SPHERICAL FUNCTIONS AND CONFORMAL DENSITIES ON SPHERICALLY SYMMETRIC $CAT(-1)$-SPACES

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ABSTRACT. Let X be a $CAT(-1)$-space which is spherically symmetric around some point $x_0 \in X$ and whose boundary has finite positive $s-$dimensional Hausdorff measure. Let $\mu = (\mu_x)_{x \in X}$ be a conformal density of dimension $d > s/2$ on ∂X. We prove that μ_{x_0} is a weak limit of measures supported on spheres centered at x_0. These measures are expressed in terms of the total mass function of μ and of the $d-$dimensional spherical function on X. In particular, this result proves that μ is entirely determined by its dimension and its total mass function. The results of this paper apply in particular for symmetric spaces of rank one and semi-homogeneous trees.

0. INTRODUCTION

Let X be a $CAT(-1)$–space and let ∂X be its boundary. A conformal density of dimension d on ∂X is a family $\mu = (\mu_x)_{x \in X}$ of finite positive Borel measures on ∂X, such that for every x and y in X, μ_y is absolutely continuous with respect to μ_x, with Radon-Nikodým derivative given by $\frac{d\mu_y}{d\mu_x}(\xi) = j^d(x, y, \xi)$ for all $\xi \in \partial X$. Here, $j(x, y, \xi)$ stands for the infinitesimal distorsion at the point ξ of the visual metric on ∂X seen from y with respect to the visual metric seen from x (see Section 1 for the precise definitions). The total mass function of μ is the function $\phi_\mu : X \to \mathbb{R}_+$ defined by $\phi_\mu(x) = \mu_x(\partial X)$.

For every x in X, we denote by $\mathcal{H}_x^s$ the $s-$dimensional Hausdorff measure on ∂X with respect to the visual metric seen from X.

In this paper, we shall always assume that there exist a point $x_0 \in X$ and a real number $s \geq 0$ such that the following two properties are satisfied:

(0.1). X is *spherically symmetric* around $x_0 \in X$. This means that the group K_{x_0} of isometries of X which stabilizes x_0 acts transitively on each sphere centered at this point.

(0.2). For some (or, equivalently, for every) $x \in X$, we have

$$0 < \mathcal{H}_x^s(\partial X) < \infty.$$

We note right away that condition (0.2) implies that s is the Hausdorff dimension of ∂X with respect to any of the visual metrics.

If Γ is a discrete subgroup of X, conformal densities play a central role in the study of the ergodic properties of the action of Γ on ∂X. The results of this paper

concern conformal densities on the boundary of a space X satisfying conditions (0.1) and (0.2).

For each $d \in \mathbb{R}$, we define the *spherical function* $\sigma_d : X \times X \to \mathbb{R}_+$ by the formula

$$\sigma_d(x, y) = \int_{\xi \in \partial X} j^d(x, y, \xi) \, d\mathcal{H}_x^s(\xi)$$

for every x and y in X.

For every $R \geq 0$, let $S(x_0, R)$ denote the sphere in X of radius R centered at x_0. We define $\omega_{x_0, R}$ to be the natural measure on $S(x_0, R)$ which is induced by the Haar measure on K_{x_0}, the Haar measure being normalized to have total mass one.

Let μ be a conformal density of dimension d on ∂X. For every $R \geq 0$, we define the measure $\mu_{x_0, R}$ on $S(x_0, R)$ by the formula

$$\mu_{x_0, R} = \frac{\mathcal{H}_{x_0}^s(\partial X)}{c_{d, R}} \phi_\mu \omega_{x_0, R} \ ,$$

where $c_{d, R}$ is the value taken by the function $\sigma_d(x_0, .)$ on $S(x_0, R)$. (The fact that X is spherically symmetric around x_0 implies that the value of $\sigma_d(x_0, y)$, as a function of y, depends only on the distance from x to x_0.)

In this paper, we prove (Theorem 4.3) the following fact.

Theorem A. *Let X be a $CAT(-1)-$space satisfying* (0.1) *and* (0.2)*. Then, there exists a sequence (R_n) of real numbers, with $R_n \to \infty$ as $n \to \infty$, such that for every conformal density $\mu = (\mu_x)_{x \in X}$ of dimension d on ∂X with $d > s/2$, the measure μ_{x_0} is the limit, in the sense of weak convergence, of the sequence of measures μ_{x_0, R_n}.*

Let us remark that the conformal measure μ on ∂X is determined by its value at an arbitrary point x in X and the dimension d, so that the conformal density μ is completely determined by μ_{x_0} and d.

We now state a few results which follow from Theorem A.

From previous work on the subject, it turns out to be interesting to know under which conditions a conformal density of a given dimension is determined by its total mass function, and there are several results in this direction. For example, by a result of D. Sullivan (see [Sul3]), in the case where $X = \mathbb{H}^n$ (the hyperbolic n–space) a conformal density of dimension $\geq (n-1)/2$ is uniquely determined by its total mass function. The same result holds in the case where X is a homogeneous tree of degree k, provided that the dimension of the conformal density is $\geq \frac{1}{2} \log(k-1)$ (see [CP1]). In this paper, we obtain, as a consequence of Theorem A, the following

Corollary B. *Let X be a $CAT(-1)-$space satisfying* (0.1) *and* (0.2) *and let $\mu = (\mu_x)_{x \in X}$ and $\mu' = (\mu'_x)_{x \in X}$ be two conformal densities on ∂X having the same dimension d, with $d > s/2$. If the total mass functions of μ and μ' are equal, then $\mu = \mu'$.*

Suppose now that X is a $CAT(-1)-$space which is spherically symmetric around each of its points, that is, satisfying (0.1) for every point x_0 in X. Suppose also that X satisfies (0.2). Under these hypotheses, we have proved in [CP2] the existence of a transformation which for every conformal density μ of dimension $d < s/2$ on ∂X associates a conformal density μ^+ of dimension $s - d$ having the same total mass function as μ. As a consequence of Corollary B, we have

Corollary C. *The conformal density μ^+ is uniquely determined by the total mass function of μ.*

We shall prove also the following (Theorem 4.4)

Theorem D. *Suppose that X satisfies* (0.1) *and the following property:*
There exist four constants $r_0 > 0$, $C_1 > 0$, $C_2 > 0$ and $s \geq 0$ such that for every $\xi \in \partial X$ and for every $0 \leq r \leq r_0$, we have

$$C_1 \leq \frac{\mathcal{H}_{x_0}^s\big(B(\xi,r)\big)}{r^s} \leq C_2\,,$$

where $B(\xi, r)$ denotes the sphere in ∂X of center ξ and radius r with respect to the visual metric seen from x_0. (We note that this implies property (0.2), *with the same s.)*
Then, for every conformal density μ on ∂X of dimension $d > s/2$, we have

$$\mu_{x_0} = \lim_{R \to \infty} \mu_{x_0, R}$$

in the sense of weak convergence of measures.

We note that if a $CAT(-1)-$space satisfying (0.1) admits a discrete cocompact group Γ of isometries whose critical exponent δ_Γ is finite, then the hypotheses of Theorem D are satisfied with $s = \delta_\Gamma$ (see [Coo] and [Bou]). In particular, this is the case if X is a symmetric space of rank one or a semi-homogeneous tree. Hence we obtain

Corollary E. *Let X be a Riemannian symmetric space of rank one. Then, for every conformal density $\mu = (\mu_x)_{x \in X}$ on ∂X of dimension $d > s/2$, we have, for every point x in X,*

$$\mu_x = \lim_{R \to \infty} \mu_{x, R}$$

in the sense of weak convergence of measures.

We note that if X is a Riemannian symmetric space of rank one, the visual metrics on ∂X are Lipschitz-equivalent to the natural Carnot-Carathéodory metrics.
It may be useful to recall the classification of Riemannian symmetric spaces of rank one: These are the hyperbolic spaces $\mathbb{H}_{\mathbf{K}}^n$, where $\mathbf{K}$ denotes either the field of real numbers (with n being any integer ≥ 1), or the field of complex numbers or of quaternions (and in these cases $n \geq 2$), or finally the algebra of Cayley numbers (and here $n = 2$). In all these cases, we have $s = kn + k - 2$, where k denotes the real dimension of $\mathbf{K}$ (the formula is in [Pan], and it is a consequence of a general formula in [Mit]). In a Riemannian symmetric space of rank one, the total mass function of a conformal density of dimension d is a positive eigenfunction of the Laplacian, with eigenvalue $\lambda = s(s - d)$.
Let p and q be two integers which are ≥ 2. Recall that the *semi-homogeneous tree of type (p, q)* is the simplicial tree $T^{p,q}$ where each vertex is of order p or q (that is, belongs respectively to p or q edges), and such that each vertex of order p (resp. q) is adjacent only to vertices of order q (resp. p). The tree $T^{p,q}$ satisfies (0.2) with $s = \frac{1}{2}\big(\log(p-1) + \log(q-1)\big)$ (see the example at the end of this paper). As a consequence of Theorem D, we therefore have

Corollary F. *Let $T^{p,q}$ be a semi-homogeneous tree of type (p,q). Then, for every conformal density $\mu = (\mu_x)_{x \in X}$ of dimension $d > \frac{1}{2}\big(\log(p-1) + \log(q-1)\big)$ on ∂X, we have, for every vertex x of X,*

$$\mu_x = \lim_{R \to \infty} \mu_{x,R}$$

in the sense of weak convergence.

The outline of the paper is as follows.

Section 1 contains background material on $CAT(-1)$–spaces, with the visual metric on their boundary as defined in this setting by M. Bourdon.

Section 2 introduces conformal densities on boundaries of $CAT(-1)$–spaces.

In Section 3, we introduce spherical functions on X and we study their basic properties. In particular, we prove that they satisfy a multiplicative property (Corollary 3.6) which is analogous to a property satisfied by the classical spherical functions associated to symmetric spaces of rank one.

In Section 4, we prove all the results stated above.

In Section 5, we study spherically symmetric trees, which give an interesting class of examples of spaces satisfying the hypotheses of Theorem D.

1. Preliminaries on $CAT(-1)$–spaces

Let X be a metric space. The distance between two points x and y in X will be denoted by $|x - y|$. A *geodesic segment* (resp. a *geodesic ray*) in X is a distance–preserving map from $[a,b]$ to X (resp. from $[a, \infty[$ to X), where $[a,b]$ is a compact interval in $\mathbb{R}$.

A *triangle* T in X consists of three points in X together with three geodesic segments joining them pairwise. The segments (or their images) are called the *sides* of the triangle.

If T is a triangle in X, then a *comparison triangle* T' for T in $\mathbb{H}^2$ (the real hyperbolic plane) is a triangle in $\mathbb{H}^2$ together with a map f_T which sends each side of T isometrically onto a side of T'.

A triangle T is said to satisty the $CAT(-1)$–*inequality* if for every x and y in T, we have

$$|x - y|_X \le |f_T(x) - f_T(y)|_{\mathbb{H}^2},$$

where f_T is the map associated to a comparison triangle for T in $\mathbb{H}^2$. A metric space X is *geodesic* if any two points in X can be joined by a geodesic segment. The space is said to be *proper* if all closed balls are compact. A $CAT(-1)$–space is a proper geodesic metric space in which every triangle satisfies the $CAT(-1)$–inequality. The class of $CAT(-1)$–spaces is a vast generalization of that of simply–connected Riemannian manifolds of sectional curvature ≤ -1. It contains many singular spaces of combinatorial curvature ≤ -1 (in particular, metric trees); see [Gr] and [Bou].

$CAT(-1)$–spaces are examples of Gromov–hyperbolic spaces, of which we recall the definition. Let X be a metric space with basepoint x_0. The *Gromov product* with respect to x_0 of two points x and y in X is defined as

$$(x.y)_{x_0} = \frac{1}{2}(\,|x_0 - x| + |x_0 - y| - |x - y|\,).$$

The space X is said to be *Gromov–hyperbolic* if there exists $\delta \geq 0$ such that the following holds:

$$(1.1) \qquad (x.y)_{x_0} \geq \min \big((x.z)_{x_0}, (y.z)_{x_0} \big) - \delta$$

for all $x, y, z \in X$ and for every choice of a basepoint x_0.

Without going into the details, let us recall that a Gromov–hyperbolic space has a canonical boundary, ∂X, called its *Gromov boundary* (see [Gr], Section 1.8, or [CDP], Chapter 2). In the case where X is geodesic and proper, ∂X is realized as the set of geodesic rays up to the equivalence relation which identifies two geodesic rays $r_1 : [0, \infty[\to X$ and $r_2 : [0, \infty[\to X$ if and only if $\sup_{t \geq 0} |r_1(t) - r_2(t)| < \infty$. We denote by $r(\infty)$ the point in ∂X defined by the geodesic ray r, and we call it the *endpoint* of r.

Let X be a $CAT(-1)$–space. If x is in X and y in X (resp. in ∂X), then, up to reparametrization, there is a unique geodesic segment (resp. ray) joining x and y. This segment (resp. ray) is denoted by $[x, y]$ (resp. $[x, y[$). The Gromov product can be extended to $X \cup \partial X \times X \cup \partial X$ by the following formula:

$$(1.2) \qquad (\xi.\eta)_{x_0} = \lim_{(x,y) \to (\xi, \eta)} (x.y)_{x_0},$$

where ξ and η are points in $X \cup \partial X$, with x converging to ξ and y to η respectively on $[x_0, \xi[$ and $[x_0, \eta[$. (To see that the limit exists, we note that if the point x' (resp. y') is situated beyond the point x (resp. y) on the ray $[x_0, \xi[$ (resp. on the ray $[x_0, \eta[$), then using the triangle inequality, we find that $(x'.y')_{x_0} \geq (x.y)_{x_0}$.) The product $(\xi.\eta)_{x_0}$ is equal to $+\infty$ if and only if $\xi = \eta \in \partial X$. The hyperbolicity relation (1.1) is valid for every x and y in $X \cup \partial X$.

Recall that the *Busemann function* $h_r : X \to \mathbb{R}$ associated with a geodesic ray $r : [0, \infty[\to X$ is defined by

$$h_r(x) = \lim_{t \to \infty} (|x - r(t)| - t).$$

Given x and y in X, $\xi \in \partial X$, and a geodesic ray $r : [0, \infty[\to X$ satisfying $r(\infty) = \xi$, we define the *Busemann cocycle*, as in [Bou], p. 17, by

$$(1.3) \qquad B(x, y, \xi) = h_r(x) - h_r(y).$$

The value of $B(x, y, \xi)$ does not depend on the choice of the geodesic ray r ending at ξ. (This is easily seen using the fact that if r and r' are two geodesic rays ending at ξ, the distance between the point $r(t)$ and the image of the ray r' tends to 0 as t tends to ∞, as explained in [Bou], p. 18.)

There is a nice family of metrics $(|\ |_x)_{x \in X}$ on ∂X, which we shall call the *visual metrics*, and which have been defined, in the setting of $CAT(-1)$–spaces, by Bourdon [Bou] using the formula

$$(1.4) \qquad |\xi - \eta|_x = e^{-(\xi.\eta)_x}$$

for every x in X and for every ξ and η in ∂X.

The following formula is easy to prove (see [Bou], p. 26):

$$(1.5) \qquad |\xi - \eta|_y = |\xi - \eta|_x \, e^{\frac{1}{2} \big(B(x,y,\xi) + B(x,y,\eta) \big)}$$

for every x and y in X, and ξ and η in ∂X.

We define now, for every x and y in X and ξ in ∂X,

$$(1.6) \qquad j(x, y, \xi) = e^{B(x,y,\xi)}.$$

As we shall see in Corollary 1.2, the function $j(x, y, \xi)$ measures the infinitesimal distorsion of the metric $|\ |_y$ with respect to the metric $|\ |_x$. From the definitions and formula (1.5), we immediately deduce

Proposition 1.1. *For every x and $y \in X$ and for every ξ and $\eta \in \partial X$, we have*

$$(1.7) \qquad\qquad |\xi - \eta|_y^2 = j(x, y, \xi)j(x, y, \eta)|\xi - \eta|_x^2.$$

∎

Remark. For natural reasons, we call formula (1.7) the "formula for the change of point of view". It generalizes a formula in [Sul1] for the case of hyperbolic space $\mathbb{H}^n$ and a formula in [Coo] which holds in the case of metric trees.

Corollary 1.2. *For every x and y in X and for every ξ in ∂X, we have*

$$j(x, y, \xi) = \lim_{\substack{\eta \to \xi \\ \eta \neq \xi}} \frac{|\xi - \eta|_y}{|\xi - \eta|_x}.$$

Proof. The proof follows from Proposition 1.1, using the continuity of the function $B(x, y, \xi)$ with respect to ξ. ∎

The following formula is also useful:

Proposition 1.3. *For every x and y in X and for every ξ in ∂X, we have*

$$j(x, y, \xi) = e^{2(y.\xi)_x - |x-y|}.$$

Proof. We use the definition

$$j(x, y, \xi) = e^{h_r(x) - h_r(y)}$$

with $r : [0, \infty[\to X$ being the geodesic ray satisfying $r(0) = x$ and $r(\infty) = \xi$. Therefore, $h_r(x) = 0$. To compute $h_r(y)$, let us take a sequence of points x_n on $[x, \xi[$ converging to ξ. We have

$$\begin{aligned}
h_r(y) &= \lim_{n \to \infty} (|y - x_n| - |x - x_n|) \\
&= \lim_{n \to \infty} (-|x - x_n| - |x - y| + |y - x_n|) + |x - y| \\
&= -2(y.\xi)_x + |x - y|.
\end{aligned}$$

This proves the proposition. ∎

2. Conformal densities on ∂X

In the rest of this paper, X is a $CAT(-1)$–space and ∂X is equipped with the class of visual metrics $(|\ |_x)_{x \in X}$ defined by formula (1.4).

Let d be a real number. A *conformal density* of dimension d on ∂X is a family $\mu = (\mu_x)_{x \in X}$ of finite positive Borel measures on ∂X such that for every x and y in X, the measure μ_x is absolutely continuous with respect to μ_y, with Radon-Nikodým derivative

$$\frac{d\mu_y}{d\mu_x}(\xi) = j^d(x, y, \xi) \quad \forall \xi \in \partial X.$$

We note that a conformal density is completely determined by its dimension and its value at a given point of X, and that this value can be an arbitrary finite positive Borel measure on ∂X.

In the next proposition, we give an important example of a conformal density. We shall make the hypothesis that the $s-$dimensional Hausdorff measure $\mathcal{H}_x^s$ on ∂X (equipped with the metric $|\ |_x$) satisfies $0 < \mathcal{H}_x^s(\partial X) < \infty$, and for this we note right away that since for all x and y in X the metrics $|\ |_x$ and $|\ |_y$ on ∂X are Lipschitz–equivalent, this condition does not depend on the choice of the basepoint x.

Proposition 2.1. *Suppose that* $0 < \mathcal{H}_x^s(\partial X) < \infty$. *Then* $\mathcal{H}^s = (\mathcal{H}_x^s)_{x \in X}$ *is a conformal density of dimension s on* ∂X.

Proof. The proof follows easily from the definition of the Hausdorff measure, using Proposition 1.2 and the continuity of the infinitesimal distorsion $j(x, y, \xi)$ as a function of ξ. ∎

Given a conformal density $\mu = (\mu_x)_{x \in X}$ of dimension d, we recall that its *total mass* function $\phi_\mu : X \to \mathbb{R}_+$ is defined, for every $x \in X$, by the formula

$$\phi_\mu(x) = \mu_x(\partial X).$$

From the definition of a conformal density, we may also write, for every $y \in X$,

$$(2.1) \qquad \phi_\mu(y) = \int_{\xi \in \partial X} j^d(x, y, \xi) \, d\mu_x(\xi).$$

3. Spherical functions

Let X be a $CAT(-1)-$space satisfying $0 < \mathcal{H}_x^s(\partial X) < \infty$, for some $s \geq 0$.

We define, for each $d \in \mathbb{R}$, the *spherical function* $\sigma_d : X \times X \to \mathbb{R}_+$ by the formula

$$\sigma_d(x, y) = \int_{\xi \in \partial X} j^d(x, y, \xi) \, d\mathcal{H}_x^s(\xi).$$

In other words, for a fixed x, $\sigma_d(x, \cdot)$ is the total mass function of the unique conformal density of dimension d whose value at x is the Hausdorff measure $\mathcal{H}_x^s$.

Let x_0 be a fixed point of X and K_{x_0} the group of isometries of X that fix x_0. This group is equipped with the topology of uniform convergence on compact subsets. It is easy to see that K_{x_0} is compact. Note also that K_{x_0} acts on ∂X as a group of isometries with respect to the metric $|\ |_{x_0}$. We shall say that the space X is *spherically symmetric around* x_0 if the group K_{x_0} acts transitively on every sphere centered at x_0.

We begin with the following:

Proposition 3.1. *Suppose that* X *is spherically symmetric around* x_0. *Then, the group* K_{x_0} *acts transitively on* ∂X. *Conversely, suppose that* K_{x_0} *acts transitively on* ∂X *and that every point in* X *belongs to some geodesic ray originating at* x_0. *Then* X *is spherically symmetric around* x_0.

Proof. Suppose that X is spherically symmetric around x_0 and let ξ and η be two arbitrary points in ∂X. Consider the two geodesic rays $[x_0, \xi[$ and $[x_0, \eta[$, and for every $n = 0, 1, 2, ...$, let x_n (resp. y_n) be the point of intersection of $[x_0, \xi[$ (resp. $[x_0, \eta[)$ with the sphere $S(x_0, n)$. For each n, there exists an isometry $g_n \in K_{x_0}$ which sends x_n to y_n. Since K_{x_0} is compact, there is a subsequence of g_n converging to an isometry g in K_{x_0}, and it is clear that g sends ξ to η.

Conversely, suppose that K_{x_0} acts transitively on ∂X. Given $R > 0$, let x and y be two points on the sphere $S(x_0, R)$. Consider two geodesic rays $[x_0, \xi[$ and $[x_0, \eta[$, containing x and y respectively, and let g be an isometry in K_{x_0} sending ξ to η. Then $g(x) = y$. This completes the proof of the proposition. ∎

We now study a few elementary properties of $\sigma_d(x, .)$ that will be useful for us in the sequel.

Proposition 3.2. *For every x and y in X and for every isometry g of X, we have*

$$\sigma_d(x, y) = \sigma_d(gx, gy).$$

Proof. This is clear from the definitions. ∎

Corollary 3.3. *If X is spherically symmetric around x_0, then for every y and z in X such that $|x_0 - y| = |x_0 - z|$, we have $\sigma_d(x_0, y) = \sigma_d(x_0, z)$.* ∎

For every $R \geq 0$, let y be an arbitrary point on the sphere $S(x_0, r)$ and consider the orbit map $\pi : K_{x_0} \to S(x_0, R)$ defined by $\pi(g) = g(y)$. Let $\omega_{x_0, R}$ be the image by π of the Haar measure m_{x_0} on K_{x_0} (normalized by $m_{x_0}(K_{x_0}) = 1$). It is clear that the definition of $\omega_{x_0, R}$ does not depend on the choice of the point y. We have the following:

Proposition 3.4. *Assume that X is spherically symmetric with respect to x_0 and let y be an arbitrary point in X, with $r = |y - x_0|$. Then, for every conformal density $\mu = (\mu_x)_{x \in X}$ of dimension d on ∂X, we have*

$$\phi_\mu(x_0) = \frac{\mathcal{H}_{x_0}^s(\partial X)}{\sigma_d(x_0, y)} \int\limits_{z \in S(x_0, R)} \phi_\mu(z) d\omega_{x_0, R}(z).$$

Proof. We have

$$\int\limits_{z \in S(x_0, R)} \phi_\mu(z) d\omega_{x_0, R}(z) = \int\limits_{\gamma \in K_{x_0}} \phi_\mu(\gamma y) dm_{x_0}(\gamma)$$

$$= \int\limits_{\gamma \in K_{x_0}} \int\limits_{\xi \in \partial X} j^d(x_0, \gamma y, \xi) d\mu_{x_0}(\xi) dm_{x_0}(\gamma)$$

$$= \int\limits_{\gamma \in K_{x_0}} \int\limits_{\xi \in \partial X} j^d(x_0, y, \gamma^{-1} \xi) d\mu_{x_0}(\xi) dm_{x_0}(\gamma).$$

We now use the following lemma:

Lemma 3.5. *The measure $\mathcal{H}_{x_0}^s$ is, up to a constant factor, the unique finite positive Borel measure on ∂X which is invariant by the group K_{x_0}.*

Proof. The proof follows from general facts in measure theory. Suppose that μ_1 and μ_2 are two K_{x_0}−invariant finite positive Borel measures on ∂X, and consider the measure $\nu = \mu_1 + \mu_2$. For every Borel subset $A \subset \partial X$, we can write $\nu(A) \geq \mu_1(A)$. Therefore, μ_1 is absolutely continuous with respect to ν, and the Radon-Nikodým derivative $\frac{d\mu_1}{d\nu}$ is a function on ∂X which is invariant under the action of K_{x_0}. Since K_{x_0} acts transitively on ∂X, $\frac{d\mu_1}{d\nu}$ is constant. Therefore, the measure μ_1 is itself proportional to ν, which implies that μ_2 is proportional to μ_1. This proves Lemma 3.5. ∎

Now we can finish the proof of Proposition 3.4. The measure on ∂X defined by

$$f \mapsto \int\limits_{\gamma \in K_{x_0}} \int\limits_{\xi \in \partial X} f(\gamma^{-1}\xi) d\mu_{x_0}(\xi) dm_{x_0}(\gamma),$$

and which appears in the last integral, is $K_{x_0}-$invariant, and is therefore a constant multiple of $\mathcal{H}^s_{x_0}$. Its total mass is equal to $\mu_{x_0}(\partial X)$, since the total mass of the measure m_{x_0} on K_{x_0} is equal to one. Therefore, we obtain

$$\int\limits_{z \in S(x_0, R)} \phi_\mu(z) d\omega_{x_0, R}(z) = \frac{\mu_{x_0}(\partial X)}{\mathcal{H}^s_{x_0}(\partial X)} \int\limits_{\xi \in \partial X} j^d(x_0, y, \xi) d\mathcal{H}^s_{x_0}(\xi)$$

$$= \frac{\phi_\mu(x_0)}{\mathcal{H}_{x_0}(\partial X)} \sigma_d(x_0, y).$$

This proves Proposition 3.4. $\blacksquare$

Corollary 3.6 (the multiplicative property of spherical functions). *We have, for every x in X and for every $y \in S(x_0, R)$,*

$$\int\limits_{z \in S(x_0, R)} \sigma_d(x, z) d\omega_{x_0, R}(z) = \frac{1}{\mathcal{H}^s_{x_0}(\partial X)} \sigma_d(x, x_0) \sigma_d(x_0, y).$$

Proof. Let μ be the conformal density of dimension d on ∂X which at the point x is equal to the measure $\mathcal{H}^s_x$. Then $\phi_\mu(w)$ is equal to $\sigma_d(x, w)$ for every $w \in X$, and the proof follows from Proposition 3.4. $\blacksquare$

Remark. Let G denote the isometry group of X. We suppose that we have a $G-$invariant subset V of X such that whenever (x, y) and (x', y') are two elements in $V \times V$ with $|x - y| = |x' - y'|$, then there exists an element g in G such that $g(x) = x'$ and $g(y) = y'$. This implies in particular that G acts transitively on V, and therefore the value $\mathcal{H}^s_{x_0}(\partial X)$ does not depend on the choice of $x_0 \in V$. Thus, we can normalize this value to be equal to one. Let us note also that by Proposition 3.2, the value of $\sigma_d(x, y)$, for x and y in V, depends only on the distance $|x - y|$. Assume also that X is spherically symmetric around some (and hence any) point in V. Homogeneous trees and symmetric spaces of rank one satisfy all the required properties, with V being the set of vertices in the first case and with V being the whole space in the second case. Let x_0 be a basepoint in V. Consider the function $f : G \to \mathbb{R}_+$ defined, for all $g \in G$, by

$$f(g) = \sigma_d(x_0, gx_0).$$

Let us note that $f(g) = f(g^{-1})$ for every g in G, since $\sigma_d(\cdot, \cdot)$ on $V \times V$ depends only on the distance between the two variables. Note also that f is $K_{x_0}-$bi-invariant. By Corollary 3.6, we have (taking $x = g^{-1}x_0$ and $y = hx_0$)

$$\int\limits_{\gamma \in K_{x_0}} \sigma_d(g^{-1}x_0, \gamma hx_0) dm_{x_0}(\gamma) = f(g)f(h),$$

which implies

$$\int\limits_{\gamma \in K_{x_0}} \sigma_d(x_0, g\gamma hx_0) dm_{x_0}(\gamma) = f(g)f(h),$$

and finally

$$\int\limits_{\gamma \in K_{x_0}} f(g\gamma h) = f(g)f(h),$$

for every g and h in G.

In the case where X is a symmetric space of rank one, taking $V = X$, we obtain therefore a formula in [Hel], Chapter IV, Proposition 2.2, which characterizes spherical functions.

The following properties of our symmetric functions will also be useful.

Lemma 3.7. *For every x, y and z in X, we have*

$$\sigma_d(x, y) \geq e^{-d|y-z|}\sigma_d(x, z).$$

Proof. Let ξ be a point in ∂X and h the Busemann function associated with some geodesic ray ending at ξ. Using the triangle inequality, we see that

$$h(y) \leq h(z) + |y - z|.$$

Therefore,

$$j(x, y, \xi) = e^{h(x)-h(y)}$$
$$\geq e^{-|y-z|}j(x, z, \xi).$$

It follows that

$$\sigma_d(x, y) = \int\limits_{\xi \in \partial X} j^d(x, y, \xi)\, d\mathcal{H}_x^s(\xi)$$

$$\geq e^{-d|y-z|}\int\limits_{\xi \in \partial X} j^d(x, z, \xi)\, d\mathcal{H}_x^s(\xi)$$

$$= e^{-d|y-z|}\sigma_d(x, z).$$

This proves the lemma.   ∎

Proposition 3.8. *Asume that X is spherically symmetric around x_0. Then, for every y in X and for every ξ in ∂X, we have*

$$\sigma_d(x_0, y) = \mathcal{H}_{x_0}^s(\partial X)\int\limits_{\gamma \in K_{x_0}} j^d(x_0, \gamma y, \xi)\, dm_{x_0}(\gamma).$$

Proof. By Proposition 3.2, we can write, for every γ in K_{x_0},

$$\sigma_d(x_0, y) = \int\limits_{\xi \in \partial X} j^d(x_0, \gamma y, \xi) d\mathcal{H}_{x_0}^s(\xi).$$

Integrating with respect to the measure m_{x_0} on K_{x_0}, we obtain

$$\sigma_d(x_0, y) = \int\limits_{\gamma \in K_{x_0}} \int\limits_{\xi \in \partial X} j^d(x_0, \gamma y, \xi)\, d\mathcal{H}_{x_0}^s(\xi) dm_{x_0}(\gamma)$$

$$= \int\limits_{\xi \in \partial X} \int\limits_{\gamma \in K_{x_0}} j^d(x_0, \gamma y, \xi)\, dm_{x_0}(\gamma) d\mathcal{H}_{x_0}^s(\xi).$$

Using the transitivity of the action of K_{x_0} on ∂X and the left-invariance of the Haar measure, we see that the value of the integral

$$\int_{\gamma \in K_{x_0}} j^d(x_0, \gamma y, \xi)\, dm_{x_0}(\gamma)$$

does not depend on the choice of the point ξ. Therefore, we can separate the variables, and we obtain

$$\sigma_d(x_0, y) = \int_{\gamma \in K_{x_0}} j^d(x_0, \gamma y, \xi)\, dm_{x_0}(\gamma) \int_{\xi \in \partial X} d\mathcal{H}^s_{x_0}(\xi),$$

which proves the lemma. ■

4. Proof of the results

In this section, we prove the results stated in the introduction. In all the section, we suppose that X is a $CAT(-1)$−space which is spherically symmetric around some point x_0, and that the s-dimensional Hausdorff measure $\mathcal{H}^s_{x_0}$ of ∂X satisfies

$$0 < \mathcal{H}^s_{x_0}(\partial X) < \infty,$$

for some $s \geq 0$.

Proposition 4.1. *There exists a sequence (y_n) of points in X with $|x_0 - y_n| \to \infty$ as $n \to \infty$, such that for every $d \in \mathbb{R}$ there is a constant $C = C(d)$ such that for every $n = 0, 1, 2, ...$, we have*

$$\sigma_d(x_0, y_n) \geq C\, e^{-(s-d)|x_0-y_n|}.$$

Proof. By Proposition 5.1 of [Fal], there exists a constant $C' > 0$ such that for $\mathcal{H}^s_{x_0}$−almost every ξ_0 in ∂X, we have

(4.1) $$\limsup_{r \to 0} \frac{\mathcal{H}^s_{x_0}\big(B(\xi_0, r)\big)}{r^s} \geq C',$$

where $B(\xi_0, r)$ is as usual the closed ball centered at ξ_0 with radius r, with respect to the metric $|\ |_{x_0}$.

Note that by the symmetry hypothesis, (4.1) is valid for every point ξ_0 in ∂X. Now let ξ_0 be a fixed point in ∂X. Then there exists a sequence of real numbers $r_n > 0$ ($n = 0, 1, 2, ...$), with $r_n \to 0$, such that for every n we have

$$\frac{\mathcal{H}^s_{x_0}\big(B(\xi_0, r_n)\big)}{r_n^s} \geq C'',$$

where C'' is a positive constant (slightly smaller than C').

We suppose now, without loss of generality, that $r_n < 1$ for every n, and we consider the sequence of points y_n on the geodesic ray $[x_0, \xi_0[$ satisfying, for every $n = 0, 1, 2, ...$,

$$|x_0 - y_n| = -\log(r_n).$$

By Proposition 1.3, we can write, for every $\xi \in B(\xi_0, r_n)$,

(4.2) $$j(x_0, y_n, \xi) = e^{2(y_n.\xi) - |x_0 - y_n|}.$$

Let us find an upper bound for $(y_n.\xi)$. As the space X is Gromov-hyperbolic, there exists $\delta \geq 0$ such that

$$(4.3) \qquad (y_n.\xi) \geq \min\left((y_n.\xi_0), (\xi.\xi_0)\right) - \delta.$$

From the definition of the metric $|\ |_{x_0}$, we have

$$(\xi.\xi_0) \geq -\log r_n = |x_0 - y_n|$$

which implies, using the definition of the point y_n,

$$(y_n.\xi_0) = |x_0 - y_n| \leq (\xi.\xi_0).$$

Therefore, the min in formula (4.3) is attained by $(y_n.\xi_0)$ and we can write

$$(y_n.\xi) \geq (y_n.\xi_0) - \delta = |x_0 - y_n| - \delta.$$

Thus, formula (4.2) implies

$$j(x_0, y_n, \xi) \geq e^{|x_0 - y_n| - 2\delta}.$$

Summing up, we have, for all $n \geq 0$,

$$\sigma_d(x_0, y_n) \geq \int_{\xi \in B(\xi_0, r_n)} j^d(x_0, y_n, \xi) d\mathcal{H}^s_{x_0}(\xi)$$

$$\geq e^{d|x_0 - y_n| - 2\delta d} \int_{\xi \in B(\xi_0, r_n)} d\mathcal{H}^s_{x_0}(\xi)$$

$$\geq e^{d|x_0 - y_n| - 2\delta d} C'' r_n^s$$

$$= e^{d|x_0 - y_n| - 2\delta d} C'' e^{-s|x_0 - y_n|}.$$

Therefore, we have

$$(4.4) \qquad \sigma_d(x_0, y_n) \geq C e^{-(s-d)|x_0 - y_n|},$$

where C is the constant $C'' e^{-2\delta d}$. This proves Proposition 4.1. ∎

Proposition 4.2. *Let d be a real number satisfying $d > s/2$, and $f : X \cup \partial X \to \mathbb{R}$ a continuous function. For every $n = 0, 1, 2, \ldots$, define the function $g_n : \partial X \to \mathbb{R}$ by*

$$g_n(\xi) = \frac{\mathcal{H}^s_{x_0}(\partial X)}{\sigma_d(x_0, y_n)} \int_{\gamma \in K_{x_0}} j^d(x_0, \gamma y_n, \xi) f(\gamma y_n) dm_{x_0}(\gamma),$$

where (y_n) is the sequence of points defined by Proposition 4.1. Then the sequence (g_n) converges uniformly to f on ∂X, as $n \to \infty$.

Proof. By Proposition 3.8, we have

$$\frac{\mathcal{H}^s_{x_0}(\partial X)}{\sigma_d(x_0, y_n)} \int_{\gamma \in K_{x_0}} j^d(x_0, \gamma y_n, \xi) dm_{x_0}(\gamma) = 1.$$

Therefore, for every ξ in ∂X, the value of $g_n(\xi) - f(\xi)$ is equal to

$$\frac{\mathcal{H}_{x_0}^s(\partial X)}{\sigma_d(x_0, y_n)} \left(\int_{\gamma \in K_{x_0}} f(\gamma y_n) j^d(x_0, \gamma y_n, \xi) dm_{x_0}(\gamma) \right.$$

$$\left. - \int_{\gamma \in K_{x_0}} f(\xi) j^d(x_0, \gamma y_n, \xi) dm_{x_0}(\gamma) \right)$$

$$= \frac{\mathcal{H}_{x_0}^s(\partial X)}{\sigma_d(x_0, y_n)} \int_{\gamma \in K_{x_0}} \big(f(\gamma y_n) - f(\xi)\big) j^d(x_0, \gamma y_n, \xi) dm_{x_0}(\gamma).$$

The function f being continuous on $X \cup \partial X$, we can find $K \geq 0$ such that

(4.5) $|f(\gamma y_n) - f(\xi)| \leq \dfrac{\epsilon}{2}$ whenever $(\gamma y_n . \xi)_{x_0} \geq K.$

Now define the subset $W \subset K_{x_0}$ as

$$W = \{\gamma \in K_{x_0} \text{ such that } (\gamma y_n . \xi)_{x_0} \geq K\}.$$

Then we have

$$|g_n(\xi) - f(\xi)| \leq \frac{\mathcal{H}_{x_0}^s(\partial X)}{\sigma_d(x_0, y_n)} \int_{\gamma \in K_{x_0}} |f(\gamma y_n) - f(\xi)| \, j^d(x_0, \gamma y_n, \xi) \, dm_{x_0}(\gamma)$$

$$= \frac{\mathcal{H}_{x_0}^s(\partial X)}{\sigma_d(x_0, y_n)} \int_{\gamma \in W} |f(\gamma y_n) - f(\xi)| \, j^d(x_0, \gamma y_n, \xi) \, dm_{x_0}(\gamma)$$

$$+ \frac{\mathcal{H}_{x_0}^s(\partial X)}{\sigma_d(x_0, y_n)} \int_{\gamma \in K_{x_0} \setminus W} |f(\gamma y_n) - f(\xi)| \, j^d(x_0, \gamma y_n, \xi) \, dm_{x_0}(\gamma).$$

Let us estimate separately each of the two terms in the last expression.
For each $\gamma \in W$, we have

$$|f(\gamma y_n) - f(\xi)| \leq \frac{\epsilon}{2}.$$

Therefore, we can write

$$\frac{\mathcal{H}_{x_0}^s(\partial X)}{\sigma_d(x_0, y_n)} \int_{\gamma \in W} |f(\gamma y_n) - f(\xi)| \, j^d(x_0, \gamma y_n, \xi) dm_{x_0}(\gamma)$$

$$\leq \frac{\epsilon}{2} \frac{\mathcal{H}_{x_0}^s(\partial X)}{\sigma_d(x_0, y_n)} \int_{\gamma \in W} j^d(x_0, \gamma y_n, \xi) dm_{x_0}(\gamma)$$

$$\leq \frac{\epsilon}{2} \frac{\mathcal{H}_{x_0}^s(\partial X)}{\sigma_d(x_0, y_n)} \int_{\gamma \in K_{x_0}} j^d(x_0, \gamma y_n, \xi) dm_{x_0}(\gamma).$$

Using Proposition 3.8, we obtain

$$\frac{\mathcal{H}_{x_0}^s(\partial X)}{\sigma_d(x_0, y_n)} \int_{\gamma \in W} |f(\gamma y_n) - f(\xi)| \, j^d(x_0, \gamma y_n, \xi) dm_{x_0}(\gamma) \leq \frac{\epsilon}{2}.$$

To estimate the other term, let

$$||f||_\infty = \sup_{X \cup \partial X} |f|.$$

Then,

$$|f(y) - f(\xi)| \le 2||f||_\infty$$

for all y in $X \cup \partial X$ and $\xi \in \partial X$.

For every $\gamma \in K_{x_0} \setminus W$, we have, using Proposition 1.3,

$$j(x_0, \gamma y_n, \xi) \le e^{2K - |x_0 - \gamma y_n|} = e^{2K - |x_0 - y_n|}.$$

Therefore, we obtain

$$|g_n(\xi) - f(\xi)| \le \frac{\epsilon}{2} + \frac{2\mathcal{H}_{x_0}^s(\partial X)}{\sigma_d(x_0, y_n)} ||f||_\infty e^{d(2K - |x_0 - y_n|)}.$$

Using Proposition 4.1, we obtain

$$|g_n(\xi) - f(\xi)| \le \frac{\epsilon}{2} + \frac{2}{C} \mathcal{H}_{x_0}^s(\partial X) ||f||_\infty e^{d(2K - |x_0 - y_n|) + (s-d)|x_0 - y_n|}$$

$$\le \frac{\epsilon}{2} + \frac{2}{C} \mathcal{H}_{x_0}^s(\partial X) ||f||_\infty e^{2Kd} e^{(s-2d)|x_0 - y_n|}.$$

As $s - 2d < 0$ and since $|x_0 - y_n| \to \infty$ as $n \to \infty$, we have, $e^{(s-2d)|x_0 - y_n|} \to 0$ as $n \to \infty$. Therefore there exists an integer N_0 such that for all $n \ge N_0$, we have

$$|g_n(\xi) - f(\xi)| \le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This completes the proof of Proposition 4.2.   ■

Let μ be a conformal density on ∂X, whose dimension d satisfies $d > s/2$. For every $R \ge 0$, define the measure $\mu_{x_0, R}$ on the sphere $S(x_0, R)$ by the formula

$$\mu_{x_0, R} = \frac{\mathcal{H}_{x_0}^s(\partial X)}{c_{d,R}} \phi_\mu \omega_{x_0, R} \,,$$

where $c_{d,R}$ is the common value taken by the function $\sigma_d(x_0, \cdot)$ on $S(x_0, R)$. (Note that ϕ_μ is a continuous function on X and that the formula above means that $\mu_{x_0, R}$ is absolutely continuous with respect to the measure $\omega_{x_0, R}$, with Radon-Nikodým derivative $\frac{\mathcal{H}_{x_0}^s(\partial X)}{c_{d,R}} \phi_\mu$.)

Theorem 4.3. *There exists a sequence $(R_n)_{n \ge 0}$ with $R_n > 0$ and $R_n \to \infty$ as $n \to \infty$ such that for every conformal density μ of dimension $d > s/2$ on ∂X, we have*

$$\mu_{x_0} = \lim_{n \to \infty} \mu_{x_0, R_n}$$

in the sense of weak convergence of measures.

Proof. Consider the sequence of points (y_n) given in Proposition 4.1, and let $R_n = |x_0 - y_n|$. For any continuous function $f : X \cup \partial X \to \mathbb{R}$, let $g_n : \partial X \to \mathbb{R}$ be the

associated sequence of functions defined by Proposition 4.2. We have, for every $n \in \mathbb{N}$,

$$
\begin{aligned}
\mu_{x_0, R_n}(f) &= \frac{\mathcal{H}^s_{x_0}(\partial X)}{\sigma_d(x_0, y_n)} \int_{\gamma \in K_{x_0}} \phi_\mu(\gamma y_n) \, f(\gamma y_n) \, dm_{x_0}(\gamma) \\
&= \frac{\mathcal{H}^s_{x_0}(\partial X)}{\sigma_d(x_0, y_n)} \int_{\gamma \in K_{x_0}} \int_{\xi \in \partial X} j^d(x_0, \gamma y_n, \xi) \, d\mu_{x_0}(\xi) \, f(\gamma y_n) \, dm_{x_0}(\gamma) \\
&= \frac{\mathcal{H}^s_{x_0}(\partial X)}{\sigma_d(x_0, y_n)} \int_{\xi \in \partial X} \int_{\gamma \in K_{x_0}} j^d(x_0, \gamma y_n, \xi) \, f(\gamma y_n) \, dm_{x_0}(\gamma) d\mu_{x_0}(\xi) \\
&= \int_{\xi \in \partial X} g_n(\xi) \, d\mu_{x_0}(\xi) \\
&= \mu_{x_0}(g_n).
\end{aligned}
$$

Now, by Proposition 4.2, the sequence (g_n) converges uniformly to f on ∂X, as $n \to \infty$. Therefore, $\mu_{x_0, R_n}(f)$ converges to $\mu_{x_0}(f)$. Thus, the sequence μ_{x_0, R_n} converges weakly to μ_{x_0}. This proves Theorem 4.3. ∎

Remark. Let A be an $\epsilon-$dense subset of $\mathbb{R}_+$. Using Lemma 3.7, we can find a sequence of points (y_n) satisfying the properties of Proposition 4.1 with $|x_0 - y_n| \in A$ for all n. Hence, in Theorem 4.3, we can take R_n in A for all n.

Theorem 4.4. *Suppose that X is a $CAT(-1)-$space which is spherically symmetric around x_0 and such that the following condition holds: For some $s \geq 0$, there exist three constants $r_0 > 0$, $C_1 > 0$, and $C_2 > 0$ such that for every $\xi \in \partial X$ and for every $0 < r \leq r_0$, we have*

$$
\text{(4.6)} \qquad C_1 \leq \frac{\mathcal{H}^s_{x_0}\big(B(\xi, r)\big)}{r^s} \leq C_2.
$$

Then, for every conformal density $\mu = (\mu_x)_{x \in X}$ of dimension $d > s/2$, we have

$$
\mu_{x_0} = \lim_{R \to \infty} \mu_{x_0, R}
$$

in the sense of weak convergence of measures.

Remark. Note that condition (4.6) clearly implies that $0 < \mathcal{H}^s_{x_0}(\partial X) < \infty$. Conversely, the condition $0 < \mathcal{H}^s_{x_0}(\partial X) < \infty$ implies that there exist $r_0 > 0$ and $C_2 > 0$ such that for every $0 < r \leq r_0$, we have

$$
\frac{\mathcal{H}^s_{x_0}\big(B(\xi, r)\big)}{r^s} \leq C_2
$$

for $\mathcal{H}^s_{x_0}-$almost every $\xi \in \partial X$ (see [Fal], Proposition 5.1).

Proof of Theorem 4.4. The same proof as for Theorem 4.3 goes through with the following modifications: Instead of Proposition 4.1, we use Proposition 4.5 below, and the statement of Proposition 4.2 is replaced by the corresponding statement valid for every sequence of points y_n tending to infinity. ∎

Proposition 4.5. *Suppose that X satisfies the hypotheses of Theorem 4.4. Then, for every $d \in \mathbb{R}$, there exists a constant $C = C(d)$ such that for every $y \in X$, we have*

$$\sigma_d(x_0, y) \geq C \, e^{-(s-d)|x_0 - y|}.$$

Proof of Proposition 4.5. The proof follows that of Proposition 4.1, with the difference that here, inequality (4.1) is, by hypothesis, valid for all r in $[0, r_0]$. ∎

We deduce from Theorem 4.4 the following corollary:

Corollary 4.6. *Let X be $CAT(-1)-$space which is spherically symmetric around some point x_0, and admitting a cocompact isometric action of a group Γ such that the critical exponent δ_Γ is finite. Then, for every conformal density $\mu = (\mu_x)_{x \in X}$ of dimension $d > \delta_\Gamma / 2$ on ∂X, we have, for every x in X,*

$$\mu_x = \lim_{R \to \infty} \mu_{x, R}$$

in the sense of weak convergence of measures.

Proof. By a result in [Coo] and [Bou], the space X satisfies condition (4.6) with $s = \delta_\Gamma$. ∎

As we said in §0, the hypotheses of Corollary 4.6 are satisfied by Riemannian symmetric spaces of rank one and semi-homogeneous trees.

5. Spherically symmetric trees

We conclude the paper with a description of a class of examples satisfying the hypotheses of Theorem 4.4:

Consider a simplicial locally finite tree X which is spherically symmetric around some vertex x_0. The degree of a vertex v of X therefore depends only on the distance $|x_0 - v|$. Let p_0 denote the degree of x_0 and let $1 + p_n$ be the degree of a vertex at distance $n \geq 1$ from x_0. It is known (see [Lyo], p. 935) that the Hausdorff dimension of ∂X is equal to $\lim_{n \to \infty} \frac{1}{n+1} \sum_{k=0}^{n} \log p_i$.

Proposition 5.1. *Condition (4.6) of Theorem 4.4 is satisfied if and only if the following holds:*

(5.1)   *There exist $C_1 > 0$ and $C_2 > 0$ such that $C_1 \leq \dfrac{e^{ns}}{p_0 p_1 ... p_n} \leq C_2$,*

for every $n = 0, 1, ...$

Proof. Suppose first that (4.6) is satisfied. We know that the Hausdorff measure $\mathcal{H}_{x_0}^s$ on ∂X, which by hypothesis is finite and positive, is $K_{x_0}-$invariant. For every vertex v in X, let $B(v)$ denote the set of ξ in ∂X such that $v \in [x_0, \xi[$. Note that $B(v)$ is a closed ball of radius $e^{-(n+1)}$, where $n + 1 = |x_0 - v|$, with respect to the visual metric $|\ |_{x_0}$. By symmetry, the $\mathcal{H}_{x_0}^s-$mass of $B(v)$ depends only on the distance $|x_0 - v|$. Therefore, the value of $\mathcal{H}_{x_0}^s\big(B(v)\big)$ is equal to $\mathcal{H}_{x_0}^s(\partial X)/k_n$, where $k_n = p_0 ... p_n$ is the number of points at distance $n + 1$ from x_0. Condition (4.6) now clearly implies (5.1).

Conversely, suppose that condition (5.1) is satisfied. There is a canonical measure μ on ∂X which is characterized by the fact that for a vertex v at distance $n + 1$ from x_0, we have $\mu\big(B(v)\big) = \frac{1}{p_0 p_1 ... p_n}$. Condition (5.1) implies now that there exist

two constants $C_1' > 0$ and $C_2' > 0$ such that for every closed ball $B(\xi, r)$ in $(X, | \ |_{x_0})$ centered at ξ and of radius r, we have, for every $r \leq 1$,

$$C_1' \leq \frac{\mu\big(B(\xi, r)\big)}{r^s} \leq C_2'.$$

Let us fix a real number $\epsilon \in]0, 1]$, and consider a covering of $(\partial X, | \ |_{x_0})$ by a set of closed balls (B_i) of radii $r_i \leq \epsilon$. By the ultrametric property of the space $(\partial X, | \ |_{x_0})$, we can suppose (up to taking a subcover) that all the B_i's are disjoint. We can now write

$$\frac{1}{C_2'}\mu(\partial X) = \frac{1}{C_2'}\sum_i \mu(B_i) \leq \sum_i r_i^s \leq \frac{1}{C_1'}\sum_i \mu(B_i) = \frac{1}{C_1'}\mu(\partial X).$$

Therefore, we have $0 < \frac{1}{C_2'}\mu(\partial X) \leq \mathcal{H}_{x_0}^s \leq \frac{1}{C_1'}\mu(\partial X) < \infty$, which implies, using Lemma 3.5, that $\mathcal{H}_{x_0}^s$ is, up to a constant factor, equal to the measure μ. Therefore, condition (4.6) is satisfied.

An interesting special case of simplicial trees satisfying condition (5.1) is the case where the sequence $p_0, ..., p_n$ is preperiodic. By this we mean that there exist two integers $r \geq 1$ and $l \geq 1$ such that for every $k \geq r$, we have $p_{k+l} = p_k$. Then condition (5.1) is satisfied with $s = \frac{1}{l}\sum_{i=1}^{l}\log(p_{r+i})$. In particular, a semi-homogeneous tree of degree (p, q) (using the notation of §0) satisfies (5.1) with $s = \frac{1}{2}\big(\log(p-1) + \log(q-1)\big)$. Semi-homogeneous trees admit many discrete co-compact group actions, but the construction above provides us also with examples of simplicial trees which satisfy condition (5.1) and which do not admit any cocompact group action. For example, consider spherically symmetric trees such that the sequence $p_0, p_1, p_2...$ is preperiodic, with x_0 being the unique vertex of X having order p_0. In this case, the isometry group of X is reduced to K_{x_0}. In fact, it is possible to classify spherically symmetric trees which admit discrete cocompact group actions. Indeed, one can easily see that for such a tree, the sequence $p_1, p_2, ...$ is periodic of period N with $p_N = p_0 - 1$ and $p_{N-k} = p_k$ for all $k = 1, ..., N-1$.

References

[Bou]　M. Bourdon, Actions quasi-convexes d'un groupe hyperbolique, flot géodésique, Thèse, Université de Paris-Sud, centre d'Orsay, (1993).

[Coo]　M. Coornaert, Mesures de Patterson–Sullivan sur le bord d'un espace hyperbolique au sens de Gromov, Pacific Journal of Mathematics, **159**, No. 2, 241-270 (1993). MR **94m:**57075

[CDP]　M. Coornaert, T. Delzant, A. Papadopoulos, "Géométrie et théorie des groupes", Lecture Notes in Mathematics, vol. 1441, Springer-Verlag (1990). MR **92f:**57003

[CP1]　M. Coornaert and A. Papadopoulos, Positive p eigenfunctions of the Laplacian and conformal densities on homogeneous trees, J. London Math. Soc. (2) **55** (1997) no. 3, 609–624. MR **98j:**58089

[CP2]　M. Coornaert and A. Papadopoulos, Sur une formule de transformation pour les densités conformes au bord des $CAT(-1)$–espaces, C. R. Acad. Sci. Paris Sér. I Math. 320 (1995), no. 10, 1231–1236. MR **96f:**58088

[Fal]　K. Falconer, Fractal geometry, Mathematical Foundations and Applications, J. Wiley & Sons Ltd. (1990). MR **92j:**28008

[Gr]　M. Gromov, "Hyperbolic groups", MSRI Publications **8**, p. 75-263, Springer Verlag (1987). MR **89e:**20070

[Hel]　S. Helgason, "Groups and geometric analysis ", Academic Press, (1984). MR **86c:**22017

[Lyo]　R. Lyons, Random walks and percolation on trees, Annals of probability, vol. **18** No. 3, p. 931-958 (1990). MR **91i:**60179

[Mit]　J. Mitchell, Carnot-Carathéodory metrics, J. Differential Geometry, **21**, p. 35-45 (1985). MR **87d:**53086

[Pan] P. Pansu, Dimension conforme et sphère à l'infini des variétés à courbure négative, Annales
       Acad. Scient. Fennicae, Series A. I. Mathematica, vol. **14**, p. 177-212 (1989). MR **90k:**53079
[Sul1] D. Sullivan, The density at infinity of a discrete group of hyperbolic motions, Publ. Math.
       IHES **50**, 171-202 (1979). MR **81b:**58031
[Sul2] D. Sullivan, Discrete conformal groups and measurable dynamics, Bull. Amer. Math. Soc.
       (N.S.) **6**, 57-73 (1982). MR **83c:**58066
[Sul3] D. Sullivan, Related aspects of positivity in Riemannian manifolds, J. Diff. Geom. **25**,
       327-351 (1987). MR **88d:**58132

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