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THE SINGULAR LIMIT OF A VECTOR–VALUED REACTION–DIFFUSION PROCESS

LIA BRONSARD AND BARBARA STOTH

Abstract. We study the asymptotic behaviour of the solution to the vector–valued reaction–diffusion equation

$$\varepsilon\partial_t\varphi - \varepsilon\triangle\varphi + \frac{1}{\varepsilon}\tilde{W}_{,\varphi}(\varphi) = 0 \quad \text{in } \Omega_T,$$

where $\varphi_\varepsilon = \varphi : \Omega_T := (0,T)\times\Omega \longrightarrow \mathbf{R}^2$. We assume that the the potential $\tilde{W}$ depends only on the modulus of φ and vanishes along two concentric circles. We present a priori estimates for the solution φ, and, in the spatially radially symmetric case, we show rigorously that in the singular limit as $\varepsilon \to 0$, two phases are created. The interface separating the bulk phases evolves by its mean curvature, while φ evolves according to a harmonic map flow on the respective circles, coupled across the interfaces by a jump condition in the gradient.

1. Introduction

In this paper, we study the singular limit of the vector–valued reaction–diffusion equation for $\varphi_\varepsilon = \varphi : \Omega_T := (0,T)\times\Omega \longrightarrow \mathbf{R}^2$,

$$\varepsilon\partial_t\varphi - \varepsilon\triangle\varphi + \frac{1}{\varepsilon}\tilde{W}_{,\varphi}(\varphi) = 0 \quad \text{in } \Omega_T,$$

with Neumann condition on $(\partial\Omega)_T$, where Ω is a smooth bounded domain in $\mathbb{R}^n, n \geq 2$. We assume that the potential $\tilde{W}$ depends only on the modulus of φ, that it is smooth, that it vanishes along two circles and that it is positive elsewhere. A typical example is given by $\tilde{W}(\varphi) = (|\varphi| - a)^2(b - |\varphi|)^2$. In the singular limit as $\varepsilon \to 0$, one expects the domain Ω to be divided in bulk regions where $|\varphi|$ is close to a or b. We present a priori estimates for the solution φ and, in the radial case, we prove that the interfaces separating the bulk regions evolve normally according to their curvature as $\varepsilon \to 0$. Moreover we establish the diffusion equation satisfied by φ in the bulk regions as $\varepsilon \to 0$, coupled across the interfaces by a jump condition in the gradient. This jump condition is new and was not previously derived in the formal asymptotic analysis of this system (cf. [RSK]).

Rubinstein, Sternberg and Keller [RSK] introduced this vector–valued reaction–diffusion equation for $\varphi \in \mathbf{R}^N, N \geq 2$, to model certain chemical reactions. Indeed in chemical reactions, the (nonconserved) order parameter φ represents a vector of concentration of reactants, and the law of mass action leads to the above reaction–diffusion equation with the potential $\tilde{W}$ vanishing on one or more manifolds. In their paper [RSK], Rubinstein, Sternberg and Keller presented formal asymptotic

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expansions when the potential vanishes on two manifolds. Convergence proofs were given in [RSK] and [C] when the potential vanishes on one sphere, and in [CS] when it vanishes on a general connected manifold. In fact, Chen [C] and Chen–Struwe [CS] used this parabolic system to study existence and singularity formation of the harmonic map flow when the target manifold is respectively a sphere and a general connected manifold.

We study rigorously the radial case with $\varphi \in \mathbf{R}^2$, and with a potential that only depends on the modulus $|\varphi|$ and vanishes on two concentric spheres. We always assume that the energy is initially bounded (cf. (A1)). We choose initial data that lie strictly in one half plane (cf. (A2), (A3) and (A4)). We may then represent φ through $u(\cos f, \sin f)$, where u is the modulus of φ and f is the polar angle, and rewrite the problem for φ as a system of equations for u and f (cf. Section 2, equations (E2) and (E3)). By standard techniques a subsequence of $u = u_\varepsilon$ converges to a limit u_0 (cf. Remark 3.2). We also show that a subsequence of $f = f_\varepsilon$ converges to a limit f_0 (cf. Lemma 3.5).

The results of Rubinstein, Sternberg and Keller [RSK] suggest that the interfaces Γ separating the regions Ω_a and Ω_b, where u_0 is either a or b, evolve normally by their curvature. We prove this result in the radial case by presenting the convergence of (E2) to a weak formulation of curvature flow (cf. Section 5, Proposition 5.2 and Theorem 5.5). In addition, the results in [RSK] suggest that the limit f_0 of the angle f satisfies a harmonic map flow in the bulk phases. We rigorously verify this assertion in the radial case by passing to the limit in the equation (E3) satisfied by f (cf. Section 6, Theorem 6.3). We show that f_0 is a generalized solution of

$$u_0^2 \partial_t f_0 - \text{ div } (u_0^2 \nabla f_0) = 0.$$

This equation couples the harmonic flow $\partial_t f_0 - \triangle f_0 = 0$ in the bulk regions Ω_a and Ω_b to the jump condition

$$[u_0^2 \nabla f_0]_{|\Gamma} \cdot \nu = 0$$

on the interfaces Γ. This condition, which was not explicitly derived in [RSK], arises as a byproduct of our analysis. Of course, with the benefit of hindsight, one can formally derive this jump condition, and we include a brief description of this formal derivation in Remark 6.5 in the context of [RSK].

To prove these results, we extend the energy method developed in [BSt], which is based on ideas introduced in [BK1, 2] and [St1, 2] to treat scalar–valued reaction–diffusion equations. One advantage of the energy method is that it does not rely on comparison principles (e.g. the maximum principle), and in this paper we show that we can extend it to treat systems. The new feature of the method, once specialized to (E1), is the derivation of the necessary a priori estimates on the solutions u and f (cf. Section 3, Lemma 3.4, Lemma 3.6 and Lemma 3.8, and Section 4, Proposition 4.1), but the underlying principles of the method remain unaltered. In particular we prove that u_ε is well approximated by a stationary wave solution as in the case of the scalar Allen–Cahn equation. Our main result is contained in the following theorem.

Theorem. *Let $\tilde{W}$ be a smooth double well potential satisfying (W1) and (W2) of Section 2. Let $\varphi_\varepsilon^0 = u_\varepsilon^0(\cos f_\varepsilon^0, \sin f_\varepsilon^0)$ be a smooth, radially symmetric function satisfying (A1)-(A4) of Section 3, and for any $\varepsilon > 0$ let $\varphi_\varepsilon = u_\varepsilon(\cos f_\varepsilon, \sin f_\varepsilon)$ be the smooth solution of equation (E1) of Section 2 with Neumann condition on the spatial boundary and initial values φ_ε^0.*

Then there exist a subsequence $\varepsilon \to 0$ *and limits* $u_0 \in L^\infty(0,T;BV(\Omega)) \cap BV(\Omega_T)$ *and* $f_0 \in L^\infty(\Omega_T) \cap L^2(0,T;H^{1,2}(\Omega))$*, such that* $u_\varepsilon \to u_0$ *in* $L^1(\Omega_T)$ *and* $f_\varepsilon \to f_0$ *in* $L^1(\Omega_T)$*.*

The order parameter u_0 *takes only the values* a *and* b *almost everywhere in* Ω_T*, and the free boundary* $\partial\{u_0 = a\}$ *consists of a collection of graphs* $R^j : [t^j, T^j) \to (0,1)$ *of class* $H^{1,2}_{loc}([t^j, T^j))$ *that evolve according to their curvature:*

$$\frac{d}{dt} R^j = -\frac{n-1}{R^j} \quad \text{in } (t^j, T^j).$$

The limit angle f_0 *satisfies a harmonic map flow in bulk, a jump condition on the interface* $\partial\{u_0 = a\}$ *and a Neumann condition on the boundary:*

$$u_0^2 \partial_t f_0 - \operatorname{div}(u_0^2 \nabla f_0) = 0 \quad \text{in the sense of Theorem 6.3.}$$

Next, we remark on the hypothesis of radial symmetry, which is essential for our approach. At the present time there are few analytical methods for treating non-radial problems, and in fact we know of no method which has been used for rigorous analysis of singularly perturbed systems such as (E1). Moreover, our rigorous radial results already give a more profound understanding of the formal asymptotic results in [RSK]. In fact, by interpreting the limiting problem in an analytically rigorous way (as a weak solution) we were able to identify the correct jump condition, which their formal analysis did not reveal.

Finally, if the potential W vanishes on two manifolds but does not have the symmetry assumed here, the results in [RSK] suggest that the interfaces will still evolve by their mean curvature, while the bulk flows will be given by harmonic map flows taking values in each manifold. However it is not known (even formally) how the flows in the bulk regions are coupled across the interfaces. We expect the following: define the distance between the two manifolds as in Sternberg [S1]. It is given by the minimal energy of stationary waves connecting pair of points on the manifolds (see [S2], [BR]). If there is exactly one stationary wave of minimal energy, we expect that the boundary values of the bulk flows at the interface will coincide with the endpoints of this stationary wave. On the other hand, if the distance between the manifolds is attained at a continuum of points (as in the radially symmetric case), then we expect that there will also be a jump condition in the gradient across the interface. In order to prove this result, the order parameter φ must be decomposed into a quantity that determines the interface, and another quantity that determines the bulk flow. In the present paper this is achieved by representing φ as $u\, e^{if}$.

2. Formulation of the problem

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, $T > 0$ and $\Omega_T := (0,T) \times \Omega$.

We always assume that

$$\varphi_\varepsilon = \varphi : \Omega_T \longrightarrow \mathbf{R}^2$$

is a smooth function that satisfies

$$\textbf{(E1)} \qquad \varepsilon \partial_t \varphi - \varepsilon \triangle \varphi + \frac{1}{\varepsilon} \tilde{W}_{,\varphi}(\varphi) = 0 \quad \text{in } \Omega_T,$$

with Neumann condition on $(\partial\Omega)_T$ and with initial values $\varphi(0,\cdot) = \varphi^0 = \varphi^0_\varepsilon$, that we assume to lie strictly in the right half plane.

We make the following assumptions on the potential $\tilde{W}$:

(**W1**) $$\tilde{W}(\varphi) = W(|\varphi|),$$

where $|\cdot|$ is the euclidean norm and $W : (0,\infty) \to [0,\infty)$ is a smooth function that vanishes exactly at a and b;

(**W2**) W' vanishes exactly at $a,\ b$ and some $d \in (a,b)$.

A typical example is $W(\lambda) = (\lambda - a)^2(b\ -\lambda)^2$, in which case $d = (a+b)/2$.

Lemma 2.1.

$$\max_{\overline{\Omega_T}} |\varphi| \leq \max(b, \sup_{\overline{\Omega}} |\varphi^0|).$$

If $\varphi_1^0 > 0$ *in* $\overline{\Omega}$*, then*

$$\min_{\overline{\Omega_T}} \frac{\varphi_1}{|\varphi|} \geq \min_{\overline{\Omega}} \frac{\varphi_1^0}{|\varphi^0|},$$

and

$$\min_{\overline{\Omega_T}} \varphi_1 \geq \min_{\overline{\Omega}} (\varphi_1^0, \frac{a\varphi_1^0}{|\varphi^0|}).$$

Proof. We again introduce $u := |\varphi|$ and $e := \frac{\varphi}{|\varphi|}$ (then $e_1 = \frac{\varphi_1}{|\varphi|}$), as long as $|\varphi|$ does not attain zero. Then u satisfies the differential equation

$$\partial_t u - \triangle u + \frac{1}{\varepsilon^2} W'(u) = -u|\nabla e|^2,$$

and e satisfies

$$u^2 \partial_t e -\ \operatorname{div}\ (u^2 \nabla e) = e\, u^2 |\nabla e|^2.$$

The first claim now follows from the maximum principle for the first equation. The second follows because

$$M := \{(u,e) \mid e_1 \geq c_0 \text{ and } \varphi_1 = u e_1 \geq c_1\}$$

is an invariant region for the above flow. Here $c_0 > 0$ and $0 < c_1 < a\, c_0$, where a is the first minimum of W (cf. W2). As a consequence $u = |\varphi|$ never attains zero, and thus e is a well defined quantity in all of Ω_T. In addition, φ ranges stricty in the right half plane. □

Remark and Formulation 2.2. Assume that the initial data satisfy $\varphi_1^0 > 0$ in $\overline{\Omega}$.

Since by assumption and as a consequence of Lemma 2.1 φ remains strictly in the right half plane, it follows that $u_\varepsilon = u = |\varphi|$ and $e_\varepsilon = e = \frac{\varphi}{|\varphi|}$ are well defined and e only ranges in the right half–sphere.

Thus there exists a smooth lift $f_\varepsilon = f : \overline{\Omega}_T \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, such that

$$\varphi = u(\cos f, \sin f).$$

By direct calculation, using the fact that $\partial_i e \cdot e = 0$, we get

$$\nabla u \cdot \nabla e = (\nabla u \cdot \nabla f) \cdot e^\perp \quad \text{and} \quad \triangle e = -|\nabla f|^2\, e + \triangle f\, e^\perp.$$

The terms in the equation for φ become

$$\begin{aligned} \partial_t \varphi &= \partial_t u\ e + u\ \partial_t e, \\ \triangle \varphi &= \triangle u\ e + 2\nabla u \cdot \nabla e + u \triangle e, \\ \tilde{W}_{,\varphi}(\varphi) &= W'(u) e. \end{aligned}$$

Taking the inner product of equation (E1) with e and $ue^{\perp}$ resp., we find for $(t,x) \in \Omega_T$ the system

$$\textbf{(E2)} \qquad \varepsilon\partial_t u - \varepsilon\triangle u + \varepsilon u|\nabla f|^2 + \frac{1}{\varepsilon}W'(u) = 0,$$

$$\textbf{(E3)} \qquad u^2\partial_t f - \operatorname{div}(u^2\nabla f) = 0,$$

together with Neumann conditions for u and f on the boundary $(\partial\Omega)_{T_\varepsilon}$ and initial conditions $u^0_\varepsilon = u^0 := |\varphi^0|$ and $f^0_\varepsilon = f^0$, where u^0 and f^0 are defined through

$$\varphi^0 = u^0\,(\cos\ f^0, \sin\ f^0).$$

3. Estimates

We define the energy

$$E_\varepsilon(\varphi) := \int_\Omega \left(\frac{\varepsilon}{2}|\nabla\varphi|^2 + \frac{1}{\varepsilon}W(|\varphi|)\right)\,dx,$$

where we adopt the notation $|\nabla\varphi|^2 = \sum_i |\nabla\varphi_i|^2$ and $|\partial_t\varphi|^2 = \sum_i |\partial_t\varphi_i|^2$.

Assumptions on the initial data. We assume that the initial data φ^0_ε are given in the form $u^0_\varepsilon(\cos\ f^0_\varepsilon, \sin\ f^0_\varepsilon)$ and that there exist C_0 and c_0, such that for all ε,

$$\textbf{(A1)} \qquad E_\varepsilon(\varphi^0_\varepsilon) \le C_0,$$

$$\textbf{(A2)} \qquad \inf_{\overline{\Omega}} u^0_\varepsilon(x) \ge c_0,$$

$$\textbf{(A3)} \qquad \sup_{\overline{\Omega}} u^0_\varepsilon(x) \le C_0$$

and

$$\textbf{(A4)} \qquad \sup_{\overline{\Omega}} |f^0_\varepsilon(x)| < \frac{\pi}{2}.$$

Throughout this paper we will denote by C any constant that only depends on n, Ω, T, W as well as on c_0 and C_0.

Remark 3.1. Since E_ε is a Lyapunov functional for equation (E1), we conclude that (A1) is satisfied also at any positive time $t \le T$ and

$$\text{(energy bound)} \qquad \varepsilon\int_{t_1}^{t_2}\int_\Omega |\partial_t\varphi_\varepsilon|^2\,dx\,dt + E_\varepsilon(\varphi_\varepsilon)\Big|_{t_1}^{t_2} = 0$$

for all $0 \le t_1 \le t_2 \le T$. We note that in terms of u_ε and f_ε the energy becomes

$$E_\varepsilon(\varphi_\varepsilon) = \int_\Omega \left(\frac{\varepsilon}{2}\left(|\nabla u_\varepsilon|^2 + u_\varepsilon^2|\nabla f_\varepsilon|^2\right) + \frac{1}{\varepsilon}W(u_\varepsilon)\right)dx$$

and

$$\int_\Omega |\partial_t\varphi_\varepsilon|^2\,dx = \int_\Omega \left(|\partial_t u_\varepsilon|^2 + u_\varepsilon^2|\partial_t f_\varepsilon|^2\right)dx.$$

Lemma 2.1 implies that (A2)–(A4) are satisfied for any positive time $t \le T$. Thus we have uniform L^∞–bounds for $E_\varepsilon(\varphi_\varepsilon)$, u_ε, $\frac{1}{u_\varepsilon}$ and f_ε.

Remark 3.2 (Compactness of u_ε). The energy bound implies that

$$\int_0^T\!\!\int_\Omega |\partial_t g(u_\varepsilon)|\,dx\,dt \quad \text{and} \quad \sup_t \int_\Omega |\nabla g(u_\varepsilon)|\,dx$$

are uniformly bounded, where g is defined by $g'(\lambda) = \sqrt{2W(\lambda)}$ and $g(0) = 0$. (This follows from $(\partial_t, \nabla)g(u_\varepsilon) = \sqrt{2W(u_\varepsilon)}(\partial_t, \nabla)u_\varepsilon$ and the Cauchy–Schwartz inequality.) Thus we may select a subsequence $\varepsilon \to 0$, such that

$$\begin{aligned} u_\varepsilon &\longrightarrow u_0 && \text{pointwise and strongly in } L^1(\Omega_T), \\ (\partial_t, \nabla)g(u_\varepsilon) &\overset{*}{\rightharpoonup} g_0(\partial_t, \nabla)u_0 && \text{in the weak } * \text{ topology of } (C^0(\Omega_T))', \end{aligned}$$

where $g_0 := g(b) - g(a)$.

We restrict any further discussion to this particular subsequence.

Lemma 3.3. *For any $T \geq t_2 > t_1 \geq 0$*

$$\frac{\varepsilon}{2}\int_\Omega |\partial_t \varphi_\varepsilon|^2(\cdot, x)\,dx\Big|_{t_1}^{t_2} \leq \frac{C}{\varepsilon^2} E_\varepsilon(\varphi_\varepsilon)\Big|_{t_2}^{t_1}.$$

Proof. We differentiate the equation (E1) for φ_ε with respect to t, multiply by $\partial_t\varphi_\varepsilon$ and integrate:

$$\begin{aligned} \frac{\varepsilon}{2}\int_\Omega |\partial_t\varphi_\varepsilon|^2\,dx\Big|_{t_1}^{t_2} &+ \varepsilon\int_{t_1}^{t_2}\int_\Omega |\partial_t \nabla\varphi_\varepsilon|^2\,dx\,dt \\ &= -\frac{1}{\varepsilon}\int_{t_1}^{t_2}\int_\Omega \left(W''(u_\varepsilon)|\partial_t u_\varepsilon|^2 + u_\varepsilon W'(u_\varepsilon)|\partial_t f_\varepsilon|^2\right)dx\,dt. \end{aligned}$$

Then we use the energy bound and the boundedness of u_ε and its inverse to conclude. □

Lemma 3.4.

$$\int_0^T\!\!\int_\Omega u_\varepsilon^2 |\nabla f_\varepsilon|^2\,dx\,dt \leq C.$$

Proof. Since $g(u_\varepsilon)$ is uniformly bounded in $BV(\Omega_T)$ and g is monotone increasing, we conclude that for some $\lambda_\varepsilon \in (a + \frac{b-a}{4}, b - \frac{b-a}{4})$ the step function

$$\chi_\varepsilon := \begin{cases} a, & \text{if } u_\varepsilon \leq \lambda_\varepsilon, \\ b, & \text{if } u_\varepsilon > \lambda_\varepsilon, \end{cases}$$

is uniformly bounded in $BV(\Omega_T)$. We multiply the differential equation (E3) by f_ε and integrate:

$$\int_0^T\!\!\int_\Omega u_\varepsilon^2|\nabla f_\varepsilon|^2\,dx\,dt = -\frac{1}{2}\int_0^T\!\!\int_\Omega u_\varepsilon^2 \partial_t f_\varepsilon^2\,dx\,dt.$$

We write u_ε^2 in the term on the right hand side as $\chi_\varepsilon^2 + (u_\varepsilon^2 - \chi_\varepsilon^2)$. We integrate the χ_ε^2–term by parts and find that the result is bounded as a consequence of the estimates of Remark 3.1. Moreover we note that

$$|u_\varepsilon^2 - \chi_\varepsilon^2| \leq C|u_\varepsilon - \chi_\varepsilon| \leq C\sqrt{W(u_\varepsilon)}.$$

Thus the term with the difference $u_\varepsilon^2 - \chi_\varepsilon^2$ can be estimated by

$$C\left(\frac{1}{\varepsilon}\int_0^T\!\!\int_\Omega W(u_\varepsilon)\,dx\,dt\right)^{\frac{1}{2}}\left(\varepsilon\int_0^T\!\!\int_\Omega |\partial_t f_\varepsilon|^2\,dx\,dt\right)^{\frac{1}{2}}$$

and is therefore bounded. This yields the result. □

Lemma 3.5 (Compactness for f_ε). *We may select a further subsequence such that*

$$f_\varepsilon \longrightarrow f_0 \quad \textit{pointwise and in } L^1(\Omega_T),$$

$$\nabla f_\varepsilon \rightharpoonup \nabla f_0 \quad \textit{weakly in } L^2(\Omega_T).$$

Proof. We use the same auxiliary function χ_ε as in the proof of Lemma 3.4. We approximate χ_ε by a smooth function $\tilde\chi_\varepsilon$ such that

$$\int_0^T\int_\Omega |(\partial_t,\nabla)\tilde\chi_\varepsilon|\,dx\,dt \le 2\int_0^T\int_\Omega |(\partial_t,\nabla)\chi_\varepsilon|\,dx\,dt,$$

and

$$\int_0^T\int_\Omega |\chi_\varepsilon - \tilde\chi_\varepsilon|\,dx\,dt \le \varepsilon.$$

The differential equation for f_ε implies that

$$\partial_t(\tilde\chi_\varepsilon^2 f_\varepsilon) = \partial_t\tilde\chi_\varepsilon^2 f_\varepsilon + (\tilde\chi_\varepsilon^2 - u_\varepsilon^2)\partial_t f_\varepsilon + \operatorname{div}(u_\varepsilon^2\nabla f_\varepsilon).$$

In order to prove the compactness, we follow ideas of Alt and Luckhaus [AL]. We use the fact that we have good control of $\partial_t(\tilde\chi_\varepsilon^2 f_\varepsilon)$ in some weak norm, and good control of $\nabla(\tilde\chi_\varepsilon^2 f_\varepsilon)$ in some strong norm.

We first show that for any $\max(2,n) < p < \infty$

$$(*) \qquad \int_0^{T-h} ||(\tilde\chi_\varepsilon^2 f_\varepsilon)(t+h,\cdot) - (\tilde\chi_\varepsilon^2 f_\varepsilon)(t,\cdot)||_{W^{-1,p'}(\Omega)}\,dt \le C\cdot h.$$

For this we note that the term in question is equal to

$$\int_0^{T-h}\sup_{||\zeta||_{W_0^{1,p}(\Omega)}\le 1}\int_\Omega\int_t^{t+h}\left(\left(\partial_t\tilde\chi_\varepsilon^2 f_\varepsilon + (\tilde\chi_\varepsilon^2 - u_\varepsilon^2)\partial_t f_\varepsilon\right)\zeta - u_\varepsilon^2\nabla f_\varepsilon\cdot\nabla\zeta\right)\,dx\,d\tau\,dt.$$

Since $p > n$ and $p > 2$, we may estimate this by

$$\int_0^{T-h}\int_t^{t+h}\int_\Omega\left(|\partial_t\tilde\chi_\varepsilon^2 f_\varepsilon| + |(\tilde\chi_\varepsilon^2 - u_\varepsilon^2)\partial_t f_\varepsilon| + u_\varepsilon^4|\nabla f_\varepsilon|^2\right)\,dx\,d\tau\,dt.$$

We then proceed as in the proof of Lemma 3.4, using the estimates on $\partial_t\tilde\chi_\varepsilon$ and on f_ε to obtain $(*)$.

Next we use the following interpolation inequality: Assume that $1 < p < \infty$. Then there exists some $C > 0$ such that for any smooth function w and any positive ρ

$$||w||_{L^1(\Omega)} \le C\left(\rho\,||\nabla w||_{L^1(\Omega)} + \frac{1}{\rho}\,||w||_{W^{-1,p'}(\Omega)} + \rho\,||w||_{L^\infty(\Omega)}\right).$$

For a proof of this interpolation inequality we refer to [Lu].

We apply this to $w = (\tilde{\chi}_\varepsilon^2 f_\varepsilon)(t+h,\cdot) - (\tilde{\chi}_\varepsilon^2 f_\varepsilon)(t,\cdot)$, integrate in time and use the above estimate $(*)$ as well as the bound on $\nabla\tilde{\chi}_\varepsilon$ to conclude that

$$\int_0^{T-h}\int_\Omega |(\tilde{\chi}_\varepsilon^2 f_\varepsilon)(t+h,x) - (\tilde{\chi}_\varepsilon^2 f_\varepsilon)(t,x)|\,dx\,dt \le C(\rho + \frac{h}{\rho}).$$

We choose $\rho = \sqrt{h}$.

We now may use the Kolmogorov theorem to deduce that $\tilde{\chi}_\varepsilon^2 f_\varepsilon$ is precompact in L^1, and thus that there exists a further subsequence such that $\tilde{\chi}_\varepsilon^2 f_\varepsilon$ converges in L^1 and pointwise. Since $\tilde{\chi}_\varepsilon$ is an L^1–approximation of u_ε and bounded below, and since u_ε converges pointwise (cf Remark 3.2), we infer that f_ε itself has a pointwise limit. Using Lemma 3.4, we may as well assume that ∇f_ε converges weakly. □

Lemma 3.6. *Assume that Ω is the unit ball and that the initial data φ_ε^0 are radially symmetric. Then for all $T > t_2 > t_1 > 0$ and all $1 > r > 0$*

$$\varepsilon^2 |u_\varepsilon^2 f_\varepsilon'(t_2,r)|^2 \le \frac{C}{r^{n-1}}\left(E_\varepsilon(\varphi_\varepsilon)\Big|_{t_2}^{t_1} + \varepsilon^3\int_\Omega |\partial_t\varphi_\varepsilon(t_1,x)|^2\,dx\right)^{\frac{1}{2}}.$$

Proof. We integrate (E3) over the set $\{r < |x| < s\}$ and then take the mean value over $s \in (r, r+\delta)$:

$$\begin{aligned}
|\omega_n r^{n-1} u_\varepsilon^2 f_\varepsilon'(t_2,r)|^2 &= \left|\int_{|x|=r} (u_\varepsilon^2\nabla f_\varepsilon)(t_2,x)\cdot\frac{x}{|x|}\,d\mathcal{H}^{n-1}(x)\right|^2 \\
&\le \left(\frac{1}{\delta}\int_r^{r+\delta}\int_{|x|=s}(u_\varepsilon^2|\nabla f_\varepsilon|)(t_2,x)d\mathcal{H}^{n-1}(x)\,ds\right. \\
&\qquad \left. + \frac{1}{\delta}\int_r^{r+\delta}\int_{r<|x|<s}(u_\varepsilon^2|\partial_t f_\varepsilon|)(t_2,x)\,dx\,ds\right)^2 \\
&\le C r^{n-1}\left(\frac{1}{\delta}\int_\Omega (u_\varepsilon^4|\nabla f_\varepsilon|^2)(t_2,x)\,dx + \delta\int_\Omega (u_\varepsilon^4|\partial_t f_\varepsilon|^2)(t_2,x)\,dx\right).
\end{aligned}$$

But u_ε is uniformly bounded in $L^\infty(\Omega_T)$, so that (due to the energy estimate) the first term is uniformly bounded in time by $\frac{C}{\delta\varepsilon}$, and by Lemma 3.3

$$\varepsilon\int_\Omega (u_\varepsilon^4|\partial_t f_\varepsilon|^2)(t_2,x)\,dx \le \frac{C}{\varepsilon^2}\,E_\varepsilon(\varphi_\varepsilon)|_{t_2}^{t_1} + \varepsilon\int_\Omega |\partial_t\varphi_\varepsilon(t_1,x)|^2\,dx.$$

Thus choosing $\frac{1}{\delta^2} = \frac{C}{\varepsilon^2}E_\varepsilon(\varphi_\varepsilon)\Big|_{t_2}^{t_1} + \varepsilon\int_\Omega |\partial_t\varphi_\varepsilon|^2(t_1,x)\,dx$ proves the lemma, since u_ε is uniformly bounded below. □

Remark 3.7. As a by-product of the above proof and Lemma 3.4 we obtain

$$\int_0^T |u_\varepsilon^2 f_\varepsilon'|^2(t,r)\,dt \le \frac{C}{r^{n-1}}\frac{1}{\sqrt{\varepsilon}},$$

by choosing $\delta = \sqrt{\varepsilon}$.

Lemma 3.8. *Assume that Ω is a ball and that the initial data are radially symmetric. Then*

$$\varepsilon\int_0^T\int_0^1 |\nabla\varphi_\varepsilon|^2 r^{n-3}\,dr\,dt + \frac{1}{\varepsilon}\int_0^T\int_0^1 W(|\varphi_\varepsilon|)r^{n-3}\,dr\,dt \le C.$$

Proof. The differential equation (E1) written in the radial variable reads

$$\varepsilon\partial_t\varphi_\varepsilon - \varepsilon\varphi_\varepsilon'' - \varepsilon\frac{n-1}{r}\varphi_\varepsilon' + \frac{1}{\varepsilon}\tilde{W}_{,\varphi}(\varphi_\varepsilon) = 0.$$

We multiply this by $-r^{n-2}\varphi_\varepsilon'$, integrate the resulting identity over $(0,s)\subset(0,1)$ and finally integrate by parts. This yields

$$\begin{aligned}\varepsilon\frac{n}{2}\int_0^s(\varphi_\varepsilon')^2r^{n-3}\,dr + \frac{\varepsilon}{2}|\varphi_\varepsilon'|^2s^{n-2} + \frac{n-2}{\varepsilon}\int_0^s W(u_\varepsilon)r^{n-3}\,dr \\ = \frac{1}{\varepsilon}W(u_\varepsilon)s^{n-2} + \varepsilon\int_0^s\partial_t\varphi_\varepsilon\varphi_\varepsilon' r^{n-2}\,dr.\end{aligned}$$

We use Hölder's inequality for the last term, and obtain

$$\begin{aligned}\varepsilon\frac{n-1}{2}\int_0^s(\varphi_\varepsilon')^2r^{n-3}\,dr + \frac{n-2}{\varepsilon}\int_0^s W(u_\varepsilon)r^{n-3}\,dr \\ \le \frac{1}{\varepsilon}W(u_\varepsilon)s^{n-2} + \varepsilon\int_0^s|\partial_t\varphi_\varepsilon|^2r^{n-1}\,dr.\end{aligned}$$

Now we take the mean value of the right hand side over $s\ge\frac{1}{2}$ and bound the left hand side from below by the integral over $(0,\frac{1}{2})$, and obtain

$$\begin{aligned}\varepsilon\frac{n-1}{2}\int_0^T\int_0^{\frac{1}{2}}(\varphi_\varepsilon')^2r^{n-3}\,dr\,dt + \frac{n-2}{\varepsilon}\int_0^T\int_0^{\frac{1}{2}}W(u_\varepsilon)r^{n-3}\,dr\,dt \\ \le 2\int_0^T\int_{\frac{1}{2}}^1\frac{1}{\varepsilon}W(u_\varepsilon)s^{n-2}\,ds\,dt + \varepsilon\int_0^T\int_0^1|\partial_t\varphi_\varepsilon|^2r^{n-1}\,dr\,dt \le C.\end{aligned}$$

This implies the assertion of the lemma. □

4. Definition of the ε–interfaces and an approximation of the solution

In this and all the following sections we will assume that the domain Ω is the unit ball and all functions are radially symmetric. We will denote by $'$ the derivative with respect to the radial variable.

The energy $E_\varepsilon(\varphi_\varepsilon)$ is a uniformly bounded and decreasing function of time. Therefore a subsequence $E_\varepsilon(\varphi_\varepsilon)$ converges pointwise to a limit E_0, which is also decreasing. So for any $\eta>0$, the set

$$N(\eta) := \{t\in(0,T) \mid \operatorname{ess}\inf_{s<t}E_0(s) - \operatorname{ess}\sup_{s>t}E_0(s) > \eta\},$$

which is the set of points where E_0 has a jump of at least height η, is finite.

Let c_{crit} be a small positive number and $a<\lambda_{crit}^-<d<\lambda_{crit}^+<b$ two level values such that in particular

$$W(\lambda) > c_{crit} \qquad \text{for all } \lambda_{crit}^- < \lambda < \lambda_{crit}^+,$$

and such that c_{crit} and $d-\lambda_{crit}^-$ and λ_{crit}^+-d are as close to zero as required in Proposition 4.9 and Proposition 4.10. (We recall that d is the maximum point of W in the spinodal region and was introduced in hypothesis (W1).)

Proposition 4.1. *For any* $\frac{1}{2}>R_0>0$ *there is an* $\eta(R_0)$ *such that for any* $0<t_0\notin N(\eta(R_0))$*, there exist* $T_0=T_0(t_0,R_0)$ *and* $\varepsilon_0=\varepsilon_0(t_0,R_0)$ *such that*

$$\sup_{t_0-T_0<t<t_0+T_0}\left\|\varepsilon^2|u_\varepsilon'|^2 - 2W(u_\varepsilon)\right\|_{L^\infty(R_0,1)}(t) \le c_{crit} \qquad \textit{for all } \varepsilon\le\varepsilon_0.$$

Proof. If $0 < t_0 \notin N(\eta)$, then there exists $T_0 = T_0(t_0, \eta)$ such that $E_\varepsilon(\varphi_\varepsilon)|_{t_0-T_0}^{t_0+T_0} \le \eta$ for all $\varepsilon \le \varepsilon_0 = \varepsilon_0(t_0, \eta)$. To see this, define T_ε by $E_\varepsilon(\varphi_\varepsilon)|_{(t_0-T_\varepsilon)+}^{t_0+T_\varepsilon} = \eta$ or, if there is no such T_ε, by $+\infty$. Then by the definition of $N(\eta)$ we find that T_ε is bounded below. Define T_0 to be this lower bound.

Let $t \in (t_0 - \frac{T_0}{2}, t_0 + T_0)$.

We now use the differential equation (E1), written in the radial variable:

$$\varepsilon \partial_t \varphi_\varepsilon - \varepsilon \varphi_\varepsilon'' - \varepsilon \frac{n-1}{r} \varphi_\varepsilon' + \frac{1}{\varepsilon} \tilde{W}_{,\varphi}(\varphi_\varepsilon) = 0.$$

Choose $r > R_0$ arbitrarily. We multiply the above differential equation by φ_ε', and for any $s \in (R_0, 1)$ we integrate over either (r, s) or (s, r). Next take absolute values and integrate over $s \in (R_0, 1)$. Then we use Hölder's inequality and the bound on the energy to arrive at

$$\begin{aligned}
&\left| \varepsilon^2 |\varphi_\varepsilon'|^2 - 2\tilde{W}(\varphi_\varepsilon) \right| (t, r) \\
\le & C(R_0) \left(\varepsilon \int_\Omega \left(\frac{\varepsilon}{2} |\nabla \varphi_\varepsilon|^2 \ + \ \frac{1}{\varepsilon} W(|\varphi_\varepsilon|) \right) dx + \left(\varepsilon^4 \int_\Omega |\partial_t \varphi_\varepsilon|^2 \, dx \int_\Omega |\nabla \varphi_\varepsilon|^2 \, dx \right)^{\frac{1}{2}} \right. \\
& \left. \qquad + \varepsilon^2 \int_\Omega |\nabla \varphi_\varepsilon|^2 \, dx \right) (t) \\
\le & C(R_0) \left(\varepsilon + \left(\varepsilon^3 \int_\Omega |\partial_t \varphi_\varepsilon|^2 (t, x) \, dx \right)^{\frac{1}{2}} \right).
\end{aligned}$$

As a consequence

$$\begin{aligned}
&\left| \varepsilon^2 |u_\varepsilon'|^2 - 2W(u_\varepsilon) \right| (t, r) \\
&\qquad \le C(R_0) \left(\varepsilon + \left(\varepsilon^3 \int_\Omega |\partial_t \varphi_\varepsilon|^2 (t, x) \right)^{\frac{1}{2}} \right) dx + \varepsilon^2 (u_\varepsilon^2 |f_\varepsilon'|^2)(t, r).
\end{aligned}$$

We use Lemma 3.3 with $t_2 = t > t_1$ to estimate the $\partial_t \varphi_\varepsilon$–term and Lemma 3.6 to estimate the f_ε'–term. We find that

$$\begin{aligned}
&\left| \varepsilon^2 |u_\varepsilon'|^2 - 2W(u_\varepsilon) \right| (t, r) \\
&\qquad \le C(R_0) \left(\varepsilon + \left(E_\varepsilon(\varphi_\varepsilon)|_t^{t_1} \right)^{\frac{1}{2}} + \left(\varepsilon^3 \int_\Omega |\partial_t \varphi_\varepsilon|^2 (t_1, x) \, dx \right)^{\frac{1}{2}} \right)
\end{aligned}$$

for all $r > R_0$ and all $t > t_1$.

Since $t > t_0 - \frac{T_0}{2}$ we may choose $t_1 \in (t_0 - T_0, t_0 - \frac{T_0}{2})$, and by the definition of T_0 we obtain

$$\left| \varepsilon^2 |u_\varepsilon'|^2 - 2W(u_\varepsilon) \right| (t, r) \le C(R_0) \left(\varepsilon + \eta^{\frac{1}{2}} + \left(\varepsilon^3 \int_\Omega |\partial_t \varphi_\varepsilon|^2 (t_1, x) \, dx \right)^{\frac{1}{2}} \right).$$

Taking the mean value over $t_1 \in (t_0 - T_0, t_0 - \frac{T_0}{2})$, using Hölder's inequality and the energy bound for $\partial_t \varphi_\varepsilon$ finally yields

$$\left| \varepsilon^2 |u_\varepsilon'|^2 - 2W(u_\varepsilon) \right| (t, r) \le C(R_0) \left(\varepsilon + \eta^{\frac{1}{2}} + \left(\frac{2}{T_0} \right)^{\frac{1}{2}} \varepsilon \right).$$

We define $\eta(R_0)$ by $C(R_0)\eta^{\frac{1}{2}} = \frac{c_{crit}}{2}$ and choose $\varepsilon_0(t_0, R_0)$ small enough in order to conclude. □

Remark 4.2. As a byproduct of the above proof we obtain

$$\int_0^T \left\|\varepsilon^2 |u_\varepsilon'|^2 - 2W(u_\varepsilon)\right\|_{L^\infty(R_0,1)}(t)\,dt \le C(R_0)\varepsilon.$$

Definition 4.3. For any $R_0 > 0$ we redefine $N(R_0)$ by $N(\eta(R_0))$, and we let

$$A(R_0) := \bigcup_{0<t\notin N(R_0)} (t_0 - T_0(t_0,R_0), t_0 + T_0(t_0,R_0))$$

and note that its complement in $[0,T]$ consists of only finitely many points. We choose an increasing sequence of open sets $B = B_m$ that are compact in $A(R_0)$ and such that $A(R_0) = \bigcup B_m$.

Furthermore we define for any $R_0 > 0$ and any $\lambda \in (\lambda_{crit}^-, \lambda_{crit}^+)$ the ε–free boundary

$$\Gamma_\varepsilon(R_0,\lambda) := \{(t,r) \in (0,T) \times (R_0,1] \mid u_\varepsilon(t,r) = \lambda\}.$$

Corollary and Definition 4.4. *For any $R_0 > 0$ there exists $\lambda_\varepsilon = \lambda_\varepsilon(R_0) \in (\lambda_{crit}^-, \lambda_{crit}^+)$, such that for any $B = B_m$ as in Definition 4.3 and for any $\varepsilon \le \varepsilon_0(B,R_0)$ the set*

$$\Gamma_\varepsilon(R_0,\lambda_\varepsilon) \cap B \times (R_0,1]$$

consists of a collection of graphs

$$r_\varepsilon^i \,:\, I_\varepsilon^i \subset B \longrightarrow (R_0,1) \qquad \text{for } i = 1,\dots,M_\varepsilon,$$

with $r_\varepsilon^i > r_\varepsilon^{i+1}$ and $r_\varepsilon^i = R_0$ on $\partial I_\varepsilon^i \cap B$. In addition, $M_\varepsilon \le C(R_0)$ and

$$\int_B \sum_i \left|\frac{d}{dt} r_\varepsilon^i(t)\right|^2 dt \le C(R_0).$$

Proof. By Proposition 4.1 all level sets with value in the region $(\lambda_{crit}^-, \lambda_{crit}^+)$ are graphs. So we may define $r_\varepsilon^i(t,\lambda)$ by $u_\varepsilon(t, r_\varepsilon^i(t,\lambda)) = \lambda$. Using the co–area formula, the bound for $\frac{1}{|u_\varepsilon'|}$ given by Proposition 4.1 and the energy bound for $\partial_t u_\varepsilon$, we can select a particular value $\lambda_\varepsilon \in (\lambda_{crit}^-, \lambda_{crit}^+)$ so that $r_\varepsilon^i(t,\lambda_\varepsilon)$ is uniformly bounded in $H^{1,2}$ as follows:

$$\begin{aligned}\int_B \int_{\lambda_{crit}^-}^{\lambda_{crit}^+} \sum_i |\partial_t r_\varepsilon^i(t,\lambda)|^2 \,d\lambda\,dt &\le \int_B \int_{W(u_\varepsilon)>c_{crit}} \frac{|\partial_t u_\varepsilon(t,r)|^2}{|u_\varepsilon'(t,r)|}\,dr\,dt \\ &\le \frac{1}{||u_\varepsilon'||_{L^\infty}} \int_0^T\!\!\int_\Omega |\partial_t u_\varepsilon|^2\,dx\,dt \le C(R_0) \qquad \text{for } \varepsilon \le \varepsilon_0(B,R_0).\end{aligned}$$

Then we define for any R_0 the set $\Lambda(B) := \{\lambda \mid \sum_i \int_B |\dot r_\varepsilon^i(t,\lambda)|\,dt \le C(R_0)$ for all $\varepsilon \le \varepsilon_0(B,R_0)\}$ and show that $\Lambda(B') \subset \Lambda(B)$ if $B \subset B'$ and that $|\Lambda(B)| \ge c_0(R_0)$. Thus we may choose $\lambda_\varepsilon \in \bigcap_B \Lambda(B)$, and we define $r_\varepsilon^i(t) := r_\varepsilon^i(t,\lambda_\varepsilon)$.

Proposition 4.1 and the Neumann condition for u_ε imply furthermore that this level set will never hit $\partial\Omega$. In addition, combining Proposition 4.1 and the bound for the energy, we find that the number of interfaces is uniformly bounded. □

Definition 4.5. Let R_0 and $B = B_m$ be as in Definition 4.3. For $r > R_0$ and $t \in B$ we define

$$z := \frac{r}{\varepsilon}, \quad U_\varepsilon(t,z) := u_\varepsilon(t,r) \quad \text{and} \quad z_\varepsilon^i(t) := \frac{r_\varepsilon^i(t)}{\varepsilon}\ (i = 1,\dots,M_\varepsilon).$$

Next we define $\tau = \tau_\varepsilon$ to be either 0 or 1, according to whether u_ε is close to a or b on $\partial\Omega$. This is locally constant in time. In addition we define the stationary wave solution $Q : \mathbf{R} \to \mathbf{R}$ by

$$Q'' = W'(Q), \quad Q(-\infty) = a, \quad Q(+\infty) = b, \quad Q(0) = d,$$

and $Q_\varepsilon(z) := Q(z + \mu_\varepsilon)$, where μ_ε is such that $Q(\mu_\varepsilon) = \lambda_\varepsilon$. We choose a smooth, increasing function Ξ with $\Xi(z) = 1$ if $z > 1$ and $\Xi(z) + \Xi(-z) = 1$. Furthermore we define the first order approximation

$$V_\varepsilon(t,z) := \sum_{i=1}^{M_\varepsilon} \Xi_\varepsilon^i(t,z) V_\varepsilon^i(t,z) := \sum_{i=1}^{M_\varepsilon} \Xi_\varepsilon^i(t,z) Q_\varepsilon\left((-1)^{i+\tau_\varepsilon}[z - z_\varepsilon^i(t)]\right),$$

where Ξ_ε^i is a partition of unity, given by

$$\Xi_\varepsilon^i(t,z) := \begin{cases} \Xi\left(z - \dfrac{z_\varepsilon^i(t) + z_\varepsilon^{i+1}(t)}{2}\right), & \text{if } z \le z_\varepsilon^i(t), \\ \Xi\left(\dfrac{z_\varepsilon^i(t) + z_\varepsilon^{i-1}(t)}{2} - z\right), & \text{if } z \ge z_\varepsilon^i(t), \end{cases}$$

for $i = 2, ..., M_\varepsilon - 1$, and by

$$\Xi_\varepsilon^1(t,z) := \begin{cases} \Xi\left(z - \dfrac{z_\varepsilon^1(t) + z_\varepsilon^2(t)}{2}\right), & \text{if } z \le z_\varepsilon^1(t), \\ 1, & \text{if } z \ge z_\varepsilon^1(t), \end{cases}$$

$$\Xi_\varepsilon^{M_\varepsilon}(t,z) := \begin{cases} 1, & \text{if } z \le z_\varepsilon^{M_\varepsilon}(t), \\ \Xi\left(\frac{z_\varepsilon^{M_\varepsilon}(t) + z_\varepsilon^{M_\varepsilon - 1}(t)}{2} - z\right), & \text{if } z \ge z_\varepsilon^{M_\varepsilon}(t). \end{cases}$$

Lemma 4.6 below implies that this is well defined.

Finally we define the difference

$$Y_\varepsilon := U_\varepsilon - V_\varepsilon.$$

In the remainder of this section we will show that Y_ε is small.

Lemma 4.6. *Let R_0 and $B = B_m$ be as in Definition 4.3. Then there exists a constant $M(c_{crit})$, that converges to zero as c_{crit} goes to zero, such that for all $\varepsilon \le \varepsilon_0(B, R_0)$*

$$\sup_B \|Y_\varepsilon\|_{L^\infty(R_0/\varepsilon, 1/\varepsilon)} \le M(c_{crit})$$

and

$$\sup_B |z_\varepsilon^i - z_\varepsilon^{i+1}| \ge \frac{1}{M(c_{crit})}$$

for $i = 1, ..., M_\varepsilon - 1$. In addition, for almost all $t \in B$

$$|z_\varepsilon^i - z_\varepsilon^{i+1}| \longrightarrow \infty.$$

Proof. Proposition 4.1 implies a differential equation for U_ε:

$$|U_\varepsilon'| = \sqrt{2W(U_\varepsilon)} + K_\varepsilon,$$

where $||K_\varepsilon||_{L^\infty} \le c_{crit}$ and $\int_B ||K_\varepsilon||_{L^\infty} \le C(R_0)\varepsilon$. Lemma 4.6 follows by direct integration of this equation. $\square$

Remark and Definition 4.7. The difference Y_ε satisfies the differential equation

$$-Y_\varepsilon'' + W''(V_\varepsilon)Y_\varepsilon = G_\varepsilon - H_\varepsilon - \int_{V_\varepsilon}^{V_\varepsilon+Y_\varepsilon} W'''(\lambda)\,(V_\varepsilon + Y_\varepsilon - \lambda)\ d\lambda,$$

where

$$G_\varepsilon(t,z) := \left[-\varepsilon^2\partial_t u_\varepsilon + \varepsilon^2\frac{n-1}{r}u_\varepsilon' - \varepsilon^2 u_\varepsilon|f_\varepsilon'|^2\right](t,\varepsilon z)$$

and

$$H_\varepsilon := \sum_i \Bigg((\Xi_\varepsilon^i)''V_\varepsilon^i + 2(\Xi_\varepsilon^i)'(V_\varepsilon^i)' + \Xi_\varepsilon^i W''(V_\varepsilon^i)(V_\varepsilon - V_\varepsilon^i) + \Xi_\varepsilon^i \int_{V_\varepsilon^i}^{V_\varepsilon} W'''(\lambda)(V_\varepsilon - \lambda)\,d\lambda\Bigg).$$

In addition, $Y_\varepsilon(t, z_\varepsilon^i(t)) = 0$ for $i = 1, ..., M_\varepsilon$.

Lemma 4.8. *Let R_0 and $B = B_m$ be as in Definition 4.3. Then for $\varepsilon \le \varepsilon_0(B, R_0)$*

$$\int_B \int_{R_0/\varepsilon}^{1/\varepsilon} |G_\varepsilon|^2\,dz\,dt \le C(R_0)\varepsilon^{3/2},$$

and for $i = 2, ..., M_\varepsilon$

$$\int_B \int_{z_\varepsilon^i(t)}^{z_\varepsilon^{i-1}(t)} |H_\varepsilon|^2\,dz\,dt \le C(R_0)\int_B \frac{\varepsilon^2}{(r_\varepsilon^{i-1} - r_\varepsilon^i)^2(t)}\,dt,$$

$$\int_B \int_{z_\varepsilon^i(t)}^{z_\varepsilon^{i-1}(t)} \int_{V_\varepsilon}^{V_\varepsilon+Y_\varepsilon} |W'''(\lambda)\,(V_\varepsilon + Y_\varepsilon - \lambda)|\ d\lambda\ |Y_\varepsilon|\ dz\,dt \le C(R_0)M(c_{crit})\int_B \int_{z_\varepsilon^i(t)}^{z_\varepsilon^{i-1}(t)} |Y_\varepsilon|^2\,dz\,dt,$$

and the same is true for $i = 1$ and $i = M_\varepsilon + 1$, when formally setting $r_\varepsilon^0 = 1$ and $r_\varepsilon^{M_\varepsilon+1} = R_0$, and hence $z_\varepsilon^0 = \frac{1}{\varepsilon}$ and $z_\varepsilon^{M_\varepsilon+1} = \frac{R_0}{\varepsilon}$.

Proof. The first inequality follows from the bounds derived in Remarks 3.1 and 3.7.

The second inequality follows directly from the precise knowledge of the stationary wave solution Q and of the partition of unity Ξ. Indeed the sum is only taken over two integers at a time, and in the support of $(\Xi_\varepsilon^i)' \cap (z_\varepsilon^i, z_\varepsilon^{i-1})$ the difference $|V_\varepsilon^i - V_\varepsilon^{i-1}|$ is bounded by $\frac{C}{z_\varepsilon^{i-1} - z_\varepsilon^i}$.

The last inequality is straightforward, since $\|Y_\varepsilon\|_{L^\infty(B\times(R_0/\varepsilon,1/\varepsilon))} \le M(c_{crit})$ by Lemma 4.6. ☐

Proposition 4.9. *Let R_0 and $B = B_m$ be as in Definition 4.3. If c_{crit} is sufficiently small and λ_{crit}^- and λ_{crit}^+ are sufficiently close to d, then there are positive constants η_1 and η_2 such that for any $i = 2, ..., M_\varepsilon$*

$$\int_{z_\varepsilon^i(t)}^{z_\varepsilon^{i-1}(t)} [-Y_\varepsilon'' + W''(V_\varepsilon)Y_\varepsilon]\,Y_\varepsilon\,dz \ge \eta_1 \int_{z_\varepsilon^i(t)}^{z_\varepsilon^{i-1}(t)} |Y_\varepsilon'|^2\,dz + \eta_2 \int_{z_\varepsilon^i(t)}^{z_\varepsilon^{i-1}(t)} Y_\varepsilon^2\,dz$$

and

$$\int_{z_\varepsilon^1(t)}^{1/\varepsilon} \left[-Y_\varepsilon'' + W''(V_\varepsilon)Y_\varepsilon\right] Y_\varepsilon \, dz \geq \eta_1 \int_{z_\varepsilon^1(t)}^{1/\varepsilon} |Y_\varepsilon'|^2 \, dz + \eta_2 \int_{z_\varepsilon^1(t)}^{1/\varepsilon} Y_\varepsilon^2 \, dz - (Y_\varepsilon Y_\varepsilon')(\frac{1}{\varepsilon})$$

and

$$\int_{R_0/\varepsilon}^{z_\varepsilon^{M_\varepsilon}(t)} \left[-Y_\varepsilon'' + W''(V_\varepsilon)Y_\varepsilon\right] Y_\varepsilon Z \, dz$$
$$\geq \eta_1 \int_{R_0/\varepsilon}^{z_\varepsilon^{M_\varepsilon}(t)} |Y_\varepsilon'|^2 Z \, dz + \eta_2 \int_{R_0/\varepsilon}^{z_\varepsilon^{M_\varepsilon}(t)} Y_\varepsilon^2 Z \, dz - \frac{1}{2} \int_{R_0/\varepsilon}^{z_\varepsilon^{M_\varepsilon}(t)} Y_\varepsilon^2 Z'' \, dz$$

for any smooth, positive test function Z with $Z(\frac{R_0}{\varepsilon}) = Z'(\frac{R_0}{\varepsilon}) = 0$.

Proof. This proposition follows from an idea of Berger and Fraenkel (1970). We sketch the proof for the first inequality: Define $I_\varepsilon := \{z \mid W''(V_\varepsilon) \leq \eta_2\}$. If c_{crit}, $d - \lambda_{crit}^-$ and $\lambda_{crit}^+ - d$ as well as η_2 are sufficiently small, then by Definition 4.5 and Lemma 4.6

$$I_\varepsilon = \bigcup_i (z_\varepsilon^i(t), z_\varepsilon^i(t) + l^+) \cup (z_\varepsilon^{i-1}(t) - l^-, z_\varepsilon^{i-1}(t)) + o(1),$$

where $(-l^-, l^+) := \{z \mid W''(Q) \leq 0\}$. The $o(1)$–term converges to zero with c_{crit}, $d - \lambda_{crit}^-$ and $\lambda_{crit}^+ - d$ and η_2 independently of ε. Consequently

$$\int \left(|Y_\varepsilon'|^2 + W''(V_\varepsilon)Y_\varepsilon^2\right) dz \geq \int |Y_\varepsilon'|^2 \, dz - (\eta_2 + |\min W''|) \int_I Y_\varepsilon^2 \, dz + \eta_2 \int Y_\varepsilon^2 \, dz$$
$$\geq \left(1 - \left(\frac{1}{2}\max{}^2(l^-, l^+) + o(1)\right)(\eta_2 + |\min W''|)\right) \int |Y_\varepsilon'|^2 \, dz + \eta_2 \int Y_\varepsilon^2 \, dz,$$

where we have used the fact that by construction $Y_\varepsilon(t, z_\varepsilon^{i-1}(t)) = Y_\varepsilon(t, z_\varepsilon^i(t)) = 0$. It turns out that $1 - \frac{1}{2}\max^2(l^-, l^+)|\min W''|$ is positive, and thus the proposition follows. For more details, we refer to the appendix of [BSt]. □

Proposition 4.10. *Let R_0 and $B = B_m$ be as in Definition 4.3 and assume that c_{crit} is sufficiently small and λ_{crit}^- and λ_{crit}^+ are sufficiently close to d. Then for $i = 2, ..., M_\varepsilon$ and any $B' \subset B$*

$$\int_{B'} \|Y_\varepsilon(t, \cdot)\|^2_{H^{1,2}(z_\varepsilon^i(t), z_\varepsilon^{i-1}(t))} \, dt \leq C(R_0) \left(\int_{B'} \frac{\varepsilon^2}{|r_\varepsilon^{i-1}(t) - r_\varepsilon^i(t)|^2} \, dt + \varepsilon^{3/2} \right),$$

and

$$\int_{B'} \|Y_\varepsilon(t, \cdot)\|^2_{H^{1,\infty}(z_\varepsilon^i(t), z_\varepsilon^{i-1}(t))} \, dt \leq C(R_0) \left(\int_{B'} \frac{\varepsilon^2}{|r_\varepsilon^{i-1}(t) - r_\varepsilon^i(t)|^2} \, dt + \varepsilon^{3/2} \right).$$

The same is true, if a) $(z_\varepsilon^i(t), z_\varepsilon^{i-1}(t))$ is replaced by $(z_\varepsilon^1(t), 1/\varepsilon)$ and the right hand side by $C(R_0, \delta)\varepsilon^{\frac{3}{2}}$ and B' is any set where $\frac{1}{\varepsilon} - z_\varepsilon^1(t) \geq \frac{\delta}{\varepsilon}$, or b) $(z_\varepsilon^i(t), z_\varepsilon^{i-1}(t))$ is replaced by $(\frac{R_0+\delta}{\varepsilon}, z_\varepsilon^{M_\varepsilon}(t))$ and the right hand side by $C(R_0, \delta)\varepsilon^{\frac{3}{2}}$ and B' is any set where $z_\varepsilon^{M_\varepsilon}(t) - \frac{R_0+2\delta}{\varepsilon} \geq 0$.

Proof. We multiply the differential equation in Remark 4.7 by Y_ε and integrate over $(z_\varepsilon^i(t), z_\varepsilon^{i-1}(t))$ or $(z_\varepsilon^1(t), \frac{1}{\varepsilon})$ or $(\frac{R_0+\delta}{\varepsilon}, z_\varepsilon^{M_\varepsilon}(t))$, resp. In the first case we use the first estimate of Proposition 4.9 to obtain the lower bound of the left hand side. Hölder's inequality and Lemma 4.8 imply the bound for the right hand side. In the two other cases we apply the second or third inequality of Proposition 4.9, resp. In the second case the extra boundary term is exponentially small. In the third case the extra term (we choose Z as a smooth cut–off function with supp $Z' \subset (\frac{R_0}{\varepsilon}, \frac{R_0+\delta}{\varepsilon})$) is bounded by $C\frac{\varepsilon^2}{\delta^2}\int_{B'}\int_{R_0/\varepsilon}^{(R_0+\delta)/\varepsilon}|Y_\varepsilon|^2\,dz\,dt$. But in the set of integration V_ε is exponentially close to a or b, which in return implies that $|Y_\varepsilon| = |U_\varepsilon - V_\varepsilon| \le C\sqrt{W(U_\varepsilon)}$. This implies that the extra term is of order ε^2.

This proves the first part of this proposition.

Since R_0 is positive the problem is one dimensional, and the $H^{1,\infty}$–norm can be controlled by the $H^{2,2}$–norm. But this norm can be calculated using the equation and the estimates already obtained in this proposition. □

5. Passage to the limit in equation (E2)

In this section we derive the limit equation of (E2). First we define the limit interface.

Definition 5.1. We choose a sequence $R_0 \to 0$. For any $R_0 > 0$ in this sequence we define the limit interface $\Gamma(R_0)$ as follows.

Let $A(R_0)$ be as in Definition 4.3. Then by Corollary 4.4, we know that for a subsequence $\varepsilon \to 0$ there exist limits

$$\bar{r}^i \;:\; I^i \subset A(R_0) \longrightarrow (R_0, 1) \qquad \text{for } i = 1, ..., M_0,$$

such that $r_\varepsilon^i \longrightarrow \bar{r}^i$ strongly in $C^0_{loc}(A(R_0))$ and $\frac{d}{dt}r^i_\varepsilon \rightharpoonup \frac{d}{dt}\bar{r}^i$ weakly in $L^2_{loc}(A(R_0))$ for $i = 1, ..., M_0$. In addition, $\bar{r}^i \ge \bar{r}^{i+1}$ and $\bar{r}^i = R_0$ on $\partial I^i \cap A(R_0)$. So we set

$$\Gamma(R_0) := \left\{(t, r) \in A(R_0) \times (R_0, 1) \mid r = \bar{r}^i(t) \text{ for some } i \in \{1, ..., M_0\}\right\}.$$

(We note that by a diagonal argument the above subsequence can be chosen independently of R_0.) Now let $(t_0, r_0) \in \Gamma(R_0)$ be arbitrary. Then there exists a minimal k such that $r_0 = \bar{r}^k(t_0)$. We define the multiplicity $m_0(t_0, r_0)$ to be the number of ε–interfaces $r^i_\varepsilon(t_0)$, which converge to r_0. Because the ε–interfaces are locally uniformly bounded in $H^{1,2}$, there exists a neighbourhood (t_1, t_2) of t_0, such that for $i = k, ..., k + m_0 - 1$,

$$r^i_\varepsilon \to \bar{r}^i \quad \text{uniformly in } (t_1, t_2) \text{ and weakly in } H^{1,2}(t_1, t_2)$$

and for some $\delta > 0$ and $\varepsilon \le \varepsilon_0(t_0, r_0)$

$$r^k_\varepsilon - r^{k+1}_\varepsilon \ge \delta \quad \text{and} \quad r^{k+m_0}_\varepsilon - r^{k+m_0-1}_\varepsilon \ge \delta \quad \text{uniformly in } (t_1, t_2).$$

If $m_0(t_0, r_0)$ is odd, we define the normal $\nu(t_0, r_0)$ by $\operatorname{sgn} u'_\varepsilon(t_0, r^k_\varepsilon(t_0))$, which for $\varepsilon \le \varepsilon_0(t_0, r_0)$ does not depend on ε as a consequence of Proposition 4.1 and since u_ε has a limit in L^1 (cf. Remark 3.2). If $m_0(t_0, r_0)$ is even, we define $\nu(t_0, r_0) := 0$. We note that ν gives the direction of the jump in the whole neighbourhood of (t_1, t_2).

Proposition 5.2. *Let $(t_0, r_0) \in \Gamma(R_0)$ be an interfacial point, and let k, t_1 and t_2 as well as $m_0(t_0, r_0)$ be as in Definition 5.1. Then*

$$\sum_{i=k}^{k+m_0-1}\left(\frac{d}{dt}\bar{r}^i + \frac{n-1}{\bar{r}^i}\right) = 0 \qquad \textit{almost everywhere in } (t_1, t_2).$$

Proof. We write the differential equation (E2) in the rescaled variable z, multiply by U'_ε and integrate. Using the approximation of Section 4, we can pass to the limit in all terms. For the f_ε–term, we use the fact that ∇f_ε is uniformly bounded in L^2. For more details we refer to the Appendix. □

Corollary 5.3. *The graphs $\bar{r}^i : I^i \subset A(R_0) \longrightarrow (R_0, 1)$ with $i = 1, ..., M_0$ as in Definition 5.1 move by mean curvature:*

$$\frac{d}{dt}\bar{r}^i + \frac{n-1}{\bar{r}^i} = 0 \qquad \textit{almost everywhere in } I^i.$$

For any $i = 1, ..., M_0$ the multiplicity $m_0(\cdot, \bar{r}^i)$ and the normal $\nu(\cdot, \bar{r}^i)$ are constant along the i–th interface in any connected component of $A(R_0)$.

We call interfaces with $\nu \neq 0$ true interfaces, whereas those with $\nu = 0$ are "phantom"-interfaces, since they do not separate different phases of u_0.

Proof. Consider a set of time points where a fixed collection $i = j, .., l$ of ε–interfaces have the same limit: $\bar{r}^j = ... = \bar{r}^l$. Then almost everywhere in this set, the derivatives with respect to time of the limit interfaces $\bar{r}^i$ coincide and Proposition 5.2 implies the first result. Since all the interfaces move by mean curvature, they can never meet, and so the multiplicity and the normals have to be locally constant. □

Lemma and Definition 5.4. *Let $t_0 \notin A(R_0)$. Then for all $i \in \{1, ..., M_0\}$ with $\partial I^i \cap A(R_0) = \emptyset$ and $\lim_{s \nearrow t_0} \nu(s, \bar{r}^i(s)) \neq 0$, there exists $j \in \{i, ..., M_0\}$ with $\lim_{s \searrow t_0} \nu(s, \bar{r}^j(s)) \neq 0$ such that*

$$\lim_{s < t_0, s \to t_0} \bar{r}^i(s) = \lim_{s > t_0, s \to t_0} \bar{r}^j(s),$$

i.e. any true interface has a continuation across points $t_0 \notin A(R_0)$. Therefore we may define continuous

$$R^j : D^j \subset (0, T) \longrightarrow (R_0, 1) \qquad \textit{for } j = 1, ..., M'_0,$$

that are given through renaming

$$R^j(t) := \bar{r}^i(t).$$

Here $i = i(t, j)$ is piecewise constant in t, the domains of definition D^j are connected open intervals, $\nu^j := \nu(t, R^j(t))$ are constant for $t \in D^j$ and $R^j(t) < R^{j-1}(t)$ in the intersection of D^j and D^{j-1}.

We define the collection of true interfaces

$$\Gamma'(R_0) := \left\{(t, r) \in (0, T) \times (R_0, 1) \mid r = R^j(t) \textit{ for some } j = 1, ..., M'_0\right\}.$$

(We note that Γ' does not depend on the subsequence selected in the preceding Definitions and Lemmata, but only on the limiting order parameter u_0 selected in Remark 3.2.) Finally we obtain $\partial_t u_0^2 \in L^2(0, T; [C^0((R_0, 1])]^*)$ and for almost all t the formula

$$\langle \partial_t u_0^2(t, \cdot), \zeta \rangle = \sum_{j=1}^{M'_0} (b^2 - a^2) \frac{d}{dt} R^j(t) (R^j)^{n-1}(t) \zeta(R^j(t))$$

for any $\zeta \in C^0((R_0, 1])$.

Proof. The energy estimate implies the following weak Hölder continuity of the limiting order parameter u_0 (cf. Remark 3.2):

$$\int_\Omega |u_0(t,x) - u_0(s,x)|\, dx \le C|t-s|^{\frac{1}{2}}.$$

This implies that true interfaces have continuous continuations at all time points. ☐

Theorem 5.5. *For $j = 1, ..., M_0'$, let R^j be as in Lemma 5.4. Then*

$$\frac{d}{dt}R^j + \frac{n-1}{R^j} = 0 \qquad \text{in } D^j.$$

Proof. Continuous functions that are piecewise $H^{1,2}$ are globally $H^{1,2}$, and thus this theorem follows from Corollary 5.3 and Lemma 5.4. ☐

6. Passage to the limit in equation (E3)

In this section we derive the limit equation of (E3).

Remark 6.1. We want to describe a local situation that contains at most one interface. Let ζ be a smooth test function with compact support in $(0,T)\times(0,1]$. For any such ζ, there exist $R_0 > 0$ and some $0 \le T_1 < T_2 \le T$ such that $\operatorname{supp}\zeta \subset (T_1,T_2)\times(R_0,1)$. Since $(0,T)\backslash A(R_0)$ is finite, we will assume without loss of generality that there is at most one $t_0 \in (0,T)\backslash A(R_0)\cap(T_1,T_2)$. Moreover, we may assume without loss of generality, that $\Gamma'(R_0)\cap\operatorname{supp}\zeta = \{(t,r) \mid r = R^j(t)\}\cap\operatorname{supp}\zeta$ for some $j = 1, ..., M_0'$, and that $\operatorname{dist}\big(\Gamma'(R_0)\backslash \text{ graph } R^j; \operatorname{supp}\zeta\big) > 0$.

Lemma 6.2. *Let ζ and R_0, T_1, T_2 and t_0 be as in Definition 6.1. Let t_1 and t_2 be any two points with $T_1 < t_1 < t_0 < t_2 < T_2$. Then there exists $\lambda_\varepsilon \in (a+\frac{b-a}{4}, b-\frac{b-a}{4})$ such that*

$$\chi_\varepsilon(t,x) := \begin{cases} a, & \text{if } u_\varepsilon \le \lambda_\varepsilon,\\ b, & \text{if } u_\varepsilon > \lambda_\varepsilon,\end{cases}$$

is uniformly bounded in $BV(\Omega_T)$ and

$$\int_{t_1}^{t_2}\int_\Omega |(\partial_t,\nabla)\chi_\varepsilon|\, dx\, dt \le C(t_2-t_1)^{\frac{1}{2}}.$$

Proof. This is a consequence of the estimate

$$\int_{t_1}^{t_2}\int_\Omega |(\partial_t,\nabla)g(u_\varepsilon)|\, dx\, dt \le C(t_2-t_1)^{\frac{1}{2}},$$

which itself follows from the energy bound (cf. Remark 3.2). We stress that λ_ε may depend on t_1 and t_2, but C does not. ☐

Theorem 6.3. *Let ζ be a smooth test function with support in $(0,T)\times[0,1]$. Then*

$$-\int_0^T \langle \partial_t u_0^2, f_0\zeta\rangle\, dt - \int_0^T\!\!\int_\Omega u_0^2 f_0 \partial_t\zeta\, dx\, dt + \int_0^T\!\!\int_\Omega u_0^2 \nabla f_0\cdot\nabla\zeta\, dx\, dt = 0.$$

Proof. First we assume that $\operatorname{supp}\zeta \subset (0,T)\times(0,1]$ and R_0, T_1, T_2, t_0 and j are as in Remark 6.1. We multiply equation (E3) by ζ:

$$\int_0^T\!\!\int_\Omega u_\varepsilon^2 \partial_t f_\varepsilon \zeta\, dx\, dt - \int_0^T\!\!\int_\Omega \operatorname{div}(u_\varepsilon^2\nabla f_\varepsilon)\, \zeta\, dx\, dt = 0.$$

We integrate the second term by parts and pass to the limit as $\varepsilon \to 0$.

Next let t_1 and t_2 be any two points with $T_1 < t_1 < t_0 < t_2 < T_2$. For the first term we split the integral over (T_1, T_2) into three parts: $(T_1, t_1) \cup (t_1, t_2) \cup (t_2, T_2)$.

We start with the interior integral over (t_1, t_2). We write u_ε^2 as $\chi_\varepsilon^2 + (u_\varepsilon^2 - \chi_\varepsilon^2)$, where χ_ε is as in Lemma 6.2. Since $|u_\varepsilon^2 - \chi_\varepsilon^2| \le C\sqrt{W(u_\varepsilon)}$, the energy bound implies that the term with $(u_\varepsilon^2 - \chi_\varepsilon^2)$ is uniformly bounded by $C(t_1 - t_2)^{\frac{1}{2}}$. We integrate the χ_ε^2–term by parts:

$$\begin{aligned}
&\int_{t_1}^{t_2} \int_\Omega \chi_\varepsilon^2 \partial_t f_\varepsilon \zeta \, dx \, dt \\
&= - \int_{t_1}^{t_2} \int_\Omega \partial_t \chi_\varepsilon^2 f_\varepsilon \zeta \, dx \, dt - \int_{t_1}^{t_2} \int_\Omega \chi_\varepsilon^2 f_\varepsilon \partial_t \zeta \, dx \, dt + \int_\Omega \chi_\varepsilon^2 f_\varepsilon \zeta \, dx \Big|_{t_1}^{t_2}.
\end{aligned}$$

By Lemma 6.2 the first term is bounded by $C(t_2 - t_1)^{\frac{1}{2}}$. The second is bounded by $C(t_2 - t_1)$. We will combine the boundary terms with the corresponding terms of the other time intervals (see below).

Now we study the integral over (t_2, T_2). We write u_ε^2 as $v_\varepsilon^2 + (u_\varepsilon^2 - v_\varepsilon^2)$, where $v_\varepsilon(t, r) := V_\varepsilon(t, z)$ is given by the approximation V_ε of U_ε defined in section 4. The energy bound implies that the term with $(u_\varepsilon^2 - v_\varepsilon^2)$ is bounded by

$$C \left(\iint |U_\varepsilon - V_\varepsilon|^2 \, dz \, dt \right)^{\frac{1}{2}},$$

where integration is over the rescaled support of ζ. Then Proposition 4.10 implies that this term converges to zero. We integrate the v_ε^2–term by parts:

$$\begin{aligned}
&\omega_n \int_{t_2}^{T_2} \int_{R_0}^1 v_\varepsilon^2 \partial_t f_\varepsilon \zeta r^{n-1} \, dr \, dt \\
&\quad = -\omega_n \int_{t_2}^{T_2} \int_{R_0}^1 \partial_t v_\varepsilon^2 f_\varepsilon \zeta r^{n-1} \, dr \, dt - \omega_n \int_{t_2}^{T_2} \int_{R_0}^1 v_\varepsilon^2 f_\varepsilon \partial_t \zeta r^{n-1} \, dr \, dt \\
&\qquad + \omega_n \int_{R_0}^1 v_\varepsilon^2 f_\varepsilon \zeta r^{n-1} \, dr \Big|_{t_2}^{T_2}.
\end{aligned}$$

Since f_ε converges strongly in $L^2(\Omega_T)$ (cf. Lemma 3.5), f'_ε is uniformly in $L^2(\Omega_T)$, and v_ε is given explicitly, we may conclude that the first term converges to

$$\begin{aligned}
&- \omega_n \int_{t_2}^{T_2} (b^2 - a^2) \left(-\frac{d}{dt} R^j(t) \right) \left(R^j(t) \right)^{n-1} (f_0 \zeta)(t, R^j(t)) \, dt \\
&\quad = - \int_{t_2}^{T_2} \langle \partial_t u_0^2, f_0 \zeta \rangle \, dt.
\end{aligned}$$

The details of this convergence are in the same spirit as the arguments of the proof of Prop. 5.2 presented in the Appendix. The second term converges to

$$-\omega_n \int_{t_2}^{T_2} \int_{R_0}^1 u_0^2 f_0 \partial_t \zeta r^{n-1} \, dr \, dt = - \int_{t_2}^{T_2} \int_0^1 u_0^2 f_0 \partial_t \zeta \, dr \, dt.$$

We combine the boundary term with the corresponding term from the intergration over (t_1, t_2). The result converges to zero for almost every t_2, since $v_\varepsilon - \chi_\varepsilon$ converges to zero almost everywhere.

The same arguments apply for the integral over (T_1, t_1).

Finally, choosing a suitable sequence of points t_1 and t_2 with $t_2 - t_1 \to 0$ proves the theorem for any ζ with compact support in $(0,T) \times (0,1]$.

In case of an arbitrary test function ζ we approximate it through $\zeta\eta_{R_0}$, where η_{R_0} is a smooth cut–off function with support in $B_1(0)\backslash B_{R_0}(0)$ and gradient such that $\nabla\eta_{R_0}$ is only non–zero in $B_{2R_0}\backslash B_{R_0}(0)$. We find that

$$-\int_0^T \langle \partial_t u_0^2, f_0\zeta\eta_{R_0}\rangle\, dt - \int_0^T\!\!\int_\Omega u_0^2 f_0 \partial_t(\zeta\eta_{R_0})\, dx\, dt + \int_0^T\!\!\int_\Omega u_0^2 \nabla f_0 \cdot \nabla(\zeta\eta_{R_0})\, dx\, dt = 0.$$

Since $\nabla f_0 \in L^2(\Omega_T)$ and $\nabla(\zeta\eta_{R_0}) \longrightarrow \nabla\zeta$ in $L^2(\Omega_T)$, the third term converges to the expected limit as $R_0 \to 0$. The second term converges also, since η_{R_0} does not depend on time. In order to handle the first term we use Lemma 3.8. It implies that

$$g_0 \int_0^T \int_{B_{2R_0}} |\partial_t u_0|\, dx\, dt \le \liminf_{\varepsilon\to 0} \int_0^T \int_{B_{2R_0}} |\partial_t g(u_\varepsilon)|\, dx\, dt \le C R_0^2,$$

and consequently $\int_0^T \langle \partial_t u_0^2, f_0\zeta\eta_{R_0}\rangle\, dt \longrightarrow \int_0^T \langle \partial_t u_0^2, f_0\rangle\, dt$. This finishes the proof. □

Remark 6.4. The differential equation of Theorem 6.3 is a suitable distributional formulation of

$$u_0^2 \partial_t f_0 - \text{ div } (u_0^2 \nabla f_0) = 0,$$

together with the Neumann–condition for f_0.

If $\partial_t f_0 \in L^2$, then this distributional formulation implies the jump condition $[u_0^2 \nabla f_0]_{|\Gamma} \cdot \nu = 0$ on the interface Γ.

Remark 6.5. It is also possible to derive the jump condition in the formal context of [RSK]. In fact, in taking the tangential component of (4.9) in [RSK], which is obtained by setting $\varphi = R\, e$ with R being u and with $e = (\cos f, \sin f)$ in our notation, and using the η–time scale of [RSK] (which is the one studied in the present paper), we arrive at

$$\varepsilon R \partial_t f = \varepsilon(2\nabla R \cdot \nabla f + R\Delta f).$$

The idea now is to multiply this by R and to write the right hand side as a derivative:

$$\varepsilon R^2 \partial_t f = \varepsilon \text{ div } (R^2 \nabla f).$$

Now it is a well established strategy to expand R and f in the interfacial region as $R(t,x) = R^0(t,z,s) + \varepsilon R^1(t,z,s) + \dots$ and $f(t,x) = f^0(t,z,s) + \varepsilon f^1(t,z,s) + \dots$, where $z = \frac{d(t,x)}{\varepsilon}$ is the rescaled signed distance $d(t,x) = \pm\,\text{dist}(x,\Gamma_t)$ to the interface Γ at time t, and $s = s(t,x)$ is a tangential variable. In doing so, we assume that the variation of R and f in the transition region is only large in the orthogonal direction to the interface. The construction implies that to leading order in ε the spatial derivative $\nabla_x d$ is the normal to the interface Γ and thus independent of z with $|\nabla_x d|^2 = 1$, whereas ∇s is tangential to the interface Γ, so that $\nabla_x d \cdot \nabla_x s = 0$. Since f does not jump across the interface, f^0 does not depend on z. We proved this in Lemma 3.4 even without the assumption of radial symmetry. Thus the leading order term of the left hand side is of order ε, so that the order 1 term of the right hand side has to vanish:

$$\partial_z \left((R^0)^2 \left(\partial_s f^0 \nabla_x s + \partial_z f^1 \nabla_x d\right)\right) \cdot \nabla_x d = 0.$$

Using the identities for $\nabla_x s$ and $\nabla_x d$, this simplifies into $\partial_z \left((R^0)^2 \partial_z f^1\right) = 0$. Integration implies

$$\left((R^0)^2 \partial_z f^1\right)\Big|_{z=-\infty}^{z=+\infty} = 0.$$

Matching the interfacial solution to some outer expansion implies that $R^0(\pm\infty)$ are the bulk values of R from either side of the interface Γ, whereas $\partial_z f^1(\pm\infty)$ are the values of $\nabla f \cdot \nu$. Thus we find the jump condition on the interface.

Conclusion

We have studied the singular limit of a vectorial Allen–Cahn equation with a potential vanishing on concentric spheres and the order parameter φ ranging in a half plane of $\mathbb{R}^2$. Assuming radial symmetry in space, we have established a priori bounds for the modulus u and the polar angle f of φ. In addition we have shown an error estimate for the difference between the exact solution u and a properly rescaled travelling wave solution. This enabled us to rigorously determine the limit problem: the mean curvature flow for the interface and a harmonic map flow for the polar angle.

Many of the above results remain true for higher dimensional order parameters. Writing $\varphi = u\, e$ with a unit vector e, the original equation transforms into

$$\begin{aligned} \varepsilon \partial_t u - \varepsilon \Delta u + \frac{1}{\varepsilon} W'(u) &= -\varepsilon u |\nabla e|^2, \\ u^2 \partial_t e - \operatorname{div} u^2 \nabla e &= e\, u^2 |\nabla e|^2. \end{aligned}$$

It is again possible to show that any half space is an invariant region. Assuming that the data initially lie in one half space, we consequently obtain that u never vanishes and e is well defined. With this assumption we can introduce polar coordinates and derive all estimates of sections 3, 4 and 5 in this higher dimensional case. In particular, we can deduce the mean curvature flow for the interfaces.

In section 6 we strongly used the fact that the harmonic map flow is a linear equation in the polar angle φ. This is no longer true in the higher dimensional case. One has to deal either with the nonlinear term $eu^2|\nabla e|^2$ or (after multiplication by a suitable tangent vector) with products of the form $u^2 \partial_t e^i\, e^j$. Since we have only very little control on the time derivative of e, it has not been possible to pass to the limit in any of these terms.

Appendix. Proof of Proposition 5.2

Proof of Proposition 5.2. We write the differential equation (E2) in the rescaled variable z:

$$\varepsilon \partial_t U_\varepsilon - \frac{n-1}{\varepsilon z} U_\varepsilon' = \frac{1}{\varepsilon}\left(U_\varepsilon'' - W'(U_\varepsilon)\right) - \frac{1}{\varepsilon} U_\varepsilon |F_\varepsilon'|^2.$$

We multiply this by U_ε' and integrate over $(\alpha, \beta) := (z_\varepsilon^k(t) + \frac{\delta}{4\varepsilon}, z_\varepsilon^{m_0+k-1}(t) - \frac{\delta}{4\varepsilon})$, where δ is as in Definition 5.1. We multiply the result by a smooth test function of time ξ with compact support in B:

$$\int_B \varepsilon\xi \int_\alpha^\beta \partial_t U_\varepsilon U_\varepsilon' \, dz\, dt - \int_B \xi \int_\alpha^\beta \frac{n-1}{\varepsilon z} |U_\varepsilon'|^2 \, dz\, dt$$

$$(*) \qquad = \int_B \xi \, \frac{1}{\varepsilon} \left(\frac{1}{2}|U_\varepsilon'|^2 - W(U_\varepsilon) \right)\Bigg|_\alpha^\beta dt - \int_B \xi \frac{1}{\varepsilon} \int_\alpha^\beta U_\varepsilon U_\varepsilon' |F_\varepsilon'|^2 \, dz\, dt.$$

We claim that the limit of the right hand side vanishes. Indeed, the first term on the right hand side converges to 0 because Proposition 4.10 implies that at α and β we may replace U_ε by V_ε, thereby only making an error of order $\varepsilon\sqrt{\varepsilon}$. We rewrite the second term on the right hand side in the original variable, then integrate the result by parts and use the differential equation $(E3)$:

$$\frac{1}{\varepsilon} \int_\alpha^\beta U_\varepsilon U_\varepsilon' |F_\varepsilon'|^2 \, dz = \varepsilon \int u_\varepsilon u_\varepsilon' |f_\varepsilon'|^2 \, dr = \varepsilon \int (u_\varepsilon^2 f_\varepsilon')' f_\varepsilon' \, dr - \frac{\varepsilon}{2} u_\varepsilon^2 f_\varepsilon' \Big|_{\cdot\cdot}^{\cdot\cdot}$$

$$= \varepsilon \int u_\varepsilon^2 \partial_t f_\varepsilon f_\varepsilon' \, dr - \int u_\varepsilon^2 (f_\varepsilon')^2 \frac{n-1}{r} \, dr - \frac{\varepsilon}{2} u_\varepsilon^2 f_\varepsilon' \Big|_{\cdot\cdot}^{\cdot\cdot},$$

where integration is over $(r_\varepsilon^{m_0+k-1}(t) - \frac{\delta}{4}, r_\varepsilon^k(t) + \frac{\delta}{4})$. Thus the second term of the right hand side of $(*)$ converges to 0 as a consequence of the energy estimate and Remark 3.7.

The second term on the left hand side of $(*)$ will give the curvature term. First we may replace U_ε by V_ε, because

$$\int_B \int_\alpha^\beta |(U_\varepsilon')^2 - (V_\varepsilon')^2| \, dz\, dt$$

$$\le \left(\int_B \int_\alpha^\beta |U_\varepsilon' - V_\varepsilon'| \, dz\, dt \right)^{\frac{1}{2}} \left(\int_B \int_\alpha^\beta |U_\varepsilon' + V_\varepsilon'| \, dz\, dt \right)^{\frac{1}{2}},$$

and by the estimate of Proposition 4.1. We introduce the following notation:

$$(\alpha, \beta) = \left(\alpha, \frac{z_\varepsilon^{m_0+k-2} + z_\varepsilon^{m_0+k-1}}{2} - 1 \right) \cup \bigcup_{i=k-1}^{m_0+k-1} \left(\frac{z_\varepsilon^i + z_\varepsilon^{i+1}}{2} - 1, \frac{z_\varepsilon^i + z_\varepsilon^{i+1}}{2} + 1 \right)$$

$$\cup \bigcup_{i=k-1}^{m_0+k-2} \left(\frac{z_\varepsilon^i + z_\varepsilon^{i+1}}{2} + 1, \frac{z_\varepsilon^{i-1} + z_\varepsilon^i}{2} - 1 \right) \cup \left(\frac{z_\varepsilon^k + z_\varepsilon^{k+1}}{2} + 1, \beta \right)$$

$$=: T_{m_0+k-1} \cup \bigcup_{i=k-1}^{m_0+k-1} C_i \quad \cup \bigcup_{i=k-1}^{m_0+k-2} T_i \quad \cup T_k.$$

Thus in T_i the solution V_ε is well approximated by the stationary wave solution, whereas C_i is the region where two stationary wave solutions are matched together. We divide the integral over (α, β) as above. Then the integral over any C_i gives 0 in the limit, because V_ε^i converges pointwise to 0 in C_i and is uniformly bounded. In any of the T_i we may replace $\frac{n-1}{\varepsilon z}$ by $\frac{n-1}{r_\varepsilon^i(t)}$, and since r_ε^i converges strongly to $\bar{r}^i$, and $\int_{T_i} |V_\varepsilon'|^2 \, dz$ converges strongly to the constant g_0, the limit of the second term of the left hand side of $(*)$ is

$$-\int_B \xi(t) g_0 \sum_{i=k}^{m_0+k-1} \frac{n-1}{\bar{r}^i(t)} \, dt.$$

In the first term of the left hand side of (∗), the strategy is to replace U_ε by V_ε everywhere, thereby only making small errors. We first replace U_ε' by V_ε', which is possible due to the energy bound and Proposition 4.10. We integrate the result by parts with respect to time. Clearly the term involving $\partial_t\xi$ converges to 0, and so does the term involving $\dot\beta$ and $\dot\alpha$, because V_ε' is exponentially close to 0 at α and β. So we are left with determining the limit of $\int_B \varepsilon\xi \int_\alpha^\beta U_\varepsilon \partial_t V_\varepsilon' \, dz\, dt$. We integrate this by parts with respect to z. The resulting boundary term converges to 0 because V_ε' is exponentially small at α and β. In the remaining term we may replace U_ε' by V_ε', as a consequence of Proposition 4.1.

In order to calculate the limit of $\int_B \varepsilon\xi \int_\alpha^\beta V_\varepsilon \partial_t V_\varepsilon \, dz\, dt$, we divide the integral over (α, β) again according to the above partition. We find that

$$\begin{aligned}\int_B \varepsilon\xi \int_{T_i} V_\varepsilon' \partial_t V_\varepsilon \, dz\, dt &= -\int_B \xi \frac{d}{dt} r_\varepsilon^i \int_{T_i} |(V_\varepsilon^i)'|^2 \, dz\, dt \\ &\longrightarrow -\int_B \xi g_0 \frac{d}{dt}\bar r^i \, dt,\end{aligned}$$

because $\frac{d}{dt} r_\varepsilon^i$ converges weakly to $\frac{d}{dt}\bar r^i$, and $\int_{T_i} |(V_\varepsilon^i)'|^2\, dz$ converges strongly to g_0. For $z \in C_i$ we find the formula $V_\varepsilon = \Xi_\varepsilon^i V_\varepsilon^i + \Xi_\varepsilon^{i+1} V_\varepsilon^{i+1} = \Xi_\varepsilon^i (V_\varepsilon^i - V_\varepsilon^{i+1}) + V_\varepsilon^{i+1}$, and thus

$$\varepsilon\partial_t V_\varepsilon = -\Xi_\varepsilon^i (V_\varepsilon^i)' \frac{d}{dt} r_\varepsilon^i - \Xi_\varepsilon^{i+1} (V_\varepsilon^{i+1})' \frac{d}{dt} r_\varepsilon^{i+1} - (\Xi_\varepsilon^i)' \frac{1}{2}\frac{d}{dt}(r_\varepsilon^i + r_\varepsilon^{i+1})(V_\varepsilon^i - V_\varepsilon^{i+1}).$$

Thus $\int_{C_i} |\varepsilon\partial_t V_\varepsilon|^2 \, dz$ converges strongly to 0, because in the partition region $B \times C_i$ the difference $V_\varepsilon^i - V_\varepsilon^{i+1}$ and the derivatives $(V_\varepsilon^i)'$ and (V_ε^{i+1}) converge strongly to 0 in L^2, whereas the $\frac{d}{dt} r_\varepsilon^i$ are uniformly bounded in $L^2(B)$. This implies the proposition. □

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Department of Mathematics, McMaster University, Hamilton, Ont. L8S 4K1, Canada
E-mail address: bronsard@math.mcmaster.ca

IAM, Universität Bonn, 53115 Bonn, Deutschland
E-mail address: bstoth@iam.uni-bonn.de