated to a line field on an odd-dimensional manifold (see Definition 2.3). Then we state the index formula for the self-duality operator with an elliptic boundary condition when the manifold is compact with boundary and the line field is transverse to the boundary (see Theorem 2.9). The proof of this formula is postponed to the final section. This index formula has several other significant applications besides those discussed in Section 1. It yields the formulas for the virtual dimension of the moduli space of anti-self-dual connections on a 5-cobordism in [11], and the virtual dimension of the moduli space of Seiberg-Witten-type monopoles on a 5-cobordism in [12] (see Corollaries 2.11 and 2.12). In Section 3, we give an application of the index formula to a general nowhere zero vector field (see Corollary 3.1).

In Section 4 we define the self-duality cohomology groups associated to a general line field on a compact odd-dimensional manifold and get many topological inequalities about the dimensions of these self-duality cohomology groups. In Section 5 we focus our attention to nearly-self-dual vector fields again. By using the index formula of self-duality operator we get formulas for all the dimensions of the self-duality cohomology groups associated to a nearly-self-dual vector field (see Theorem 5.6). In the final section we present all the postponed proofs.

We want to emphasize that the line fields in all sections *except Sections 1 and 5* are *general* ones. Sections 1 and 5 concern the applications of self-duality operators to nearly-self-dual line fields. Finally note that this paper is the published version of the author's preprint "Self-dual vector fields on odd dimensional manifolds".

1. Applications to nearly-self-dual line fields I

In this section we first discuss the applications of self-duality operators to nearly-self-dual line fields. Let L be a smooth tangent line field on a smooth $(2m+1)$-manifold M. Define

$$(1.1) \qquad Q^* \stackrel{def}{=} \{\omega \in T^*M | \langle \omega, L \rangle = 0\}.$$

Then Q^* is a real $2m$-dimensional subbundle of T^*M. A fiberwise Riemannian metric g_q on Q^* and an orientation on Q^* determine an operator

$$(1.2) \qquad *_q : \bigwedge^j Q^* \to \bigwedge^{2m-j} Q^*$$

by $\alpha \wedge *_q \beta = g_q(\alpha, \beta) vol(g_q)$ for any $\alpha, \beta \in \bigwedge^j Q^*$, where $vol(g_q)$ is the oriented volume element of g_q.

Let L_V denote the Lie derivative with respect to a vector V. For any section V of the line field L, define

$$(1.3) \qquad \Gamma_V \stackrel{def}{=} L_V *_q - *_q L_V : \bigwedge^m Q^* \to \bigwedge^m Q^*.$$

The following definition is a special case of ([10], Definition 6.6).

Definition 1.1. A tangent line field L on a smooth $(2m+1)$-manifold M is called *self-dual* if it is smooth and there is a smooth fiberwise Riemannian metric g_q on Q^* and an orientation on Q^* such that $\Gamma_V \stackrel{def}{=} L_V *_q - *_q L_V = 0$ on sections of $\bigwedge^m Q^*$ for any smooth section V of L. Such a metric g_q on Q^* is called *L-compatible*. A vector field is called *self-dual* if it is smooth, nowhere zero and the line field generated by the vector field is self-dual.

The vanishing of Γ_V means that the operator $*_q$ commutes with the Lie derivative L_V, that is, all self-dual sections of $\bigwedge^m Q^*$ are preserved by the flow generated by V. This is the reason why we call such a line field self-dual. Note that Γ_V is a *zero-th order* operator on M, that is, $\Gamma_V(f\omega) = f\Gamma_V\omega$ for any smooth function f and any smooth section ω of $\bigwedge^m Q^*$ over M. In addition $\Gamma_{fV} = f\Gamma_V$ on $\bigwedge^m Q^*$ for any function f and section V of L over M.

In fact what we need is a notion of a nearly-self-dual vector field, which requires the commutativity of $*_q$ and L_V only on certain smooth sections of $\bigwedge^m Q^*$. But the definition technically gets more involved, so we defer it to Definition 5.1. For this section we only need to know

Lemma 1.2. *(1) Any self-dual line field is nearly self-dual.*

(2) If a vector field V is self-dual (nearly-self-dual resp.) over an open subset O then fV is self-dual (nearly-self-dual resp.) over O for any nowhere zero smooth function f over O.

The first assertion is obvious by Definitions 1.1 and 5.1. The second assertion follows from the fact that V and fV generate the same line field over O. Recall that a vector field V is *conformal* if V is nowhere zero, smooth and the 1-dimensional foliation generated by V is conformal. Any Riemannian vector field is conformal. The vector field induced by a locally free smooth S^1-action is Riemannian. (See Vaisman [21] and Appendix A by Y. Carriere in Molino [17] or Section 6 of [10].)

Lemma 1.3. *If a vector field on an odd-dimensional orientable manifold is conformal, then it is self-dual.*

For the proof of this fact see ([10], Remark 6.8). For a $(2m+1)$-manifold M and a subspace $K \subset M$, the *half-characteristic* and *relative half-characteristic* are respectively defined as

$$(1.4) \qquad \chi_{\frac{1}{2}}(M) = \sum_{j=0}^{m-1} (-1)^j b^j(M), \qquad \chi_{\frac{1}{2}}(M, K) = \sum_{j=0}^{m-1} (-1)^j b^j(M, K).$$

Here $b^j(M)$ denotes the j-th Betti number of M and $b^j(M, K)$ denotes the j-th relative Betti number of (M, K). Note that $\chi_{\frac{1}{2}}$ is different from the Kervaire semi-characteristic.

Definition 1.4. A vector field on a smooth compact manifold with boundary is called *proper* if it is smooth, nowhere zero and transverse to the boundary.

Now we can state a main result in this paper.

Theorem 1.5. *Let M be a compact, connected oriented smooth $(2m+1)$-manifold with nonempty boundary ∂M and $\dim M \geq 5$. Let Y be a closed connected oriented smooth $(2m+1)$-manifold such that $\chi_{\frac{1}{2}}(Y) < 1$ in the case m is odd and $\chi_{\frac{1}{2}}(Y) > 1$ in the case m is even. Then for all sufficiently large k, $M \sharp kY$ doesn't admit any nearly-self-dual proper vector field.*

This theorem should be compared with the following result.

Proposition 1.6. *Let M be a compact, connected oriented smooth $(2m+1)$-manifold with nonempty boundary ∂M, and let N be the union of some boundary components of M (N may be empty). Suppose that $\chi(N) = \chi(\partial M - N)$. Then for any given closed oriented smooth $(2m+1)$-manifold Y, the connected sum $M \sharp Y$ always admits a proper vector field which is outward on N and inward on $\partial M - N$.*

Thus in odd dimensions the existence of a proper vector field is a *stable* phenomena. But by contrast, Theorem 1.5 reveals that in all odd dimensions ≥ 5 the existence of a proper nearly-self-dual vector field is a *non-stable* phenomena.

Theorem 1.5 follows from Theorems 1.10 and 1.11 and Lemma 1.8. Proposition 1.6 follows from the fact that the condition $\chi(N) = \chi(\partial M - N)$ in Theorem 1.12 depends only on the *boundary*. Theorem 1.5 also can be applied to a closed odd-dimensional manifold.

Theorem 1.7. *Let M be a closed, connected oriented smooth $(2m+1)$-manifold of $\dim M \geq 5$. Let Y be as in Theorem 1.5. Then for all sufficiently large k, $M \sharp kY$ doesn't admit any nearly-self-dual vector field which is transverse to a smoothly imbedded closed $2m$-manifold C in M, and in particular, $M \sharp kY$ doesn't admit any locally free smooth S^1-action which is transverse to a smoothly imbedded closed $2m$-manifold C in M.*

Indeed if such a nearly-self-dual vector field existed, then we could cut M along C and got a situation in contradiction to Theorem 1.5. (A locally free smooth S^1-action transverse to C just means that the vector field induced by it is so.)

It should be emphasized that there is a qualitative difference between the invariants $\chi_{\frac{1}{2}}(M, K)$ and $\chi(M, K)$. For an odd-dimensional compact oriented smooth manifold M with boundary ∂M and $K \subset \partial M$,

$$(1.5) \qquad\qquad \chi(M, K) = \frac{1}{2}\chi(\partial M) - \chi(K),$$

which depends only on the *boundary* ∂M. Thus $\chi(M, K)$ remains *unchanged* when M is replaced by $M \sharp k Y$ for any closed oriented manifold Y of the same dimension. However, $\chi_{\frac{1}{2}}(M, K)$ is *changed* in general if we do so in odd dimensions ≥ 5.

Lemma 1.8. *Let M be a compact connected oriented $(2m+1)$-manifold. Let Y be a closed connected oriented $(2m+1)$-manifold. Then*

$$(1.6) \qquad \chi_{\frac{1}{2}}(M \sharp k Y) = \chi_{\frac{1}{2}}(M) + k(\chi_{\frac{1}{2}}(Y) - 1).$$

Examples of Y with $\chi_{\frac{1}{2}}(Y) \neq 1$:

(1) Let Σ^n denote an oriented rational homology n-sphere, that is, a closed oriented smooth n-manifold such that $H_j(\Sigma^n, \mathbb{Q}) = H_j(S^n, \mathbb{Q})$ for all j. Some examples are n-spheres, exotic n-spheres, odd-dimensional real projective spaces, lens spaces, oriented quotients of any homotopy n-sphere by a free action of a finite group. For any $2m + 1 \geq 7$, $1 \leq j \leq m - 1$, we have

$$(1.7) \qquad \chi_{\frac{1}{2}}(\Sigma^j \times \Sigma^{2m-j+1}) = 1 + (-1)^j.$$

Note that $S^j \times S^{2m-j+1}$ for $j \geq 2$ is simply-connected. So the operation of the connected sum with $S^j \times S^{2m-j+1}$ for $j \geq 2$ preserves the fundamental group.

(2) For any $2m$-manifold V we have

$$(1.8) \qquad \chi_{\frac{1}{2}}(S^1 \times V) = (-1)^{m-1} b^{m-1}(V).$$

Thus $\chi_{\frac{1}{2}}(S^1 \times V) \neq 1$ unless m is odd and $b^{m-1}(V) = 1$. For example, for $2m+1 \geq 5$ and $4k + 1 \geq 5$, we have

$$(1.9) \qquad \chi_{\frac{1}{2}}(S^1 \times \Sigma^{2m}) = \chi_{\frac{1}{2}}(S^1 \times \mathbb{C}P^{2k}) = 0.$$

Now we state the following main theorem about necessary conditions for the existence of a nearly-self-dual proper vector field. It will be proved in the final section by using self-duality cohomology groups. Let $N^c \stackrel{def}{=} \partial M - N$ and $K^c \stackrel{def}{=} \partial M - K$.

Theorem 1.9. *Let M be a compact connected oriented smooth $(2m + 1)$-manifold M with nonempty boundary and N be the union of some components of ∂M. Suppose M admits a nearly-self-dual proper vector field outward on N and inward on $\partial M - N$. Then*

$$(1.10)$$
$$(-1)^m \chi_{\frac{1}{2}}(M, K) \leq \frac{(-1)^m}{4}[\chi(\partial M) - 2\chi(K)] - \frac{1}{2}|\sigma(N \cap K) + \sigma(N^c \cap K^c)|.$$

for any union K of components of ∂M such that $K \neq \partial M$. Here the signature σ is with respect to the outward orientation. In particular, we have

$$(1.11) \qquad (-1)^m \chi_{\frac{1}{2}}(M) \leq \frac{(-1)^m}{4} \chi(\partial M) - \frac{1}{2}|\sigma(N)|.$$

For convenience in applications, we restate Theorem 1.9 separately in the case of odd m and in the case of even m.

Theorem 1.10. *Let M be a compact connected orientable smooth $(4k+3)$-manifold M with nonempty boundary and with integer $k \geq 0$. Suppose M admits a nearly-self-dual proper vector field V. Then*

$$(1.12) \qquad \chi_{\frac{1}{2}}(M, K) \geq \frac{1}{4}\chi(\partial M) - \frac{1}{2}\chi(K)$$

for any union K of boundary components of M such that $K \neq \partial M$. In particular,

$$(1.13) \qquad\qquad \chi_{\frac{1}{2}}(M) \geq \frac{1}{4}\chi(\partial M).$$

Theorem 1.11. *Let M be a compact connected oriented smooth $(4k+1)$-manifold M with nonempty boundary and with integer $k \geq 1$. Suppose M admits a nearly-self-dual proper vector field V which is outward on N and inward on $\partial M - N$, where N is a union of some boundary components of M (N may be empty). Then*

$$(1.14) \qquad \chi_{\frac{1}{2}}(M, K) \leq \frac{1}{4}\chi(\partial M) - \frac{1}{2}\chi(K) - \frac{1}{2}|\sigma(N \cap K) + \sigma(N^c \cap K^c)|$$

for any union K of boundary components of M such that $K \neq \partial M$. In particular,

$$(1.15) \qquad\qquad \chi_{\frac{1}{2}}(M) \leq \frac{1}{4}\chi(\partial M) - \frac{1}{2}|\sigma(N)|$$

Note that the right sides of the inequalities in Theorems 1.9, 1.10 and 1.11 depend only on the *boundary* ∂M, while the left sides depend on the *whole manifold* M.

To make comparisons, we recall the classical Hopf Theorem (in its simplest version) states that there is a nowhere zero smooth vector field on a closed smooth manifold M if and only if $\chi(M) = 0$. The following is the relative version of this theorem.

Theorem 1.12 (Relative Hopf Theorem). *Let M be a compact connected oriented smooth n-manifold with boundary ∂M. Let N be the union of some boundary components of M (N may be empty). Then there is a proper vector field which is outward on N and inward on $\partial M - N$ if and only if $\chi(M, N) = 0$. In the case $dim M$ is odd, $\chi(M, N) = 0$ is equivalent to $\chi(N) = \chi(\partial M - N)$.*

This relative version can be proved in the same way as the original Hopf Theorem on closed smooth manifolds. (1) The sum of indices of isolated singularities of vector fields outward on N and inward on $\partial M - N$ is independent of such vector fields. (This follows from the facts that the indices are degrees of the Gauss maps and degrees of maps are homotopy invariant.) (2) There is a Morse function on M which is 0 on N and 1 on $\partial M - N$. The index of the gradient vector field of the Morse function at a critical point is $(-1)^\lambda$ where λ is the Morse index of the critical point. The sum of them is $\chi(M, N)$. For the converse statement, take any vector field which points outward on N and inward on $\partial M - N$ and has only isolated singularities. First move all the isolated singularities into a coordinate chart by a one-parameter of diffeomorphisms which preserve a collar of ∂M. Secondly perturb the vector field so that all indices are either $+1$ or -1. Thirdly join two singularities of degrees $+1$ and -1 by disjoint arcs. (In dimension 2, use an easy surgery to achieve this.) Finally isotope the vector field near the arc to make both singularities disappear. For details of these steps see (Hirsch [14], pp. 136-137, p. 161).

The second statement in Theorem 1.12 follows from the fact that $\chi(M) = \frac{1}{2}\chi(\partial M)$ for any odd-dimensional compact oriented smooth manifold M with boundary ∂M.

Remark 1.13. (1) If one regards a nearly-self-dual (or self-dual) vector field as a *geometric structure* in odd dimensions, just as one regards a nondegenerate closed 2-form as a symplectic structure in even dimensions, then one may compare the non-stable phenomena of the existence of nearly-self-dual proper vector fields in

odd dimensions ≥ 5 with the non-stable phenomena in the differential topology in dimension 4. For example, the connected sum of two closed smooth 4-manifolds with $b_+^2 > 1$ doesn't admit any Kähler structure (a theorem of Donaldson, [9]). In fact, such a connected sum doesn't even admit any symplectic structure (a theorem of Taubes, [20]). These results in dimension 4 are proved by using the (anti-self-dual) Yang-Mills equation and the Seiberg-Witten equation, which linearizations are essentially the self-duality operator in dimension 4 (and the Dirac operator). Our proof for Theorem 1.5 is based on an analogue of the self-duality operator in odd dimensions. Of course, the Yang-Mills equation and Seiberg-Witten equation are much deeper than their linearized equations. On the other hand, the Yang-Mills equation and Seiberg-Witten equation in dimension 4 have been generalized to dimension 5 in [11] and [12]. The consequences of these nonlinear equations on 5-manifolds remain to be investigated further.

(2) For any contact structure on an odd-dimensional smooth manifold, there is a naturally associated vector field, which is called the *characteristic vector field* of the contact structure. The characteristic vector field of any K-contact structure is Riemannian. A special class of K-contact structures are so-called Sasakian structures, which are regarded as the odd-dimensional analogue of Kähler structures in even dimensions. Much work has been done about the relations between the underlying topology and Sasakian structures and K-contact structures. (See Blair [6], pp.64-80.) It is natural to hope these studies can be extended to *self-dual contact structures*, that is, the contact structures whose characteristic vector fields are self-dual.

(3) In the case $(M; N_0, N_1)$ is an oriented compact smooth *rational homology h-cobordism* triad, i.e., $H_j(M, N_i; \mathbb{Q}) = 0$ for $i = 0, 1$ and all j, then *all inequalities in Theorems 1.10 and 1.11 are satisfied* with $N = N_1$. Indeed in such a case we have $b^j(M) = b^j(N_1) = b^j(N_0)$ for all j and thus $\chi_{\frac{1}{2}}(M, N_1) = \chi_{\frac{1}{2}}(M, N_0) = 0$ and $\chi(N_1) = \chi(N_0) = \frac{1}{2}\chi(\partial M)$. Moreover when $dim M = 4k + 3$, we have $\chi_{\frac{1}{2}}(M) = \frac{1}{4}\chi(\partial M) + \frac{1}{2}b^{2k+1}(N)$. When $dim M = 4k + 1$, we have $\chi_{\frac{1}{2}}(M) = \frac{1}{4}\chi(\partial M) - \frac{1}{2}b^{2k}(N) \leq \frac{1}{4}\chi(\partial M) - \frac{1}{2}|\sigma(N)|$. Due to this fact, it is natural to ask

Question 1.14. Is there any obstruction for the existence of a nearly-self-dual proper vector field V on a compact oriented smooth odd-dimensional h-cobordism such that V is outward on one boundary component and inward on the other boundary component ?

Of course, if $(M; N_0, N_1)$ is an oriented compact smooth s-cobordism triad of dimension > 5, then the s-cobordism Theorem implies that it is a smooth product, and therefore there is an obvious self-dual vector field as required. The most interesting cases are probably in dimensions 3 and 5, for the relation of this problem to low-dimensional topology may turn out to be important.

When $dim M = 3$, V being self-dual is equivalent to V being conformal, and is also equivalent to the fact that V is transversely holomorphic. Transversely holomorphic vector fields which are transverse to some codimension-one foliation on a closed oriented 3-manifold are classified by Brunella and Ghys [7].

In this paper we can deduce the following result about nearly-self-dual vector fields on 3-manifolds from Theorem 5.6. Its full proof will be presented in the final section.

Theorem 1.15. *Let M be a compact connected orientable 3-manifold with $k > 0$ many boundary components. If $k \neq 2$ and one of the boundary components of M is not torus, then M doesn't admit any nearly-self-dual proper vector field.*

As an example of Theorem 1.15, a compact connected orientable 3-manifold M with three boundary components S^2, S^2 and T^2 (a torus) always admits a proper vector field which is outward on one S^2 and inward on the other S^2 and T^2. But it never admits any nearly-self-dual proper vector field.

In dimension 4, the Yang-Mills equation and the Seiberg-Witten equation can be applied to get lower bounds for the genus of smoothly imbedded surfaces in smooth 4-manifolds. (Kronheimer and Mrowka [15], [16], Morgan, Szabo and Taubes [18].) The self-duality operator in odd-dimensions can also be applied to get lower bounds of *middle-dimensional Betti numbers* and $b_{\pm}$-*invariants* of a smoothly imbedded codimension-one closed manifold which is transverse to a self-dual proper vector field.

Recall for any closed oriented $2m$-manifold K with even m, $b_+^m(K)$ ($b_-^m(K)$ resp.) is the dimension of maximal positive (negative resp.) subspace for the intersection form on $H^m(K)$. For any closed oriented $2m$-manifold K with odd m, $b_{\pm}^m(K) \overset{def}{=} \frac{1}{2}b^m(K)$. The following theorem follows immediately from Theorem 5.7, which will be proved by using self-duality cohomology groups. (See Remark 5.8.)

Theorem 1.16. *Let M be a compact connected oriented smooth $(2m+1)$-manifold with nonempty boundary and with a nearly-self-dual proper vector field V and integer $m \geq 1$. Let C be a smoothly imbedded closed $2m$-manifold in M which is transverse to V, and let C be so oriented that $V \wedge$ orientation of C agrees with the orientation of M. Let N denote the union of components where V is outward. Suppose either (1) C is a part of the boundary of M or (2) V is self-dual. Then we have*

$$(1.16) \qquad b_{\pm}^m(C) \geq (-1)^m \chi_{\frac{1}{2}}(M, \partial M) + \frac{(-1)^m}{4}[\chi(\partial M) \pm 2\sigma(N)].$$

Here N and ∂M takes the outward orientation induced from the orientation of M.

Theorems 1.9 and 1.16 can be applied to a closed odd-dimensional manifold. Recall for any closed *oriented* $2m$-manifold C we have

$$(1.17) \qquad\qquad\qquad \chi^{\pm}(C) \overset{def}{=} \frac{1}{2}[\chi(C) \pm \sigma(C)].$$

Let $-C$ denote C with the opposite orientation.

Theorem 1.17. *Let M be a closed, connected oriented smooth $(2m+1)$-manifold M with a nearly-self-dual vector field V and integer $m \geq 1$. Let C be a smoothly imbedded connected closed $2m$-manifold in M which is transverse to V, and let C be so oriented that $V \wedge$ orientation of C agrees with the orientation of M.*

(1) In the case C doesn't separate M, we can cut M along C and get a manifold $\hat{M}$ such that $\partial\hat{M} = C \sqcup -C$. Then we have

$$(1.18) \qquad\qquad\qquad (-1)^m \chi^{\pm}(C) \geq (-1)^m \chi_{\frac{1}{2}}(\hat{M}),$$

$$(1.19) \qquad\qquad\qquad 0 \geq (-1)^m \chi_{\frac{1}{2}}(\hat{M}, C),$$

$$(1.20) \qquad\qquad\qquad 0 \geq (-1)^m \chi_{\frac{1}{2}}(\hat{M}, -C),$$

$$(1.21) \qquad b^m_\pm(C) + \frac{(-1)^{m+1}}{2}\chi^\pm(C) \geq (-1)^m \chi_{\frac{1}{2}}(\hat{M}, C \sqcup -C).$$

(2) In the case C separates M, we can cut M along C and get a manifold M_1 with $\partial M_1 = C$ where V is outward, and another manifold M_2 with $\partial M_2 = -C$ where V is inward. Then for $i = 1, 2$, we have

$$(1.22) \qquad\qquad \chi(C) = 0, \qquad \sigma(C) = 0,$$

$$(1.23) \qquad\qquad 0 \geq (-1)^m \chi_{\frac{1}{2}}(M_i),$$

$$(1.24) \qquad\qquad \frac{b^m(C)}{2} \geq (-1)^m \chi_{\frac{1}{2}}(M_i, C).$$

Proof. Inequality (1.18) follows from (1.11) applied to $\hat{M}$ with $N = C$ and (1.17). Inequality (1.19) ((1.20) resp.) follows from (1.10) applied to $\hat{M}$ with $N = C$ and $K = C$ ($K = -C$ resp.). Inequality (1.21) follows from (1.16) applied to $\hat{M}$ and (1.17).

For (1.22), $\chi(C) = 0$ follows from Theorem 1.12. $\sigma(C) = 0$ since C bounds M_1. Inequality (1.23) follows from (1.11) applied to M_i and (1.22) and (1.17). Inequality (1.24) follows from (1.16) applied to M_i and (1.22), where we note that $0 = \sigma(C) = b^m_+(C) - b^m_-(C)$ implies $b^m_+(C) = b^m_-(C) = \frac{b^m(C)}{2}$. $\qquad\square$

Note that the left sides of (1.18), (1.21) and (1.24) depend *only* on C, but the right sides of (1.18), (1.19), (1.20), (1.21), (1.23) and (1.24) depend on the whole M.

2. THE SELF-DUALITY OPERATOR OF A GENERAL LINE FIELD

Let L be any given smooth tangent line field on a smooth $(2m+1)$-manifold M. Recall (1.1) for the definition of Q^*. It can be canonically identified as the dual of the intrinsic quotient bundle TM/L in the following way. Let $[Y]$ denote the equivalent class of Y in TM/L for $Y \in TM$. Because the paring $\langle \omega, [Y] \rangle \overset{def}{=} \langle \omega, Y \rangle$ is well-defined for any $\omega \in Q^*$, it induces an identification Q^* with $Hom_\mathbb{R}(TM/L, \mathbb{R})$ in the obvious way. In the theory of foliations, one usually use the intrinsic quotient bundle. However, in this paper we would like to regard Q^* as a subbundle of T^*M so that the whole calculus of differential forms is at our disposal.

Definition 2.1. Let M be an orientable smooth $(2m+1)$-manifold. By the term data (L, g, o_q) on M, we mean that L is a smooth orientable tangent line field on M, g is a smooth Riemannian metric on M, and o_q is an orientation on the dual codimension-one cotangent plane field $Q^* \overset{def}{=} L^\perp$ over M, where $\perp$ is with respect to the paring of a vector and covector.

For a given data (L, g, o_q), let $g_q \overset{def}{=} g|_{Q^*}$, which is a smooth fiberwise Riemannian metric on Q^*. An orientation on Q^* and a metric on Q^* are called a *quotient-orientation* and a *quotient-metric* respectively.

Let L^* be the g-orthogonal complement of Q^* in T^*M. It is a cotangent line field. Let $g_l \overset{def}{=} g|_{L^*}$, which is a smooth fiberwise Riemannian metric on L^*. Obviously we have $g_l \oplus g_q = g$ and the following orthogonal decomposition:

$$(2.1) \qquad\qquad T^*M = L^* \oplus Q^*$$

A section of $\bigwedge^j Q^*$ is called a *quotient j-form*.

Choose an orientation o on M. Then the orientation o_l on L^* is defined by $o_l \wedge o_q = o$. It induces a unique orientation of L by requiring that $\langle \nu, V \rangle > 0$ for any positively oriented nowhere zero sections ν of L^* and V of L. This orientation on L is still denoted as o_l.

For the sake of brevity we make the following convention throughout this paper.

Convention 2.2. m is a positive integer ≥ 1. When m is *odd*, all forms are complex forms; $\Omega^j(M)$ is the space of sections of $\bigwedge^j T_c^* M$ where $T_c^* M \overset{def}{=} \mathbb{C} \otimes_{\mathbb{R}} T^* M$; $\Omega_q^j(M)$ is the space of sections of $\bigwedge^j Q_c^*$ where $Q_c^* \overset{def}{=} \mathbb{C} \otimes_{\mathbb{R}} Q^*$; E is a complex vector bundle over M; $\Omega^j(E)$ is the space of sections of $\bigwedge^j T_c^* M \otimes_{\mathbb{C}} E$; $\Omega_q^j(E)$ is the space of sections of $\bigwedge^j Q_c^* \otimes_{\mathbb{C}} E$; *dim* means *complex* dimensions; $H^j(M)$ means $H^j(M, \mathbb{C})$; and s is a nonzero complex number.

When m is *even*, all forms are real forms; $\Omega^j(M)$ is the space of sections of $\bigwedge^j T^* M$; $\Omega_q^j(M)$ is the space of sections of $\bigwedge^j Q^*$; E is a real or complex vector bundle over M; $\Omega^j(E)$ is the space of sections of $\bigwedge^j T^* M \otimes_{\mathbb{R}} E$; $\Omega_q^j(E)$ is the space of sections of $\bigwedge^j Q^* \otimes_{\mathbb{R}} E$; *dim* means *real* dimensions; $H^j(M)$ means $H^j(M, \mathbb{R})$; and s is a nonzero real or complex number according to the bundle E is real or complex.

To make notations simpler, we omit mention of E in the notations such as $dim \mathcal{H}_+^m(A, K)$ when the bundle E is fixed in our discussions. All forms and metrics in this paper are smooth, here the word "smooth" means class C^k where k is sufficiently large.

For those readers who have read [10], it should be reminded that the notations in this paper are different from those in [10]. The correspondence is as follows: L, L^*, o_l, g_l, Q^*, o_q, g_q and Γ_V in this paper correspond to H, H^*, o_h, g_h, L^*, o_l, g_l and T_V in [10] respectively; quotient-metrics, quotient-orientations and quotient-forms in this paper correspond to level-metrics, level-orientations and level-forms in [10] respectively. We change notations of [10] because it seems more appropriate to use L to denote a tangent line field and Q^* to denote the dual cotangent plane field of L, since Q^* can be canonically identified as the dual of the intrinsic quotient bundle TM/L of L as mentioned in the beginning of this section.

Now from (2.1) we have

$$
(2.2) \quad
\begin{aligned}
\bigwedge^j T^* M &= L^* \wedge \bigwedge^{j-1} Q^* \oplus \bigwedge^j Q^*, \\
\Omega^j(M) &= \overline{L^*} \wedge \Omega_q^{j-1}(M) \oplus \Omega_q^j(M), \\
\Omega^j(E) &= \overline{L^*} \wedge \Omega_q^{j-1}(E) \oplus \Omega_q^j(E),
\end{aligned}
$$

where $\overline{L^*}$ denotes the space of all smooth sections of L^*. Let π_l be the projection to the first factor and π_q be the projection to the second factor.

Recall (1.2) that a fiberwise Riemannian metric g_q on Q^* and an orientation o_q on Q^* induce the quotient-star operator

$$
*_q : \bigwedge^j Q^* \to \bigwedge^{2m-j} Q^*.
$$

When m is even we have $*_q^2 = 1$ on $\bigwedge^m Q^*$. When m is odd we have $*_q^2 = -1$ on $\bigwedge^m Q^*$. In such a case, we let $Q_c^* = \mathbb{C} \otimes_{\mathbb{R}} Q^*$ and identify Q^* as $\mathbb{R} \otimes_{\mathbb{R}} Q^*$ by the map

which sends α to $1 \otimes \alpha$. Also we shall write $c \otimes \alpha$ briefly as $c\alpha$ for $\alpha \in Q^*$ and $c \in \mathbb{C}$. Q_c^* has an obvious complex structure J defined as $J(\alpha) = i\alpha$ and $J(i\alpha) = -\alpha$ for $\alpha \in Q^*$. We can extend $*_q$ *complex-linearly* to $\bigwedge^j Q_c^* \overset{def}{=} \mathbb{C} \otimes_{\mathbb{R}} \bigwedge^j Q^*$ for all j. The Hermitian quotient-metric g_q^c on Q_c^* is defined as

$$\alpha \wedge *_q \overline{\beta} = g_q^c(\alpha, \beta) vol(g_q),$$

where $\overline{\beta}$ is the complex conjugate of β with respect to the complex structure J. Since $*_q$ extends complex-linearly, we still have $*_q^2 = -1$ on $\bigwedge^m Q_c^*$.

Let $\bigwedge_{\pm}^m Q^*$ be the ± 1-eigenbundle of $*_q$ for even m, and let $\bigwedge_{\pm}^m Q_c^*$ be the $\pm i$-eigenbundle of $*_q$ for odd m.

Let $\Omega_{\pm}^m(M)$ be the space of sections of $\bigwedge_{\pm}^m Q^*$ when m is even, and be the space of sections $\bigwedge_{\pm}^m Q_c^*$ when m is odd.

Let $\Omega_{\pm}^m(E)$ be the space of sections of $\bigwedge_{\pm}^m Q^* \otimes_{\mathbb{R}} E$ when m is even, and be the space of sections $\bigwedge_{\pm}^m Q_c^* \otimes_{\mathbb{C}} E$ when m is odd.

Then we have the following orthogonal decompositions

$$\overset{m}{\bigwedge} Q^* = \overset{m}{\underset{+}{\bigwedge}} Q^* \oplus \overset{m}{\underset{-}{\bigwedge}} Q^* \quad \text{if } m \text{ is even,}$$

(2.3)
$$\overset{m}{\bigwedge} Q_c^* = \overset{m}{\underset{+}{\bigwedge}} Q_c^* \oplus \overset{m}{\underset{-}{\bigwedge}} Q_c^* \quad \text{if } m \text{ is odd,}$$

$$\Omega_q^m(M) = \Omega_+^m(M) \oplus \Omega_-^m(M),$$
$$\Omega_q^m(E) = \Omega_+^m(E) \oplus \Omega_-^m(E).$$

Here note that $\Omega_q^m(M)$ and $\Omega_q^m(E)$ have been defined in Convention 2.2. The sections of $\bigwedge_+^m Q^*$ and $\bigwedge_+^m Q_c^*$ are called *self-dual quotient* m-forms, and those of $\bigwedge_-^m Q^*$ and $\bigwedge_-^m Q_c^*$ called *anti-self-dual quotient* m-forms. Let π_q^+ and π_q^- denote the projection to the first and second factor in (2.3) respectively. Define

(2.4)
$$\Omega_{++}^m(E) \overset{def}{=} \overline{L^*} \wedge \Omega_q^{m-1}(E) \oplus \Omega_+^m(E),$$
$$\Omega_+^{m+1}(E) \overset{def}{=} \overline{L^*} \wedge \Omega_+^m(E).$$

Let π^{++} be the projection from $\Omega^m(E)$ to $\Omega_{++}^m(E)$. Let π^+ be the projection from $\Omega^{m+1}(E)$ to $\Omega_+^{m+1}(E)$. (Note the notational difference between the terms π^+ and π_q^+.) Define

(2.5)
$$d_A^{++} \overset{def}{=} \pi^{++} d_A, \qquad d_A^+ \overset{def}{=} \pi^+ d_A.$$

Now we want to introduce a parameter s into these operators. The reason to introduce the parameter s is as follows. Computations show that when $|s|^2 = 2$ the symbol of the self-duality operator in Definition 2.3 defines a Clifford module structure. (See Proposition 6.6 and Lemma 6.7.) But in relevant papers [11] and [12] where we need to quote results from this paper, we find it is more natural to work with $s = 1$. So we would like to work with a general nonzero parameter s in this paper and define

(2.6)
$$d_A^{++s} \overset{def}{=} l_s d_A^{++}, \qquad d_A^{+s} \overset{def}{=} d_A^+ h_s,$$

where l_s and h_s are automorphisms of $\Omega^m_{++}(E)$ given by $l_s(\theta + \alpha) = \theta + s\alpha$ and $h_s(\theta + \alpha) = s\theta + \alpha$, where $\theta \in \overline{L^*} \wedge \Omega^{m-1}_q(E)$, $\alpha \in \Omega^m_+(M)$, and s is a nonzero real or complex number. Obviously we have $l_s h_s = h_s l_s = s$ on $\Omega^m_{++}(E)$.

Given a Riemannian or Hermitian fiberwise metric on E, we can define formal adjoints $(d_A^{++s})^*$ and $(d_A^{+s})^*$ in the usual fashion. It is easy to see

$$(2.7) \qquad (d_A^{++s})^* = d_A^* l_{\bar{s}}, \qquad (d_A^{+s})^* = h_{\bar{s}} \pi^{++} d_A^*,$$

where $\bar{s}$ is the conjugate of s.

Now we are ready to construct the self-duality operator. Let

$$(2.8) \qquad \Omega^{od}_+(E) \overset{def}{=} \sum_{j=0}^{\frac{m-3}{2}} \Omega^{2j+1}(E) \oplus \Omega^m_{++}(E) \quad \text{for odd } m;$$

$$(2.9) \qquad \Omega^{ev}_+(E) \overset{def}{=} \sum_{j=0}^{\frac{m-1}{2}} \Omega^{2j}(E) \oplus \Omega^{m+1}_+(E) \quad \text{for odd } m;$$

$$(2.10) \qquad \Omega^{od}_+(E) \overset{def}{=} \sum_{j=0}^{\frac{m-2}{2}} \Omega^{2j+1}(E) \oplus \Omega^{m+1}_+(E) \quad \text{for even } m;$$

$$(2.11) \qquad \Omega^{ev}_+(E) \overset{def}{=} \sum_{j=0}^{\frac{m-2}{2}} \Omega^{2j}(E) \oplus \Omega^m_{++}(E) \quad \text{for even } m.$$

Let ω_j denote forms of degree j. All forms of negative degree are defined to be zero.

Definition 2.3. Let M be a smooth orientable $(2m+1)$-manifold. The self-duality operator T_A, associated to data (L, g, o_q) in Definition 2.1 and a connection A on a vector bundle E, is defined as

$$T_A : \Omega^{od}_+(E) \to \Omega^{ev}_+(E),$$

$$T_A\Big(\sum_{j=0}^{\frac{m-3}{2}} \omega_{2j+1} + \omega_m\Big) = \sum_{j=0}^{\frac{m-3}{2}} (d_A \omega_{2j-1} + d_A^* \omega_{2j+1})$$
$$+ d_A \omega_{m-2} + (d_A^{++s})^* \omega_m + d_A^{+s} \omega_m \text{ for odd } m;$$

$$T_A\Big(\sum_{j=0}^{\frac{m-2}{2}} \omega_{2j+1} + \omega_{m+1}\Big) = \sum_{j=0}^{\frac{m-2}{2}} (d_A \omega_{2j-1} + d_A^* \omega_{2j+1})$$
$$+ d_A^{++s} \omega_{m-1} + (d_A^{+s})^* \omega_{m+1} \text{ for even } m.$$

It is easy to see its formal adjoint is

$$T_A^* : \Omega^{ev}_+(E) \to \Omega^{od}_+(E),$$

$$T_A^*\Big(\sum_{j=0}^{\frac{m-1}{2}} \omega_{2j} + \omega_{m+1}\Big) = \sum_{j=0}^{\frac{m-3}{2}} (d_A \omega_{2j} + d_A^* \omega_{2j+2})$$
$$+ d_A^{++s} \omega_{m-1} + (d_A^{+s})^* \omega_{m+1} \text{ for odd } m,$$

$$T_A^*(\sum_{j=0}^{\frac{m-2}{2}} \omega_{2j} + \omega_m) = \sum_{j=0}^{\frac{m-4}{2}} (d_A\omega_{2j} + d_A^*\omega_{2j+2})$$
$$+ d_A\omega_{m-2} + (d_A^{++s})^*\omega_m + d_A^{+s}\omega_m \text{ for even } m.$$

Define $T = T_A$ for $d_A = d$, when we replace E by M in the above.

Remark 2.4. Although we use o_l to define T_A in Definitions 2.3, the o_l and $-o_l$ give the same T_A because $*$ appears twice in d_A^*. Thus the o_l is *not* a piece of data for the definition of T_A.

For both even m and odd m, let

(2.12) $$\Omega_+^{total}(E) \overset{def}{=} \bigoplus_{j=0}^{m-1} \Omega^j(E) \oplus \Omega_{++}^m(E) \oplus \Omega_+^{m+1}(E),$$

(2.13) $$\overline{T}_A \overset{def}{=} T_A \oplus T_A^* : \Omega_+^{total}(E) \to \Omega_+^{total}(E).$$

Explicitly,

$$\overline{T}_A(\sum_{j=0}^{m-1} \omega_j + \omega_m + \omega_{m+1})$$
$$= \sum_{j=0}^{m-2} (d_A\omega_{j-1} + d_A^*\omega_{j+1}) + d_A\omega_{m-2} + (d_A^{++s})^*\omega_m$$
$$+ d_A^{++s}\omega_{m-1} + (d_A^{+s})^*\omega_{m+1} + d_A^{+s}\omega_m$$
$$= \sum_{j=0}^{m-2} (d_A\omega_{j-1} + d_A^*\omega_{j+1}) + d_A\omega_{m-2} + d_A^* l_{\bar{s}}\omega_m$$
$$+ l_s\pi^{++}d_A\omega_{m-1} + h_{\bar{s}}\pi^{++}d_A^*\omega_{m+1} + \pi^+ d_A h_s\omega_m.$$

A Riemannian metric g is called *L-admissible* if $L \perp_g \partial M$ with respect to g.

Lemma 2.5. *If a tangent line field L on a smooth manifold M is smooth and transverse to ∂M at any point of ∂M, then L-admissible smooth Riemannian metrics always exist on M.*

Indeed, we can choose a smooth Riemannian metric $g_{\partial M}$ on the boundary ∂M and a smooth fiberwise Riemannian metric g_l on L over M. Then for any extension metric of $g_{\partial M} \oplus g_l|_{\partial M}$ over M we have $L \perp_g \partial M$.

Now let K be the union of some connected components of ∂M. (It may be empty.) For any data (L, g, o_q) such that $L \perp_g \partial M$, define the boundary operator b as follows:

(2.14) $$b = \pi_q \quad \text{on } K , \qquad b = \pi_l \quad \text{on } \partial M - K.$$

The boundary condition on forms $\omega \in \Omega_+^{total}(E)$ is

(2.15) $$b\omega = 0 \quad \text{on } \partial M.$$

By using Green's formula, one can easily prove that the boundary operator b on $\Omega_+^{total}(E)$ is *self-adjoint*, in the sense that $(T_A\omega, \eta) = (\omega, T_A^*\eta)$ for any $\omega \in \Omega_+^{od}(E)$ and $\eta \in \Omega_+^{ev}(E)$ such that $b\omega = 0$ on ∂M and $b\eta = 0$ on ∂M. We shall review

the definition of $\mathbb{C} - \mathbb{R}_+ - \mathbb{R}_-$ ellipticity in the final section. (Here $\mathbb{C}$ is the field of complex numbers. $\mathbb{R}_+$ and $\mathbb{R}_-$ are the set of positive and negative real numbers respectively.)

Theorem 2.6. *Let T_A be the self-duality operator in Definition 2.3 and let b be the boundary operator in (2.14). Then $(T_A \oplus T_A^*, b)$ is $\mathbb{C} - \mathbb{R}_+ - \mathbb{R}_-$-elliptic and $(T_A \oplus T_A^*)_b$ is self-adjoint for any nonzero number s.*

Theorem 2.6 will be proved in the final section. Now define

$$
\begin{aligned}
H_+^{od}(A,K) &\overset{def}{=} Ker(T_A)_b, & H_+^{ev}(A,K) &\overset{def}{=} Ker(T_A^*)_b \cong CokerT_A, \\
b_+^{od}(A,K) &\overset{def}{=} dimH_+^{od}(A,K), & b_+^{ev}(A,K) &\overset{def}{=} dimH_+^{ev}(A,K).
\end{aligned}
\tag{2.16}
$$

Here recall that dim refers to complex dimensions when m is odd and to real dimensions when m is even. Let

$$
Ind(T_A)_b \overset{def}{=} b_+^{od}(A,K) - b_+^{ev}(A,K)
\tag{2.17}
$$

for a data (L, g_q, o_q) as in Definition 2.1 such that $H \perp_g \partial M$ and (E, A).

Convention 2.7. In this paper when $d_A = d$ we replace the term A by M, $A|_K$ by K, and T_A by T in all suitable notations. When K is empty we omit mention of the term K. When the quotient orientation o_q changes to its opposite $-o_q$, we change $+$ signs to $-$ signs in all the corresponding notations.

The most basic formula we obtain in this paper is the index formula for $Ind(T_A)_b$. It has many applications we shall discuss in the subsequent sections. For the statement of this index formula we need to review some notations on even-dimensional manifolds. Let N be a connected, oriented closed Riemannian $2m$-manifold with orientation o_N and let A_1 be a smooth connection on a real or complex vector bundle E_1 over N. When m is odd, a complex m-form ω over N is called *self-dual* if $*_N\omega = i\omega$. When m is even, a real m-form ω over N is called *self-dual* if $*_N\omega = \omega$. Let $\Omega_+^m(N, E_1)$ be the space of E_1-valued complex (real resp.) self-dual m-forms when m is odd (even resp.). Let $\Omega^j(N, E_1)$ be the space of E_1-valued complex (real resp.) j-forms when m is odd (even resp.). Define

$$
\begin{aligned}
& b^j(A_1) \overset{def}{=} dim(Ker\, d_{A_1} \cap Ker\, d_{A_1}^*) \text{ in } \Omega^j(N, E_1) \text{ for } 0 \le j \le m-2, \\
& b_+^{m-1}(A_1) \overset{def}{=} dim(Ker\, \pi_q^+ d_{A_1} \cap Ker\, d_{A_1}^*) \text{ in } \Omega^{m-1}(N, E_1), \\
& b_+^m(A_1) \overset{def}{=} dimKer\, d_{A_1}^* \text{ in } \Omega_+^m(N, E_1).
\end{aligned}
\tag{2.18}
$$

Convention 2.8. When $d_{A_1} = d$, we replace the term A_1 by N. When the orientation of N changes to the opposite, we change $+$ signs to $-$ signs.

Next we need to make clear about the difference between the o_q-orientation and the outward orientation o_{out} on ∂M. The o_q-orientation of ∂M is induced by the orientation o_q on L and $T_*(\partial M) = L|_{\partial M}$. It is well-defined as long as the quotient-orientation o_q on L is given, *even if M is not oriented.* The outward orientation o_{out} of ∂M is defined by requiring that at points of ∂M the outward normal vector $e_{out} \wedge o_{out} = o_M$, the orientation of M.

Let K be the union of some components of ∂M and A be a connection over M. The restriction of A to K is denoted as $A|_K$. Define

$$(2.19) \qquad \chi_{o_q}^+(A|_K) = \sum_{j=0}^{m-1} (-1)^j b^j(A|_K) + (-1)^m b_+^m(A|_K),$$

$$(2.20) \qquad \chi_{o_q}^-(A|_K) = \sum_{j=0}^{m-1} (-1)^j b^j(A|_K) + (-1)^m b_-^m(A|_K),$$

$$(2.21) \qquad \chi_{o_q}^+(K) = \sum_{j=0}^{m-1} (-1)^j b^j(K) + (-1)^m b_+^m(K),$$

$$(2.22) \qquad \chi_{o_q}^-(K) = \sum_{j=0}^{m-1} (-1)^j b^j(K) + (-1)^m b_-^m(K),$$

where $+$ and $-$ are with respect to the o_q-*orientation* on K. They make sense *even if M is not oriented.* Note that

$$(2.23) \qquad \chi_{o_q}^+(K) = \frac{1}{2}(\chi(K) + \sigma_{o_q}(K)), \qquad \chi_{o_q}^-(K) = \frac{1}{2}(\chi(K) - \sigma_{o_q}(K)),$$

where χ denotes the Euler characteristic and $\sigma_{o_q}(K)$ denotes the signature of K with the o_q-orientation. If m is even, then $\sigma_{o_q}(K) = b_+^m(K) - b_-^m(K)$, where $b_+^m(K)$ ($b_-^m(K)$ resp.) is the dimension of maximal positive (negative resp.) subspace of the intersection form on $H^m(K)$ where K takes the o_q-orientation. If m is odd, then $\sigma_{o_q}(K) = 0$ and $b_{\pm}^m(K) = \frac{1}{2}b^m(K)$, which is *independent* of orientations of K. Thus

$$(2.24) \qquad \chi_{o_q}^{\pm}(K) = \frac{1}{2}\chi(K) \quad \text{for odd } m.$$

Theorem 2.9. *Let M be a compact orientable smooth $(2m+1)$-manifold with nonempty boundary ∂M. Let T_A be the self-duality operator in Definition 2.3 associated to any given data (L, g, o_q) and any (E, A) and any nonzero number s such that $L \perp_g \partial M$. Let b be the boundary operator in (2.14). Then for all $m \geq 1$ we have*

$$(2.25) \qquad Ind(T_A)_b = \frac{1}{2}[\chi_{o_q}^+(A|_K) - \chi_{o_q}^+(A|_{\partial M - K})],$$

$$(2.26) \qquad Ind\, T_b = \frac{1}{2}[\chi_{o_q}^+(K) - \chi_{o_q}^+(\partial M - K)].$$

In particular for odd $m \geq 1$ we have

$$(2.27) \qquad Ind\, T_b = \frac{1}{4}[\chi(K) - \chi(\partial M - K)].$$

Recall that we use T to denote T_A when $d_A = d$. (2.25) becomes (2.26) when $d_A = d$. Theorem 2.9 will be proved in the final section by using an index theorem ([13], Theorem 1.9.3). Before we are going to discuss the special cases and applications of it, we want to say something more about $\chi_{o_q}^+(A|_K)$. When M is *oriented,*

we let

$$(2.28) \quad \chi^+(A|_K) \overset{def}{=} \sum_{j=0}^{m-1} (-1)^j b^j(A|_K) + (-1)^m b_+^m(A|_K),$$

$$\chi^-(A|_K) \overset{def}{=} \sum_{j=0}^{m-1} (-1)^j b^j(A|_K) + (-1)^m b_-^m(A|_K),$$

where $+$ and $-$ are with respect to the *outnormal orientation* o_{out} on K. To illustrate the difference between $\chi^+_{o_q}(A|_K)$ and $\chi^+(A|_K)$, consider the product case $M = N \times [0,1]$, where N is an oriented closed Riemannian $2m$-manifold with orientation o_N and $[0,1]$ is oriented upward. Since $o_q = o_N = o_{out}$ on $N \times \{1\}$ and $o_q = o_N = -o_{out}$ on $N \times \{0\}$, we have $\chi^+_{o_q}(A|_{\partial M}) = 2\chi^+(A|_N)$, but $\chi^+(A|_{\partial M}) = \chi^+(A|_N) + \chi^-(A|_N)$.

Now we discuss the index formulas (2.25), (2.26) and (2.27) in the cases of 3-manifolds and 5-manifolds.

Let M be a compact orientable 3-manifold with boundary, and let A be a $U(r)$-connection on a $U(r)$-bundle E over M. Let N be a connected component of ∂M. The metric $g_q|_N$ and the orientation o_q on $Q^*|_N = T^*N$ give a complex structure on N by $J = -*_q$ on T^*N. Let D denote $A|_N$ for a moment. Then $\bigwedge^1_+ Q^*_c|_N = T^{0,1}N$ and $\pi^+_q d_D = \overline{\partial}_D$. Thus the Riemann-Roch Theorem gives

$$\chi^+_{o_q}(A|_N) = dim Ker \overline{\partial}_D - dim Coker \overline{\partial}_D = c_1(E|_N) + r(1 - g(N)).$$

For the union K of some components of ∂M, let $c_1(E|_K)$ and $g(K)$ be the sum of $c_1(E|_N)$ and $g(N)$ over all components N of K respectively.

Corollary 2.10. *Let* $(M, L, g, o_q, T_A, b, s)$ *be as in Theorem 2.9 such that* $dim M = 3$ *and* A *is a* $U(r)$-*connection on a* $U(r)$-*bundle* E *over* M. *Then*

$$(2.29) \quad \begin{aligned} Ind(T_A)_b = &\frac{1}{2} c_1(E|_K) - \frac{1}{2} c_1(E|_{\partial M - K}) \\ &+ \frac{1}{2} r[b^0(K) - g(K) - b^0(\partial M - K) + g(\partial M - K)]. \end{aligned}$$

Now suppose M is a compact orientable smooth 5-manifold with boundary. Let G be a compact semi-simple Lie group, A be a G-connection on a principal G-bundle P over M, and adP be the adjoint bundle of Lie algebras associated to the P. By (2.19), $-\chi^+_{o_q}(A|_N)$ is just the virtual dimension of the moduli space of irreducible *anti*-self-dual G-connections on $P|_N$, where N is with o_q-*orientation*. Thus (Atiyah, Hitchin and Singer [5], Theorem 6.1) gives

$$(2.30) \quad -\chi^+_{o_q}(A|_N) = -p_1(adP|_N) - \frac{1}{2} dim G[\chi(N) + \sigma_{o_q}(N)]$$

For the union K of some components of ∂M, let $p_1(adP|_K)$, $c_2(E|_K)$ and $p_1(E|_K)$ be the sum of $p_1(adP|_N)$, $c_2(E|_N)$ and $p_1(E|_N)$ over all components N of K. We have

Corollary 2.11. *Let* $(M, L, g, o_q, T_A, b, s)$ *be as in Theorem 2.9 such that* $dim M = 5$ *and* A *is a connection on a principal* G-*bundle* P *over* M, *where* G *is a compact*

semi-simple group. Then

$$(2.31) \qquad Ind(T_A)_b = -\frac{1}{2}p_1(adP|_{\partial M - K}) + \frac{1}{2}p_1(adP|_K)$$
$$-\frac{1}{4}dimG[\chi(\partial M - K) + \sigma_{o_q}(\partial M - K) - \chi(K) - \sigma_{o_q}(K)].$$

In particular if $G = SU(2)$ and E is the complex bundle associated to P and the standard representation $\mathbb{C}^2$ of $SU(2)$, then

$$(2.32) \qquad Ind(T_A)_b = 4c_2(E|_{\partial M - K}) - 4c_2(E|_K)$$
$$-\frac{3}{4}[\chi(\partial M - K) + \sigma_{o_q}(\partial M - K) - \chi(K) - \sigma_{o_q}(K)].$$

If $G = SO(3)$ and E is the real bundle associated to P and the standard representation $\mathbb{R}^3$ of $SO(3)$, then

$$(2.33) \qquad Ind(T_A)_b = -p_1(E|_{\partial M - K}) + p_1(E|_K)$$
$$-\frac{3}{4}[\chi(\partial M - K) + \sigma_{o_q}(\partial M - K) - \chi(K) - \sigma_{o_q}(K)].$$

Corollary 2.12. *If $(M; N_0, N_1)$ is a compact connected oriented smooth 5-dimensional rational homology h-cobordism triad such that $\partial M = N_1 \sqcup N_0$ as oriented manifolds and the L points outward on N_1 and inward on N_0,*

$$(2.34) \qquad Ind(T_A)_b = -p_1(adP|_{N_1}) - \frac{1}{2}dimG[\chi(N_1) + \sigma(N_1)].$$

Indeed, $o_q = o_{out}$ on N_1 and $o_q = -o_{out}$ on N_0. Thus $\sigma_{o_q}(\partial M) = \sigma(N_1) - \sigma(N_0) = 2\sigma(N_1)$ and $p_1(adP|_{\partial M}) = 2p_1(adP|_{N_1})$. Note also that since $(M; N_0, N_1)$ is a rational homology h-cobordism, we have $\chi(\partial M) = \chi(N_0) + \chi(N_1) = 2\chi(N_1)$.

Remark 2.13. $Ind(T_A)_b$ is the virtual dimension of the moduli space of anti-self-dual G-connections on P over M studied in [11]. In the situation as in Corollary 2.12, it is the *same* as the dimension of the moduli space of anti-self-dual G-connections on $P|_{N_1}$ over the 4-manifold N_1, which is the basic object in the Donaldson theory of 4-manifolds. It should be important to study the *degree* of the restriction mapping from the moduli space of P over M to the moduli space of $P|_{N_1}$ over N_1.

3. An application to nowhere zero vector fields

An immediate consequence of the index formula in Theorem 2.9 is as follows.

Corollary 3.1. *Let M be a compact oriented smooth $(2m + 1)$-manifold with nonempty boundary ∂M and N be the union of some components of ∂M. If M admits a nowhere zero smooth vector field pointing outward on N and inward on $\partial M - N$, then $\chi(N) \equiv \chi(\partial M - N) \pmod{4}$.*

Proof. Give an orientation on M and let N be the union of the boundary components of M where L points outward. Let $K^c \overset{def}{=} \partial M - K$ and $N^c \overset{def}{=} \partial M - N$. Then $o_q = o_{out}$ on N and $o_q = -o_{out}$ on $\partial M - N$. Thus we have $\sigma_{o_q}(N) = \sigma(N) = \sigma_{o_q}(N^c)$. Note that $\sigma(N) + \sigma(N^c) = \sigma(\partial M) = 0$. So we have

$$\sigma(N \cap K^c) + \sigma(N^c \cap K^c) + \sigma(N \cap K) + \sigma(N^c \cap K) = 0.$$

Since

$$\sigma_{o_q}(K^c) = \sigma(N \cap K^c) - \sigma(N^c \cap K^c), \quad \sigma_{o_q}(K) = \sigma(N \cap K) - \sigma(N^c \cap K),$$

we have $\sigma_{o_q}(K) - \sigma_{o_q}(K^c) = 2\sigma(N \cap K) + 2\sigma(N^c \cap K^c)$. So by (2.23) we get

$$(3.1) \quad \frac{1}{2}(\chi_{o_q}^+(K) - \chi_{o_q}^+(K^c)) = \frac{1}{4}[\chi(K) - \chi(K^c) + 2\sigma(N \cap K) + 2\sigma(N^c \cap K^c)].$$

Note that $\chi(K) - \chi(K^c) = 2\chi(K) - \chi(\partial M)$, $\chi(K) \equiv \sigma(K) \pmod 2$ and $\chi(N) \equiv \sigma(N) \pmod 2$. Therefore when $mod \, 4$, the following identity holds:

$$\begin{aligned}
0 &\equiv 2\chi(K) - \chi(\partial M) + 2\sigma(N \cap K) + 2\sigma(N^c \cap K^c) \\
&\equiv 2\sigma(K) - 2\sigma(N^c \cap K^c) + 2\sigma(N \cap K) - \chi(\partial M) \\
&\equiv 2\sigma(N \cap K^c) + 2\sigma(N \cap K) - \chi(\partial M) \\
&= 2\sigma(N) - \chi(\partial M) \equiv \chi(N) - \chi(\partial M - N).
\end{aligned}$$

According to Theorem 2.9, the right side of (3.1) must be an integer. Thus we have $\chi(N) \equiv \chi(\partial M - N) \pmod 4$. $\qquad\square$

Corollary 3.1 can be regarded as "*mod* 4" version of the necessity part of Relative Hopf Theorem 1.12 in odd-dimensions, which we shall discuss in Section 3. While the classical proof of Relative Hopf Theorem 1.12 uses essentially homology theory, the proof of Corollary 3.1 in this paper only uses index theory. This is similar to the method in (Atiyah [2]).

4. The self-duality cohomology groups of a general line field

Definition 4.1. Let $0 \le j \le m - 2$. Define

$\mathcal{H}^j(A, K)$ as the space of $\omega \in Ker\, d_A \cap Ker\, d_A^* \subset \Omega^j(E)$ with $b\omega = 0$ on ∂M,

$\mathcal{H}_+^{m-1}(A, K)$ as the space of $\omega \in Ker\, d_A^{++s} \cap Ker\, d_A^* \subset \Omega^{m-1}(E)$ with $b\omega = 0$ on ∂M,

$\mathcal{H}_{+s}^m(A, K)$ as the space of $\omega \in Ker\, d_A^{+s} \cap Ker(d_A^{+s})^* \subset \Omega_{++}^m(E)$ with $b\omega = 0$ on ∂M,

$\mathcal{H}_+^{m+1}(A, K)$ as the space of $\omega \in Ker(d_A^{+s})^* \subset \Omega_+^{m+1}(E)$ with $b\omega = 0$ on ∂M.

They are called the *self-duality cohomology groups* associated to (L, g, o_q, o_l). Let $b^j(A, K)$, $b_+^{m-1}(A, K)$, $b_{+s}^m(A, K)$ and $b_+^{m+1}(A, K)$ be $dim\mathcal{H}^j(A, K)$, $dim\mathcal{H}_+^{m-1}(A, K)$, $dim\mathcal{H}_{+s}^m(A, K)$ and $dim\mathcal{H}_+^{m+1}(A, K)$ respectively.

Since $d_A^{++s} = l_s d_A^{++}$ and $(d_A^{+s})^* = h_{\bar{s}}\pi^{++}d_A^*$, we see that $\mathcal{H}_+^{m-1}(A, K)$ and $\mathcal{H}_+^{m+1}(A, K)$ are independent of s.

Condition 4.2. In the case $m = 2$, assume $d_A^{++}d_A = 0$ on $\Omega^0(E)$. In the case $m \ge 3$, assume $F_A = 0$. In the case $m = 1$, A can be any connection.

Note that Condition 4.2 clearly holds if A is a flat connection, in particular if $d_A = d$.

If Condition 4.2 holds, then by using the Green's formula, we have $d_A\omega_{2j-1} \perp d_A^*\omega_{2j+1}$, $d_A\omega_{2j} \perp d_A^*\omega_{2j+2}$ and $d_A\omega_{m-2} \perp (d_A^{++s})^*\omega_m$ for those j's in Definition 2.3. Moreover, $d_A^{++s}\omega_{m-1} = 0$ and $(d_A^{+s})^*\omega_{m+1} = 0$ trivially imply that $d_A^{++s}\omega_{m-1} + (d_A^{+s})^*\omega_{m+1} = 0$. (Note that the converse is not necessarily true in general.) Therefore according to (2.16) and Definitions 2.3 and 4.1, we can have

Proposition 4.3. *If $d_A = d$ or Condition 4.2 holds, then:*

In the case m is odd, we have

$$(4.1) \qquad H_+^{od}(A, K) = \sum_{j=0}^{\frac{m-3}{2}} \mathcal{H}^{2j+1}(A, K) \oplus \mathcal{H}_{+s}^m(A, K),$$

$$(4.2) \qquad H_+^{ev}(A,K) \supset \sum_{j=0}^{\frac{m-3}{2}} \mathcal{H}^{2j}(A,K) \oplus \mathcal{H}_+^{m-1}(A,K) \oplus \mathcal{H}_+^{m+1}(A,K),$$

$$(4.3) \qquad b_+^{od}(A,K) = \sum_{j=0}^{\frac{m-3}{2}} b^{2j+1}(A,K) + b_{+s}^m(A,K),$$

$$(4.4) \qquad b_+^{ev}(A,K) \geq \sum_{j=0}^{\frac{m-3}{2}} b^{2j}(A,K) + b_+^{m-1}(A,K) + b_+^{m+1}(A,K).$$

In the case of m is even, we have

$$(4.5) \qquad H_+^{ev}(A,K) = \sum_{j=0}^{\frac{m-2}{2}} \mathcal{H}^{2j}(A,K) \oplus \mathcal{H}_{+s}^m(A,K),$$

$$(4.6) \qquad H_+^{od}(A,K) \supset \sum_{j=0}^{\frac{m-4}{2}} \mathcal{H}^{2j+1}(A,K) \oplus \mathcal{H}_+^{m-1}(A,K) \oplus \mathcal{H}_+^{m+1}(A,K),$$

$$(4.7) \qquad b_+^{ev}(A,K) = \sum_{j=0}^{\frac{m-2}{2}} b^{2j}(A,K) \oplus b_{+s}^m(A,K),$$

$$(4.8) \qquad b_+^{od}(A,K) \geq \sum_{j=0}^{\frac{m-4}{2}} b^{2j+1}(A,K) \oplus b_+^{m-1}(A,K) \oplus b_+^{m+1}(A,K).$$

The inequalities in Proposition 4.3 become equalities under the following condition, which is stronger than Condition 4.2.

Condition 4.4. In the case $m = 1$, we assume $d_A^{+s} d_A^{++s} = 0$ on $\Omega^0(E)$. In the case $m = 2$, we assume $d_A^{++s} d_A = 0$ on $\Omega^0(E)$ and $d_A^{+s} d_A^{++s} = 0$ on $\Omega^1(E)$. In the case $m \geq 3$, we assume $F_A = 0$ and $d_A^{+s} d_A^{++s} = 0$ on $\Omega^{m-1}(E)$. In addition we assume $L \perp_g \partial M$.

If Condition 4.4 holds, we can easily show that $d_A \omega_{m-2} \perp (d_A^{++s})^* \omega_m$ in Definition 2.3, and thus $d_A \omega_{m-2} + (d_A^{++s})^* \omega_m = 0$ imply $d_A \omega_{m-2} = 0$ and $(d_A^{++s})^* \omega_m = 0$. Therefore, we get

Proposition 4.5. *If Condition 4.4 holds, then the $\supset$'s and $\geq$'s in Proposition 4.3 become equal signs, and for all $m \geq 1$ the following identity holds*

$$(4.9) \qquad \begin{aligned} Ind(T_A)_b &= \sum_{j=0}^{m-2} (-1)^{j+1} b^j(A,K) + (-1)^m b_+^{m-1}(A,K) \\ &\quad + (-1)^{m+1} b_+^m(A,K) + (-1)^m b_+^{m+1}(A,K). \end{aligned}$$

By Theorem 2.6, $(T_A)_b$ is a Fredholm operator for all nonzero numbers s. Indices of Fredholm operators are deformation invariant. Thus *under Condition 4.4,* $b_{+s}^m(A,K)$ is *independent* of s, and can be denoted as $b_+^m(A,K)$.

Theorem 4.6. *Let M be a compact connected orientable smooth $(2m+1)$-manifold with nonempty boundary with data (L, g, o_q) such that $L \perp_g \partial M$. Let K be the union of some components of ∂M. Then*

(1) For any $m \geq 1$, we have

$$(4.10) \qquad \mathcal{H}_+^{m-1}(M, K) \cong H^{m-1}(M, K), \qquad b_+^{m-1}(M, K) = b^{m-1}(M, K).$$

In the case m is odd, we further have

$$(4.11) \qquad H_+^{od}(M, K) = \sum_{j=0}^{\frac{m-3}{2}} \mathcal{H}^{2j+1}(M, K) \oplus \mathcal{H}_{+s}^m(A, K),$$

$$(4.12) \qquad H_+^{ev}(M, K) \supset \sum_{j=0}^{\frac{m-1}{2}} \mathcal{H}^{2j}(M, K) \oplus \mathcal{H}_+^{m+1}(M, K),$$

$$(4.13) \qquad b_+^{od}(M, K) = \sum_{j=0}^{\frac{m-3}{2}} b^{2j+1}(M, K) + b_{+s}^m(M, K),$$

$$(4.14) \qquad b_+^{ev}(M, K) \geq \sum_{j=0}^{\frac{m-1}{2}} b^{2j}(M, K) + b_+^{m+1}(M, K),$$

$$(4.15) \qquad b_+^{od}(M, K) = b_+^{ev}(M, K) + \frac{1}{4}[\chi(K) - \chi(\partial M - K)].$$

In the case m is even, we further have

$$(4.16) \qquad H_+^{ev}(M, K) = \sum_{j=0}^{\frac{m-2}{2}} \mathcal{H}^{2j}(M, K) \oplus \mathcal{H}_{+s}^m(M, K),$$

$$(4.17) \qquad H_+^{od}(M, K) \supset \sum_{j=0}^{\frac{m-2}{2}} \mathcal{H}^{2j+1}(M, K) \oplus \mathcal{H}_+^{m+1}(M, K),$$

$$(4.18) \qquad b_+^{ev}(M, K) = \sum_{j=0}^{\frac{m-2}{2}} b^{2j}(M, K) + b_{+s}^m(M, K),$$

$$(4.19) \qquad b_+^{od}(M, K) \geq \sum_{j=0}^{\frac{m-2}{2}} b^{2j+1}(M, K) + b_+^{m+1}(M, K),$$

$$(4.20) \qquad b_+^{ev}(M, K) = b_+^{od}(M, K) - \frac{1}{2}[\chi_{o_q}^+(K) - \chi_{o_q}^+(\partial M - K)].$$

(2) In the case $K \neq \partial M$, for any $m \geq 1$ we have

$$(4.21) \qquad \mathcal{H}_+^{m+1}(M, K) = 0, \qquad b^{m+1}(M, K) = 0.$$

Proof. According to ([10], Corollary 7.7), we have

$$\mathcal{H}_+^{m+1}(M, K) = 0 \quad \text{and} \quad b_+^{m+1}(M, K) = 0$$

in the case $K \neq \partial M$. According to ([10], Corollary 7.5), we have $\mathcal{H}_+^{m-1}(M, K) = \mathcal{H}^{m-1}(M, K) \cong H^{m-1}(M, K)$ and $b_+^{m-1}(M, K) = b^{m-1}(M, K)$ for any K. Then we can apply $IndT_b \overset{def}{=} b_+^{od}(M, K) - b_+^{ev}(M, K)$ and Theorem 2.9 and Proposition 4.3 to complete the proof. $\square$

By (4.13), (4.14) and (4.15) in the case of odd m and (4.18), (4.19) and (4.20) in the case of even m and (4.21), we have

Corollary 4.7. *Let (M, L, o_q, g, K) be as in Theorem 4.6. If $K \neq \partial M$, then*

$$(4.22) \qquad b_{+s}^m(M, K) \geq \chi_{\frac{1}{2}}(M, K) + \frac{1}{4}[\chi(K) - \chi(\partial M - K)] \quad \text{for odd } m,$$

$$(4.23) \quad b_{+s}^m(M, K) \geq -\chi_{\frac{1}{2}}(M, K) - \frac{1}{2}[\chi_{o_q}^+(K) - \chi_{o_q}^+(\partial M - K)] \quad \text{for even } m.$$

5. Applications to nearly-self-dual line fields II

First we need to give the definition of a nearly-self-dual line field.

Definition 5.1. A tangent line field L on a smooth $(2m+1)$-manifold M is called *nearly-self-dual* if it is smooth and there is a smooth fiberwise Riemannian metric g_q on Q^* and an orientation on Q^* such that $\Gamma_V \overset{def}{=} L_V *_q - *_q L_V = 0$ on $d_q^-\Omega^{m-1}(M) \cup d_q^+\Omega^{m-1}(M)$ for any smooth section V of L and for a splitting $TM = L \oplus Q$ with Q tangent to ∂M. Such a metric g_q on Q^* is called *L-compatible*. A vector field is called *nearly-self-dual* if it is nowhere zero, smooth and the line field generated by the vector field is nearly-self-dual. Here $d_q^\pm = \pi_q^\pm \pi_q d$.

Definition 5.2. A smooth Riemannian metric g on a smooth $(2m+1)$-manifold M with a nearly-self-dual (self-dual resp.) line field L is called *L-compatible* if (1) $L \perp_g \partial M$ at any point of ∂M and (2) the induced metric $g_q \overset{def}{=} g|_{Q^*}$ on Q^* over M is L-compatible in the sense of Definition 5.1 (Definition 1.1 resp.).

Note that when ∂M is empty the condition (1) is automatically satisfied.

Lemma 5.3. *If a tangent line field L on a smooth $(2m+1)$-manifold M is nearly-self-dual (self-dual resp.) and transverse to ∂M at any point of ∂M, then L-compatible smooth Riemannian metrics always exist on M.*

Proof. For nearly-self-dual case it is clear. We only need to prove for self-dual case. According to Lemma 2.5, there is a smooth Riemannian metric g' on M such that $L \perp_{g'} \partial M$ at any point of ∂M. Let $L^* \overset{def}{=} (Q^*)^{\perp_{g'}}$, the g'-orthogonal complement of Q^* in T^*M. Then $T^*M = L^* \oplus Q^*$. Define $g \overset{def}{=} g'|_{L^*} \oplus g_q$, where g_q is a L-compatible smooth fiberwise Riemannian metric on Q^* in Definition 1.1. This g is L-compatible, which is proved as follows. We only need to prove $L \perp_g \partial M$. For any vector $Y \in TM$, let Y_g ($Y_{g'}$ resp.) denote the covector dual to Y with respect to the metric g (g' resp.).

Let $V \in L|_{\partial M}$ and $W \in T(\partial M)$ at the same point in ∂M. By (1.1) and $V \in L$ we have $0 = \langle V, Q^* \rangle = g(V_g, Q^*)$. Thus $V_g \in (Q^*)^{\perp_g} = L^*$.

Since $W \in T(\partial M) \perp_{g'} L$, we have $0 = g'(W, L) = \langle W_{g'}, L \rangle$ and thus $W_{g'} \in Q^* = (L^*)^{\perp_{g'}}$ by (1.1) and definition of L^*. Thus $0 = g'(W_{g'}, L^*) = \langle W, L^* \rangle = g(W_g, L^*)$. So $W_g \in (L^*)^{\perp_g}$, and therefore $0 = g(V_g, W_g) = g(V, W)$ since $V_g \in L^*$, shown in the above.   □

As usual let i_V denote the interior contraction by vector field V. Define

$$(5.1) \qquad \nabla_{A_V} \overset{def}{=} i_V d_A, \qquad F_{A_V} \overset{def}{=} i_V F_A, \qquad d_{A_q} = \pi_q d_A, \qquad d_{A_q}^{\pm} = \pi_q^{\pm} d_{A_q}.$$

When $d_A = d$ we omit mention of A. Now we quote a formula from [10]. It can also be directly checked.

Proposition 5.4 ([10], Proposition 3.4). *Let m be any integer ≥ 1. Let A be any connection on a real or complex vector bundle E. Then*

$$(5.2) \qquad d_A^{+s} d_A^{++s} \omega = s\tau \wedge [\frac{\epsilon}{2} \Gamma_V d_{A_q}^- \omega + \pi_q^+ (F_{A_V} \wedge \alpha) + \pi_q^+ (F_{A_q} \wedge \beta)]$$

for all $\omega \in \Omega^{m-1}(E|_U)$, where V is a section of L over U, τ is a section of L^* over U such that $\langle \tau, V \rangle = 1$, $\omega = \tau \wedge \beta + \alpha$ with $\beta \in \Omega_q^{m-2}(E|_U)$ and $\alpha \in \Omega_q^{m-1}(E|_U)$, $\epsilon = 1$ for even m and $\epsilon = -i$ for odd m.

Note that when $m = 1$, $\pi_q^+ (F_{A_q} \wedge \beta) = 0$ always holds, for $\beta \in \Omega_q^{m-2}(E|_U) = 0$. When $m = 2$, $\pi_q^+ (F_{A_q} \wedge \beta) = (\pi_q^+ F_{A_q})\beta$, since β is a $E|_U$-valued function.

Since $F_A = 0$, $F_{A_V} = 0$ and $F_{A_q} = 0$ for $d_A = d$, by Proposition 5.4, we see that the *conditions in Definition 5.1 imply Condition 4.4* for $d_A = d$, and therefore we get the following theorem (cf. [10], Theorems 1.3 and 2.2):

Theorem 5.5. *If a tangent line field L on a smooth $(2m+1)$-manifold M is nearly-self-dual, then for any L-compatible metric g_q on Q^* and any orientation o_q on Q^* and any nonzero s, the following complex is well-defined*

$$(5.3) \qquad 0 \to \Omega^0(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{m-1}(M) \xrightarrow{d^{++s}} \Omega_{++}^m(M) \xrightarrow{d^{+s}} \Omega_+^{m+1}(M) \to 0.$$

Complex (5.3) and its generalizations are the main subjects first studied in [10].

Theorem 5.6. *Let M be a compact connected orientable smooth $(2m+1)$-manifold with nonempty boundary and with data (L, g, o_q) such that L is nearly-self-dual and transverse to ∂M and g is L-compatible. Then $b_{\pm s}^m(M, K)$ are independent of s, and thus can be denoted as $b_{\pm}^m(M, K)$. Moreover all $\geq$'s and $\supset$'s in Theorem 4.6 and 4.7 become equal. Explicitly for all m:*
(1) In the case $K \neq \partial M$, we have

$$(5.4) \qquad\qquad\qquad b_{\pm}^{m-1}(M, K) = b^{m-1}(M, K),$$

$$(5.5) \qquad\qquad\qquad b_{\pm}^{m+1}(M, K) = 0,$$

$$(5.6) \qquad b_{\pm}^m(M, K) = (-1)^{m+1} \chi_{\frac{1}{2}}(M, K) + \frac{(-1)^{m+1}}{2} [\chi_{o_q}^{\pm}(K) - \chi_{o_q}^{\pm}(\partial M - K)].$$

(2) In the case $K = \partial M$, we have

$$(5.7) \qquad\qquad\qquad b_{\pm}^{m-1}(M, \partial M) = b^{m-1}(M, \partial M),$$

$$(5.8) \quad b_{\pm}^m(M, \partial M) - b_{\pm}^{m+1}(M, \partial M) = (-1)^{m+1} \chi_{\frac{1}{2}}(M, \partial M) + \frac{(-1)^{m+1}}{2} \chi_{o_q}^{\pm}(\partial M).$$

Proof. Since the conditions in Definition 5.1 imply Condition 4.4 for $d_A = d$, we can use the conclusion of Proposition 4.5 for $d_A = d$. Also, the conditions in Definition 5.1 imply the conditions of Theorem 4.6 and Corollary 4.7. In addition, ([10], Corollary 7.5) gives that $b_+^{m-1}(M, K) = b^{m-1}(M, K)$ for all K, including the case $K = \partial M$. Thus all these together imply the above formulas about the dimensions of the self-duality cohomology groups. $\square$

Theorem 5.7. *Let (M, V, C, N) be the same as in Theorem 1.16. Then we have*

$$(5.9) \quad b_\pm^m(C) \geq (-1)^m \chi_{\frac{1}{2}}(M, \partial M) + \frac{(-1)^m}{4}[\chi(\partial M) \pm 2\sigma(N)] + b_+^m(M, \partial M),$$

where $b_+^m(M, \partial M)$ is with respect to some $\langle V \rangle$-compatible smooth Riemannian metric g on M.

Proof. Since $L \perp_g C$ in the case (1) or $L_V *_q = *_q L_V$ on $\wedge^m Q^*$ in the case (2), we can use ([10], Corollary 7.10) to get

$$(5.10) \qquad\qquad b_\pm^{m+1}(M, \partial M) \leq b_\pm^m(C).$$

Using notations in Theorem 5.6, we have $\chi_{o_q}^\pm(\partial M) = \frac{1}{2}(\chi(\partial M) \pm 2\sigma(N))$. Then (5.8) and (5.10) yield (5.9). $\square$

Remark 5.8. Note that Theorem 1.16 follows directly from Theorem 5.7 and Lemma 5.3 and the fact that $b_+^m(M, \partial M) \geq 0$.

The Relative Hopf Theorem 1.12 applied to a 3-manifold yields the following result about existence of proper vector fields on a 3-manifold.

Proposition 5.9. *Let M be a compact connected oriented 3-manifold with k many boundary components.*

(1) In the case $k = 1$, M admits a proper vector field if and only if ∂M is a torus.

(2) In the case $k = 2$, let N_0 and N_1 be two boundary components of M. Then M admits a proper vector field which is outward on N_0 and inward on N_1 if and only if $g(N_0) = g(N_1)$. M admits a proper vector field outward (or inward) on both N_0 and N_1 if and only if $g(N_0) + g(N_1) = 2$.

(3) In the case $k \geq 3$, M admits a proper vector field if and only if there is a union N of boundary components of M such that $\chi(N) = \frac{1}{2}\chi(\partial M)$.

When $dim M = 3$, (5.8) gives

$$(5.11) \qquad\qquad b_+^2(M, \partial M) = b_+^1(M, \partial M) - \frac{1}{4}\chi(\partial M).$$

This and Proposition 5.9 give a lot of information on $b_+^2(M, \partial M)$ and $b_+^1(M, \partial M)$.

Proposition 5.10. *Let (M, L, g, o_q) be as in Theorem 5.6 with $dim M = 3$ and let k be the number of boundary components of M. In the case $k = 2$, we let N_1 and N_0 be two boundary components of M, arranged so that $g(N_0) \leq g(N_1)$. Then one of the following cases must hold:*

(1) $k \neq 2$, $b_+^2(M, \partial M) = b_+^1(M, \partial M) = 0$ or 1.

(2) $k = 2$, $g(N_0) = g(N_1)$, $b_+^2(M, \partial M) = b_+^1(M, \partial M) + g(N_1) - 1$, $b_+^1(M, \partial M) = 0$ or 1, and V is outward on one boundary component and inward on the other boundary component.

(3) $k = 2$, $g(N_0) = 0$, $g(N_1) = 2$, $b_+^2(M, \partial M) = b_+^1(M, \partial M) = 0$, and V is outward (or inward) on N_1 and N_0.

(4) $k = 2$, $g(N_0) = g(N_1) = 1$, $b_+^2(M, \partial M) = b_+^1(M, \partial M) = 0$ *or 1, and V is outward (or inward) on N_1 and N_0.*

Proof. Let N be a boundary component of M. Since $L \perp_g N$ and V is nearly-self-dual, we can use ([10], Corollary 7.10) to get

$$(5.12) \qquad\qquad b_+^2(M, \partial M) \leq b_+^1(N).$$

Recall that $b_\pm^1(N) = g(N)$ for closed oriented 2-manifold N.

If $k \neq 2$, then by Theorem 1.15, we have $1 = g(N) \geq b_+^2(M, \partial M) = b_+^1(M, \partial M) \geq 0$.

Now suppose $k = 2$. In the case where V points outward on N_1 and inward on N_0, we have $g(N_0) = g(N_1)$ and $\chi(\partial M) = 4 - 4g(N_1)$. Thus $g(N_1) \geq b_+^2(M, \partial M) = b_+^1(M, \partial M) - \frac{1}{4}(4 - 4g(N_1)) \geq g(N_1) - 1$. This gives the claim (2).

In the case where V points outward (inward) on N_1 and N_0, we have $g(N_0) + g(N_1) = 2$ and $\chi(\partial M) = 0$. Thus $g(N_0) \geq b_+^2(M, \partial M) = b_+^1(M, \partial M)$. Since $g(N_0) + g(N_1) = 2$ and $g(N_0) \leq g(N_1)$, we have $g(N_0) = 0$ or 1, which yields the claims (3) and (4). □

6. Proofs of Theorems 1.9, 1.15, 2.9 and 2.6

In this section we shall present in order all the proofs of Theorems 1.9, 1.15, 2.9 and 2.6.

I. *Proof of Theorem 1.9 and Theorem 1.15.*

By applying Lemma 5.3 and (5.6) and (3.1), we have

$$0 \leq b_+^m(M, K) = (-1)^{m+1} \chi_{\frac{1}{2}}(M, K)$$
$$+ (-1)^{m+1} \frac{1}{4}[\chi(K) - \chi(K^c) + 2\sigma(N \cap K) + 2\sigma(N^c \cap K^c)],$$

$$0 \leq b_-^m(M, K) = (-1)^{m+1} \chi_{\frac{1}{2}}(M, K)$$
$$+ (-1)^{m+1} \frac{1}{4}[\chi(K) - \chi(K^c) - 2\sigma(N \cap K) - 2\sigma(N^c \cap K^c)].$$

Since $\chi(K) - \chi(K^c) = -\chi(\partial M) + 2\chi(K)$, we get (1.10) and thus complete the proof of Theorem 1.9.

Now we are going to prove Theorem 1.15. Suppose M admitted a nearly-self-dual proper vector field. When $k = 1$, ∂M had to be a torus according to Proposition 5.9. This contradicts to the assumption. Suppose $k \geq 3$. Recall (5.6). Since $m = 1$, we have $\chi_{\frac{1}{2}}(M, K) = 0$, $\chi_{\frac{1}{2}}(M, K^c) = 0$ and σ of any boundary component of M is zero. THus (5.6) yields $0 \leq b_+^1(M, K) = \chi(K) - \chi(K^c)$. Switching K and K^c, we get that $\chi(K) = \chi(K^c)$ holds for any *nonempty* union K of components of ∂M such that $K \neq \partial M$. Let $N_0, N_1, N_2, \ldots, N_{k-1}$ be connected components of ∂M, $k \geq 3$, and be arranged in order: $\chi_0 \leq \chi_1 \leq \ldots \leq \chi_{k-1} \leq 2$, where $\chi_j = \chi(N_j)$.

Since $\chi_0 = \sum_{j=1}^{k-1} \chi_j$ and $\chi_{k-1} = \sum_{j=0}^{k-2} \chi_j$, we get $\chi_0 = \chi_{k-1}$. Thus $\chi_0 = \chi_j$ for all $j = 1, \ldots, k-1$. Therefore $\chi_0 = (k-2)\chi_0$. Since $k \neq 2$, we got $0 = \chi_0 = \chi_j$ for all j. This contradicts to the assumption.

II. *Proof of Theorem 2.9.*

First let us recall an index theorem from [13]. Let F_1 and F_2 be complex vector bundles over M with $dimF_1 = dimF_2$. Let F_1' and F_2' be complex vector bundles over ∂M with $2dimF_i' = dimF_i$, $i = 1, 2$. Let $P : \Gamma(F_1) \to \Gamma(F_2)$ be an elliptic first

order differential operator with a zero-th order boundary operator $B : \Gamma(F_1|_{\partial M}) \to \Gamma(F_1')$. The operator

$$(P, B) : Dom(P) \to Rang(P) \oplus Rang(B)$$

is given by $(P, B)f = (Pf, B(f|_{\partial M}))$. The operator

$$P_B : \ \{f \in Dom(P), \ B(f|_{\partial M}) = 0\} \ \to Rang(P)$$

is defined by $P_B f = Pf$. If B is *surjective* then $IndP_B = Ind(P, B)$. Let P^* be the formal adjoint of P. The adjoint boundary operator $B^* : \Gamma(F_2|_{\partial M}) \to \Gamma(F_2')$ is defined by the requirement that $(Pf, f') = (f, P^*f')$ when $B(f|_{\partial M}) = 0$ and $B^*(f'|_{\partial M}) = 0$. Note $P \oplus P^* : \Gamma(F_1 \oplus F_2) \to \Gamma(F_1 \oplus F_2)$ and $B \oplus B^* : \Gamma(F_1 \oplus F_2|_{\partial M}) \to \Gamma(F_1' \oplus F_2')$. The definitions of $\mathbb{C} - \mathbb{R}_+ - \mathbb{R}_-$ ellipticity will be reviewed later.

Theorem 6.1 ([13], Theorem 1.9.3)**.** *If $(P \oplus P^*, B \oplus B^*)$ is $\mathbb{C} - \mathbb{R}_+ - \mathbb{R}_-$-elliptic and $(P \oplus P^*)_{B \oplus B^*}$ is self-adjoint, then*

$$IndP_B = \int_M a_n(P)vol(M) + \int_{\partial M} a_{n-1}(P, B)vol(\partial M).$$

Moreover, (1) $a_n(P) = 0$ if $n = dimM$ is odd, and (2) $a_{n-1}(P, B)$ is a scalar valued local invariant defined on Y and depends functorially on a finite number of jets of symbols of P and B on the boundary ∂M.

Note that $a_n(P)$ is the interior local invariant for the index of an elliptic operator P on the double of M, and thus *is zero if n is odd.*

For the boundary operator (2.14), let $K = K_1 \sqcup \cdots \sqcup K_l$ and $\partial M - K = N_1 \sqcup \cdots \sqcup N_l$, where all N_j's and K_j's are components of ∂M. Since $dimM = 2m + 1$ is odd, Theorem 6.1 gives

(6.1)
$$\begin{aligned} Ind(T_A)_b &= \int_{\partial M} a_{2m}(T_A, b)vol(\partial M) \\ &= \sum_{j=1}^h \int_{N_j} a_{2m}(T_A, \pi_l)vol(N_j) + \sum_{k=1}^l \int_{K_j} a_{2m}(T_A, \pi_q)vol(K_j). \end{aligned}$$

Thus we only need to determine

$$\int_{N_j} a_{2m}(T_A, \pi_l)vol(N_j) \quad \text{and} \quad \int_{K_j} a_{2m}(T_A, \pi_q)vol(K_j).$$

The key property of $a_{2m}(T_A, b)$ is that it is scalar valued *local invariant* depends *functorially* on a finite number of jets of symbols of T_A and b on the *boundary* of M. The operators T_A and b in Definition 2.3 and (2.14) *depend only on the data* (L, g, o_q, A, E) *and don't depend on the orientation o_l* on L. (See Remark 2.4.) Thus we have

Proposition 6.2. *Given two sets of data (L, g, o_q, A, E) and (L', g', o_q', A', E'), where g is L-admissible and g' is L'-admissible. Suppose C and C' are boundary components of compact oriented $(2m+1)$-manifolds M and M' respectively. If there is an isometry ϕ from a collar of C in M onto a collar of C' in M' such that ϕ maps the data (L, g, o_q, A, E) to the data (L_*', g', o_q', A', E'), then $a_{2m}(T_A, b)$ at point y of C is the same as $a_{2m}(T_A', b')$ at point $\phi(y)$ of C' and*

$$\int_C a_{2m}(T_A, b)vol(C) = \int_{C'} a_{2m}(T_A', b')vol(C').$$

We should emphasize that the above ϕ is permissible to map o_l to *either o'_l or $-o'_l$*.

Now we compute the index in the simplest case. Let N be a connected, oriented closed $2m$-manifold with orientation o_N. Let $M = N \times [0,1]$ with the obvious line field L, i.e., the field of tangent lines of copies of $[0,1]$ in M. $Q^* \stackrel{def}{=} L^\perp$ is the pull-back of T^*N by the projection $\pi_N : M \to N$. The orientation on N gives an orientation o_q on Q^*.

Let M take the orientation $o_M = dt \wedge o_N$, where $t \in [0,1]$. Then $\partial M = N \times \{1\} \sqcup N \times \{0\}$ gets the outward orientation o_{out} by $e_{out} \wedge o_{out} = o_M = dt \wedge o_N$, where e_{out} is a covector field on ∂M pointing out of M. On $N \times \{1\}$ we have $o_q = o_{out}$. On $N \times \{0\}$ we have $o_q = -o_{out}$. We can identify $N \times \{1\}$ with N by $(x,1) \to x$ and identify $N \times \{0\}$ with N^- by $(x,0) \to x$, where N^- is the oppositely oriented N.

Let g_0 be the product of a metric on N and the standard metric on $[0,1]$, and let A_1 be a smooth connection on a real or complex vector bundle E_1 over N. The pull back of A_1 to the bundle $E = \pi_N^* E_1$ over M is called the *pulled-back connection* A of A_1 over N. $d_A = d_{A_1} + \frac{\partial}{\partial t} dt$. For such an A, we have

$$(6.2) \qquad\qquad \pi_l F_A = 0, \qquad F_A = \pi_q F_A = F_{A_1}.$$

Let $*_N$ be the star operator on N. Recall that when $m = 2$, A_1 is called an anti-self-dual connection if $*_N F_{A_1} = -F_{A_1}$.

Lemma 6.3. *Suppose A_1 is a flat smooth connection over N when $m \geq 3$; A_1 is an anti-self-dual smooth connection over N when $m = 2$; and A_1 is any given smooth connection when $m = 1$. Then the pulled-back connection A of A_1 satisfies Condition 4.4.*

Proof. Since $M = N \times [0,1]$, we can let $\tau = dt$ and $V = \frac{\partial}{\partial t}$ with $t \in [0,1]$. Since g is a product metric, we have $\Gamma_V \stackrel{def}{=} L_V *_q - *_q L_V = 0$ on any quotient forms. By using (6.2), we have $F_{A_V} = 0$. Then by using Proposition 5.4 we get $d_A^{+s} d_A^{++s} = 0$ for the A. In addition, by using (6.2), we have $d_A^{+s} d_A = l_s d_A^{++} d_A = l_s(\pi_l F_A + \pi_q^+ F_A) = 0$. $\qquad\square$

It is not hard to compute $b_+^j(A, K)$ in this product case $M = N \times [0,1]$ with the pulled-back connection A of the connection A_1 over N. To avoid repeating work we here quote the results from ([10], Sections 7 and 8) directly.

If K is empty, then $b = \pi_l$ on ∂M. By using ([10], Corollaries 8.6 and 8.11) for the pulled-back A of the connection A_1, we have

$$b^j(A) = b^j(A_1), \quad b_+^{m-1}(A) = b_+^{m-1}(A_1), \quad b_+^m(A) = b_+^m(A_1), \quad b_+^{m+1}(A) = 0$$

for $0 \leq j \leq m - 2$. Therefore (4.9) yields

$$ind(T_A)_{\pi_l} = -\chi_{o_q}^+(A_1),$$

where $+$ refers to the o_q-*orientation* on $N \times \{1\} = N$, which is the same as the given orientation on N. Now (6.1) implies

$$ind(T_A)_{\pi_l} = \int_N a_{2m}(T_A, \pi_l) vol(N) + \int_{N^-} a_{2m}(T_A, \pi_l) vol(N^-).$$

Let ϕ be the isometry from $N \times (\frac{2}{3}, 1]$ to $N \times [0, \frac{1}{3})$ by $(x, t) \to (x, 1 - t)$ which preserves (L, g, o_q, A). By using Proposition 6.2, we get

(6.3)
$$\int_N a_{2m}(T_A, \pi_l) vol(N) = \int_{N^-} a_{2m}(T_A, \pi_l) vol(N^-) = \frac{1}{2} ind(T_A)_{\pi_l} = -\frac{1}{2} \chi_{o_q}^+(A_1).$$

Now let $K = N \times \{1\} = N$. Thus $b = \pi_q$ on $N \times \{1\} = N$ and $b = \pi_l$ on $N \times \{0\} = N^-$. By using ([10], Corollaries 8.6 and 8.11), we get

$$0 = b^j(A, N) = b_+^{m-1}(A, N) = b_+^m(A, N) = b_+^{m+1}(A, N)$$

for all $0 \le j \le m - 2$. Thus

$$ind(T_A, \pi_q) = 0.$$

By (4.9), (6.1) and (6.3), we get

(6.4)
$$\int_N a_{2m}(T_A, \pi_q) vol(N) = -\int_{N^-} a_{2m}(T_A, \pi_l) vol(N^-) = -\frac{1}{2} ind(T_A)_{\pi_l} = \frac{1}{2} \chi_{o_q}^+(A_1).$$

Now by using Proposition 6.2 and (6.1), (6.3) and (6.4), we have

(6.5) $$Ind(T_A)_b = \frac{1}{2} [\chi_{o_q}^+(A|_K) - \chi_{o_q}^+(A|_{\partial M - K})],$$

in the case the restriction of g to a collar of each boundary component of M is isometric to the product metric g_0.

Because indices are deformation invariant, and because any two L-admissible metrics g_0 and g_1 can be deformed to each other by L-admissible metrics $tg_0 + (1 - t)g_1$ and any two connections on the same bundle can be deformed to each other, we see that (6.5), which is the same as (2.25), holds for all data (L, g, o_q) such that g is L-admissible and any connection A on E. Thus the proof of Theorem 2.9 is completed.

III. *Proof of Theorem 2.6.*

First let us recall briefly the definition of $\mathbb{C} - \mathbb{R}_+ - \mathbb{R}_-$ ellipticity of a first order operator with a zero-th order boundary operator from ([13], Section 1.9). Let P be a first order formally self-adjoint elliptic differential operator $\Gamma(F) \to \Gamma(F)$, where F is a complex vector bundle over a manifold M with boundary ∂M and $\Gamma(F)$ is the space of smooth sections of F. Let β be a zero-th order boundary operator $F|_{\partial M} \to F'$ where F' is a complex vector bundle over ∂M with $2dimF' = dimF$. Let $p(x, \xi)$ be the principal symbol of P at (x, ξ) where ξ is a covector at $x \in M$. Near the boundary ∂M, $x = (x_0, x')$ where x' is a point in ∂M and x_0 is the normal distance to the boundary ∂M, and $\xi = (\xi_0, \xi')$ according to the splitting $T^*M = T^*\mathbb{R} \oplus T^*(\partial M)$ induced by the identification of a neighborhood of ∂M in M with a collar $[0, \epsilon) \times \partial M$ for some $\epsilon > 0$. This identification is induced by inward normal geodesic flows. Thus

$$p(x, \xi) = p(x_0, x', \xi_0, \xi').$$

Consider the ordinary differential equation:

$$p(0, x', \xi', -i\frac{d}{dt})f(t) = \lambda f(t), \qquad lim_{t \to \infty} f(t) = 0,$$

where $(\xi', \lambda) \ne (0, 0) \in T^*(\partial M) \times (\mathbb{C} - \mathbb{R}_+ - \mathbb{R}_-)$. The boundary problem (P, β) is called $\mathbb{C} - \mathbb{R}_+ - \mathbb{R}_-$ *elliptic* if (1) $det(p(x, \xi) - \lambda) \ne 0$ for all $x \in M - \partial M$ and

$(\xi, \lambda) \neq (0,0) \in T^*M \times (\mathbb{C} - \mathbb{R}_+ - \mathbb{R}_-)$, and (2) there is a unique solution $f(t)$ to the above ordinary differential equation such that $\beta f(0) = f'$ for any given $f' \in F'$.

The symbol $p(x, \xi)$ can be expressed uniquely as follows.

$$p(x, \xi) = p(x_0, x', \xi_0, \xi') = p_0(x')\xi_0 + p(0, x', 0, \xi').$$

Define

(6.6) $$\tau(x', \xi', \lambda) \overset{def}{=} ip_0(x')^{-1}(p(x', 0, \xi', 0) - \lambda).$$

Lemma 6.4 ([13], Lemma 1.9.4). *If $p(x, \xi)$ is self-adjoint and elliptic, then $\tau(x', \xi', \lambda)$ has no eigenvalue of zero real part for $(\xi', \lambda) \in T^*(\partial M) \times (\mathbb{C} - \mathbb{R}_+ - \mathbb{R}_-)$ and $(\xi', \lambda) \neq (0, 0)$.*

Therefore, over $T^*(\partial M) \times (\mathbb{C} - \mathbb{R}_+ - \mathbb{R}_-)$ there is a direct sum decomposition $F = F_+(\tau) \oplus F_-(\tau)$, where $F_+(\tau)$ ($F_-(\tau)$ resp.) is fiberwisely spanned by eigenvectors of $\tau(x', \xi', \lambda)$ with eigenvalues of positive (negative resp.) real parts.

Lemma 6.5 ([13], Lemma 1.9.5). *Suppose $P : \Gamma(F) \to \Gamma(F)$ is a first order formally self-adjoint elliptic differential operator with zero-th order boundary operator $\beta : F|_{\partial M} \to F'$. Then (1) (P, β) is $\mathbb{C} - \mathbb{R}_+ - \mathbb{R}_-$-elliptic if and only if $\beta : F_+(\tau) \to F'$ is an isomorphism for $(\xi', \lambda) \neq (0, 0) \in T^*(\partial M) \times (\mathbb{C} - \mathbb{R}_+ - \mathbb{R}_-)$; (2) P_β is self-adjoint if and only if $p_0(x')Ker\beta$ is perpendicular to $Ker\beta$ in F for $x' \in \partial M$.*

Now we are going to study the symbol of the self-duality operator $T_A \oplus T_A^*$ in Theorem 2.6. Since $\mathbb{C} - \mathbb{R}_+ - \mathbb{R}_-$ ellipticity depends only on principal symbols, we need only to prove Theorem 2.6 for $d_A = d$. Recall that we use T to denote T_A for $d_A = d$.

Let the symbol of $T \oplus T^*$ at (x, ξ) be denoted by $\sigma_\xi(T \oplus T^*)$, where ξ is a covector at point x in M. For the sake of brevity, let

(6.7) $$\xi \circ \overset{def}{=} -i\sigma_\xi(T \oplus T^*).$$

Recall that $T_c^*M \overset{def}{=} \mathbb{C} \otimes_\mathbb{R} T^*M$ and $Q_c^* = \mathbb{C} \otimes_\mathbb{R} Q^*$. When m is odd, let $\bigwedge_{++}^m T_c^*M \overset{def}{=} L^* \wedge \bigwedge^{m-1} Q_c^* \oplus \bigwedge_+^m Q_c^*$ and let $\bigwedge^{m+1} T_c^*M \overset{def}{=} L^* \wedge \bigwedge_+^m Q_c^*$. When m is even, let $\bigwedge_{++}^m T^*M \overset{def}{=} L^* \wedge \bigwedge^{m-1} Q^* \oplus \bigwedge_+^m Q^*$ and let $\bigwedge^{m+1} T^*M \overset{def}{=} L^* \wedge \bigwedge_+^m Q^*$. Let

$\quad F \overset{def}{=} \sum_{j=0}^{m-1} \bigwedge^j T_c^*M \oplus \bigwedge_{++}^m T_c^*M \oplus \bigwedge_+^{m+1} T_c^*M$ for odd m,

$\quad F_q \overset{def}{=} \sum_{j=0}^{m-1} \bigwedge^j Q_c^* \oplus \bigwedge_+^m Q_c^*$ for odd m,

$\quad F \overset{def}{=} \mathbb{C} \otimes_\mathbb{R} (\sum_{j=0}^{m-1} \bigwedge^j T^*M \oplus \bigwedge_{++}^m T^*M \oplus \wedge_+^{m+1}T^*M)$ for even m,

$\quad F_q \overset{def}{=} \mathbb{C} \otimes_\mathbb{R} (\sum_{j=0}^{m-1} \bigwedge^j Q^* \oplus \bigwedge_+^m Q^*)$ for even m.

Then $F = L^* \wedge F_q \oplus F_q$ and $\xi \circ : F \to F$. Let e^0 be an oriented unit basis of L over M and let $(e^1, \dots, e^{2m})$ be an o_q-oriented g_q-orthonormal basis of $Q^*|_U$, where U is an open neighborhood of x in M. Let J be an ordered subset of $\{1, \dots, 2m\}$ and e^J be the wedge of e^j in the order of J where j runs through J. Let $|J|$ be the length of J.

Proposition 6.6. *Let s be any nonzero complex number, $1 \le j < k \le 2m$.*

(1) When $\dim M = 3$, let $e_s^1 \overset{def}{=} \frac{\sqrt{2}}{|s|}e^1$ and $e_s^2 \overset{def}{=} \frac{\sqrt{2}}{|s|}e^2$, then we have

$$(e^0 \circ)^2 = -1, \qquad (e_s^1 \circ)^2 = -1, \qquad (e_s^2 \circ)^2 = -1,$$

$$e^0 \circ e^1_s \circ + e^1_s \circ e^0 \circ = 0, \qquad e^0 \circ e^2_s \circ + e^2_s \circ e^0 \circ = 0, \qquad e^1_s \circ e^2_s \circ + e^2_s \circ e^1_s \circ = 0.$$

(2) When $\dim M \geq 3$, we have

$$(e^0 \circ)^2 = -1, \quad e^0 \circ e^j \circ + e^j \circ e^0 \circ = 0 \quad \text{for any nonzero } s;$$

$$(e^j \circ)^2 = -1, \quad e^j \circ e^k \circ + e^k \circ e^j \circ = 0 \quad \text{if } |s|^2 = 2.$$

(3) When $\dim M \geq 5$, we have

$$(e^j \circ)^2 = -N_{j,s}^2, \quad e^j \circ e^k \circ + e^k \circ e^j \circ = C_{jk,s} \quad \text{for any nonzero } s.$$

(4) For any nonzero s and all $\dim M \geq 3$, we have that $e^0 \circ = e^0 \wedge$ on F_q, $e^0 \circ = -i_{e^0}$ on $L^* \wedge F_q$, $e^0 \circ$ interchanges $L^* \wedge F_q$ and F_q, and $e^j \circ$ preserves $L^* \wedge F_q$ and also preserves F_q for all $1 \leq j \leq 2m$.

This proposition will be proved at the end of this section. Note the above (3) means $\xi \circ$ is a representation of the Clifford multiplication when $|s|^2 = 2$.

$N_{j,s}$ and $C_{jk,s}$ are defined as follows. Let $m \geq 2$ and $0 \leq j \leq 2m$. Define
$G_j = \langle e^0 \wedge e^J, e^J : |J| = m - 1 \text{ and } j \notin J \rangle \oplus \bigwedge_+^m Q_c^*$ for odd m,
$G_j = \langle e^0 \wedge e^J, e^J : |J| = m - 1 \text{ and } j \notin J \rangle \oplus \mathbb{C} \otimes_{\mathbb{R}} \bigwedge_+^m Q^*$ for even m.
Let $G_j^\perp$ be the orthogonal complement of G_j in $F|_U$. Define

(6.8)
$$N_{j,s} = \frac{|s|}{\sqrt{2}} \text{ on } G_j, \qquad N_{j,s} = 1 \text{ on } G_j^\perp.$$

Let $m \geq 2$ and $0 \leq j < k \leq 2m$. Define
$G_{jk} = \langle e^0 \wedge e^{jR}, e^0 \wedge e^{kR}, e^{jR}, e^{kR} : |R| = m - 2 \text{ and } j, k \notin R \rangle$.
We define $C_{jk,s} = 0$ on $G_{jk}^\perp$. On G_{jk}, we define $C_{jk,s}$ as the mapping:

(6.9)
$$e^{jR} \rightarrow (-1 + \frac{|s|^2}{2}) e^{kR},$$

$$e^{kR} \rightarrow (\frac{|s|^2}{2} - 1) e^{jR},$$

$$e^0 \wedge e^{jR} \rightarrow (-1 + \frac{|s|^2}{2}) e^0 \wedge e^{kR},$$

$$e^0 \wedge e^{kR} \rightarrow (\frac{|s|^2}{2} - 1) e^0 \wedge e^{jR}.$$

Lemma 6.7. (1) If $|s|^2 = 2$ then all $N_{j,s} = 1$ and $C_{jk,s} = 0$ on $F|_U$ for $0 \leq j < k \leq 2m$.
(2) $N_{j,s}$ are positive definite real diagonal matrices with respect to the above basis for $G_j \oplus G_j^\perp = F|_U$ for all $0 \leq j \leq 2m$ and any nonzero s.
(3) $C_{jk,s}$ are real skew-symmetric matrices with respect to the above basis of $G_{jk} \oplus G_{jk}^\perp = F|_U$ for all $0 \leq j < k \leq 2m$ and any nonzero s.

Lemma 6.8. If N is a positive definite Hermitian matrix and C a skew-Hermitian matrix, then $\det(N + C) \neq 0$.

Proof. If $\det(N + C) = 0$, then there is a vector $v \neq 0$ such that $Nv = -Cv$. Since N is Hermitian and C is skew-Hermitian, we have $\langle Nv, v \rangle = \langle v, Nv \rangle = -\langle v, Cv \rangle = \langle Cv, v \rangle = -\langle Nv, v \rangle$, and thus $\langle Nv, v \rangle = 0$, which contradicts the fact that N is positive definite. $\square$

By applying Proposition 6.6, we can follow the same arguments for the $\mathbb{C} -$ $\mathbb{R}_+ - \mathbb{R}_-$ ellipticity of De Rham operator $d + d^*$ with the classical relative and absolute boundary conditions. (See [13], pp.243-244.) In the following proof, when $dim M = 3$, let e^1 and e^2 be replaced respectively by e_s^1 and e_s^2 and let all $N_{j,s} = 1$ and $C_{jk,s} = 0$. Let F' be F_q on K and $L^* \wedge F_q$ on $\partial M - K$. Our $T \oplus T^*$ is obviously a first order formally self-adjoint operator. It has the symbol $p(x, \xi) = i\xi \circ = \xi_0 i e^0 \circ + \sum_{j=1}^{2m} \xi_j i e^j \circ$ where $\xi = \xi_0 e^0 + \sum_{j=1}^{2m} \xi_j e^j$. Applying Proposition 6.6, we get

$$p(x, \xi)^2 = \xi_0^2 + \sum_{j=1}^{2m} \xi_j^2 N_{j,s}^2 + \sum_{j<k} \xi_j \xi_k C_{jk,s},$$

where $0 \leq j, k \leq 2m$. By Lemma 6.7, $\xi_0^2 + \sum_{j=1}^{2m} \xi_j^2 N_{j,s}^2$ is a positive definite Hermitian matrix for $\xi \neq 0$ and $\sum_{j<k} \xi_j \xi_k C_{jk,s}$ is a skew-Hermitian matrix. Thus by Lemma 6.8, $det\, p(x, \xi)^2 \neq 0$ for all $\xi \neq 0$ and $T \oplus T^*$ is elliptic.

Let $\xi' = \sum_{j=1}^{2m} \xi_j e^j$. According to (6.6), we have

$$\tau(x', \xi', \lambda) = i(ie^0 \circ)^{-1} (\sum_{j=1}^{2m} \xi_j i e^j \circ - \lambda) = -i\lambda q_0 + \sum_{j=1}^{2m} \xi_j q_j,$$

where $q_0 = ie^0$ and $q_j = -ie^0 \circ e^j \circ$ for $1 \leq j \leq 2m$. Let τ denote $\tau(x', \xi', \lambda)$ in this proof, and let $\alpha = 1$ on $L^* \wedge F_q$ and $\alpha = -1$ on F_q. Using (4) of Proposition 6.6, we see that all q_j interchanges $L^* \wedge F_q$ and F_q. Thus $\alpha q_j = -q_j \alpha$ and $\alpha\tau = -\tau\alpha$. If μ is an eigenvalue for τ such that $\tau(\omega) = \mu\omega$ with $\omega \neq 0$, then $-\mu$ is also an eigenvalue for τ, since $\tau(\alpha(\omega)) = -\mu\alpha(\omega)$. Lemma 6.4 states that any eigenvalue of τ is not zero and has no zero real part. Thus $dim F_+(\tau) = dim F_-(\tau)$ and $dim F_+(\tau) = dim F' = \frac{1}{2} dim F$. Therefore we need only to prove $b|_{F_+(\tau)} : F_+(\tau) \rightarrow F'$ is injective.

Since $ker\, b = L^* \wedge E_q$ over K and $ker\, b = F_q$ over $\partial M - K$, we have $\alpha = \pm 1$ on $ker\, b$. If $ker\, b|_{F_+(\tau)} = ker\, b \cap F_+(\tau)$ were not zero, then there were $\omega \neq 0$ such that $\alpha(\omega) = \pm\omega$ and $\tau(\omega) = \mu\omega$ for some $\mu \neq 0$. Using $\alpha\tau = -\tau\alpha$, we get $\mu\omega = -\mu\omega$ and $\mu = 0$, which is a contradiction. This completes the $\mathbb{C} - \mathbb{R}_+ - \mathbb{R}_-$ ellipticity of $(T \oplus T^*, b)$ by Lemma 6.5. Directly using (2.14) and using Green's formula, one can easily see that $(T \oplus T^*, b)$ is self-adjoint. Thus the proof of Theorem 2.6 is completed.

IV. *Proof of Proposition 6.6.*

The case of $dim M = 3$ is the simplest case. Recall e^0 is an oriented unit basis of L over M, and (e^1, e^2) is an oriented g_q-orthonormal basis of $Q^*|_U$ where U is an open subset of M. Since $*_q e^1 = e^2$ and $*_q e^2 = -e^1$, $\overline{\delta} = e^1 - ie^2$ is a complex basis for $\bigwedge_+^1 Q_c^*|_U$ and $\delta = e^1 + ie^2$ is a complex basis for $\bigwedge_-^1 Q_c^*|_U$. Thus $e^1 = \frac{\delta + \overline{\delta}}{2}$ and $e^1 = \frac{\delta - \overline{\delta}}{2i}$. Definition 2.3 applying to the case $m = 1$ yields that

$$\xi \circ : \mathbb{C} \oplus \bigwedge_{++}^1 T_c^* M \oplus \bigwedge_+^2 T_c^* M \rightarrow \mathbb{C} \oplus \bigwedge_{++}^1 T_c^* M \oplus \bigwedge_+^2 T_c^* M,$$

$$\xi \circ (\omega_0 + \omega_1 + \omega_2) = -i_\xi l_{\bar{s}} \omega_1 + l_s \pi^{++}(\xi \wedge \omega_0) - h_{\bar{s}} \pi^{++}(i_\xi \omega_2) + \pi^+(\xi \wedge h_s \omega_1).$$

$e^0\circ$ maps

$$e^0 \to -i_{e^0} l_{\bar{s}} e^0 + \pi^+(e^0 \wedge h_s e^0) = -1,$$
$$\overline{\delta} \to -i_{e^0} l_{\bar{s}} \overline{\delta} + \pi^+(e^0 \wedge h_s \overline{\delta}) = e^0 \wedge \overline{\delta},$$
$$1 \to l_s \pi^{++} e^0 = e^0,$$
$$e^0 \wedge \overline{\delta} \to -h_{\bar{s}} i_{e^0}(e^0 \wedge \overline{\delta}) = -\overline{\delta}.$$

$e^1\circ$ maps

$$e^0 \to -i_{e^1} l_{\bar{s}} e^0 + \pi^+(e^1 \wedge h_s e^0) = -\frac{s}{2} e^0 \wedge \overline{\delta},$$
$$\overline{\delta} \to -i_{e^1} l_{\bar{s}} \overline{\delta} + \pi^+(e^1 \wedge h_s \overline{\delta}) = -\overline{s},$$
$$1 \to l_s \pi^{++} e^1 = \frac{s}{2}\overline{\delta},$$
$$e^0 \wedge \overline{\delta} \to -h_{\bar{s}} i_{e^1}(e^0 \wedge \overline{\delta}) = \overline{s} e^0.$$

$e^2\circ$ maps

$$e^0 \to -i_{e^2} l_{\bar{s}} e^0 + \pi^+(e^2 \wedge h_s e^0) = \frac{s}{2i} e^0 \wedge \overline{\delta},$$
$$\overline{\delta} \to -i_{e^2} l_{\bar{s}} \overline{\delta} + \pi^+(e^2 \wedge h_s \overline{\delta}) = \overline{s}i,$$
$$1 \to l_s \pi^{++} e^2 = -\frac{s}{2i}\overline{\delta},$$
$$e^0 \wedge \overline{\delta} \to -h_{\bar{s}} i_{e^2}(e^0 \wedge \overline{\delta}) = -i\overline{s} e^0.$$

The claim (1) of Proposition 6.6 follows from the above tables. Now we prove claims (1)-(4) of Proposition 6.6 for $dim M \geq 5$. For an ordered subset J of $\{1,\ldots,2m\}$, let $*_q J$ be the ordered subset such that $J *_q J$ is an even permutation of ordered $\{1,\ldots,2m\}$. Let $\epsilon = 1$ if m is even and $\epsilon = -i$ if m is odd in this proof. Thus $*_q e^J = e^{*_q J}$ and $\{ \delta_{\pm}^J \stackrel{def}{=} e^{1J} \pm \epsilon e^{*_q(1J)}$ for $|J| = m-1$ and $1 \notin J \}$ is an orthogonal basis of the same length for $\bigwedge_{\pm}^m Q_c^*$ if m is odd, and for $\mathbb{C} \otimes_{\mathbb{R}} \bigwedge_{\pm}^m Q^*$ if m is even. By Definition 2.3, we see that $\xi\circ : F \to F$ is given by

$$(6.10) \quad \xi \circ \left(\sum_{j=0}^{m-1} \omega_j + \omega_m + \omega_{m+1}\right) = \sum_{j=0}^{m-2} c(\xi)(\omega_{j-1}) + \xi \wedge \omega_{m-2} - i_\xi l_{\bar{s}} \omega_m$$
$$+ l_s \pi^{++}(\xi \wedge \omega_{m-1}) - h_{\bar{s}} \pi^{++}(i_\xi \omega_{m+1}) + \pi^+(\xi \wedge h_s \omega_m)$$

where $c(\xi) = \xi \wedge -i_\xi$ is the usual Clifford multiplication. Thus $\xi\circ = c(\xi)$ on $\sum_{j=0}^{m-2} T_c^* M$.

We also have $\xi\circ = c(\xi)$ on $L^* \wedge \bigwedge^{m-2} Q_c^*$. Indeed, let $\omega_{m-1} \in L^* \wedge \bigwedge^{m-2} Q^*$, then $\xi \wedge \omega_{m-1} \in L^* \wedge \bigwedge^{m-1} Q^*$ on which $l_s \pi^{++}$ is identity. Thus $\xi \circ \omega_{m-1} = l_s \pi^{++}(\xi \wedge \omega_{m-1}) - i_\xi \omega_{m-1} = \xi \wedge \omega_{m-1} - i_\xi \omega_{m-1} = c(\xi)\omega_{m-1}$.

Therefore we need only to prove Proposition 6.6 for covectors in $\bigwedge^{m-1} Q_c^* \oplus \bigwedge_{++}^m T_c^* M \oplus \bigwedge_+^{m+1} T_c^* M$ when m is odd and in $\bigwedge^{m-1} Q^* \oplus \bigwedge_{++}^m T^* M \oplus \bigwedge_+^{m+1} T^* M$ when m is even. This space is $L^* \wedge H_q \oplus H_q$, where $H_q \stackrel{def}{=} \bigwedge^{m-1} Q_c^* \oplus \bigwedge_+^m Q_c^*$ when m is odd, and $H_q \stackrel{def}{=} \bigwedge^{m-1} Q^* \oplus \bigwedge_+^m Q^*$ when m is even.

In the following tables, I, J, T, P, R, S are ordered subsets of $\{1,\ldots,2m\}$ such that

$$|I| = m-1, \quad |J| = m-1 \text{ and } 1 \notin J, \quad |P| = m-1 \text{ and } 1,2 \notin P,$$

$|R| = m-2$ and $1 \notin R$, $|S| = m-2$ and $1,2 \notin S$, $|W| = m-3$ and $1,2 \notin W$.

When $|R| = 0$ or $|S| = 0$ in the case $m = 2$, or $|W| = 0$ in the case $m = 3$, we regard R, S and W as empty sets and define $e^{empty} = 1$. When $|W| = -1$ in the case $m = 2$, we delete the corresponding entry from the following tables. In addition, in the following, L^*, H_q and F means $L^*|_U$, $H_q|_U$ and $F|_U$ respectively.

On $L^* \wedge H_q$, $e^0 \circ$ maps

$$e^0 \wedge e^I \to \pi^+(e^0 \wedge h_s(e^0 \wedge e^I)) - i_{e^0}l_{\bar{s}}(e^0 \wedge e^I) = -e^I,$$
$$e^0 \wedge \delta_+^J \to -h_{\bar{s}}\pi^{++}i_{e^0}(e^0 \wedge \delta_+^J) = -\delta_+^J.$$

On H_q, $e^0 \circ$ maps

$$e^I \to l_s\pi^{++}(e^0 \wedge e^I) - i_{e^0}e^I = e^0 \wedge e^I,$$
$$\delta_+^J \to \pi^+(e^0 \wedge h_s\delta_+^J) - i_{e^0}l_{\bar{s}}\delta_+^J = e^0 \wedge \delta_+^J.$$

On $L^* \wedge H_q$, $e^1 \circ$ maps

$$e^0 \wedge e^{1R} \to \pi^+(e^1 \wedge h_s(e^0 \wedge e^{1R})) - i_{e^1}l_{\bar{s}}(e^0 \wedge e^{1R}) = e^R,$$
$$e^0 \wedge e^J \to \pi^+(e^1 \wedge h_s(e^0 \wedge e^J)) - i_{e^1}l_{\bar{s}}(e^0 \wedge e^J) = -s\pi^+(e^0 \wedge e^{1J}) = \frac{-s}{2}e^0 \wedge \delta_+^J,$$
$$e^0 \wedge \delta_+^J \to -h_{\bar{s}}\pi^{++}i_{e^1}(e^0 \wedge \delta_+^J) = -h_{\bar{s}}\pi^{++}(-e^0 \wedge e^J) = \bar{s}e^0 \wedge e^J.$$

On H_q, $e^1 \circ$ maps

$$e^{1R} \to l_s\pi^{++}(e^1 \wedge e^{1R}) - i_{e^1}e^{1R} = -e^R,$$
$$e^J \to l_s\pi^{++}(e^1 \wedge e^J) - i_{e^1}e^J = \frac{s}{2}\delta_+^J,$$
$$\delta_+^J \to \pi^+(e^1 \wedge h_s\delta_+^J) - i_{e^1}l_{\bar{s}}\delta_+^J = \pi^+(e^1 \wedge \epsilon e^{*_q(1J)}) - \bar{s}\frac{1}{2}e^J = -\frac{\bar{s}}{2}e^J.$$

The above tables imply $(e^0\circ)^2 = -1$, $e^0 \circ e^1 \circ + e^1 \circ e^0 \circ = 0$ and $(e^1\circ)^2 = -N_{1,s}^2$ for any nonzero s. By $SO(2m)$-*symmetry* of Q^*, we get $e^0 \circ e^j \circ + e^j \circ e^0 \circ = 0$ and $(e^j\circ)^2 = -N_{j,s}^2$ for any nonzero s.

On H_q, $e^1 \circ$ maps

$$e^{12W} \to l_s\pi^{++}(e^1 \wedge e^{12W}) - i_{e^1}e^{12W} = -e^{2W},$$
$$e^{1S} \to l_s\pi^{++}(e^1 \wedge e^{1S}) - i_{e^1}e^{1S} = -e^S,$$
$$e^{2S} \to l_s\pi^{++}(e^1 \wedge e^{2S}) - i_{e^1}e^{2S} = \frac{s}{2}\delta_+^{2S},$$
$$e^P \to l_s\pi^{++}(e^1 \wedge e^P) - i_{e^1}e^P = \frac{s}{2}\delta_+^P,$$
$$\delta_+^{2S} \to \pi^+(e^1 \wedge h_s\delta_+^{2S}) - i_{e^1}l_{\bar{s}}\delta_+^{2S} = -\bar{s}e^{2S},$$
$$\delta_+^P \to \pi^+(e^1 \wedge h_s\delta_+^P) - i_{e^1}l_{\bar{s}}\delta_+^P = -\frac{\bar{s}}{2}e^P.$$

In the following, $[*_q(1P)/2] \overset{def}{=} x$ such that $2x = *_q1P$, that is, $e^{1P2x} = vol(g_q)$, and $[*_q(2P)/1] \overset{def}{=} y$ such that $1y = *_q2P$, that is, $e^{2P1y} = vol(g_q)$. We have $[*_q(1P)/2] = -[*_q(2P)/1]$ since $(-1)^{|P|}e^{12Px} = e^{1P2x} = vol(g_q) = e^{2P1y} = (-1)^{|P|+1}e^{12Py}$.

On H_q, $\quad e^2\circ$ maps

$$e^{12W} \to l_s\pi^{++}(e^2 \wedge e^{12W}) - i_{e^2}(e^1 \wedge e^{2W}) = e^{1W},$$

$$e^{1S} \to l_s\pi^{++}(e^2 \wedge e^{1S}) - i_{e^2}e^{1S} = -\frac{s}{2}\delta_+^{2S},$$

$$e^{2S} \to l_s\pi^{++}(e^2 \wedge e^{2S}) - i_{e^2}e^{2S} = -e^S,$$

$$e^P \to l_s\pi^{++}(e^2 \wedge e^P) - i_{e^2}e^P = s\frac{e^{2P} + \epsilon e^{*_q(2P)}}{2}$$

$$= s\epsilon\frac{e^{*_q(2P)} + \epsilon e^{*_q*_q(2P)}}{2} = s\epsilon\delta^{[*_q(2P)/1]},$$

$$\delta_+^{2S} \to \pi^+(e^2 \wedge h_s\delta_+^{2S}) - i_{e^2}l_{\bar{s}}\delta_+^{2S} = \pi^+(e^1 \wedge \epsilon e^{*_q(12R)}) - \overline{s}\frac{1}{2}e^{2S} = -\overline{s}e^{1S},$$

$$\delta_+^P \to \pi^+(e^2 \wedge h_s\delta_+^P) - i_{e^2}l_{\bar{s}}\delta_+^P = -\overline{s}\epsilon i_{e^2}\frac{e^{1P} + \epsilon e^{*_q(1P)}}{2} = -\overline{s}\epsilon e^{[*_q(1P)/2]}.$$

On H_q, $\quad e^1 \circ e^2\circ$ maps

$$e^{12W} \to -e^{2W} \to e^W,$$

$$e^{1S} \to -e^S \to -e^{2S},$$

$$e^{2S} \to \frac{s}{2}\delta_+^{2S} \to \frac{1}{2}|s|^2 e^{1S},$$

$$e^P \to \frac{s}{2}\delta_+^P \to -\frac{1}{2}|s|^2\epsilon e^{[*_q(1P)/2]},$$

$$\delta_+^{2S} \to \overline{s}e^{2S} \to \overline{s}e^S,$$

$$\delta_+^P \to -\frac{1}{2}\overline{s}e^P \to -\frac{1}{2}|s|^2\epsilon\delta^{[*_q(2P)/1]}.$$

On H_q, $\quad e^2 \circ e^1\circ$ maps

$$e^{12W} \to e^{1W} \to -e^W,$$

$$e^{1S} \to -\frac{s}{2}\delta_+^{2S} \to \frac{1}{2}|s|^2 e^{2S},$$

$$e^{2S} \to -e^S \to -e^{1S},$$

$$e^P \to s\epsilon\delta^{[*_q(2P)/1]} \to -\frac{1}{2}|s|^2\epsilon e^{[*_q(2P)/1]},$$

$$\delta_+^{2S} \to \overline{s}e^{1S} \to -\overline{s}e^S,$$

$$\delta_+^P \to -\frac{1}{2}\overline{s}\epsilon e^{[*_q(1P)/2]} \to -\frac{1}{2}|s|^2\epsilon\delta^{[*_q(1P)/2]}.$$

Recall $e^0\circ = e^0\wedge$ on $H_q \subset F_q$ and $e^0 \circ e^j\circ = -e^j \circ e^0$ for all $0 \le j \le 2m$. Thus the above tables for $e^1\circ e^2\circ$ and $e^1\circ e^2\circ$ on H_q also give the corresponding tables for $e^1 \circ e^2\circ$ and $e^1\circ e^2\circ$ on $L^* \wedge H_q$. By these tables, we have $e^1 \circ e^2 \circ + e^2 \circ e^1 = C_{12,s}$ on F. By $SO(2m)$-symmetry of Q^* we get $e^j \circ e^k \circ + e^k \circ e^j = C_{jk,s}$ on F for $0 \le j < k \le 2m$. The proof of Proposition 6.6 is completed.

REFERENCES

[1] V.I.Arnold, *Problems on singularity and dynamical system,* preprint, 1992.

[2] M.F.Atiyah, *Vector fields on manifolds,* Arbeitsgemeinschaft für Forschung des Landes Nordrhein-Westfalen, 200, 1970. MR **41**:7707

[3] M.F.Atiyah, *The index theorem for manifolds with boundary,* Appendix I in seminar on the Atiyah-Singer index theorem, Ann. Math. Stud. **57**(1965), Princeton Univ. Press, 337-351. MR **33:**6649

[4] M.F.Atiyah and R.Bott, *The index theorem for manifolds with boundary,* Differential Analysis (papers collected at the Bombay Colloquium 1964), Oxford Univ. Press, 175-186. MR **32:**3069

[5] M.F.Atiyah, N.J. Hitchin, and I.M.Singer, *Self-duality in four dimensional Riemannian geometry,* Proc. Royal Soc. London, Ser.A **362**, (1978). 425-461. MR **80d:**53023

[6] D.E.Blair, *Contact manifolds in Riemannian geometry,* Springer-Verlag, 1976. MR **57:**7444

[7] M.Brunella and E.Ghys, *Umbilical foliations and transversely holomorphic flows,* J. Diff. Geom. **41**(1995), 1-19. MR **95k:**53039

[8] S.K.Donaldson, *Irrationality and the h-cobordism conjecture,* J. Diff. Geom. **26**(1986), 141-168. MR **88j:**57035

[9] S.K.Donaldson and P.B.Kronheimer, *The geometry of four-manifolds*, Oxford Science Publications, Oxford, 1990. MR **92a:**57036

[10] H.H.Fan, *Half De Rham complexes and line fields on odd dimensional manifolds,* Trans. Amer. Math. Soc. **348**(1996), no.8, 2947-2982. MR **96k:**58204

[11] H.H.Fan, *Gauge fields and line fields on five-manifolds,* preprint, Yale, 1995.

[12] H.H.Fan, *Seiberg-Witten-type equation on five-manifolds,* preprint, Yale, 1995.

[13] P.B.Gilkey, *Invariance theory, the heat equation, and the Atiyah-Singer index theorem,* Publish or Perish, 1984. MR **86j:**58144

[14] M.W.Hirsch, *Differential topology,* Springer-Verlag, 1976. MR **56:**6669

[15] P.B.Kronheimer and T.S.Mrowka, *The genus of embedded surfaces in the projective plane,* Math. Res. Letters **1**, 797-808 (1994) MR **96a:**57073

[16] P.B.Kronheimer and T.S.Mrowka, *Embedded surfaces and the structure of Donaldson's Polynomial Invariants,* J. Diff. Geom. **3** (1995) 573-734. MR **96e:**57019

[17] P.Molino, *Riemannian foliations*, Birkhäuser, Boston, 1988 MR **89b:**53054

[18] J.W.Morgan, Z.Szabó and C.H.Taubes. *The generalized Thom conjecture,* preprint, 1995.

[19] D.B.Ray and I.M.Singer, *R-torsion and the Laplacian on Riemannian manifolds,* Advan. Math. **7** (1971), 145-210. MR **45:**4447

[20] C.H.Taubes, *The Seiberg-Witten invariants and symplectic forms,* Math. Res. Letters **1**, 809-822 (1994). MR **95j:**57039

[21] I.Vaisman, *Conformal foliations,* Kodai Math. J. **2** (1979), 26-37. MR **80g:**57038

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