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CONJUGACY CLASSES OF $SU(h, \mathcal{O}_S)$ IN $SL(2, \mathcal{O}_S)$

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Abstract. Let K be a quadratic extension of a global field F, of characteristic not two, and $\mathcal{O}_S$ the integral closure in K of a Dedekind ring of S-integers $\mathfrak{O}_S$ in F. Then $PSL(2, \mathcal{O}_S)$ is isomorphic to the spinorial kernel $O'(L)$ for an indefinite quadratic $\mathfrak{O}_S$-lattice L of rank 4. The isomorphism is used to study the conjugacy classes of unitary groups $PSU(h, \mathcal{O}_S)$ of primitive odd binary hermitian matrices h under the action of $PSL(2, \mathcal{O}_S)$.

1. Introduction

Let $\mathcal{O}_d$ be the ring of integers in $\mathbb{Q}(\sqrt{-d})$, where d is a square-free integer. It was shown in Theorem 2.1 of James and Maclachlan [4] that the Bianchi group $PSL(2, \mathcal{O}_d)$, for $d > 0$ and $d \equiv 1, 2 \mod 4$, is isomorphic to the spinorial kernel $O'(L)$ of an integral orthogonal group $O(L)$. Here

$$L = \mathbb{Z}r \perp \mathbb{Z}s \perp (\mathbb{Z}u + \mathbb{Z}v) \tag{1}$$

is a lattice on the quadratic space V with quadratic form $q : V \to \mathbb{Q}$ and associated bilinear form $f(x, y) = q(x + y) - q(x) - q(y)$, with $q(r) = 1$, $q(s) = d$, and u and v isotropic with $f(u, v) = d$. The extended Bianchi group B_d is isomorphic to $PSO(L)$. For $d \equiv 3 \mod 4$, L must be replaced by $L + \mathbb{Z}2^{-1}(r - s)$.

Much of the proof in [4] remains valid when $d < 0$. In particular, there is a homomorphism Φ from the Hilbert modular group $SL(2, \mathcal{O}_d)$ into the group $O'(L)$ with kernel the center $\pm I$. In [4] this map was shown surjective only for $d > 0$ by using the extended Bianchi group as the maximal discrete extension of $PSL(2, \mathcal{O}_d)$ in $PSL(2, \mathbb{C})$. We now give a local-global number theoretic treatment in the more general setting of a quadratic extension of global fields K/F with $\mathcal{O}_d$ replaced by a ring of integers $\mathcal{O}_S$ in K. Here $\mathcal{O}_S$ is the integral closure in K of a Dedekind ring $\mathfrak{O}_S$ of S-integers in F (see [7]). We prove $PSL(2, \mathcal{O}_S)$ is isomorphic to the spinorial kernel $O'(L)$ for a suitable $\mathfrak{O}_S$-lattice L on a quadratic space V over F. When $\mathfrak{O}_S = \mathbb{Z}$ and $d \equiv 1, 2 \mod 4$, L is the $\mathbb{Z}$-lattice given in (1). For $F = \mathbb{F}(X)$ a function field over a finite field, of characteristic not two, $\mathfrak{O}_S = \mathbb{F}[X]$ and $K = F(\sqrt{-d})$ with d a square-free polynomial, L is the corresponding $\mathbb{F}[X]$-lattice. However, in general, only the localizations L_p are explicitly determined.

The results in [4] also gave a classification of the non-elementary maximal Fuchsian subgroups of the Bianchi group up to conjugacy. A Fuchsian subgroup stabilizes a circle in the complex plane. The conjugacy classes of the projective special unitary groups $PSU(h, \mathcal{O}_S)$ of primitive binary hermitian matrices h over $\mathcal{O}_S$ are

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classified in the final sections. Relating h to a circle in the complex plane then gives a geometric connection between the two problems for the Bianchi groups (see also [5], [6], [9] and [10]). Some examples from cyclotomic fields are also given.

2. $SL(2,K)$ and quadratic forms

In this section, the relationship between $SL(2,K)$ and the orthogonal group of the related quadratic form is summarized when K is the quadratic extension of a field F with characteristic not two. Let $K = F(\sqrt{-d})$ where $-d \in F$ (we keep the negative sign to match the notation in [4]). Let $\bar{a}$ denote the conjugate of $a \in K$ under the non-trivial galois automorphism of K fixing F.

Let A denote the quaternion algebra $\mathbb{M}(2,K)$ with standard basis I, i, j, ij where

$$i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad j = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then A admits a conjugate-linear involution τ defined by

$$\tau(a_0 I + a_1 i + a_2 j + a_3 ij) = \bar{a_0} I - \bar{a_1} i - \bar{a_2} j - \bar{a_3} ij$$

whose fixed point set V is a 4-dimensional space over F. With the restriction of the norm form, denoted by q , V is a regular quadratic space with orthogonal group $O(V)$. Let $f : V \times V \to F$ denote the associated symmetric bilinear form. In V, fix a basis $\{r, s, u, v\}$ with $q(r) = 1$, $q(s) = d$, $q(u) = q(v) = 0$ and $f(u, v) = d$ by choosing $r = I$, $s = (\sqrt{-d})j$, $u = \frac{1}{2}(\sqrt{-d})(-i + ij)$ and $v = \frac{1}{2}(\sqrt{-d})(i + ij)$.

Define the group A_F^* by

$$A_F^* = \{\beta \in A^* \mid \det\ \beta \in F^*\}.$$

For $\beta \in A_F^*$ define $\phi_\beta : V \to V$ by $\phi_\beta(t) = (\det\ \beta)^{-1}\beta t \tau(\beta)$. Then $\phi_\beta \in O(V)$. Setting $\Phi(\beta) = \phi_\beta$ defines a homomorphism

$$\Phi : A_F^* \to O(V).$$

As in [4] we give a description of Φ in terms of the basis $\{r, s, u, v\}$. Thus if $\beta = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$, then $\Phi(\beta)$ is $(\det\ \beta)^{-1}$ times the 4×4 matrix

$$\begin{pmatrix} \mathcal{R}(x\bar{w} - y\bar{z}) & -d\mathcal{I}(x\bar{w} + y\bar{z}) & -d\mathcal{I}(x\bar{z}) & d\mathcal{I}(y\bar{w}) \\ \mathcal{I}(x\bar{w} - y\bar{z}) & \mathcal{R}(x\bar{w} + y\bar{z}) & \mathcal{R}(x\bar{z}) & -\mathcal{R}(y\bar{w}) \\ 2\mathcal{I}(x\bar{y}) & 2\mathcal{R}(x\bar{y}) & x\bar{x} & -y\bar{y} \\ 2\mathcal{I}(w\bar{z}) & -2\mathcal{R}(w\bar{z}) & -z\bar{z} & w\bar{w} \end{pmatrix}. \tag{2}$$

The notation here is that, if $\alpha = a + b\sqrt{-d}$ with $a, b \in F$, then $\mathcal{R}(\alpha) = a$ and $\mathcal{I}(\alpha) = b$. It follows that the kernel of Φ is $F^* I$. Let

$$\theta : SO(V) \to F^*/{F^*}^2$$

denote the spinor norm, with kernel $O'(V)$. This group is also the commutator subgroup of $SO(V)$, and also the subgroup generated by all Eichler transformations (see [2]). Since $q(u) = 0$, we can define for each $t \in V$ with $f(u, t) = 0$ the Eichler transformation $E(u, t)$ by

$$E(u, t)(w) = w - f(u, w)t + f(t, w)u - q(t)f(u, w)u.$$

Let $\beta \in SL(2, K)$ have the form $\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$ where $\alpha = a + b\sqrt{-d}$. Then, as in [4],

$$\Phi(\beta) = E(u, ad^{-1}s - br). \tag{3}$$

Also $\Phi(\beta^t) = E(v, -ad^{-1}s - br)$. Since $SL(2,K)$ is generated by all β, β^t it follows that $\Phi(SL(2,K)) \subseteq O'(V)$. In fact, since $E(u,t)$ and $E(v,t)$ generate $O'(V)$, the following sequence is exact:

$$I \to \{\pm I\} \to SL(2,K) \xrightarrow{\Phi} O'(V) \to I.$$

Now let $\beta = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ with $a \in F^*$. Then $\Phi(\beta)$ fixes r, s and maps $u \to au$, $v \to a^{-1}v$. Thus $\Phi(\beta) \in SO(V)$ and $\theta(\Phi(\beta)) = aF^{*2}$. It follows that $\Phi(A_F^*) = SO(V)$, and the kernel of Φ consists of aI with $a \in F^*$. Hence, for $\beta \in A_F^*$, the spinor norm of $\Phi(\beta)$ is $(\det\ \beta)F^{*2}$.

3. S-lattices and integral groups

Now let $K = F(\sqrt{-d})$ be a quadratic extension of a global field F with characteristic not two, where d is an algebraic integer in F. Let S be a Dedekind set of primes for F (see [7]), $\mathfrak{O}_S$ the corresponding ring of integers in F, and $\mathcal{O}_S$ its integral closure in K. We show that

$$\Phi(SL(2,\mathcal{O}_S)) = O'(L)$$

for a suitably defined $\mathfrak{O}_S$-lattice L in V. Put

$$H = \mathfrak{O}_S u + \mathfrak{O}_S v.$$

For $p \in S$, denote by $\mathfrak{O}_p$ the localization of $\mathfrak{O}_S$ at p (without completion). We also denote by p a prime element of $\mathfrak{O}_p$. If p does not split in K, let $\mathcal{O}_p$ denote the localization of $\mathcal{O}_S$ at the unique extension of p to K. Then $\mathcal{O}_p = \mathfrak{O}_p + \omega_p \mathfrak{O}_p$ for some $\omega_p \in \mathcal{O}_p$. In fact, whenever $2d$ is a unit or a non-dyadic prime in $\mathfrak{O}_p$, we can take $\omega_p = \sqrt{-d}$. In this case put

$$L_p = \mathfrak{O}_p r \perp \mathfrak{O}_p s \perp H_p, \tag{4}$$

an $\mathfrak{O}_p$-lattice on V. Then $\Phi(SL(2,\mathcal{O}_p)) \subseteq O'(L_p)$, since in (2) all the matrix entries are in $\mathfrak{O}_p$.

When $p \in S$ splits in K, let $\mathcal{O}_{p_1}$ and $\mathcal{O}_{p_2}$ denote the localizations of $\mathcal{O}_S$ at the two conjugate extensions p_1 and p_2 of p to K. Then, for p non-dyadic and with d a unit in $\mathfrak{O}_p$, we have $\mathcal{O}_{p_1} \cap \mathcal{O}_{p_2} = \mathfrak{O}_p[\sqrt{-d}]$, a semilocal ring. Again take L_p as in (4). Then, from (2),

$$\Phi(SL(2,\mathfrak{O}_p[\sqrt{-d}])) \subseteq O'(L_p).$$

Hence, for all but a finite number of $p \in S$, L_p has been chosen as the localization of

$$L' = \mathfrak{O}_S r \perp \mathfrak{O}_S s \perp H.$$

For a non-dyadic prime p where $\operatorname{ord}_p\ d \geq 2$, take $\mu_p \in p\mathfrak{O}_p$ such that $d\mu_p^{-2}$ is either a unit or a prime in $\mathfrak{O}_p$. If p does not split in K, put $\omega_p = \mu_p^{-1}\sqrt{-d}$ so that $\mathcal{O}_p = \mathfrak{O}_p[\omega_p]$. Now take

$$L_p = \mathfrak{O}_p r \perp \mathfrak{O}_p \mu_p^{-1} s \perp \mu_p^{-1} H_p \tag{5}$$

so that, in essence, d has been replaced by $\mu_p^{-2}d$. Then it again follows that $\Phi(SL(2,\mathcal{O}_p)) \subseteq O'(L_p)$. The non-dyadic split case is similar with $\mathcal{O}_{p_1} \cap \mathcal{O}_{p_2} = \mathfrak{O}_p[\mu_p^{-1}\sqrt{-d}]$. Define $\mathcal{O}_p = \mathcal{O}_{p_1} \cap \mathcal{O}_{p_2}$ in all the split cases. Of course, (5) includes (4) by putting $\mu_p = 1$.

It remains to consider dyadic primes $p \in S$. Let $e = \mathrm{ord}_p\ 2$. There are four possibilities (see [1, §5]).

1. The dyadic prime $p \in S$ has two conjugate extensions p_1 and p_2 to K—the split case. Then $-d\mu_p^{-2} \equiv 1 \mod 4p$ for some $\mu_p \in \mathfrak{O}_p$. Here $\mathcal{O}_p = \mathcal{O}_{p_1} \cap \mathcal{O}_{p_2} = \mathfrak{O}_p[\omega_p]$ where $\omega_p = (1 + \mu_p^{-1}\sqrt{-d})/2$.
2. The extension K/F is unramified at p. Now, for some $\mu_p \in \mathfrak{O}_p$, $-d\mu_p^{-2} \equiv 1 + 4\delta \mod 4p$ with $\delta \in \mathfrak{O}_p$ a unit. Then $\mathcal{O}_p = \mathfrak{O}_p[\omega_p]$ where $\omega_p = (1 + \mu_p^{-1}\sqrt{-d})/2$.
3. The extension K/F is ramified at p with $\mathrm{ord}_p\ d = 2m+1$ odd—the ramified prime case. Then $\mathcal{O}_p$ is generated over $\mathfrak{O}_p$ by 1 and $\omega_p = p^{-m}\sqrt{-d}$.
4. The extension K/F is ramified at p and $\mathrm{ord}_p\ d$ is even—the ramified unit case. Then $-d\mu_p^{-2} \equiv 1 - p^{2k+1}\delta \mod 4p$ for some $\mu_p \in \mathfrak{O}_p$, unit $\delta \in \mathfrak{O}_p$, and rational integer k with $0 \le k < e$. Now $\mathcal{O}_p$ is generated over $\mathfrak{O}_p$ by 1 and $\omega_p = (1 + \mu_p^{-1}\sqrt{-d})p^{-k}$.

In the ramified prime case take L_p as in (5) above with $\mu_p = p^m$. In the three remaining cases take

(6) $$L_p = (\mathfrak{O}_p r + \mathfrak{O}_p p^{-k}(r - \mu_p^{-1}s)) \perp \mu_p^{-1}H_p$$

where $k = e$ in the split and unramified cases (so p^k is essentially 2). This case is the same as (5) when $k = 0$. In the split and unramified cases, $q(L_p) = \mathfrak{O}_p$ and L_p is an even unimodular $\mathfrak{O}_p$-lattice. In the ramified unit case, $q(L_p) = \mathfrak{O}_p$ but L_p is not unimodular. Again, after a computation using (2),

$$\Phi(SL(2, \mathfrak{O}_p[\omega_p])) \subseteq O'(L_p).$$

By [7, 81:14], there now exists an $\mathfrak{O}_S$-lattice L on V that localizes to the chosen L_p at each $p \in S$. When $\mathcal{O}_S = \mathfrak{O}_S[\sqrt{-d}]$, we have $L = L'$. Moreover, in all cases, $\Phi(SL(2, \mathcal{O}_S)) \subseteq O'(L)$, since if $\beta \in SL(2, \mathcal{O}_S)$ with $\Phi(\beta) = \phi_\beta$, then $\phi_\beta(L_p) = L_p$ for all $p \in S$ (including those p that split in K). Hence $\phi_\beta(L) = L$ by [7, 101:6]. In the next section we show that $\Phi(SL(2, \mathcal{O}_S)) = O'(L)$.

4. Generators for $O'(L_p)$ and $O'(L)$

Let $\mathcal{E}$ denote the subgroup of $O'(L)$ generated by the integral Eichler transformations $E(u,t)$ and $E(v,t)$, and let $\mathcal{E}_p$ be the corresponding local subgroup in $O'(L_p)$. For the lattice L_p in (4), $E(u,t)$ is integral when $t \in \mathfrak{O}_p r \perp \mathfrak{O}_p d^{-1}s$. See [2] for many relations involving these transformations. In particular, for $q(t) \neq 0$,

$$\Psi(t)\Psi(u - v) = T(-dq(t))E(v,t)E(u, (q(t)d)^{-1}t)E(v,t)$$

where $T(c)$ is the isometry fixing r and s while sending u to cu and v to $c^{-1}v$, and $\Psi(t)$ is the symmetry $x \to x - f(x,t)q(t)^{-1}t$. Taking $t = d^{-1}s$ it follows that $\Psi(s) \in O(H_p)\mathcal{E}_p$ when L_p is as in (4). Similarly, $\Psi(r) \in O(H_p)\mathcal{E}_p$ when d is a unit in $\mathfrak{O}_p$.

For $p \in S$, let

$$J_p = \{x \in L_p \mid q(x) \in d\mu_p^{-2}\mathfrak{O}_p\}$$

and

$$M_p = \{x \in L_p \mid f(x, J_p) \subseteq 2d\mu_p^{-2}\mathfrak{O}_p\}.$$

For L_p as in (5), we have

$$\mu_p J_p = \mathfrak{O}_p d\mu_p^{-1} r \perp \mathfrak{O}_p s \perp H_p$$

and

$$\mu_p M_p = \mathfrak{O}_p \mu_p r \perp \mathfrak{O}_p s \perp 2H_p.$$

For L_p as in (6), since $d\mu_p^{-2}$ is now a unit, we have $J_p = L_p$ and

$$M_p = (\mathfrak{O}_p p^k r + \mathfrak{O}_p(r - \mu_p^{-1} s)) \perp 2\mu_p^{-1} H_p.$$

Therefore J_p and M_p are sublattices of L_p, and both are invariant under the action of $O(L_p)$.

Theorem 4.1. *Let L_p be as in* 5 *or* 6*. Then the local group $O(L_p)$ is generated by $\mathcal{E}_p, O(H_p)$ and either $\Psi(r)$, or by $\Psi(dr + s)$ in the dyadic ramified unit case. Moreover, $O'(L_p) = \mathcal{E}_p$.*

Proof. This is a special case of Theorem 1 in [2] when L_p is an even unimodular lattice, for example when $2d$ is a unit, or in the unramified and split dyadic cases (and then $\Psi(r)$ is not needed). We modify this argument in the remaining cases where we may assume that $\mu_p = 1$, and d is a prime except in the dyadic ramified unit case treated in the next paragraph. Take $\phi \in O(L_p)$ and let $\phi(r) = ar + bs + 2cu + 2c'v \in M_p$. Then $1 = a^2 + db^2 + 4dcc'$. Hence $a \equiv 1 \mod p$ for non-dyadic p, possibly after changing ϕ by $\Psi(r)$. Also, in the dyadic ramified prime case, $1 \equiv a^2 + db^2 \mod 4p$ so that $a \equiv 1 \mod 2p$ and $2|b$. Since $E(u, r)$ changes the coefficient of u in $\phi(r)$ to $2(c + a - c'd)$, we may assume c (or equivalently c') is a unit. Then $E(u, -cr)E(v, w)\phi$, with $2cdw = (a - 1)r + bs$, fixes r and it suffices to consider $\phi \in O(H_p \perp \mathfrak{O}_p s)$. Similarly, ϕ can be modified, possibly using $\Psi(s) \in O(H_p)\mathcal{E}_p$, to also fix s and hence lies in $O(H_p)$.

Now consider the dyadic ramified unit case with $d + 1 = p^{2k+1}\delta$ and δ a unit. Let $\phi \in O'(L_p)$ with $\phi(r - s) = a(r - s) + bp^k r + 2cu + 2c'v \in M_p$. Then $(1 - a^2)(d + 1) \equiv 2abp^k + b^2 p^{2k} \mod 4$ and it follows that $2p^{-k}|b$ and $a \equiv 1 \mod 2p^{-k}$. As above, we may assume c is a unit, and then $E(u, -cr)E(v, w)\phi$, with $2cdw = (a - 1)(r - s) + bp^k r$, fixes $r - s$. Similarly, ϕ can be modified so that it also fixes $dr + s \in M_p$, possibly also using $\Psi(dr + s)$.

Finally, $O'(L_p) = \mathcal{E}_p$ follows with the help of the relation

$$E(v, as)E(u, bs)T(c)^2 = E(u, bc^{-1}s)E(v, acs)$$

where $a, b \in d^{-1}\mathfrak{O}_p$ and $c = 1 - abd^2$ is a unit; thus $T(c)^2 \in \mathcal{E}_p$. The group $O(H_p)$ is generated by $\Psi(u - v)$ and the isometries $T(c)$. Also, $\Psi(u - v)\Psi(r)$ with d prime, and $\Psi(u - v)\Psi(dr + s)$ in the ramified unit case, have spinor norms of the form $p.(\text{unit})F^{*2}$, while $T(c)$ has spinor norm cF^{*2}. Hence the result. □

Theorem 4.2. *For L as defined in §3, the sequence*

$$I \to \{\pm I\} \to SL(2, \mathcal{O}_S) \xrightarrow{\Phi} O'(L) \to I$$

is exact. Moreover, $O'(L) = \mathcal{E}$, except when $F = \mathbb{Q}$ and $d > 0$.

Proof. Only the surjectivity of Φ remains to be shown in the sequence. We already have the exact sequence

$$I \to \{\pm I\} \to SL(2, K) \xrightarrow{\Phi} O'(V) \to I.$$

Then $\Phi(SL(2, \mathcal{O}_p)) = \mathcal{E}_p = O'(L_p)$, since, by (3), each integral Eichler transformation is the image of an integral elementary matrix. Fix $\phi \in O'(L)$. Then ϕ can be extended to L_p, and hence there exist exactly two isometries $\pm\sigma \in SL(2, \mathcal{O}_p) \subseteq$

$SL(2,K)$ with $\Phi(\pm\sigma) = \phi$. Letting p vary over all the extensions of $p \in S$ to K, since $\bigcap_p \mathcal{O}_p = \mathcal{O}_S$ and $\pm\sigma$ cannot change, it follows that $\sigma \in SL(2, \mathcal{O}_S)$.

By Vaserstein [8], $SL(2, \mathcal{O}_S)$ is generated by integral elementary matrices except when $F = \mathbb{Q}$ and $d > 0$. Hence, from (3), $O'(L)$ is generated by integral Eichler transformations. □

5. Unitary groups

The non-elementary maximal Fuchsian subgroups of a Bianchi group were shown in [4, §3] to be in one-one correspondence with certain stabilizer subgroups of $O'(L)$. We will now relate the projective special unitary groups in $PSL(2, \mathcal{O}_S)$ to similar stabilizer subgroups. Let $\Phi(SL(2, \mathcal{O}_S)) = O'(L)$.

For $b \in \mathcal{O}_S$ and $a, c \in \mathfrak{O}_S$ with $D = b\bar{b} - ac \neq 0$, the matrix

$$h = \begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix} \tag{7}$$

is hermitian with discriminant $D \in \mathfrak{O}_S$. Call the matrix h *primitive* when $(a, b\bar{b}, c) = (a, c, D) = \mathfrak{O}_S$. Let $SU(h, \mathcal{O}_S) \subset SL(2, \mathcal{O}_S)$ be the special unitary group of h.

There are two types of local hermitian forms at a ramified dyadic prime p. The matrix h is locally *odd* at p when there exists $g \in \mathcal{O}_p \times \mathcal{O}_p$ with $gh\bar{g}^t$ a unit in $\mathfrak{O}_p$; this is equivalent to a or c being a unit (since trace$(\mathcal{O}_p) \subseteq p\mathfrak{O}_p$). Otherwise, h is *even* at p. The matrix h is globally *odd* when it is odd at all ramified dyadic primes. In particular,

$$h_0 = \begin{pmatrix} 1 & 0 \\ 0 & -D \end{pmatrix}$$

is odd. Let $\beta = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in SL(2, \mathcal{O}_S)$. Then $\beta \in SU(h_0, \mathcal{O}_S)$ if and only if $\beta h_0 = h_0(\bar{\beta}^t)^{-1}$, or $x = \bar{w}$ and $z = D\bar{y}$. Define $\Phi(\beta) = \phi \in O'(L)$ as in §2, so that from (2),

$$\phi(u) = -d\mathcal{I}(x\bar{z})r + \mathcal{R}(x\bar{z})s + x\bar{x}u - z\bar{z}v$$

and

$$\phi(v) = d\mathcal{I}(y\bar{w})r - \mathcal{R}(y\bar{w})s - y\bar{y}u + w\bar{w}v.$$

Hence $\beta \in SU(h_0, \mathcal{O}_S)$ if and only if $\phi(u + Dv) = u + Dv$. Therefore,

$$\Phi^*(PSU(h_0, \mathcal{O}_S)) = \{\phi \in O'(L) \mid \phi(u + Dv) = u + Dv\},$$

where $\Phi^* : PSL(2, \mathcal{O}_S) \to O'(L)$ is the isomorphism induced by Φ.

For $t \in V$, define the stabilizer

$$Stab(L, t) = \{\phi \in O'(L) \mid \phi(t) = t\},$$

with $Stab(L_p, t)$ the corresponding local group. Then $\phi \in Stab(L, t)$ if and only if $\phi \in Stab(L_p, t)$ for all $p \in S$. If $\sigma \in O(L)$, then

$$\sigma Stab(L, t)\sigma^{-1} = Stab(L, \sigma(t)).$$

For $a \neq 0$, put $\gamma_a = \begin{pmatrix} a & 0 \\ \bar{b} & 1 \end{pmatrix} \in A_F^*$, so that $\Phi(\gamma_a) \in SO(V)$. Let $b = b_1 + b_2\sqrt{-d}$ where $b_1, b_2 \in F$. Computation then gives

$$\Phi(\gamma_a)(u + Dv) = -db_2r + b_1s + au - cv = t \tag{8}$$

where $q(t) = dD$. Since $\gamma_a h_0 \bar{\gamma}_a^t = ah$, it follows that

$$\gamma_a SU(h_0, \mathcal{O}_S)\gamma_a{}^{-1} \subset SU(h, K),$$

and also, when $a \neq 0$,

$$\Phi(SU(h,K)) = Stab(V,t). \tag{9}$$

A similar argument with the same t holds for $\gamma_c = \begin{pmatrix} b & -1 \\ c & 0 \end{pmatrix}$ when $c \neq 0$. Put $\delta_g = \begin{pmatrix} 1 & gb \\ 0 & 1 \end{pmatrix}$ with $g \in \mathcal{O}_S$. Then $\sigma_g = \Phi(\delta_g) \in O'(L)$. When $a = c = 0$, put $h' = \delta_1 h \bar{\delta}_1^t = \begin{pmatrix} 2D & b \\ \bar{b} & 0 \end{pmatrix}$. Then, from (3), $\Phi(SU(h',K)) = Stab(V,\sigma_1(t))$, so that again (9) holds.

Theorem 5.1. *The group $SU(h,\mathcal{O}_S)$ is commensurable in $GL(2,K)$ to a conjugate of $SU(h_0,\mathcal{O}_S)$. Moreover, with h primitive and odd, and t as in* (8),

$$\Phi^*(PSU(h,\mathcal{O}_S)) = Stab(L,t).$$

Proof. Assume $a \neq 0$; let $SU(h_0, a\mathcal{O}_S)$ be the congruence subgroup of $SU(h_0,\mathcal{O}_S)$ consisting of those $\beta = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \equiv I \mod a\mathcal{O}_S$. Thus $x - w, y, z \in a\mathcal{O}_S$, and then $\gamma_a \beta \gamma_a^{-1}$ is integral. Hence, modifying [5], we obtain

$$SU(h, a^2\mathcal{O}_S) \subseteq \gamma_a SU(h_0, a\mathcal{O}_S)\gamma_a^{-1} \subseteq SU(h,\mathcal{O}_S)$$

and $\gamma_a SU(h_0,\mathcal{O}_S)\gamma_a^{-1}$ and $SU(h,\mathcal{O}_S)$ are commensurable subgroups of $GL(2,K)$. Also, when a is a unit in $\mathfrak{O}_p$, we have $\gamma_a \in GL(2,\mathcal{O}_p)$ and hence $\gamma_a SU(h_0,\mathcal{O}_p)\gamma_a^{-1} = SU(h,\mathcal{O}_p)$. Therefore,

$$\Phi(SU(h,\mathcal{O}_p)) = \Phi(\gamma_a) Stab(L_p, u + Dv)\Phi(\gamma_a)^{-1} = Stab(L_p,t).$$

A similar argument with the same t holds for γ_c when $c \neq 0$.

It remains to show that $\Phi(SU(h,\mathcal{O}_p)) = Stab(L_p,t)$ for all $p \in S$ with $a, c \in p\mathcal{O}_p$ and consequently b is a unit in $\mathcal{O}_p$. Since h is assumed odd, p is not ramified dyadic. The $(1,1)$-entry in $h' = \delta_g h \bar{\delta}_g^t$ is congruent to $(g+\bar{g})b\bar{b}$ modulo p. Hence, as above, if $g + \bar{g}$ is a unit, then $\Phi(SU(h',\mathcal{O}_p)) = Stab(L_p, \sigma_g(t))$ and $\Phi(SU(h,\mathcal{O}_p)) = Stab(L_p,t)$. Take $g = (1 + \mu_p^{-1}\sqrt{-d})/2$ in the unramified and split dyadic cases, and when 2 is a unit in $\mathcal{O}_p$, take $g = 1$. □

We analyse t more carefully. For L_p as in (5) and $b = b_1 + b_2\omega_p$ with $b_1, b_2 \in \mathfrak{O}_p$ and $(a, b\bar{b}, c)_p = (a, b_1, b_2, c)_p = \mathfrak{O}_p$, it follows from (8) that

$$t = -d\mu_p^{-1} b_2 r + b_1 s + au - cv \in \mu_p J_p$$

and $f(t, \mu_p M_p) = 2d\mathfrak{O}_p$. For dyadic L_p as in (6) and $b = b_1 + b_2\omega_p$, we have

$$t = -d\mu_p^{-1} b_2 p^{-k} r + (b_1 + b_2 p^{-k})s + au - cv \in \mu_p J_p$$

and $f(t, \mu_p M_p) = 2d\mathfrak{O}_p$. Hence $t \in L_p$ for all $p \in S$, so that $t \in L$.

Define $L(D)$ to be the set of all $t \in L$ with $q(t) = dD$, and $t \in \mu_p J_p$ and $f(t, \mu_p M_p) = 2d\mathfrak{O}_p$ for all $p \in S$. This is a generalization of the definition of $L(D)$ given in [4]. The group $O'(L)$ acts on $L(D)$ and we set $N(L,D)$ to be the number of orbits under this action. We have now shown

Theorem 5.2. *The map Φ^* induces an injection from the conjugacy classes of the projective special unitary groups $PSU(h,\mathcal{O}_S)$, of odd primitive hermitian matrices $h \in \mathbb{M}(2,\mathcal{O}_S)$ with discriminant $D \neq 0$, under the action of $PSL(2,\mathcal{O}_S)$, into the orbits in $L(D)$ under the action of $O'(L)$.*

Note, for $\gamma \in SL(2,\mathcal{O}_S)$ and $\gamma SU(h,\mathcal{O}_S)\gamma^{-1} = SU(h',\mathcal{O}_S)$, where $h' = \gamma h\bar{\gamma}^t$, it does not follow that h' is also primitive (for example, $\mathfrak{O}_S = \mathbb{Z}, d = 5$, $h = \begin{pmatrix}1 & b\\ \bar{b} & 0\end{pmatrix}$ with $b = 1+\sqrt{-5}$, and $\gamma = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$). Each conjugacy class considered in Theorem 5.2 need only involve at least one primitive hermitian matrix. However, the restriction that h is odd and primitive means that h locally represents a unit at each $p \in S$, and h' inherits this key property, which could have been used as a conjugacy invariant definition of primitivity.

We now study the image of the induced injective map Φ^* in the set of orbits in $L(D)$ under $O'(L)$. Denote by $n(D)$ the size of this image. Then $n(D) \le N(L,D)$. Let $t = db_2r + b_1s + au - cv \in L(D)$. Then $t \in \mu_pJ_p$ so that $a, c \in \mathfrak{O}_p$ for all $p \in S$, and hence $a, c \in \mathfrak{O}_S$. Put $b = b_1 - b_2\sqrt{-d}$. Then $b\bar{b} = D + ac \in \mathfrak{O}_S$. If we show that $b + \bar{b} = 2b_1 \in \mathfrak{O}_S$, it then follows that $b \in \mathcal{O}_S$ since $\mathcal{O}_S$ is the integral closure of $\mathfrak{O}_S$ in K. For type (5) we have $b_1 \in \mathfrak{O}_p$. For type (6) it follows from $t \in \mu_pJ_p$ that $2b_1 \in \mathfrak{O}_p$. Hence $2b_1 \in \mathfrak{O}_S$. If $h = \begin{pmatrix}a & b\\ \bar{b} & c\end{pmatrix}$ is primitive, the conjugacy class of $PSU(h,\mathcal{O}_S)$ then corresponds to the orbit of t. The matrix h is locally primitive at any p where D is locally a unit, and hence it suffices to consider only those $p \in S$ where $p|D$. However, unlike the corresponding situation in [4], if the orbit of t lies in the image of Φ^*, then t must also be primitive in μ_pL_p for all $p|D$. For if $t \in p\mu_pL_p$, then this also holds for all elements in the orbit of t; it follows that $a, c \in p\mathfrak{O}_p$, and when $p|D$ all the h corresponding to elements in the orbit of t are not primitive. Therefore, when computing $n(D)$ from the local information about $N(L,D)$ given in [4], these orbits must be excluded. Similarly, since the condition analogous to h odd is not assumed in [4], all the ramified dyadic orbits corresponding to $t \in M_p$ must also be excluded for the current situation.

We return to the question of the primitivity of the h constructed above under the additional assumptions that t is locally primitive in μ_pL_p for all $p|D$, and that either a or c is locally a unit for all ramified dyadic $p \in S$. Since $f(t,\mu_pM_p) = 2d\mathfrak{O}_p$, it follows that $(a, b_1, b_2, c)_p = \mathfrak{O}_p$ where now we have locally rewritten $b = b_1 + b_2\omega_p \in \mathfrak{O}_p[\omega_p]$. Then h is locally primitive whenever a, c or D is a local unit. It remains to consider those $p \in S$ dividing a, c and D; in particular, p is not ramified dyadic. Then $p|b\bar{b}$ so that b_2 is necessarily a local unit. Moreover, since t is now assumed to be primitive in μ_pL_p, either b_1 or $d\mu_p^{-2}$ is also a unit, and hence both are units when p is non-dyadic since $p|D$. Using the Strong Approximation Theorem (see [7, 21:2]), take $gd \in \mathfrak{O}_S$ with $gd \equiv 1 \mod p$ for all non-dyadic $p|a, c, D$, and $gd \in p\mathfrak{O}_p$ for the remaining $p|D$. Also, choose $g' \in K$ with $g'\mu_p \in \mathfrak{O}_p$ for all $p \in S$, such that $g'\mu_p \equiv 1 \mod p$ for all dyadic $p|a, c, D$, and such that $g'\mu_p \in p\mathfrak{O}_p$ for all the remaining $p|D$. Then $E(u, g'r + gs) \in O'(L)$. Put $t' = E(u, g'r + gs)(t)$. Either the coefficient of u or of v in t' is now a unit for all $p|D$, and hence the corresponding hermitian matrix h' primitive. (In fact, it would suffice for the proof above, to find suitable t'_p and h'_p for each $p|D$, one at a time.) Therefore the $O'(L)$-orbit of t is in the image of Φ^*.

6. Quadratic and cyclotomic fields

We now relate Theorem 5.2 above with Theorem 3.1 and other results in [4]. There, using the Strong Approximation Theorem for rotations (see [7, 104:4]), the number

$$N(L, D) = \prod_{p|2d} N(L_p, D)$$

is computed for the Bianchi groups $PSL(2, \mathcal{O}_d)$, $d > 0$, when $D > 0$ (the localization used in [4] includes the completion, but this is not significant). The condition analogous to h odd, that is, excluding the dyadic orbits of $t \in M_2$, is not assumed in [4]. The number of dyadic orbits needed for Theorem 5.2 is still $N(L_2, D)$ when $d \equiv 3 \mod 4$ since 2 is not ramified, or when $D \equiv 0 \mod 2$ or $d \equiv 1 \equiv -D \mod 4$, since then no $t \in M_2$ exist. However, in the remaining cases the values of $N(L_2, D)$ in [4] must be slightly modified to count the conjugacy classes of $PSU(h, \mathcal{O}_S)$ in Theorem 5.2. Also, the non-dyadic orbits corresponding to $t \in p\mu_p L_p$ must be excluded, as explained above. Of course, when $\mathfrak{O}_S = \mathbb{Z}$ is a unique factorization domain, we start with d square-free and then $\mu_p = 1$ at all non-dyadic primes.

These calculations also apply for the Hilbert modular groups where $F = \mathbb{Q}$ and $d < 0$, and determine $n(D)$ for any $D \neq 0$, since the analogue of Theorem 4.1 in [4] is now valid in this generality (see also Theorem 1.1 in [3]). The calculations also apply when $F = \mathbb{F}(X)$, under the restriction that the quadratic space $V \perp \langle -dD \rangle$ has local Witt index two at the infinite prime. For a polynomial $D \in \mathbb{F}[X]$, let $m = m(D) \geq 0$ be the number of monic irreducible factors of the g.c.d.(d, D) (with degree at least one).

Theorem 6.1. *Let $K = F(\sqrt{-d})$ where $F = \mathbb{F}(X)$ is a function field and d is a square-free polynomial in $\mathbb{F}[X]$. Assume the Hilbert symbol $(D, -d)_\infty = 1$ at the infinite prime. Then there are $n(D) = 2^m$ conjugacy classes of projective special unitary groups $PSU(h, \mathcal{O}_S)$, of primitive hermitian h with discriminant $D \neq 0$, under the action of $PSL(2, \mathcal{O}_S)$.*

Proof. The Witt index condition needed for the Strong Approximation Theorem is equivalent to $(D, -d)_\infty = 1$ at the infinite prime. The result then follows by modifying the data in Theorem 5.1 of [4] by excluding the orbits coming from t that are not primitive in L. □

If D and d are monic polynomials in $\mathbb{F}[X]$, then $(D, -d)_\infty = 1$ if and only if d has odd degree, or D has even degree, or $-1 \in \mathbb{F}^{*2}$.

The hermitian matrices h as in (7) are the starting point for Vulakh's treatment in [9] and [10] of the conjugacy classes of the maximal non-elementary Fuchsian subgroups of Bianchi groups. He relates h to the circle $\mathcal{C}$ in the complex plane with discriminant D given by

$$aZ\bar{Z} + bZ + \bar{b}\bar{Z} + c = 0$$

where $Z = X + iY \in \mathbb{C}$. Instead of a primitivity condition, an equivalence relation on rational hermitian h is introduced. A different treatment is given in [4] and [6] where it is shown, using the underlying hyperbolic geometry, that the maximal Fuchsian subgroup $\mathcal{F}$ corresponding to the transformations

$$Z' = (xZ + z)(yZ + w)^{-1}$$

that stabilize $\mathcal{C}$ then corresponds to $\{\phi \in O'(L) \mid \phi(t) = \pm t\}$, with $t \in L(D)$ as before, and that the conjugacy classes of these $\mathcal{F}$ with discriminant D are in one-to-one correspondence with the orbits in $L(D)$ under the action of $\hat{O}'(L)$, the group generated by $O'(L)$ and $-I$. The ambiguity in sign is introduced because $\pm h$, or $\pm t$, both determine the same circle $\mathcal{C}$. This problem of Fuchsian subgroups is closely related to our classification problem here, but distinct since the primitivity condition used for h is stronger.

Now let $K = \mathbb{Q}(\zeta)$, where ζ is a primitive l-th root of unity, and let $F = K \cap \mathbb{R} = \mathbb{Q}(\zeta + \bar{\zeta})$. Take $\mathfrak{O}$ to be the ring of algebraic integers in F. For $l = 4m + 3$ prime, $K = F(\sqrt{-l})$ and l is totally ramified in $\mathfrak{O}$. Since $-l \equiv 1 \mod 4$, the extension K/F, viewed dyadically, is either split or unramified at each $p'|2$, with $\mathcal{O}_{p'} = \mathfrak{O}_{p'}[\omega]$ for $\omega = (1+\sqrt{-l})/2$. Thus $L_{p'}$, as in (6) with $\mu_{p'} = 1$, is an even unimodular lattice and $N(L_{p'}, D) = 1$ by Theorem 5.3(1) in [4]. Let p be the unique prime over l in F. Then $\mathcal{O}_p = \mathfrak{O}_p[p^{-m}\sqrt{-l}]$ since $[F : \mathbb{Q}] = 2m + 1$ and l is totally ramified. Take L_p as in (4) with $q(s) = f(u, v) = lp^{-2m}$ a prime. Then, generalizing Theorem 4.1 in [4], $N(L, D) = N(L_p, D)$ when h has a totally positive discriminant D. Excluding the local orbits that are not primitive in Theorem 5.1 in [4], we get

Theorem 6.2. *Let $K = \mathbb{Q}(\zeta)$ with $l = 4m + 3$ prime, and $F = K \cap \mathbb{R}$. Then, for p the unique prime over l, and totally positive $D \in \mathfrak{O}$,*

1. *$n(D) = 1$ when D is a unit in $\mathfrak{O}_p$.*
2. *$n(D) = 2$ otherwise.*

A similar theorem holds for $l \equiv 1 \mod 4$ and prime.

When $l = 2^n \geq 8$, 2 is the only prime ramifying in $\mathfrak{O}$ and $p = \zeta + \bar{\zeta}$ is the unique prime over 2 in F. Then $K = F(\sqrt{-1}) = \mathbb{Q}(\zeta)$. For p' non-dyadic, take $L_{p'}$ as in (4) with $d = 1$. Hence $N(L, D) = N(L_p, D)$ for $D > 0$. Dyadically, K/F is a ramified unit extension with $e = 2^{n-2} \geq 2$ and $k = e - 1$ (see Lemma 7.2 in [3]). From (6) with $\mu_p = 1$, we take

$$L_p = (\mathfrak{O}_p r + \mathfrak{O}_p p^{-k}(r - s)) \perp H_p$$

where

$$d = q(s) = f(u, v) = (1 + p^{e/2} + p^{3e/4} + p^{7e/8} + \cdots + p^{(e-1)e/e})^2.$$

For $l = 8, p = \pm\sqrt{2}$ and $d = (1 \pm \sqrt{2})^2$ is a unit in $\mathfrak{O}$; then

$$L = (\mathfrak{O} r + \mathfrak{O}\sqrt{2}^{-1}(r - s)) \perp H.$$

The dyadic orbits are complicated when 2 is ramified and $n(D)$ has not been computed.

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