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# A LITTLEWOOD-RICHARDSON RULE FOR FACTORIAL SCHUR FUNCTIONS

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Abstract. We give a combinatorial rule for calculating the coefficients in the expansion of a product of two factorial Schur functions. It is a special case of a more general rule which also gives the coefficients in the expansion of a skew factorial Schur function. Applications to Capelli operators and quantum immanants are also given.

## 1. Introduction

As $\lambda$ runs over all partitions with length $l(\lambda) \leq n$, the Schur polynomials $s_\lambda(x)$ form a distinguished basis in the algebra of symmetric polynomials in the independent variables $x = (x_1, \dots, x_n)$. By definition,

$$s_\lambda(x) = \frac{\det(x_j^{\lambda_i+n-i})_{1\leq i,j\leq n}}{\prod_{i<j}(x_i - x_j)}.$$

Equivalently, these polynomials can be defined by the combinatorial formula

$$s_\lambda(x) = \sum_T \prod_{\alpha\in\lambda} x_{T(\alpha)}, \tag{1}$$

summed over semistandard tableaux $T$ of shape $\lambda$ with entries in the set $\{1, \dots, n\}$, where $T(\alpha)$ is the entry of $T$ in the cell $\alpha$.

Any product $s_\lambda(x)s_\mu(x)$ can be expanded as a linear combination of Schur polynomials:

$$s_\lambda(x)s_\mu(x) = \sum_\nu c^\nu_{\lambda\mu}\, s_\nu(x). \tag{2}$$

The classical Littlewood-Richardson rule [LR] gives a method for computing the coefficients $c^\nu_{\lambda\mu}$. These same coefficients appear in the expansion of a skew Schur function

$$s_{\nu/\lambda}(x) = \sum_\mu c^\nu_{\lambda\mu} s_\mu(x).$$

A number of different proofs and variations of this rule can be found in the literature; see, e.g. [M1, S1], and the references therein.

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To state the rule, we introduce the following notation. If $T$ is a tableau, then let $\mathrm{cw}(T)$ be the (reverse) *column word* of $T$, namely the sequence obtained by reading the entries of $T$ from top to bottom in successive columns starting from the right-most column. We will call the associated total order on the cells of $T$ *column order* and write $\alpha < \beta$ if cell $\alpha$ comes before cell $\beta$ in this order. A word $w = a_1 \cdots a_N$ in the symbols $1, \dots, n$ is a *lattice permutation* if for $1 \le r \le N$ and $1 \le i < n$ the number of occurrences of $i$ in $a_1 \cdots a_r$ is at least as large as the number of occurrences of $i+1$.

The Littlewood-Richardson rule says that the coefficient $c^{\nu}_{\lambda\mu}$ is equal to the number of semistandard tableaux $T$ of the shape $\nu/\mu$ and weight $\lambda$ such that $\mathrm{cw}(T)$ is a lattice permutation. (One usually uses *row words* in the formulation of the rule. However, it is known that these two versions are equivalent [FG].) In particular, $c^{\nu}_{\lambda\mu}$ is zero unless $\lambda, \mu \subseteq \nu$ and $|\nu| = |\lambda| + |\mu|$.

We will now state an equivalent formulation of this rule [JP, Z, KR] and establish some notation to be used in Section 3. Let $R$ denote a sequence of diagrams

$$\mu = \rho^{(0)} \to \rho^{(1)} \to \dots \to \rho^{(l-1)} \to \rho^{(l)} = \nu,$$

where $\rho \to \sigma$ means that $\rho \subset \sigma$ with $|\sigma/\rho| = 1$. Let $r_i$ denote the row number of $\rho^{(i)}/\rho^{(i-1)}$. Then the sequence $r_1 \dots r_l$ is called the *Yamanouchi symbol* of $R$. Equivalently, $R$ corresponds to a standard tableau $T$ of shape $\nu/\mu$ where $r_i$ is the row number of the entry $i$ in $T$. A semistandard tableau $T$ *fits* $\nu/\mu$ if $\mathrm{cw}(T)$ is the Yamanouchi symbol for some standard Young tableau of shape $\nu/\mu$. For example,

$$T = \begin{matrix} 1 & 1 & 2 \\ 2 & 3 & \end{matrix}$$

fits $(4,3,1)/(2,1)$ since $\mathrm{cw}(T) = 21312 = r_1 \dots r_5$ corresponds to the standard tableau

$$\begin{matrix} & & 2 & 4 \\ & 1 & 5 & \\ 3 & & & \end{matrix}$$

or equivalently to the shape sequence

$$R : \mu = (2,1) \to (2,2) \to (3,2) \to (3,2,1) \to (4,2,1) \to (4,3,1) = \nu.$$

The coefficient $c^{\nu}_{\lambda\mu}$ is then equal to the number of semistandard tableaux $T$ of shape $\lambda$ that fit $\nu/\mu$.

The factorial Schur function $s_\lambda(x|a)$ is a polynomial in $x$ and a doubly-infinite sequence of variables $a = (a_i)$. It can be defined as the ratio of two alternants (3) (see the beginning of Section 2) by analogy with the ordinary case. This approach goes back to Lascoux [L1]. The $s_\lambda(x|a)$ are also a special case of the double Schubert polynomials introduced by Lascoux and Schützenberger as explained in [L2]. The combinatorial definition (4) for the particular sequence $a$ with $a_i = i-1$ (again, see the beginning of Section 2) is due to Biedenharn and Louck [BL] while the case for general $a$ is due to Macdonald [M2] and Goulden–Greene [GG]. The equivalence of (3) and (4) was established independently in [M2] and [GG].

Specializing $a_i = 0$ for all $i$, the functions $s_\lambda(x|a)$ turn into $s_\lambda(x)$. They form a basis in the symmetric polynomials in $x$ over $\mathbb{C}[a]$ so one can define the corresponding Littlewood-Richardson coefficients $c^{\nu}_{\lambda\mu}(a)$; see (5). Our main result is Theorem 3.1 which gives a combinatorial rule for calculating a two-variable generalization $c^{\nu}_{\theta\mu}(a,b)$ of these coefficients (8), where $\theta$ is a skew diagram. In the case

$|\nu| = |\theta|+|\mu|$ the rule turns into a rule for computing the intertwining number of the skew representations of the symmetric group corresponding to the diagrams $\theta$ and $\nu/\mu$ [JP, Z]. Specializing further to $\mu = \emptyset$ (respectively $\theta = \lambda$), we get the classical Littlewood-Richardson rule in the first (respectively second) formulation above. A Pieri rule for multiplication of a double Schubert polynomial by a complete or elementary symmetric polynomial is given by Lascoux and Veigneau [V]. Lascoux has pointed out that the Newton interpolation formula in several variables [LS] can also be used to give an alternative proof of the factorial Littlewood-Richardson rule.

In Section 4 we consider the specialization $a_i = i-1$. The corresponding coefficients $c^{\nu}_{\lambda\mu}(a)$ turn out to be the structure constants for the center of the universal enveloping algebra for the Lie algebra $\mathfrak{gl}(n)$ and for an algebra of invariant differential operators in certain distinguished bases. We also obtain a formula which relates these coefficients to the dimensions of skew diagrams. This implies a symmetry property of these coefficients.

## 2. Preliminaries

Let $x = (x_1,\dots,x_n)$ be a finite sequence of variables and let $a = (a_i)$, $i \in \mathbb{Z}$, be a doubly-infinite variable sequence. The *generalized factorial Schur function* for a partition $\lambda$ of length at most $n$ can be defined as follows [M2]. Let

$$(y|a)^k = (y-a_1)\cdots(y-a_k)$$

for each $k \geq 0$. Then

$$s_\lambda(x|a) = \frac{\det\big[(x_j|a)^{\lambda_i+n-i}\big]_{1\leq i,j\leq n}}{\prod_{i<j}(x_i-x_j)}. \tag{3}$$

There is an explicit combinatorial formula for $s_\lambda(x|a)$ analogous to (1):

$$s_\lambda(x|a) = \sum_T \prod_{\alpha\in\lambda}(x_{T(\alpha)} - a_{T(\alpha)+c(\alpha)}), \tag{4}$$

where $T$ runs over all semistandard tableaux of shape $\lambda$ with entries in $\{1,\dots,n\}$, $T(\alpha)$ is the entry of $T$ in the cell $\alpha\in\lambda$ and $c(\alpha) = j-i$ is the content of $\alpha = (i,j)$.

The highest homogeneous component of $s_\lambda(x|a)$ in $x$ obviously coincides with $s_\lambda(x)$. Therefore the polynomials $s_\lambda(x|a)$ form a basis for $R[x]^{S_n}$ where $R = \mathbb{C}[a]$, and one can define Littlewood-Richardson type coefficients $c^{\nu}_{\lambda\mu}(a)$ by

$$s_\lambda(x|a)s_\mu(x|a) = \sum_\nu c^{\nu}_{\lambda\mu}(a)s_\nu(x|a). \tag{5}$$

Comparing the highest homogeneous components in $x$ on both sides and using the Littlewood-Richardson Rule for the $s_\lambda(x)$ we see that

$$c^{\nu}_{\lambda\mu}(a) = \begin{cases} c^{\nu}_{\lambda\mu} & \text{if } |\nu| = |\lambda|+|\mu|,\\ 0 & \text{if } |\nu| > |\lambda|+|\mu|.\end{cases} \tag{6}$$

Contrary to the classical case, the coefficients $c^{\nu}_{\lambda\mu}(a)$ turn out to be nonzero if $|\nu| < |\lambda|+|\mu|$ and $\lambda,\mu\subseteq\nu$. This makes it possible to compute them using induction on $|\nu/\mu|$ while keeping $\lambda$ fixed.

The starting point of our calculation is the fact that the polynomials $s_\lambda(x|a)$ possess some (characteristic) vanishing properties; see [S2, O1]. We use the following result from [O1]. For a partition $\rho$ with $l(\rho)\leq n$ define an $n$-tuple $a_\rho = (a_{\rho_1+n},\dots,a_{\rho_n+1})$.

**Theorem 2.1** (Vanishing Theorem)**.** *Given partitions $\lambda, \rho$ with $l(\lambda), l(\rho) \le n$*

$$s_\lambda(a_\rho|a) = \begin{cases} 0 & \text{if } \lambda \not\subseteq \rho, \\ \prod_{(i,j)\in\lambda}(a_{\lambda_i+n-i+1} - a_{n-\lambda^t_j+j}) & \text{if } \lambda = \rho, \end{cases}$$

*where $\lambda^t$ is the diagram conjugate to $\lambda$.*

In particular, $s_\lambda(a_\lambda|a) \neq 0$ for any specialization of the sequence $a$ such that $a_i \neq a_j$ if $i \neq j$. We reproduce the proof of the Vanishing Theorem from [O1] for completeness.

*Proof.* The $ij$-th entry of the determinant in the numerator of the right hand side of (3) for $x = a_\rho$ is

$$(a_{\rho_j+n-j+1} - a_1)\cdots(a_{\rho_j+n-j+1} - a_{\lambda_i+n-i}). \tag{7}$$

The condition $\lambda \not\subseteq \rho$ implies that there exists an index $k$ such that $\rho_k < \lambda_k$. Then for $i \le k \le j$ we have

$$1 \le \rho_j + n - j + 1 \le \rho_k + n - k + 1 \le \lambda_k + n - k \le \lambda_i + n - i,$$

and so all the entries (7) with $i \le k \le j$ are zero. Hence, the determinant is zero which proves the first part of the theorem.

Now let us set $x = a_\lambda$ in (3). Then the $ij$-th entry of the determinant is

$$(a_{\lambda_j+n-j+1} - a_1)\cdots(a_{\lambda_j+n-j+1} - a_{\lambda_i+n-i}),$$

which equals zero for $i < j$ and is nonzero for $i = j$. This means that the matrix is lower triangular with nonzero diagonal elements. Taking their product and dividing by

$$\prod_{i<j}(a_{\lambda_i+n-i+1} - a_{\lambda_j+n-j+1})$$

we get the desired equation. □

## 3. Calculating the coefficients

We will be able to prove more general results by introducing a second infinite sequence of variables denoted $b = (b_i)$, $i \in \mathbb{Z}$. Let $\theta$ and $\mu$ be skew and normal (i.e., skewed by $\emptyset$) diagrams, respectively. Define Littlewood-Richardson type coefficients $c^\nu_{\theta\mu}(a,b)$ by the formula

$$s_\theta(x|b)s_\mu(x|a) = \sum_\nu c^\nu_{\theta\mu}(a,b)s_\nu(x|a), \tag{8}$$

where $s_\theta(x|b)$ is defined as in (4) with $\lambda$ replaced by $\theta$ and $a$ replaced by $b$.

As in Section 1, consider a sequence of diagrams

$$R : \mu = \rho^{(0)} \to \rho^{(1)} \to \ldots \to \rho^{(l-1)} \to \rho^{(l)} = \nu, \tag{9}$$

and let $r_i$ be the row number of $\rho^{(i)}/\rho^{(i-1)}$. Construct the set $\mathcal{T}(\theta, R)$ of semistandard $\theta$-tableaux $T$ with entries from $\{1, \ldots, n = |x|\}$ such that $T$ contains cells $\alpha_1, \ldots, \alpha_l$ with

$$\alpha_1 < \ldots < \alpha_l \quad \text{and} \quad T(\alpha_i) = r_i,\ 1 \le i \le l,$$

where $<$ is column order. Let us distinguish the entries in $\alpha_1, \ldots, \alpha_l$ by barring each of them. For example, if $n = 2$ and

$$R : (2,1) \to (2,2) \to (3,2)$$

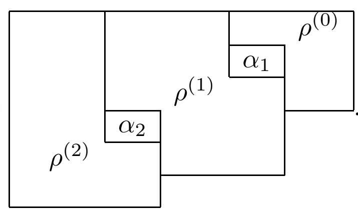
so that $r_1 r_2 = 21$, then for $\theta = (3,2)/(1)$ we have

$$\mathcal{T}(\theta, R) = \left\{ \begin{array}{ccc} & 1 & 1 \\ \overline{1} & \overline{2} & \end{array}, \begin{array}{ccc} & \overline{1} & \overline{2} \\ 1 & 2 & \end{array}, \begin{array}{ccc} & 1 & \overline{2} \\ \overline{1} & 2 & \end{array}, \begin{array}{ccc} & 1 & 2 \\ \overline{1} & \overline{2} & \end{array}, \begin{array}{ccc} & \overline{1} & \overline{2} \\ 2 & 2 & \end{array} \right\}.$$

We also let

$$\mathcal{T}(\theta, \nu/\mu) = \biguplus_R \mathcal{T}(\theta, R),$$

where the union is over all sequences $R$ of the form (9). Finally, for each cell $\alpha$ with $\alpha_i < \alpha < \alpha_{i+1}$, $0 \le i \le l$, set $\rho(\alpha) = \rho^{(i)}$. (Inequalities involving cells with out-of-range subscripts are ignored.) For instance, if $l = |\nu/\mu| = 2$, then the following schematic diagram gives the layout of the $\rho(\alpha)$

$\theta =$ 

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We are now in a position to state the Littlewood-Richardson rule for the $c^{\nu}_{\theta\mu}(a,b)$. The reader should compare the following formula with the combinatorial one for the $s_\lambda(x|a)$ in (4).

**Theorem 3.1.** *The coefficient $c^{\nu}_{\theta\mu}(a,b)$ is zero unless $\mu \subseteq \nu$. If $\mu \subseteq \nu$, then*

$$c^{\nu}_{\theta\mu}(a,b) = \sum_{T \in \mathcal{T}(\theta, \nu/\mu)} \prod_{\substack{\alpha \in \theta \\ T(\alpha) \text{ unbarred}}} \left( (a_{\rho(\alpha)})_{T(\alpha)} - b_{T(\alpha)+c(\alpha)} \right).$$

As immediate specializations of this result, note the following.

1. If $a = b$ and $\theta$ is normal, then this is a Littlewood-Richardson rule for the $s_\lambda(x|a)$.
2. If $a = b$ and $\mu$ is empty, then this is a rule for the expansion of a skew factorial Schur polynomial.
3. If $|\nu| = |\theta| + |\mu|$, then $c^{\nu}_{\theta\mu}(a,b)$ is independent of $a$ and $b$ and equals the number of semistandard tableaux of shape $\theta$ that fit $\nu/\mu$. This coincides with the number of pictures between $\theta$ and $\nu/\mu$ [JP, Z]. In particular, $c^{\nu}_{\theta\mu}(a,b) = c^{\nu}_{\theta\mu}$, an ordinary Littlewood-Richardson coefficient.
4. If $\mu = \emptyset$ and $\theta = \lambda$ is normal, then this is a rule for the re-expansion of a factorial Schur polynomial in terms of those for a different sequence of second variables. In particular,

$$s_\lambda(x|a) = \sum_{\nu \subseteq \lambda} g_{\lambda\nu}(a) s_\nu(x)$$

where

$$g_{\lambda\nu}(a) = (-1)^{|\lambda/\nu|} \sum_{T \in \mathcal{T}(\lambda, \nu)} \prod_{\substack{\alpha \in \lambda \\ T(\alpha) \text{ unbarred}}} a_{T(\alpha)+c(\alpha)}.$$

A different expression for $g_{\lambda\nu}(a)$ in terms of double Schubert polynomials is provided by the Newton interpolation formula in several variables [LS].

We present the proof of Theorem 3.1 as a chain of propositions.

Note that the first claim of the theorem follows immediately from the Vanishing Theorem. Indeed, let $\nu$ be minimal (with respect to containment) among all partitions in (8) such that $c^{\nu}_{\theta\mu}(a,b) \neq 0$. Suppose $\nu \not\supseteq \mu$. Then setting $x = a_\nu$ in (8) and using the first part of the Vanishing Theorem gives

$$0 = c^{\nu}_{\theta\mu}(a,b) s_\nu(a_\nu|a).$$

But by the Vanishing Theorem's second part we have $s_\nu(a_\nu|a) \neq 0$ and so a contradiction to $c^{\nu}_{\theta\mu}(a,b) \neq 0$.

We shall assume hereafter that $\mu \subseteq \nu$ and also write

$$|a_\rho| = a_{\rho_1+n} + \cdots + a_{\rho_n+1}.$$

**Proposition 3.2.** *If $\mu \subseteq \nu$ with $|\nu/\mu| = l$, then*

$$\frac{s_\mu(a_\nu|a)}{s_\nu(a_\nu|a)} = \sum_{\mu\to\rho^{(1)}\to\cdots\to\rho^{(l-1)}\to\nu} \frac{1}{(|a_\nu| - |a_{\rho^{(0)}}|)\cdots(|a_\nu| - |a_{\rho^{(l-1)}}|)},$$

*where $\rho^{(0)} = \mu$.*

*Proof.* Setting $x = a_\mu$ in (8) and using the Vanishing Theorem gives

$$c^{\mu}_{\theta\mu}(a,b) = s_\theta(a_\mu|b). \tag{10}$$

Further, for $\theta = (1)$ and $a = b$ relation (8) takes the form (cf. [OO, Theorem 9.1])

$$s_{(1)}(x|a)s_\mu(x|a) = s_{(1)}(a_\mu|a)s_\mu(x|a) + \sum_{\mu\to\rho} s_\rho(x|a)$$

which follows from (10), (6), and the Branching Theorem for the ordinary Schur functions.

Setting $x = a_\nu$ in the previous equation and using the Vanishing Theorem we get

$$s_{(1)}(a_\nu|a)s_\mu(a_\nu|a) = s_{(1)}(a_\mu|a)s_\mu(a_\nu|a) + \sum_{\mu\to\rho\subseteq\nu} s_\rho(a_\nu|a). \tag{11}$$

We have

$$s_{(1)}(a_\nu|a) - s_{(1)}(a_\mu|a) = |a_\nu| - |a_\mu|$$

and so (11) gives

$$\frac{s_\mu(a_\nu|a)}{s_\nu(a_\nu|a)} = \frac{1}{|a_\nu| - |a_\mu|} \sum_{\mu\to\rho\subseteq\nu} \frac{s_\rho(a_\nu|a)}{s_\nu(a_\nu|a)}.$$

Induction on $|\nu/\mu|$ completes the proof. ☐

It will be convenient to have a notation for sums like those occurring in the previous proposition. So let

$$H(\mu,\rho) = \sum_{\mu\to\rho^{(1)}\to\cdots\to\rho^{(r-1)}\to\rho} \frac{1}{(|a_\rho| - |a_{\rho^{(0)}}|)\cdots(|a_\rho| - |a_{\rho^{(r-1)}}|)}, \tag{12}$$

and

$$H'(\rho,\nu) = \sum_{\rho\to\rho^{(r+1)}\to\cdots\to\rho^{(l-1)}\to\nu} \frac{1}{(|a_\rho| - |a_{\rho^{(r+1)}}|)\cdots(|a_\rho| - |a_{\rho^{(l)}}|)}, \tag{13}$$

where $\rho^{(0)} = \mu$ and $\rho^{(l)} = \nu$.

**Proposition 3.3.** *We have the formula*

$$c^{\nu}_{\theta\mu}(a,b) = \sum_{\mu\subseteq\rho\subseteq\nu} s_\theta(a_\rho|b)H(\mu,\rho)H'(\rho,\nu).$$

*Proof.* We use induction on $|\nu/\mu|$, noting that (10) is the base case $|\nu/\mu| = 0$. Set $x = a_\nu$ in (8) and divide both sides by $s_\nu(a_\nu|a)$. By Proposition 3.2 we get

$$c^{\nu}_{\theta\mu}(a,b) = s_\theta(a_\nu|b)H(\mu,\nu) - \sum_{\sigma\subset\nu} c^{\sigma}_{\theta\mu}(a,b)H(\sigma,\nu).$$

By the induction hypotheses we can write this as

$$\begin{aligned} c^{\nu}_{\theta\mu}(a,b) &= s_\theta(a_\nu|b)H(\mu,\nu) - \sum_{\sigma\subset\nu}\sum_{\mu\subseteq\rho\subseteq\sigma} s_\theta(a_\rho|b)H(\mu,\rho)H'(\rho,\sigma)H(\sigma,\nu) \\ &= s_\theta(a_\nu|b)H(\mu,\nu) - \sum_{\mu\subseteq\rho\subset\nu} s_\theta(a_\rho|b)H(\mu,\rho)\sum_{\rho\subseteq\sigma\subset\nu} H'(\rho,\sigma)H(\sigma,\nu). \end{aligned}$$

To complete the proof we note that

$$\sum_{\rho\subseteq\sigma\subseteq\nu} H'(\rho,\sigma)H(\sigma,\nu) = 0,$$

which follows from the identity

$$\sum_{i=1}^{k} \frac{1}{(u_1-u_2)\cdots(u_1-u_i)(u_k-u_i)\cdots(u_k-u_{k-1})} = 0,$$

which holds for any variables $u_1,\ldots,u_k$ by induction on $k > 1$. (In the denominator an empty product is, as usual, equal to 1.) □

Note that a different expression for the $c^{\nu}_{\theta\mu}(a,b)$ in terms of divided differences can be deduced from the Newton interpolation formula in several variables [LS].

**Proposition 3.4.** *We have the recurrence relation*

$$c^{\nu}_{\theta\mu}(a,b) = \frac{1}{|a_\nu|-|a_\mu|}\left(\sum_{\mu\to\mu'} c^{\nu}_{\theta\mu'}(a,b) - \sum_{\nu'\to\nu} c^{\nu'}_{\theta\mu}(a,b)\right). \tag{14}$$

*Proof.* By Proposition 3.3 it suffices to check that

$$H(\mu,\rho)H'(\rho,\nu) = \frac{1}{|a_\nu|-|a_\mu|}\left(\sum_{\mu\to\mu'} H(\mu',\rho)H'(\rho,\nu) - \sum_{\nu'\to\nu} H(\mu,\rho)H'(\rho,\nu')\right).$$

This follows from the relations

$$\sum_{\mu\to\mu'} H(\mu',\rho) = (|a_\rho|-|a_\mu|)H(\mu,\rho)$$

and

$$\sum_{\nu'\to\nu} H'(\rho,\nu') = (|a_\rho|-|a_\nu|)H'(\rho,\nu). \quad \square$$

Given a sequence

$$R : \mu = \rho^{(0)} \to \rho^{(1)} \to \ldots \to \rho^{(l-1)} \to \rho^{(l)} = \nu,$$

and an index $k \in \{1, \ldots, l\}$ introduce a set of $\theta$-tableaux $\mathcal{T}_k(\theta, R)$ having entries from the set $\{1, \ldots, n\}$ as follows. Each tableau $T \in \mathcal{T}_k(\theta, R)$ contains cells $\alpha_1, \ldots, \alpha_{k-1}, \alpha_{k+1}, \ldots, \alpha_l$ such that

$$\alpha_1 < \ldots < \alpha_{k-1} < \alpha_{k+1} < \ldots < \alpha_l \quad \text{and} \quad T(\alpha_i) = r_i, \ 1 \le i \le l, i \ne k.$$

As usual, we distinguish the entries in the $\alpha_i, i \ne k$, by barring them. Now modify the $\rho(\alpha)$ for $R$ by defining, for cells with unbarred entries,

$$\rho^+(\alpha) = \begin{cases} \rho^{(k)} & \text{if } \alpha_{k-1} < \alpha < \alpha_{k+1}, \\ \rho(\alpha) & \text{otherwise,} \end{cases}$$

and

$$\rho^-(\alpha) = \begin{cases} \rho^{(k-1)} & \text{if } \alpha_{k-1} < \alpha < \alpha_{k+1}, \\ \rho(\alpha) & \text{otherwise.} \end{cases}$$

Also define corresponding weights

$$\begin{aligned} \mathcal{S}(R) &= \sum_{T \in \mathcal{T}(\theta, R)} \prod_{\substack{\alpha \in \theta \\ T(\alpha) \text{unbarred}}} \left((a_{\rho(\alpha)})_{T(\alpha)} - b_{T(\alpha)+c(\alpha)}\right), \\ \mathcal{S}_k^+(R) &= \sum_{T \in \mathcal{T}_k(\theta, R)} \prod_{\substack{\alpha \in \theta \\ T(\alpha) \text{unbarred}}} \left((a_{\rho^+(\alpha)})_{T(\alpha)} - b_{T(\alpha)+c(\alpha)}\right), \end{aligned}$$

and similarly for $\mathcal{S}_k^-(R)$. So Theorem 3.1 is equivalent to

$$c_{\theta\mu}^{\nu}(a, b) = \sum_R \mathcal{S}(R). \tag{15}$$

**Proposition 3.5.** *Given a sequence $R$ we have*

$$\mathcal{S}(R) = \frac{1}{|a_\nu| - |a_\mu|} \sum_{k=1}^{l} \left(\mathcal{S}_k^+(R) - \mathcal{S}_k^-(R)\right). \tag{16}$$

*Proof.* It suffices to show that for each $k$ we have

$$\mathcal{S}_k^+(R) - \mathcal{S}_k^-(R) = (|a_{\rho^{(k)}}| - |a_{\rho^{(k-1)}}|)\mathcal{S}(R).$$

Formula (16) will then follow from the relation

$$\sum_{k=1}^{l} (|a_{\rho^{(k)}}| - |a_{\rho^{(k-1)}}|) = |a_\nu| - |a_\mu|.$$

For a given $T \in \mathcal{T}_k(\theta, R)$ the factors in the formulas for $\mathcal{S}_k^+(R)$ and $\mathcal{S}_k^-(R)$ are identical except for the case where $\alpha_{k-1} < \alpha < \alpha_{k+1}$ and $T(\alpha) = r_k$. To see what happens when we divide $\mathcal{S}_k^+(R) - \mathcal{S}_k^-(R)$ by

$$|a_{\rho^{(k)}}| - |a_{\rho^{(k-1)}}| = (a_{\rho^{(k)}})_{r_k} - (a_{\rho^{(k-1)}})_{r_k},$$

fix $T$ and consider its contribution to the quotient. We need the following easily proved formula, where we are thinking of $u = (a_{\rho^{(k)}})_{r_k}$, $v = (a_{\rho^{(k-1)}})_{r_k}$ and $m_i = b_{T(\alpha)+c(\alpha)}$ as $\alpha$ runs over all cells of $T$ with $\alpha_{k-1} < \alpha < \alpha_{k+1}$ and $T(\alpha) = r_k$:

$$\frac{\prod_{i=1}^{s}(u - m_i) - \prod_{i=1}^{s}(v - m_i)}{u - v} = \sum_{j=1}^{s} (u - m_1) \cdots \widehat{(u - m_j)} (v - m_{j+1}) \cdots (v - m_s)$$

(a hat indicates the factor is to be omitted). The right-hand side of this expression can now be interpreted as the contribution to $\mathcal{S}(R)$ of all tableaux gotten from $T$ by barring one of the $r_k$ between $\alpha_{k-1}$ and $\alpha_{k+1}$ in column order. □

We now prove (15) by induction on $|\nu/\mu|$. Equation (10) takes care of the case $|\nu/\mu| = 0$. By the induction hypothesis,

$$\sum_{\mu\to\mu'} c^{\nu}_{\theta\mu'}(a,b) = \sum_R \mathcal{S}^+_1(R) \quad \text{and} \quad \sum_{\nu'\to\nu} c^{\nu'}_{\theta\mu}(a,b) = \sum_R \mathcal{S}^-_l(R).$$

So formulas (14), (16) and the following proposition complete the proof of (15) and hence Theorem 3.1.

**Proposition 3.6.** *We have*

$$\sum_R \sum_{k=1}^{l-1} \mathcal{S}^-_k(R) = \sum_R \sum_{k=2}^{l} \mathcal{S}^+_k(R).$$

*Proof.* We can rewrite this formula as follows:

$$\sum_{R,k,T} \mathrm{wt}^-(R,k,T) = \sum_{R',k',T'} \mathrm{wt}^+(R',k',T'), \tag{17}$$

where $T \in \mathcal{T}_k(\theta, R)$, $k = 1,\ldots,l-1$, and $T' \in \mathcal{T}_{k'}(\theta, R')$, $k' = 2,\ldots,l$, with weights defined by

$$\mathrm{wt}^-(R,k,T) = \prod_{\substack{\alpha\in\theta \\ T(\alpha)\ \text{unbarred}}} \big((a_{\rho^-(\alpha)})_{T(\alpha)} - b_{T(\alpha)+c(\alpha)}\big)$$

and similarly define $\mathrm{wt}^+(R',k',T')$. To prove (17) we will construct a bijection $(R,k,T) \longleftrightarrow (R',k',T')$ preserving the weights in the sense that $\mathrm{wt}^-(R,k,T) = \mathrm{wt}^+(R',k',T')$. There are three cases.

*Case* 1. Suppose that the skew diagram $\rho^{(k+1)}/\rho^{(k-1)}$ consists of two cells in different rows and columns. Then $R'$ is the sequence obtained from $R$ by replacing $\rho^{(k)}$ by the other diagram ${\rho'}^{(k)}$ such that $\rho^{(k-1)} \to {\rho'}^{(k)} \to \rho^{(k+1)}$ while $k' = k+1$ and $T' = T$.

*Case* 2. Let $\rho^{(k+1)}/\rho^{(k-1)}$ have two cells in the same row. Then $R' = R$, $k' = k+1$ and $T' = T$.

*Case* 3. Let $\rho^{(k+1)}/\rho^{(k-1)}$ have two cells in the same column, say in rows $r$ and $r+1$. Let $(i+1,j) = (i_1,j_1)$ be the cell of $T$ containing the correponding $\overline{r+1}$. If there is an $r$ in cell $(i,j)$ then it must be unbarred. In this case let $T'$ be $T$ with the bar moved from the $r+1$ to the $r$, $R' = R$, and $k' = k+1$. Weights are preserved since $(a_{\rho^{(k-1)}})_r = (a_{\rho^{(k+1)}})_{r+1}$ and $T(\alpha) + c(\alpha)$ is invariant under the change.

Now suppose cell $(i,j)$ of $T$ contains an entry less than $r$ or $(i,j) \notin \theta$ and let $j' = j'_1$ be the column of the left-most $r+1$ in row $i+1$. Since this subcase is more complicated than the others, the reader may wish to follow along with the example given after the end of this proof. Let $s$ be the maximum integer such that for $1 \le t \le s$ we have

1. there is an $\overline{r+t}$ in cell $(i+t, j_t)$ for some $j_t$ corresponding to a cell in the same column as those of $\rho^{(k+1)}/\rho^{(k-1)}$,

2. if $(i+t, j_t')$ contains the left-most $r+t$ in row $i+t$, then $(i+t, j_t)$ is between $(i+t-1, j_{t-1})$ and $(i+t-1, j_{t-1}')$ in column order. (Assume this is true vacuously when $t=1$.)

Note that the condition on cell $(i,j)$ implies that none of the $r+t$'s to the left of the one in $(i+t, j_t)$ can be barred.

We now form $T'$ by moving the bar in cell $(i+t, j_t)$ to cell $(i+t, j_t')$ and replacing the $r+t$'s in cells $(i+t, j_t'), (i+t, j_t'+1), \ldots, (i+t, j_t)$ by $r+t-1$'s. Note that the result will still be a semistandard tableau because of the assumption about $(i,j)$ and the choice of elements to decrease. Since the elements from the given column of $\rho^{(k+1)}/\rho^{(k-1)}$ are still added in the correct order in $T'$, it determines a valid $R'$, complete except for the step where a cell is added in row $r+s$ of that column which should be done immediately following the addition of $r+s-1$. Then $k'$ is the position of this $r+s$. Invariance of weights follows from considerations like those in the first subcase, noting that the contribution to $\mathrm{wt}^-$ of each entry decreased in $T$ is the same as that of the element on its right to $\mathrm{wt}^+$ in $T'$.

The inverse of this construction is similar and left to the reader. This completes the proof of the Theorem 3.1. □

As an example of the last subcase, suppose we have the $R$ sequence

$$(3,2,2,2) \to (3,3,2,2) \to (3,3,3,2) \to (4,3,3,2) \to (4,3,3,2,1) \to (4,3,3,3,1)$$

with Yamanouchi symbol $r_1 \ldots r_5 = 23154$. Let $k=1$ so $r=2$ and consider

$$T = \begin{array}{cccccc} 1 & 1 & 1 & 1 & \bar{1} & 1 \\ 2 & 3 & 3 & 3 & 3 & \bar{3} \\ 4 & 4 & \bar{4} & 4 & \bar{5} & \end{array}.$$

Then $(i+1, j) = (2,6)$ and $s=2$ with $r+1, r+2 = 3,4$ so

$$T' = \begin{array}{cccccc} 1 & 1 & 1 & 1 & \bar{1} & 1 \\ 2 & \bar{2} & 2 & 2 & 2 & 2 \\ \bar{3} & 3 & 3 & 4 & \bar{5} & \end{array}.$$

Column reading the barred elements of $T'$ and inserting $r+2=4$ after $r+1=3$ gives the Yamanouchi symbol 15234 corresponding to the $R'$ sequence

$$(3,2,2,2) \to (4,2,2,2) \to (4,2,2,2,1) \to (4,3,2,2,1) \to (4,3,3,2,1) \to (4,3,3,3,1)$$

and $k'=5$.

## 4. Multiplication rules for Capelli operators and quantum immanants

Let $E_{ij}$, $i,j = 1,\ldots,n$, denote the standard basis of the general linear Lie algebra $\mathfrak{gl}(n)$. Denote by $Z(\mathfrak{gl}(n))$ the center of the universal enveloping algebra $U(\mathfrak{gl}(n))$. Given $\kappa = (\kappa_1, \ldots, \kappa_n) \in \mathbb{C}^n$ consider a $\mathfrak{gl}(n)$-module $L(\kappa)$ of highest weight $\kappa$. That is, $L(\kappa)$ is generated by a nonzero vector $v$ such that

$$\begin{aligned} E_{ii} \cdot v &= \kappa_i \, v, \quad \text{for} \quad i = 1,\ldots,n; \\ E_{ij} \cdot v &= 0 \quad \text{for} \quad 1 \le i < j \le n. \end{aligned}$$

Any element $z \in Z(\mathfrak{gl}(n))$ acts as a scalar $\omega(z) = \omega_\kappa(z)$ in $L(\kappa)$ and this scalar is independent of the choice of the highest weight module $L(\kappa)$. Moreover, $\omega(z)$ is a symmetric polynomial in the shifted variables $x_1, \dots, x_n$ where $x_i = \kappa_i + n - i$. The mapping $z \mapsto \omega(z)$ defines an algebra isomorphism

$$\omega : Z(\mathfrak{gl}(n)) \to \mathbb{C}[x]^{S_n}$$

called the *Harish-Chandra isomorphism*; see e.g. Dixmier [D, Section 7.4].

For any positive integer $m$ consider the natural action of the complex Lie group $GL(n)$ in the algebra $\mathcal{P}$ of polynomials on the vector space $\mathbb{C}^n \otimes \mathbb{C}^m$. The corresponding Lie algebra $\mathfrak{gl}(n)$ then acts by differential operators

$$\pi(E_{ij}) = \sum_{a=1}^{m} x_{ia} \partial_{ja},$$

where the $x_{ia}$ are the coordinates on $\mathbb{C}^n \otimes \mathbb{C}^m$ and the $\partial_{ia}$ are the corresponding partial derivatives. This representation is uniquely extended to an algebra homomorphism

$$\pi : U(\mathfrak{gl}(n)) \to \mathcal{P}D$$

where $\mathcal{P}D$ is the algebra of polynomial coefficient differential operators in the $x_{ia}$. The image of $Z(\mathfrak{gl}(n))$ under $\pi$ is contained in the subalgebra $\mathcal{P}D^G$ of differential operators invariant with respect to the action of the group $G = GL(n) \times GL(m)$. Moreover, if $m \geq n$, then this restriction is an algebra isomorphism which can be called the *Capelli isomorphism*; see [H, HU] for further details. So if $m \geq n$, we have the triple isomorphism

$$\mathbb{C}[x]^{S_n} \xleftarrow{\omega} Z(\mathfrak{gl}(n)) \xrightarrow{\pi} \mathcal{P}D^G.$$

Distinguished bases in the three algebras which correspond to each other under these isomorphisms were constructed in [O1] (see also [N, O2, M3]).

In the algebra $\mathbb{C}[x]^{S_n}$ the basis is formed by the polynomials $s_\lambda(x|a)$ with $l(\lambda) \leq n$ and the sequence $a$ specialized to $a_i = i - 1$ for all $i \in \mathbb{Z}$. We shall denote these polynomials by $s^*_\lambda(x)$. Explicitly [BL],

$$s^*_\lambda(x) = \sum_T \prod_{\alpha \in \lambda} (x_{T(\alpha)} - T(\alpha) - c(\alpha) + 1),$$

where $T$ runs over all semistandard tableaux of shape $\lambda$ with entries in $\{1, \dots, n\}$. We shall denote by $f^\nu_{\lambda\mu}$ the coefficient $c^\nu_{\lambda\mu}(a)$ in this specialization of $a$. In other words, the $f^\nu_{\lambda\mu}$ can be defined by the formula

$$s^*_\lambda(x) s^*_\mu(x) = \sum_\nu f^\nu_{\lambda\mu}\, s^*_\nu(x).$$

To describe the basis in $Z(\mathfrak{gl}(n))$ we introduce some more notation. Given $k$ matrices $A, B, \dots, C$ of size $p \times q$ with entries from an algebra $\mathcal{A}$ we regard their tensor product $A \otimes B \otimes \cdots \otimes C$ as an element

$$\sum A_{a_1 i_1} B_{a_2 i_2} \cdots C_{a_k i_k} \otimes e_{a_1 i_1} \otimes e_{a_2 i_2} \otimes \cdots \otimes e_{a_k i_k} \in \mathcal{A} \otimes (\mathrm{Mat}_{pq})^{\otimes k},$$

where $\mathrm{Mat}_{pq}$ denotes the space of complex $p \times q$-matrices and the $e_{ai}$ are the standard matrix units. The symmetric group $S_k$ acts in a natural way in the tensor space $(\mathbb{C}^n)^{\otimes k}$, so that we can identify permutations from $S_k$ with elements of the algebra $(\mathrm{Mat}_{nn})^{\otimes k}$.

For a diagram $\lambda$ with $l(\lambda) \leq n$ denote by $T_0$ the $\lambda$-tableau obtained by filling in the cells by the numbers $1, \dots, k = |\lambda|$ from left to right in successive rows starting from the first row. We let $R_\lambda$ and $C_\lambda$ denote the row symmetrizer and column antisymmetrizer of $T_0$ respectively. By $c_\lambda(r)$ we denote the content of the cell occupied by $r$. Introduce the matrix $E = (E_{ij})$ whose $ij$-th entry is the generator $E_{ij}$ and set

$$\mathbb{S}_\lambda = \frac{1}{h(\lambda)} \operatorname{tr}(E - c_\lambda(1)) \otimes \cdots \otimes (E - c_\lambda(k)) \cdot R_\lambda C_\lambda,$$

where the trace is taken over all the tensor factors $\mathrm{Mat}_{nn}$, and $h(\lambda)$ is the product of the hook-lengths of the cells of $\lambda$:

$$h(\lambda) = \prod_{\alpha\in\lambda} h_\alpha.$$

The elements $\mathbb{S}_\lambda$ with $l(\lambda) \le n$ form a basis in the algebra $Z(\mathfrak{gl}(n))$. In [O1] they were called the *quantum immanants*.

Let us now describe the basis in the algebra $\mathcal{P}\mathcal{D}^G$. The representation $\pi$ can be written in a matrix form as follows:

$$\pi :\ E \mapsto XD^t,$$

where $X$ and $D$ are the $n \times m$ matrices formed by the coordinates $x_{ia}$ and the derivatives $\partial_{ia}$, respectively, while $D^t$ is the matrix transposed to $D$. We introduce the following differential operators:

$$\Delta_\lambda = \frac{1}{k!} \operatorname{tr} X^{\otimes k} \cdot (D^t)^{\otimes k} \cdot \chi^\lambda,$$

where $\chi^\lambda$ is the irreducible character of $S_k$ corresponding to $\lambda$. Explicitly,

$$\Delta_\lambda = \frac{1}{k!} \sum_{i_1,\dots,i_k} \sum_{a_1,\dots,a_k} \sum_{s\in S_k} \chi^\lambda(s) x_{i_1a_1}\cdots x_{i_ka_k} \partial_{i_{s(1)}a_1} \cdots \partial_{i_{s(k)}a_k}.$$

The operators $\Delta_\lambda$ with $l(\lambda) \le n$ form a basis in $\mathcal{P}\mathcal{D}^G$. They are called the *higher Capelli operators*.

The following identities were proved in [O1] (for other proofs see [N, O2, M3]):

$$\omega(\mathbb{S}_\lambda) = s^*_\lambda(x) \quad \text{and} \quad \pi(\mathbb{S}_\lambda) = \Delta_\lambda.$$

Using Theorem 3.1 we obtain the following multiplication rules for the elements $\mathbb{S}_\lambda$ and the operators $\Delta_\lambda$.

**Theorem 4.1.** *We have*

$$\mathbb{S}_\lambda\mathbb{S}_\mu = \sum_\nu f^\nu_{\lambda\mu}\, \mathbb{S}_\nu$$

*and*

$$\Delta_\lambda\Delta_\mu = \sum_\nu f^\nu_{\lambda\mu}\, \Delta_\nu$$

*where the coefficients* $f^\nu_{\lambda\mu}$ *are given by*

$$f^\nu_{\lambda\mu} = \sum_{T\in\mathcal{T}(\lambda,\nu/\mu)} \prod_{\substack{\alpha\in\theta \\ T(\alpha)\ \text{unbarred}}} \big(\rho(\alpha)_{T(\alpha)} + n - 2\,T(\alpha) - c(\alpha) + 1\big) \tag{18}$$

*with* $R$, $\mathcal{T}(\lambda,\nu/\mu)$, *and* $\rho(\alpha)$ *defined in Theorem 3.1.* □

Proposition 3.3 enables us to obtain another formula for $f^{\nu}_{\lambda\mu}$. For a skew diagram $\nu/\mu$ let

$$h(\nu/\mu) = \frac{|\nu/\mu|!}{\dim \nu/\mu},$$

where $\dim \nu/\mu$ is the number of standard $\nu/\mu$-tableaux. In particular, if $\mu$ is empty; $h(\nu)$ is the product of the hook-lengths of the cells of $\nu$.

In the specialization of the sequence $a$ under consideration we obtain from (12) and (13) that

$$H(\mu,\rho) = \frac{1}{h(\rho/\mu)} \quad \text{and} \quad H'(\rho,\nu) = \frac{(-1)^{|\nu/\rho|}}{h(\nu/\rho)}.$$

Moreover, by Proposition 3.2,

$$\frac{s_\lambda(a_\rho|a)}{s_\rho(a_\rho|a)} = \frac{1}{h(\rho/\lambda)}$$

and by the Vanishing Theorem

$$s_\rho(a_\rho|a) = h(\rho).$$

Thus Proposition 3.3 becomes the following:

**Proposition 4.2.** *One has the formula*

$$f^{\nu}_{\lambda\mu} = \sum_{\lambda,\mu\subseteq\rho\subseteq\nu} (-1)^{|\nu/\rho|} \frac{h(\rho)}{h(\nu/\rho)h(\rho/\lambda)h(\rho/\mu)}. \qquad (19) \quad \square$$

Formula (19) implies the following symmetry property of the coefficients $f^{\nu}_{\lambda\mu}$.

**Corollary 4.3.** *If* $l(\lambda^t),\ l(\mu^t),\ l(\nu^t) \le n$, *then*

$$f^{\nu^t}_{\lambda^t\mu^t} = f^{\nu}_{\lambda\mu}.$$

*Proof.* This follows immediately from the relation $h(\nu^t/\mu^t) = h(\nu/\mu)$. $\square$

*Remarks.* 1. It follows from (18) that the coefficients $f^{\nu}_{\lambda\mu}$ are integers while the summands on the right hand side of (19) need not be. In fact the numbers $h(\nu/\mu)$ need not be integers either, e.g., $h((3,2)/(1)) = 24/5$.

2. Note that since in the case of $|\nu| = |\lambda| + |\mu|$ the $f^{\nu}_{\lambda\mu}$ coincide with the classical Littlewood-Richardson coefficients $c^{\nu}_{\lambda\mu}$, the latter can be computed using (19) as well, but this does not appear to be very useful for practical purposes. For example, consider $\lambda = \mu = (1^n)$ and $\nu = (2^r 1^{n-r})$, then (19) gives

$$f^{\nu}_{\lambda\mu} = \sum_{k=0}^{r} (-1)^{r-k} \frac{(n+1)!}{k!(r-k)!(n-k+1)}$$

while, directly from (18), we get

$$f^{\nu}_{\lambda\mu} = (n-r)!. \qquad (20)$$

As a final example, take $m = n$ in the definition of $\Delta_\lambda$. Then for $\lambda = (1^n)$ we get the classical Capelli operator [C]:

$$\Delta_{(1^n)} = \det X \det D.$$

We find from (18) that the coefficients $f^{\nu}_{(1^n)(1^n)}$ are zero except for $\nu = (2^r 1^{n-r})$, $r = 0, 1, \ldots, n$ . So by (20) the square of the Capelli operator is given by

$$(\det X \det D)^2 = \sum_{r=0}^{n} (n-r)!\, \Delta_{(2^r 1^{n-r})}.$$


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