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FUNCTIONS FOR PARAMETRIZATION OF SOLUTIONS OF AN EQUATION IN A FREE MONOID

GENNADY S. MAKANIN AND TATIANA A. MAKANINA

Abstract. In this paper we introduce recursive functions

$$\mathbf{Fi}(x_1, x_2)^{\lambda_1,\ldots,\lambda_s} \qquad (s \geq 0),$$
$$\mathbf{Th}(x_1, x_2, x_3)_i^{\lambda_1,\ldots,\lambda_{2s}} \qquad (i = 1, 2, 3; s \geq 0),$$
$$\mathbf{Ro}(x_1, x_2, x_3)_i^{\mu_1,\ldots,\mu_s} \qquad (i = 1, 2, 3; s \geq 0)$$

of the word variables x_1, x_2, x_3, natural number variables λ_k and variables μ_k whose values are finite sequences of natural number variables. By means of these functions we give finite expressions for the family of solutions of the equation

$$x_1x_2x_3x_4 = \zeta(x_1, x_2, x_3)x_5,$$

where $\zeta(x_1, x_2, x_3)$ is an arbitrary word in the alphabet x_1, x_2, x_3, in a free monoid.

1. Introduction

In 1960 Lyndon [1], [2] considered equations with one unknown in a free group and proved that the family of solutions of such an equation can be represented by a finite number of parametric words. In 1967 Khmelevskii [3] considered equations with three unknowns in a free monoid and proved that the family of solutions of such an equation can be represented by a finite number of parametric words. For a short time after that it was believed that the solutions of all equations in a free group or a free monoid are parametrizable. However in 1971 Khmelevskii [4] pointed out that the solutions of Markov's equation $x_1x_3x_2 = x_2x_4x_1$ with four unknowns in a free monoid is not parametrizable by a finite number of parametric words.

Parametrizations of solutions in the Lyndon-Khmelevskii sense (now called primitive parametrization) use variables of two kinds: word variables and natural number variables. We suppose that the idea of finite parametrization of the solutions of the equations in a free group and a free monoid can be saved if we admit an additional kind of variables, namely, variables whose values are the finite sequences of natural number variables. In this paper we intend to demonstrate the possibilities of the parametrization of the solutions of the equations in a free monoid by means of parametrizing functions of variables of the three mentioned kinds on a carefully chosen example of an equation in a free monoid. This equation is not bulky and "contains" many known difficult equations.

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We introduce recursive functions

$$^{\mathbf{Fi}}(x_1, x_2)^{\lambda_1,\ldots,\lambda_s}, \qquad ^{\mathbf{Th}}(x_1, x_2, x_3)_i^{\lambda_1,\ldots,\lambda_{2s}}, \qquad ^{\mathbf{Ro}}(x_1, x_2, x_3)_i^{\mu_1,\ldots,\mu_s},$$

where $s \geq 0$, $i = 1, 2, 3$, of the word variables x_1, x_2, x_3, natural number variables λ_k, and variables μ_k whose values are finite sequences of natural number variables. We consider here an equation of the form

$$x_1x_2x_3x_4 = \zeta(x_1, x_2, x_3)x_5,$$

where $\zeta(x_1, x_2, x_3)$ is an arbitrary word in the alphabet x_1, x_2, x_3 in a free monoid. We shall give concrete expression from parametric words and parametrizing functions $^{\mathbf{Fi}}$, $^{\mathbf{Th}}$, $^{\mathbf{Ro}}$, which describe the family of solutions of the equation $x_1x_2x_3x_4 = \zeta(x_1, x_2, x_3)x_5$ in a free monoid.

We use the preprints [5] and [6].

2. Definitions and notation

Let Π be a free monoid (a free semigroup with unit) with a countable alphabet of generators

$$a_1, a_2, \ldots, a_k, \ldots \tag{1}$$

Let

$$x_1, x_2, \ldots, x_n, \ldots \tag{2}$$

be a countable alphabet of word variables.

Let

$$\lambda_1, \lambda_2, \ldots, \lambda_t, \ldots \tag{3}$$

be a countable alphabet of natural number variables (also called natural parameters).

Let

$$\mu_1, \mu_2, \ldots, \mu_u, \ldots \tag{4}$$

be a countable alphabet of variables (called second parameters) whose values are finite sequences of natural parameters.

Let

$$\nu_1, \nu_2, \ldots, \nu_v, \ldots \tag{5}$$

be a countable alphabet of variables whose values are finite sequences of second parameters.

Define inductively a *primitive parametric word* as follows: Any word on the alphabet (2) is a primitive word. If P is a primitive parametric word and λ is a natural parameter, then $(P)^\lambda$ is a primitive parametric word. If P and Q are two primitive parametric words, then PQ is a primitive parametric word.

We denote by $\mathbf{L}$ the set of linear polynomials of the form $k_0 + \sum_{i=1}^{r} k_i\lambda_i$, where $r, k_0, k_1, \ldots, k_r$ are natural numbers, and $\lambda_1, \ldots, \lambda_r$ are natural parameters.

A *primitive parametric transformation* is defined by the application

$$\begin{cases} x_i \to W_i(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_q) \qquad (i = 1, \ldots, n), \\ \lambda_i \to L_i(\lambda_1, \ldots, \lambda_q) \qquad (i = 1, \ldots, q), \end{cases}$$

where every W_i is a primitive word, and $L_i \in \mathbf{L}$. The components of the form $x_i \to x_i$ and $\lambda_i \to \lambda_i$ are often omitted.

Now we inductively define a parametric word (transformation). Any primitive parametric word (transformation) is a parametric word (transformation).

Define the function ${}^{\mathbf{Fi}}(x_1, x_2)^{\lambda_1,\ldots,\lambda_s}$ for $s \geq 0$, where x_1 and x_2 are two word variables, and $\lambda_1, \ldots, \lambda_s$ are natural parameters, inductively as follows:

$$ {}^{\mathbf{Fi}}(x_1, x_2) = 1, $$

$$ {}^{\mathbf{Fi}}(x_1, x_2)^{\lambda_1,\ldots,\lambda_s} = ({}^{\mathbf{Fi}}(x_2, x_1)^{\lambda_2,\ldots,\lambda_s} x_2)^{\lambda_1}\, {}^{\mathbf{Fi}}(x_1, x_2)^{\lambda_3,\ldots,\lambda_s} \qquad (s \geq 1). $$

In particular,

$$ {}^{\mathbf{Fi}}(x_1, x_2)^{\lambda_1} = (x_2)^{\lambda_1}, $$

$$ {}^{\mathbf{Fi}}(x_1, x_2)^{\lambda_1,\lambda_2} = ((x_1)^{\lambda_2} x_2)^{\lambda_1}, $$

$$ {}^{\mathbf{Fi}}(x_1, x_2)^{\lambda_1,\lambda_2,\lambda_3} = (((x_2)^{\lambda_3} x_1)^{\lambda_2} x_2)^{\lambda_1} (x_2)^{\lambda_3}, $$

$$ {}^{\mathbf{Fi}}(x_1, x_2)^{\lambda_1,\lambda_2,\lambda_3,\lambda_4} = ((((x_1)^{\lambda_4} x_2)^{\lambda_3} x_1)^{\lambda_2} (x_1)^{\lambda_4} x_2)^{\lambda_1} ((x_1)^{\lambda_4} x_2)^{\lambda_3}. $$

The empty sequence is denoted by $\varnothing$. Let μ be a variable whose values are finite sequences of natural parameters. The variable $\mu|$ is connected with a variable μ as follows: If $\mu = \lambda_1, \lambda_2, \ldots, \lambda_s$, where $s \geq 1$, then $\mu| = \lambda_2, \ldots, \lambda_s$. If $\mu = \varnothing$, then $\mu| = \varnothing$.

A transformation is defined by the application

$$ \begin{cases} x_1 \to {}^{\mathbf{Fi}}(x_1 P, x_2 Q)^{\mu} x_1, \\ x_2 \to {}^{\mathbf{Fi}}(x_2 Q, x_1 P)^{\mu|} x_2, \end{cases} $$

where P, Q are parametric words on the alphabet $x_3, \ldots, x_n$ and μ is a variable whose values are finite sequences of natural parameters, is a *parametric transformation* (*by the function* ${}^{\mathbf{Fi}}$).

We next define by a joint induction the functions

$$ {}^{\mathbf{Th}}(x_1, x_2, x_3)_1^{\lambda_1,\ldots,\lambda_{2s}}, $$
$$ {}^{\mathbf{Th}}(x_1, x_2, x_3)_2^{\lambda_1,\ldots,\lambda_{2s}}, $$
$$ {}^{\mathbf{Th}}(x_1, x_2, x_3)_3^{\lambda_1,\ldots,\lambda_{2s}}, $$

for $s \geq 0$, where x_1, x_2, x_3 are three word variables, and $\lambda_1, \ldots, \lambda_{2s}$ are natural parameters. Specifically, we set

$$ {}^{\mathbf{Th}}(x_1, x_2, x_3)_i = 1 \qquad (i = 1, 2, 3); $$

$$ {}^{\mathbf{Th}}(x_1, x_2, x_3)_1^{\lambda_1,\ldots,\lambda_{2s}} = ({}^{\mathbf{Th}}(x_1, x_2, x_3)_3^{\lambda_1,\ldots,\lambda_{2s}} x_3)^{\lambda_1}\, {}^{\mathbf{Th}}(x_1, x_2, x_3)_1^{\lambda_3,\ldots,\lambda_{2s}}; $$

$$ \begin{aligned} &{}^{\mathbf{Th}}(x_1, x_2, x_3)_2^{\lambda_1,\ldots,\lambda_{2s}} \\ &\quad = ({}^{\mathbf{Th}}(x_1, x_2, x_3)_2^{\lambda_3,\ldots,\lambda_{2s}} x_2\, {}^{\mathbf{Th}}(x_1, x_2, x_3)_3^{\lambda_3,\ldots,\lambda_{2s}} x_3\, {}^{\mathbf{Th}}(x_1, x_2, x_3)_1^{\lambda_3,\ldots,\lambda_{2s}} x_1 \\ &\qquad \cdot {}^{\mathbf{Th}}(x_1, x_2, x_3)_1^{\lambda_1,\ldots,\lambda_{2s}} x_1)^{\lambda_2}\, {}^{\mathbf{Th}}(x_1, x_2, x_3)_2^{\lambda_3,\ldots,\lambda_{2s}}; \end{aligned} $$

$$ \begin{aligned} &{}^{\mathbf{Th}}(x_1, x_2, x_3)_3^{\lambda_1,\ldots,\lambda_{2s}} \\ &\quad = {}^{\mathbf{Th}}(x_1, x_2, x_3)_1^{\lambda_3,\ldots,\lambda_{2s}} x_1\, {}^{\mathbf{Th}}(x_1, x_2, x_3)_2^{\lambda_3,\ldots,\lambda_{2s}} x_2\, {}^{\mathbf{Th}}(x_1, x_2, x_3)_3^{\lambda_3,\ldots,\lambda_{2s}}. \end{aligned} $$

Define inductively the auxiliary function

$$ {}^{\mathbf{Oc}}(x_1, x_2, x_3)^{\lambda_1,\dots,\lambda_{2s}} $$

for $s \geq 0$, where x_1, x_2, x_3 are three word variables, and $\lambda_1, \dots, \lambda_{2s}$ are natural parameters, by setting

$$ {}^{\mathbf{Oc}}(x_1, x_2, x_3) = 1; $$

$$ \begin{aligned} &{}^{\mathbf{Oc}}(x_1, x_2, x_3)^{\lambda_1,\dots,\lambda_{2s}} \\ &\quad = {}^{\mathbf{Oc}}(x_1, x_2, x_3)^{\lambda_3,\dots,\lambda_{2s}} ({}^{\mathbf{Th}}(x_1, x_2, x_3)_3^{\lambda_3,\dots,\lambda_{2s}} x_3 \\ &\qquad \cdot {}^{\mathbf{Th}}(x_1, x_2, x_3)_1^{\lambda_3,\dots,\lambda_{2s}} x_1 {}^{\mathbf{Th}}(x_1, x_2, x_3)_2^{\lambda_3,\dots,\lambda_{2s}} x_2)^{\lambda_1}. \end{aligned} $$

A transformation defined by the application

$$ \begin{cases} x_i \to {}^{\mathbf{Th}}(x_1, x_2, x_3)_i^{\xi} x_i, & i = 1, 2, 3, \\ x_4 \to x_4 {}^{\mathbf{Oc}}(x_1, x_2, x_3)^{\xi}, & \end{cases} $$

where ξ is a variable whose values are even sequences of natural parameters, is called a *parametric transformation* (*by the function* ${}^{\mathbf{Th}}$).

Define by a joint induction the functions

$$ {}^{\mathbf{Ro}}(x_1, x_2, x_3)_1^{\mu_1,\dots,\mu_t}, \qquad {}^{\mathbf{Ro}}(x_1, x_2, x_3)_2^{\mu_1,\dots,\mu_t}, \qquad {}^{\mathbf{Ro}}(x_1, x_2, x_3)_3^{\mu_1,\dots,\mu_t} $$

for $t \geq 0$, where x_1, x_2, x_3 are three word variables, and $\mu_1, \dots, \mu_t$ are variables whose values are finite sequences of natural parameters, as follows:

$$ {}^{\mathbf{Ro}}(x_1, x_2, x_3)_i = 1 \qquad (i = 1, 2, 3); $$

$$ \begin{aligned} &{}^{\mathbf{Ro}}(x_1, x_2, x_3)_1^{\mu_1,\dots,\mu_t} \\ &\quad = {}^{\mathbf{Fi}}({}^{\mathbf{Ro}}(x_2, x_1, x_3)_3^{\mu_2,\dots,\mu_t} x_3 {}^{\mathbf{Ro}}(x_2, x_1, x_3)_1^{\mu_2,\dots,\mu_t} x_2, \\ &\qquad \cdot {}^{\mathbf{Ro}}(x_2, x_1, x_3)_2^{\mu_2,\dots,\mu_t} x_1 {}^{\mathbf{Ro}}(x_2, x_1, x_3)_3^{\mu_2,\dots,\mu_t} x_3 \\ &\qquad \cdot {}^{\mathbf{Ro}}(x_2, x_1, x_3)_1^{\mu_2,\dots,\mu_t} x_2 ({}^{\mathbf{Ro}}(x_1, x_2, x_3)_3^{\mu_1,\dots,\mu_t} x_3)^2)^{\mu_1} \\ &\quad\ \cdot {}^{\mathbf{Ro}}(x_2, x_1, x_3)_3^{\mu_2,\dots,\mu_t} x_3 {}^{\mathbf{Ro}}(x_2, x_1, x_3)_1^{\mu_2,\dots,\mu_t} x_2 \\ &\quad\ \cdot {}^{\mathbf{Ro}}(x_2, x_1, x_3)_2^{\mu_2,\dots,\mu_t}; \end{aligned} $$

$$ \begin{aligned} &{}^{\mathbf{Ro}}(x_1, x_2, x_3)_2^{\mu_1,\dots,\mu_t} \\ &\quad = {}^{\mathbf{Fi}}({}^{\mathbf{Ro}}(x_2, x_1, x_3)_3^{\mu_2,\dots,\mu_t} x_3 {}^{\mathbf{Ro}}(x_2, x_1, x_3)_1^{\mu_2,\dots,\mu_t} x_2 \\ &\qquad \cdot ({}^{\mathbf{Ro}}(x_1, x_2, x_3)_3^{\mu_1,\dots,\mu_t} x_3)^2, {}^{\mathbf{Ro}}(x_2, x_1, x_3)_3^{\mu_2,\dots,\mu_t} x_3 \\ &\qquad \cdot {}^{\mathbf{Ro}}(x_2, x_1, x_3)_1^{\mu_2,\dots,\mu_t} x_2 {}^{\mathbf{Ro}}(x_2, x_1, x_3)_2^{\mu_2,\dots,\mu_t} x_1)^{\mu_1|} \\ &\quad\ \cdot {}^{\mathbf{Ro}}(x_2, x_1, x_3)_3^{\mu_2,\dots,\mu_t} x_3 {}^{\mathbf{Ro}}(x_2, x_1, x_3)_2^{\mu_2,\dots,\mu_t}; \end{aligned} $$

$$ {}^{\mathbf{Ro}}(x_1, x_2, x_3)_3^{\mu_1,\dots,\mu_t} = {}^{\mathbf{Ro}}(x_2, x_1, x_3)_2^{\mu_2,\dots,\mu_t} x_1 {}^{\mathbf{Ro}}(x_2, x_1, x_3)_3^{\mu_2,\dots,\mu_t}. $$

Define inductively the auxiliary function

$$ {}^{\mathbf{Re}}(x_1, x_2, x_3)^{\mu_1,\dots,\mu_t} $$

for $t \geq 0$, where x_1, x_2, x_3 are three word variables, and $\mu_1, \dots, \mu_t$ are variables for sequences of natural parameters:

$$ {}^{\mathbf{Re}}(x_1, x_2, x_3) = 1; $$

$$ {}^{\mathbf{Re}}(x_1,x_2,x_3)^{\mu_1,\dots,\mu_t} = {}^{\mathbf{Re}}(x_2,x_1,x_3)^{\mu_2,\dots,\mu_t}\,{}^{\mathbf{Ro}}(x_2,x_1,x_3)_2^{\mu_2,\dots,\mu_t}x_1. $$

A transformation defined by the application

$$ \begin{cases} x_i \to {}^{\mathbf{Ro}}(x_1,x_2,x_3)_i^{\nu}x_i, \quad i=1,2,3,\\ x_4 \to x_4{}^{\mathbf{Re}}(x_1,x_2,x_3)^{\nu}, \end{cases} $$

where ν is a variable whose values are finite sequences of second parameters, is a *parametric transformation* (*by the function* ${}^{\mathbf{Ro}}$).

A composition of parametric transformations is a parametric transformation. Any word in the right side of a parametric transformations is a parametric word.

A *coefficient transformation* is defined by the application

$$ \begin{cases} x_i \to X_i \quad (i=1,\dots,n),\\ \lambda_i \to \Lambda_i \quad (i=1,\dots,t),\\ \mu_i \to M_i \quad (i=1,\dots,u),\\ \nu_i \to N_i \quad (i=1,\dots,r), \end{cases} \tag{6} $$

where every X_i is a word in the alphabet (1), every Λ_i is a natural number, every M_i is a finite sequence of $(\lambda_1,\dots,\lambda_t)$, and every N_i is a finite sequence of $(\mu_1,\dots,\mu_u)$.

A *coefficient transformation*

$$ \begin{cases} x_i \to X_i \quad (i=1,\dots,n),\\ \lambda_i \to \Lambda_i \quad (i=1,\dots,l),\\ \mu_i \to M_i \quad (i=1,\dots,m),\\ \nu_i \to N_i \quad (i=1,\dots,p), \end{cases} $$

is called an *extension* of the coefficient transformation (6) if $l \geq t$, $m \geq u$, and $p \geq r$.

A *parametric equation* in a free monoid is given by an equality of parametric words

$$ \Phi(x_1,\dots,x_n,\lambda_1,\dots,\lambda_t) = \Psi(x_1,\dots,x_n,\lambda_1,\dots,\lambda_t). \tag{7} $$

If Φ and Ψ are empty words, the equation (7) is called the *trivial* equation, denoted by *1*.

A parametric transformation (a coefficient transformation) is called a *parametric solution* (a *solution*) of the equation (7) if the result of the application of this transformation to (7) is the trivial equation.

We will say that the parametric transformation T *contains* the coefficient transformation C by means of the auxiliary transformation I, if $TI = C$.

A finite list of parametric solutions of the equation E will be called a *general solution* of E, if every solution of E is contained in some parametric solution of this list. The general solution of E will be denoted $\langle E\rangle$.

The length of a word A in the alphabet (1) is denoted by $|A|$. The empty word is denoted by 1. The length of a finite sequence B is denoted by $|B|$.

A *condition on natural parameters* has the form

$$ L_1(\lambda_1,\dots,\lambda_q) \ <,\leq,= \ L_2(\lambda_1,\dots,\lambda_q), \tag{8} $$

where L_1, L_2 are integer polynomials. A coefficient transformation (6) is called a *solution* of the equation E with condition (8) on natural parameters, if (6) is a solution of E that satisfies

$$L_1(\Lambda_1, \ldots, \Lambda_q) \ <, \leq, = \ L_2(\Lambda_1, \ldots, \Lambda_q).$$

A *condition on the length of a solution* has the form

$$0,\ \partial(P(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_q)) \ <, \leq, = \ \partial(Q(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_q)), \tag{9}$$

where P, Q are two primitive parametric words. A coefficient transformation (6) is called a *solution* of the equation E with a condition (9) on the lengths of solutions, if (6) is a solution of E that satisfies

$$0,\ |P(X_1, \ldots, X_n, \Lambda_1, \ldots, \Lambda_q)| \ <, \leq, = \ |Q(X_1, \ldots, X_n, \Lambda_1, \ldots, \Lambda_q)|.$$

A *condition on the length of a sequence of variables* has the form

$$N_i \ <, \leq, = \ \partial(\mu_i), \tag{10}$$

where N_i is a natural number and μ_i is a second parameter. A coefficient transformation (6) is called a *solution* of the equation E with a condition (10) on the length of sequences, if (6) is a solution of E that satisfies

$$N_i \ <, \leq, = \ |M_i|.$$

A parametric transformation T is called a *parametric solution* of the equation E with a condition R, if T is a parametric solution of E and the result of the application of T to R is a true proposition for any values of the variables.

Let E be an equation with conditions and let $R_1, \ldots, R_m$ be the list of new conditions. By (E, R_i) we denote the equation E with additional condition R_i. The equation E s said to be *divided into a collection of equations* $(E, R_1), \ldots, (E, R_m)$, if every solution S of E is a solution of some (E, R_i). An equation (E, R_1) *contains* an equation (E, R_2) (and we write $(E, R_1) \supseteq (E, R_2)$), if every solution of (E, R_2) is a solution of (E, R_1).

We say that *the equation E_1 is reduced by the parametric transformation T to the equation E_2*, if $E_1 T = E_2'$, where $E_2' \supseteq E_2$, and for every solution S_1 of E_1 there exists a solution S_2 of E_2 such that $S_1 = T S_2^*$ for some extension S_2^* of S_2. We say that S_2 is the *image* of S_1 via the transformation T. We need the extension S_2^*, because T could have some variables that are not in E_2.

Lemma 1. *Let E_1 be reduced by T to E_2. Let S_1 be a solution of E_1 and S_2 its image via T. If the parametric solution Q_2 of E_2 contains a solution S_2 of E_2, then the parametric solution TQ_2 of E_1 contains the solution S_1 of E_1.*

Theorem 1. *Let the equation E_1 be reduced by the parametric transformation T to the equation E_2. If the general solution $\langle E_2 \rangle$ of E_2 is $Q_1, \ldots, Q_r$, then the general solution $\langle E_1 \rangle$ of E_1 is $TQ_1, \ldots, TQ_r$.*

Theorem 2. *Let the equation E be divided into a collection of equations with conditions $(E, R_1), \ldots, (E, R_m)$. If the general solution $\langle (E, R_i) \rangle$ of (E, R_i) is $Q_{i,1}, \ldots, Q_{i,r_i}$ $(i = 1, \ldots, m)$, then the general solution $\langle E \rangle$ of E is $Q_{1,1}, \ldots, Q_{1,r_1}, \ldots, Q_{m1}, \ldots, Q_{m,r_m}$.*

Let

$$K_\alpha(\lambda_1, \ldots, \lambda_q) \ <, \leq, = \ M_\alpha(\lambda_1, \ldots, \lambda_q) \qquad (\alpha = 1, \ldots, t), \tag{11}$$

where K_α, $M_\alpha \in \mathbf{L}$, be a system of linear Diophantine equations and inequations. A transformation

$$\lambda_i \to L_i, \quad L_i \in \mathbf{L} \qquad (i = 1, \ldots, q),$$

is called a *parametric solution* of the system (11), if

$$K_\alpha(L_1, \ldots, L_q) \ <, \leq, = \ M_\alpha(L_1, \ldots, L_q) \qquad (\alpha = 1, \ldots, t)$$

for any values of the variables.

Theorem 3. *The family of solutions of the system* (11) *is described by a finite list of parametric solution* (*see* [7]).

3. Preliminaries

The following seven propositions belong to folklore (see [4], [8], [9]). Observe that a boldface $\boldsymbol{n}$ means "the equation in Proposition $\boldsymbol{n}$".

Proposition 1. *The general solution of the equation*

1 $$x_1x_2 = x_2x_1$$

is described by the transformation

$$\begin{cases} x_1 \to x_1^\alpha, \\ x_2 \to x_1^\beta, \end{cases}$$

where α, β are natural parameters.

Proposition 2. *The general solution of the equation*

2 $$x_1x_2x_3 = x_3x_1x_2$$

is described by the transformations

$$\begin{cases} x_1 \to 1, \\ x_2 \to 1, \\ x_3 \to x_3, \end{cases} \qquad \begin{cases} x_1 \to (x_1x_2)^\alpha x_1, \\ x_2 \to (x_2x_1)^\beta x_2, \\ x_3 \to (x_1x_2)^\gamma, \end{cases}$$

where α, β, γ are natural parameters.

Proposition 3. *The general solution of the equation*

3 $$x_1x_2x_3 = x_2^\alpha x_1,$$

where α is a natural parameter, is described by the transformations

$$\begin{cases} x_1 \to x_1^\beta, \\ x_2 \to x_1^\gamma, \\ x_3 \to x_1^\delta, \end{cases}$$

where β, γ, δ are natural parameters.

Proposition 4. *The general solution of the equation*

4 $$x_1x_3 = x_2x_1$$

is described by the transformations

$$\begin{cases} x_1 \to x_1, \\ x_2 \to 1, \\ x_3 \to 1, \end{cases} \qquad \begin{cases} x_1 \to (x_1x_2)^\alpha x_1, \\ x_2 \to x_1x_2, \\ x_3 \to x_2x_1, \end{cases}$$

where α is a natural parameter.

Proposition 5. *The general solution of the equation*

5 $$x_1x_3 = x_2^\alpha x_1,$$

where α is a natural parameter, is described by the transformations

$$\begin{cases} x_1 \to x_1, \\ x_2 \to x_2, \\ x_3 \to 1, \\ \alpha \to 0, \end{cases} \qquad \begin{cases} x_1 \to x_1, \\ x_2 \to 1, \\ x_3 \to 1, \\ \alpha \to \alpha, \end{cases} \qquad \begin{cases} x_1 \to (x_1x_2)^\beta x_1, \\ x_2 \to x_1x_2, \\ x_3 \to (x_2x_1)^\alpha, \\ \alpha \to \alpha, \end{cases}$$

where β is a natural parameter.

Proposition 6. *The general solution of the equation*

6 $$x_1x_2x_3 = x_3x_4$$

is described by the transformations

$$\begin{cases} x_1 \to 1, \\ x_2 \to 1, \\ x_3 \to x_3, \\ x_4 \to 1, \end{cases} \qquad \begin{cases} x_1 \to x_1, \\ x_2 \to x_3x_2, \\ x_3 \to (x_1x_3x_2)^\alpha x_1x_3, \\ x_4 \to x_2x_1x_3, \end{cases} \qquad \begin{cases} x_1 \to x_3x_1, \\ x_2 \to x_2, \\ x_3 \to (x_3x_1x_2)^\alpha x_3, \\ x_4 \to x_1x_2x_3, \end{cases}$$

where α is a natural parameter.

Proposition 7. *The general solution of the equation*

7 $$x_1x_2x_3 = x_2x_3x_4$$

is described by the transformations

$$\begin{cases} x_1 \to 1, \\ x_2 \to x_2, \\ x_3 \to x_3, \\ x_4 \to 1, \end{cases} \qquad \begin{cases} x_1 \to x_2x_3x_1, \\ x_2 \to (x_2x_3x_1)^\alpha x_2, \\ x_3 \to (x_3x_1x_2)^\beta x_3, \\ x_4 \to x_1x_2x_3, \end{cases} \qquad \begin{cases} x_1 \to x_3x_2x_1, \\ x_2 \to (x_3x_2x_1)^\alpha x_3x_2, \\ x_3 \to (x_1x_3x_2)^\beta x_1x_3, \\ x_4 \to x_2x_1x_3, \end{cases}$$

where α, β are natural parameters.

Proposition 8. *The parametric equation*

8 $$x_1R(x_2,x_3)x_4 = (p(x_2,x_3))^{t+1}x_1Q(x_1,x_2,x_3)x_5$$

with $\partial(P(x_2,x_3)) > 0$, where P, Q, R are parametric words and t is a natural number, is reduced by the parametric transformation T:

$$x_1 \to (P(x_2,x_3,\lambda_1,\dots,\lambda_r))^\alpha x_1,$$

where α is a natural parameter, to the parametric equation E:

$$\begin{aligned} &x_1R(x_2,x_3)x_4 \\ &\quad = (P(x_2,x_3,\lambda_1,\dots,\lambda_r))^{t+1}x_1Q(P(x_2,x_3,\lambda_1,\dots,\lambda_r))^\alpha x_1,x_2,x_3,\lambda_1,\dots,\lambda_r)x_5 \end{aligned}$$

with $\partial(x_1) < \partial(P(x_2,x_3,\lambda_1,\dots,\lambda_r))$.

Proof. It is easy to verify that the substitution of the transformation T into **8** transforms **8** to some equation E' which contains E. On the other hand, let the transformation S_1:

$$\begin{cases} x_1 \to X_1, \\ x_i \to X_i \qquad (i = 2, \dots, 5), \\ \lambda_i \to \Lambda_i \qquad (i = 1, \dots, r), \end{cases}$$

where the X_i are words in the alphabet (1) and the Λ_i are natural numbers, be an arbitrary solution of the equation **8**. Since $|P(X_2, X_3, \Lambda_1, \dots, \Lambda_r)| > 0$, we have $X_1 = (P(X_2, X_3, \Lambda_1, \dots, \Lambda_r))^A Y_1$ for some word Y_1 in the alphabet (1) and some natural number A such that

$$A|P(X_2, X_3, \Lambda_1, \dots, \Lambda_r)| \leq |X_1| < (A+1)|P(X_2, X_3, \Lambda_1, \dots, \Lambda_r)|.$$

After the substitution of the solution S_1 in the equation we easily obtain that the coefficient transformation S_2:

$$\begin{cases} x_1 \to Y_1, \\ x_i \to X_i \qquad (i = 2, \dots, 5), \\ \lambda_i \to \Lambda_i \qquad (i = 1, \dots, r) \end{cases}$$

is a solution of the equation E. The extension S_2^*:

$$\begin{cases} x_1 \to Y_1, \\ x_i \to X_i \quad (i = 2, \dots, 5), \\ \lambda_i \to \Lambda_i \quad (i = 1, \dots, r), \\ \alpha \to A \end{cases}$$

satisfies $S_1 = TS_2^*$. Therefore the solution S_2 is the image of S_1 via T. Thus the parametric equation **8** is reduced by the parametric transformation T to the parametric equation E.

4. The function ${}^{\mathbf{Fi}}(x_1, x_2)^{\lambda_1, \dots, \lambda_s}$

Theorem ${}^{\mathbf{Fi}}$1. *The following identities hold:*

$$^{\mathbf{Fi}}(x_1, x_2)^{\lambda_1, \dots, \lambda_{2k}} = {}^{\mathbf{Fi}}(x_1, x_1^{\lambda_{2k}} x_2)^{\lambda_1, \dots, \lambda_{2k-1}} \qquad (k \geq 1),$$

$$^{\mathbf{Fi}}(x_1, x_2)^{\lambda_1, \dots, \lambda_{2k+1}} = {}^{\mathbf{Fi}}(x_2^{\lambda_{2k+1}} x_1, x_2)^{\lambda_1, \dots, \lambda_{2k}} x_2^{\lambda_{2k+1}} \qquad (k \geq 0).$$

Proof (Joint induction on k). If $k = 0$ or 1, the proof is obvious.

Suppose that $k > 1$. By definition, ${}^{\mathbf{Fi}}(x_1, x_2)^{\lambda_1, \dots, \lambda_{2k}}$ equals

$$({}^{\mathbf{Fi}}(x_2, x_1)^{\lambda_2, \dots, \lambda_{2k}} x_2)^{\lambda_1} \, {}^{\mathbf{Fi}}(x_1, x_2)^{\lambda_3, \dots, \lambda_{2k}}.$$

According to the induction proposition (second identity), this last expression equals

$$({}^{\mathbf{Fi}}(x_1^{\lambda_{2k}} x_2, x_1)^{\lambda_2, \dots, \lambda_{2k-1}} x_1^{\lambda_{2k}} x_2)^{\lambda_1} \, {}^{\mathbf{Fi}}(x_1, x_2)^{\lambda_3, \dots, \lambda_{2k}}.$$

According to the induction proposition (first identity), this in turn is equal to

$$({}^{\mathbf{Fi}}(x_1^{\lambda_{2k}} x_2, x_1)^{\lambda_2, \dots, \lambda_{2k-1}} x_1^{\lambda_{2k}} x_2)^{\lambda_1} \, {}^{\mathbf{Fi}}(x_1, x_1^{\lambda_{2k}} x_2)^{\lambda_3, \dots, \lambda_{2k-1}},$$

and this is equal (by definition) to

$$^{\mathbf{Fi}}(x_1, x_1^{\lambda_{2k}} x_2)^{\lambda_1, \dots, \lambda_{2k-1}}.$$

By definition, ${}^{\mathbf{Fi}}(x_1,x_2)^{\lambda_1,\dots,\lambda_{2k+1}}$ equals

$$({}^{\mathbf{Fi}}(x_2,x_1)^{\lambda_2,\dots,\lambda_{2k+1}}x_2)^{\lambda_1}\,{}^{\mathbf{Fi}}(x_1,x_2)^{\lambda_3,\dots,\lambda_{2k+1}}.$$

According to the induction proposition (first identity), this last expression equals

$$({}^{\mathbf{Fi}}(x_2,x_2^{\lambda_{2k+1}}x_1)^{\lambda_2,\dots,\lambda_{2k}}x_2)^{\lambda_1}\,{}^{\mathbf{Fi}}(x_1,x_2)^{\lambda_3,\dots,\lambda_{2k+1}},$$

and by the induction proposition (second identity), this is in turn equal to

$$({}^{\mathbf{Fi}}(x_2,x_2^{\lambda_{2k+1}}x_1)^{\lambda_2,\dots,\lambda_{2k}}x_2)^{\lambda_1}\,{}^{\mathbf{Fi}}(x_2^{\lambda_{2k+1}}x_1,x_2)^{\lambda_3,\dots,\lambda_{2k}}x_2^{\lambda_{2k+1}}.$$

But this is equal (by definition) to

$${}^{\mathbf{Fi}}(x_2^{\lambda_{2k+1}}x_1,x_2)^{\lambda_1,\dots,\lambda_{2k}}x_2^{\lambda_{2k+1}}.$$

Consider an equation of the form $x_1Px_2U = x_2Qx_1V$, where P,Q are parametric words in the alphabet $x_3,\dots,x_n$ and U,V are parametric words. Consider the sequence of parametric transformations

$$\begin{array}{ll}
1. & x_1 \to (x_2Q)^{\lambda_1}x_1,\\
2. & x_2 \to (x_1P)^{\lambda_2}x_2,\\
3. & x_1 \to (x_2Q)^{\lambda_3}x_1,\\
4. & x_2 \to (x_1P)^{\lambda_4}x_2,\\
\multicolumn{2}{l}{\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots}\\
2k-1. & x_1 \to (x_2Q)^{\lambda_{2k-1}}x_1,\\
2k. & x_2 \to (x_1P)^{\lambda_{2k}}x_2,\\
2k+1. & x_1 \to (x_2Q)^{\lambda_{2k+1}}x_1,\\
\multicolumn{2}{l}{\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots}
\end{array} \tag{12}$$

where $\lambda_1,\lambda_2,\dots$ are natural parameters.

Theorem ${}^{\mathbf{Fi}}$2. *For every natural s, the sequence* (12) *of the first s parametric transformation can be collected by the following common transformation:*

$$\begin{cases} x_1 \to {}^{\mathbf{Fi}}(x_1P,x_2Q)^{\lambda_1,\dots,\lambda_s}x_1,\\ x_2 \to {}^{\mathbf{Fi}}(x_2Q,x_1P)^{\lambda_2,\dots,\lambda_s}x_2. \end{cases}$$

Proof. If $s=0$ the proposition obviously holds. Consider two cases.

Case 1. Suppose that the sequence of the first $2k-1$ transformations can be collected by the common transformation

$$\begin{cases} x_1 \to {}^{\mathbf{Fi}}(x_1P,x_2Q)^{\lambda_1,\dots,\lambda_{2k-1}}x_1,\\ x_2 \to {}^{\mathbf{Fi}}(x_2Q,x_1P)^{\lambda_2,\dots,\lambda_{2k-1}}x_2. \end{cases}$$

Let the $2k$th transformation be of the form

$$x_2 \to x_2P^{\lambda_{2k}}x_2.$$

Then the sequence of the first $2k$ transformations can be collected by the common transformation

$$\begin{cases} x_1 \to {}^{\mathbf{Fi}}(x_1P,(x_1P)^{\lambda_{2k}}x_2Q)^{\lambda_1,\dots,\lambda_{2k-1}}x_1,\\ x_2 \to {}^{\mathbf{Fi}}((x_1P)^{\lambda_{2k}}x_2Q,x_1P)^{\lambda_2,\dots,\lambda_{2k-1}}(x_1P)^{\lambda_{2k}}x_2. \end{cases}$$

According to Theorem ${}^{\mathbf{Fi}}$1 this transformation coincides with the transformation

$$\begin{cases} x_1 \to {}^{\mathbf{Fi}}(x_1P,x_2Q)^{\lambda_1,\dots,\lambda_{2k}}x_1,\\ x_2 \to {}^{\mathbf{Fi}}(x_2Q,x_1P)^{\lambda_2,\dots,\lambda_{2k}}x_2. \end{cases}$$

Case 2. Suppose that the sequence of the first $2k$ transformations can be collected in the common transformation

$$\begin{cases} x_1 \to {}^{\mathbf{Fi}}(x_1P, x_2Q)^{\lambda_1,\dots,\lambda_{2k}} x_1, \\ x_2 \to {}^{\mathbf{Fi}}(x_2Q, x_1P)^{\lambda_2,\dots,\lambda_{2k}} x_2. \end{cases}$$

Let the $(2k+1)$st transformation be of the form

$$x_1 \to (x_2Q)^{\lambda_{2k+1}} x_1.$$

Then the sequence of the first $2k+1$ transformations can be collected by the common transformation

$$\begin{cases} x_1 \to {}^{\mathbf{Fi}}((x_2Q)^{\lambda_{2k+1}} x_1P, x_2Q)^{\lambda_1,\dots,\lambda_{2k}} (x_2Q)^{\lambda_{2k+1}} x_1, \\ x_2 \to {}^{\mathbf{Fi}}(x_2Q, (x_2Q)^{\lambda_{2k+1}} x_1P)^{\lambda_2,\dots,\lambda_{2k}} x_2. \end{cases}$$

According to Theorem ${}^{\mathbf{Fi}}1$ this transformation coincides with the transformation

$$\begin{cases} x_1 \to {}^{\mathbf{Fi}}(x_1P, x_2Q)^{\lambda_1,\dots,\lambda_{2k+1}} x_1, \\ x_2 \to {}^{\mathbf{Fi}}(x_2Q, x_1P)^{\lambda_2,\dots,\lambda_{2k+1}} x_2. \end{cases}$$

Theorem ${}^{\mathbf{Fi}}$3. *The following identities hold:*

$$\begin{aligned} {}^{\mathbf{Fi}}(x_1, x_2)^{0,\lambda_2,\dots,\lambda_s} &= {}^{\mathbf{Fi}}(x_1, x_2)^{\lambda_3,\dots,\lambda_s}, \\ {}^{\mathbf{Fi}}(x_1, x_2)^{\lambda_1,\dots,\lambda_{r-1},0,\lambda_{r+1},\dots,\lambda_s} &= {}^{\mathbf{Fi}}(x_1, x_2)^{\lambda_1,\dots,\lambda_{r-1}+\lambda_{r+1},\dots,\lambda_s}, \\ {}^{\mathbf{Fi}}(x_1, x_2)^{\lambda_1,\dots,\lambda_{s-1},0} &= {}^{\mathbf{Fi}}(x_1, x_2)^{\lambda_1,\dots,\lambda_{s-1}}. \end{aligned}$$

Proof. Follows from Theorem ${}^{\mathbf{Fi}}2$.

Proposition 9. *The parametric equation*

9 $$x_1x_3^{\alpha}x_2U(x_1, x_2, x_3, \lambda_1, \dots, \lambda_r)x_4 = x_2x_3^{\beta}x_1V(x_1, x_2, x_3, \lambda_1, \dots, \lambda_r)x_5$$

with $\partial(x_1x_3^{\alpha}) > 0$, $\partial(x_2x_3^{\beta}) > 0$, $\alpha + \beta > 0$, *where* $\alpha, \beta, \lambda_1, \dots, \lambda_r$ *are natural parameters and* U, V *are parametric words, is reduced by the transformation* T*:*

$$\begin{cases} x_1 \to {}^{\mathbf{Fi}}(x_1x_3^{\alpha}, x_2x_3^{\beta})^{\mu} x_1, \\ x_2 \to {}^{\mathbf{Fi}}(x_2x_3^{\beta}, x_1x_3^{\alpha})^{\mu|} x_2, \end{cases}$$

where μ *is a variable for sequences of natural parameters, to the equation* E*:*

$$\begin{aligned} &x_1x_3^{\alpha}x_2U({}^{\mathbf{Fi}}(x_1x_3^{\alpha}, x_2x_3^{\beta})^{\mu}x_1, {}^{\mathbf{Fi}}(x_2x_3^{\beta}, x_1x_3^{\alpha})^{\mu|}x_2, x_3, \lambda_1, \dots, \lambda_r)x_4 \\ &\quad = x_2x_3^{\beta}x_1V({}^{\mathbf{Fi}}(x_1x_3^{\alpha}, x_2x_3^{\beta})^{\mu}x_1, {}^{\mathbf{Fi}}(x_2x_3^{\beta}, x_1x_3^{\alpha})^{\mu|}x_2, x_3, \lambda_1, \dots, \lambda_r)x_5 \end{aligned}$$

with $\partial(x_1) < \partial(x_2x_3^{\beta})$, $\partial(x_2) < \partial(x_1x_3^{\alpha})$.

Proof. It is easy to verify that T reduces **9** to some equation E' which contains E.

On the other hand, let the transformation S_1:

$$\begin{cases} x_1 \to X_i \quad (i = 1, \dots, 5), \\ \lambda_1 \to \Lambda_i \quad (i = 1, \dots, r), \\ \alpha \to A, \\ \beta \to B, \end{cases}$$

where the X_i are words in the alphabet (1) and Λ_i, A, B are natural numbers, be an arbitrary solution of the equation **9**.

We prove by induction on $|X_1X_2X_3|$ that S_1 is contained in the parametric transformation T.

If $|X_1| < |X_2X_3^B|$ and $|X_2| < |X_1X_3^A|$, then the coefficient transformation S_2 :

$$\begin{cases} x_i \to X_i \quad (i = 1, \dots, 5), \\ \lambda_i \to \Lambda_i \quad (i = 1, \dots, r), \\ \alpha \to A, \\ \beta \to B, \\ \mu \to \varnothing \end{cases}$$

is a solution of the equation E, and $S_1 = TS_2$.

Let $|X_1| \geq |X_2X_3^B|$. Since $|X_2X_3^B| > 0$, we have $X_1 = X_2X_3^BY_1$ for some word Y_1 in the alphabet (1), where $|Y_1| < |X_1|$.

It is easy to see that equation **9** with the additional condition $\partial(x_1) > \partial(x_2x_3^\beta)$ is reduced by the transformation t:

$$x_1 \to x_2x_3^\beta x_1$$

to the equation E':

$$\begin{aligned} &x_1x_3^\alpha x_2U(x_2x_3^\beta x_1, x_2, x_3, \lambda_1, \dots, \lambda_r)x_4 \\ &\quad = x_2x_3^\beta x_1V(x_2x_3^\beta x_1, x_2, x_3, \lambda_1, \dots, \lambda_r)x_5 \end{aligned}$$

with $\partial(x_1x_3^\alpha) > 0$, $\partial(x_2x_3^\beta) > 0$, $\alpha + \beta > 0$.

The transformation S':

$$\begin{cases} x_1 \to Y_1, \\ x_i \to X_i \quad (i = 2, 3, 4, 5), \\ \alpha \to A, \\ \beta \to B \end{cases}$$

is a solution of the equation E'.

Since $|Y_1| < |X_1|$, one can use the inductive proposition to see that S' is contained in the parametric transformation T. Hence S_1 is contained in the parameter solution tT. Using Theorem $^{\mathbf{Fi}}2$, one can see that the transformation tT has the form

$$\begin{cases} x_1 \to {}^{\mathbf{Fi}}(x_1x_3^\alpha, x_2x_3^\beta)^{1,0,\mu}x_1, \\ x_2 \to {}^{\mathbf{Fi}}(x_2x_3^\beta, x_1x_3^\alpha)^{\mu|}x_2. \end{cases}$$

Let $|X_2| \geq |X_1X_3^A|$. Since $|X_1X_3^A| > 0$, we have $X_2 = X_1X_3^AY_2$ for some word Y_2 in the alphabet (1), where $|Y_2| < |X_2|$.

It is easy to see that equation **9** with the additional condition $\partial(x_2) > \partial(x_1x_3^\alpha)$ is reduced by the transformation t: $x_2 \to x_1x_3^\alpha x_2$ to the equation E':

$$\begin{aligned} &x_1x_3^\alpha x_2U(x_1, x_1x_3^\alpha x_2, x_3, \lambda_1, \dots, \lambda_r)x_4 \\ &\quad = x_2x_3^\beta x_1V(x_1, x_1x_3^\alpha x_2, x_3, \lambda_1, \dots, \lambda_r)x_5 \end{aligned}$$

with $\partial(x_1x_3^\alpha) > 0$, $\partial(x_2x_3^\beta) > 0$, $\alpha + \beta > 0$.

The transformation S':

$$\begin{cases} x_2 \to Y_2, \\ x_i \to X_i \quad (i = 1, 3, 4, 5), \\ \alpha \to A, \\ \beta \to B \end{cases}$$

is a solution of E'.

Since $|Y_2| < |X_2|$, one can use the inductive proposition to see that S' is contained in the parametric transformation T. Hence S_1 is contained in tT. Using Theorem $^{\mathbf{Fi}}2$, one can see that the transformation tT has form

$$\begin{cases} x_1 \to {}^{\mathbf{Fi}}(x_1 x_3^{\alpha}, x_2 x_3^{\beta})^{\mu} x_1, \\ x_2 \to {}^{\mathbf{Fi}}(x_2 x_3^{\beta}, x_1 x_3^{\alpha})^{1,\mu} x_2. \end{cases}$$

Proposition 10*. *The general solution of the equation*

10* $$x_1 x_3^{\alpha+1} x_2 = x_2 x_1 x_4 \quad \text{with } \partial(x_1) < \partial(x_2) < \partial(x_1 x_3^{\alpha+1}),$$

where α is a natural parameter, is described by the transformation

$$\begin{cases} x_2 \to x_1 (x_2 x_3)^{\beta} x_2, \\ x_3 \to x_2 x_3, \end{cases}$$
$$x_1 \to (x_3 (x_2 x_3)^{\alpha-\beta})^{\lambda} x_1,$$

followed by one of the two transformations

$$\begin{cases} x_1 \to (x_1 x_3 x_2)^{\sigma} x_1, \\ x_3 \to x_1 x_3, \\ x_4 \to x_3 (x_2 x_1 x_3)^{\alpha-\beta-\sigma} (x_1 x_3 x_2)^{\sigma} x_1 (x_2 x_1 x_3)^{\beta} x_2, \end{cases}$$
$$\begin{cases} x_1 \to (x_3 x_1 x_2)^{\tau} x_3 x_1, \\ x_2 \to x_1 x_2, \\ x_4 \to x_2 x_3 (x_1 x_2 x_3)^{\alpha-\beta-\tau-1} (x_3 x_1 x_2)^{\tau} x_3 x_1 (x_1 x_2 x_3)^{\beta} x_1 x_2, \end{cases}$$

where $\beta, \lambda, \sigma, \tau$ are natural parameters, with $\beta \le \alpha$, $\sigma + \beta \le \alpha$, $\tau + \beta + 1 \le \alpha$.

Proof. The equation **10*** can be reduced by the transformation

$$x_2 \to x_1 x_2$$

to the equation E_1:

$$x_3^{\alpha+1} x_1 x_2 = x_2 x_1 x_4 \quad \text{with } 0 < \partial(x_2) < \partial(x_3^{\alpha+1}).$$

The equation E_1 can be reduced by the transformation

$$\begin{cases} x_2 \to (x_2 x_3)^{\beta} x_2, \\ x_3 \to x_2 x_3, \end{cases}$$

where β is a natural parameter with $\beta \le \alpha$, to the equation E_2:

$$x_3 (x_2 x_3)^{\alpha-\beta} x_1 (x_2 x_3)^{\beta} x_2 = x_1 x_4 \quad \text{with } \partial(x_3) > 0.$$

According to Proposition 27, the equation E_2 is reduced by the parametric transformation

$$x_1 \to (x_3 (x_2 x_3)^{\alpha-\beta})^{\lambda} x_1,$$

where λ is a natural parameter, to the equation E_3:

$$E_2 \quad \text{with } \partial(x_1) < \partial(x_3(x_2x_3)^{\alpha-\beta}).$$

The equation E_3 can be divided into the collection of equations

(j) E_3 with $\partial(x_3x_2)^\sigma \le \partial(x_1) < \partial((x_3x_2)^\sigma x_3)$, $\sigma \le \alpha-\beta$,

(jj) E_3 with $\partial((x_3x_2)^\tau x_3) \le \partial(x_1) < \partial((x_3x_2)^{\tau+1}x_3)$, $\tau \le \alpha-\beta-1$.

The equation (j) can be reduced by the transformation

$$\begin{cases} x_1 \to (x_1x_3x_2)^\sigma x_1, \\ x_3 \to x_1x_3 \end{cases}$$

to the equation

$$x_3(x_2x_1x_3)^{\alpha-\beta-\sigma}(x_1x_3x_2)^\sigma x_1(x_2x_1x_3)^\beta x_2 = x_4.$$

The equation (jj) can be reduced by the transformation

$$\begin{cases} x_1 \to (x_3x_1x_2)^\tau x_3x_1, \\ x_2 \to x_1x_2 \end{cases}$$

to the equation

$$x_2x_3(x_1x_2x_3)^{\alpha-\beta-\tau-1}(x_3x_1x_2)^\tau x_3x_1(x_1x_2x_3)^\beta x_1x_2 = x_4.$$

Proposition 10. *The general solution of the equation*

10 $$x_1x_3^{\alpha+1}x_2 = x_2x_1x_4,$$

where α is a natural parameter, is described by the transformation

$$\begin{cases} x_1 \to {}^{\mathbf{Fi}}(x_1x_3^{\alpha+1}, x_2)^\mu x_1, \\ x_2 \to {}^{\mathbf{Fi}}(x_2, x_1x_3^\alpha)^{\mu|}x_2, \end{cases}$$

followed by one of the three transformations

$$\begin{cases} x_1 \to 1, \\ x_3 \to 1, \\ x_4 \to 1, \end{cases} \qquad \begin{cases} x_2 \to 1, \\ x_4 \to x_3^{\alpha+1}, \end{cases} \qquad \langle \mathbf{10}^* \rangle.$$

Proof. This follows directly from Propositions 9 and 10^*.

5. The function ${}^{\mathbf{Th}}(x_1, x_2, x_3)_i^{\lambda_1,\dots,\lambda_{2s}}$

Theorem ${}^{\mathbf{Th}}$1. *The following identities hold for $s \ge 0$:*

$$\begin{aligned} &{}^{\mathbf{Th}}(x_1, x_2, x_3)_1^{\lambda_1,\dots,\lambda_{2s+2}} \\ &\quad = {}^{\mathbf{Th}}((x_1x_2x_3)^{\lambda_{2s+1}}x_1, (x_2x_3x_1(x_1x_2x_3)^{\lambda_{2s+1}}x_1)^{\lambda_{2s+2}}x_2, \\ &\qquad\qquad x_1x_2x_3)_1^{\lambda_1,\dots,\lambda_{2s}}(x_1x_2x_3)^{\lambda_{2s+1}}; \end{aligned}$$

$$\begin{aligned} &{}^{\mathbf{Th}}(x_1, x_2, x_3)_2^{\lambda_1,\dots,\lambda_{2s+2}} \\ &\quad = {}^{\mathbf{Th}}((x_1x_2x_3)^{\lambda_{2s+1}}x_1, (x_2x_3x_1(x_1x_2x_3)^{\lambda_{2s+1}}x_1)^{\lambda_{2s+2}}x_2, \\ &\qquad\qquad x_1x_2x_3)_2^{\lambda_1,\dots,\lambda_{2s}}(x_2x_3x_1(x_1x_2x_3)^{\lambda_{2s+1}}x_1)^{\lambda_{2s+2}}; \end{aligned}$$

$$
\begin{aligned}
&{}^{\mathbf{Th}}(x_1,x_2,x_3)_3^{\lambda_1,\dots,\lambda_{2s+2}}\\
&\quad={}^{\mathbf{Th}}((x_1x_2x_3)^{\lambda_{2s+1}}x_1,(x_2x_3x_1(x_1x_2x_3)^{\lambda_{2s+1}}x_1)^{\lambda_{2s+2}}x_2,\\
&\qquad\qquad x_1x_2x_3)_3^{\lambda_1,\dots,\lambda_{2s}}x_1x_2;
\end{aligned}
$$

$$
\begin{aligned}
&{}^{\mathbf{Oc}}(x_1,x_2,x_3)^{\lambda_1,\dots,\lambda_{2s+2}}\\
&\quad=(x_3x_1x_2)^{\lambda_{2s+1}}\,{}^{\mathbf{Oc}}((x_1x_2x_3)^{\lambda_{2s+1}}x_1,(x_2x_3x_1(x_1x_2x_3)^{\lambda_{2s+1}}x_1)^{\lambda_{2s+2}}\\
&\qquad\qquad x_2,x_1x_2x_3)^{\lambda_1,\dots,\lambda_{2s}}.
\end{aligned}
$$

Proof. We argue by a joint induction on s.

Third identity. By definition,

$$
\begin{aligned}
&{}^{\mathbf{Th}}(x_1,x_2,x_3)_3^{\lambda_1,\dots,\lambda_{2s+2}}\\
&={}^{\mathbf{Th}}(x_1,x_2,x_3)_1^{\lambda_3,\dots,\lambda_{2s+2}}x_1\,{}^{\mathbf{Th}}(x_1,x_2,x_3)_2^{\lambda_3,\dots,\lambda_{2s+2}}x_2\,{}^{\mathbf{Th}}(x_1,x_2,x_3)_3^{\lambda_3,\dots,\lambda_{2s+2}}.
\end{aligned}
$$

According to the induction proposition, it is equal to

$$
\begin{aligned}
&{}^{\mathbf{Th}}((x_1x_2x_3)^{\lambda_{2s+1}}x_1,(x_2x_3x_1(x_1x_2x_3)^{\lambda_{2s+1}}x_1)^{\lambda_{2s+2}}x_2,\\
&\qquad\qquad x_1x_2x_3)_1^{\lambda_3,\dots,\lambda_{2s}}(x_1x_2x_3)^{\lambda_{2s+1}}x_1\\
&\cdot{}^{\mathbf{Th}}((x_1x_2x_3)^{\lambda_{2s+1}}x_1,(x_2x_3x_1(x_1x_2x_3)^{\lambda_{2s+1}}x_1)^{\lambda_{2s+2}}x_2,x_1x_2x_3)_2^{\lambda_3,\dots,\lambda_{2s}}\\
&\qquad\cdot(x_2x_3x_1(x_1x_2x_3)^{\lambda_{2s+1}}x_1)^{\lambda_{2s+2}}x_2\\
&\cdot{}^{\mathbf{Th}}((x_1x_2x_3)^{\lambda_{2s+1}}x_1,(x_2x_3x_1(x_1x_2x_3)^{\lambda_{2s+1}}x_1)^{\lambda_{2s+2}}x_2,x_1x_2x_3)_3^{\lambda_3,\dots,\lambda_{2s}}x_1x_2.
\end{aligned}
$$

This is equal by definition to

$$
{}^{\mathbf{Th}}((x_1x_2x_3)^{\lambda_{2s+1}}x_1,(x_2x_3x_1(x_1x_2x_3)^{\lambda_{2s+1}}x_1)^{\lambda_{2s+2}}x_2,x_1x_2x_3)_3^{\lambda_1,\dots,\lambda_{2s}}x_1x_2.
$$

First identity. By definition,

$$
{}^{\mathbf{Th}}(x_1,x_2,x_3)_1^{\lambda_1,\dots,\lambda_{2s+2}}=({}^{\mathbf{Th}}(x_1,x_2,x_3)_3^{\lambda_1,\dots,\lambda_{2s+2}}x_3)^{\lambda_1}\,{}^{\mathbf{Th}}(x_1,x_2,x_3)^{\lambda_3,\dots,\lambda_{2s}}.
$$

According to the induction proposition and the third identity this is equal to

$$
\begin{aligned}
&({}^{\mathbf{Th}}((x_1x_2x_3)^{\lambda_{2s+1}}x_1,(x_2x_3x_1(x_1x_2x_3)^{\lambda_{2s+1}}x_1)^{\lambda_{2s+2}}x_2,x_1x_2x_3)_3^{\lambda_1,\dots,\lambda_{2s}}x_1x_2x_3)^{\lambda_1}\\
&\cdot{}^{\mathbf{Th}}((x_1x_2x_3)^{\lambda_{2s+1}}x_1,(x_2x_3x_1(x_1x_2x_3)^{\lambda_{2s+1}}x_1)^{\lambda_{2s+2}}x_2,x_1x_2x_3)_1^{\lambda_3,\dots,\lambda_{2s}}\\
&\qquad\cdot(x_1x_2x_3)^{\lambda_{2s+1}}.
\end{aligned}
$$

This is equal by definition to

$$
\begin{aligned}
&{}^{\mathbf{Th}}((x_1x_2x_3)^{\lambda_{2s+1}}x_1,(x_2x_3x_1(x_1x_2x_3)^{\lambda_{2s+1}}x_1)^{\lambda_{2s+2}}x_2,x_1x_2x_3)_1^{\lambda_1,\dots,\lambda_{2s}}\\
&\qquad\cdot(x_1x_2x_3)^{\lambda_{2s+1}}.
\end{aligned}
$$

Second identity. By definition

$$
\begin{aligned}
&{}^{\mathbf{Th}}(x_1,x_2,x_3)_2^{\lambda_1,\dots,\lambda_{2s+2}}\\
&\quad=({}^{\mathbf{Th}}(x_1,x_2,x_3)_2^{\lambda_3,\dots,\lambda_{2s+2}}x_2\,{}^{\mathbf{Th}}(x_1,x_2,x_3)_3^{\lambda_3,\dots,\lambda_{2s+2}}x_3\\
&\quad\cdot{}^{\mathbf{Th}}(x_1,x_2,x_3)_1^{\lambda_3,\dots,\lambda_{2s+2}}x_1\,{}^{\mathbf{Th}}(x_1,x_2,x_3)_1^{\lambda_1,\dots,\lambda_{2s+2}}x_1)^{\lambda_2}\\
&\quad\cdot{}^{\mathbf{Th}}(x_1,x_2,x_3)_2^{\lambda_3,\dots,\lambda_{2s+2}}.
\end{aligned}
$$

According to the induction proposition and the first identity, this is equal to

$$\begin{aligned}({}^{\mathbf{Th}}((x_1x_2x_3)^{\lambda_{2s+1}}x_1,(x_2x_3x_1(x_1x_2x_3)^{\lambda_{2s+1}}x_1)^{\lambda_{2s+2}}x_2,x_1x_2x_3)_2^{\lambda_3,\dots,\lambda_{2s}}\\ \cdot(x_2x_3x_1(x_1x_2x_3)^{\lambda_{2s+1}}x_1)^{\lambda_{2s+2}}x_2\\ \cdot{}^{\mathbf{Th}}((x_1x_2x_3)^{\lambda_{2s+1}}x_1,(x_2x_3x_1(x_1x_2x_3)^{\lambda_{2s+1}}x_1)^{\lambda_{2s+2}}x_2,x_1x_2x_3)_3^{\lambda_3,\dots,\lambda_{2s}}\\ \cdot x_1x_2x_3\\ \cdot{}^{\mathbf{Th}}((x_1x_2x_3)^{\lambda_{2s+1}}x_1,(x_2x_3x_1(x_1x_2x_3)^{\lambda_{2s+1}}x_1)^{\lambda_{2s+2}}x_2,x_1x_2x_3)_1^{\lambda_3,\dots,\lambda_{2s}}\\ \cdot(x_1x_2x_3)^{\lambda_{2s+1}}x_1\\ \cdot{}^{\mathbf{Th}}((x_1x_2x_3)^{\lambda_{2s+1}}x_1,(x_2x_3x_1(x_1x_2x_3)^{\lambda_{2s+1}}x_1)^{\lambda_{2s+2}}x_2,x_1x_2x_3)_1^{\lambda_1,\dots,\lambda_{2s}}\\ \cdot(x_1x_2x_3)^{\lambda_{2s+1}}x_1)^{\lambda_2}\\ \cdot{}^{\mathbf{Th}}((x_1x_2x_3)^{\lambda_{2s+1}}x_1,(x_2x_3x_1(x_1x_2x_3)^{\lambda_{2s+1}}x_1)^{\lambda_{2s+2}}x_2,x_1x_2x_3)_2^{\lambda_3,\dots,\lambda_{2s}}\\ \cdot(x_2x_3x_1(x_1x_2x_3)^{\lambda_{2s+1}}x_1)^{\lambda_{2s+2}}.\end{aligned}$$

This is equal by definition to

$$\begin{aligned}{}^{\mathbf{Th}}((x_1x_2x_3)^{\lambda_{2s+1}}x_1,(x_2x_3x_1(x_1x_2x_3)^{\lambda_{2s+1}}x_1)^{\lambda_{2s+2}}x_2,x_1x_2x_3)_2^{\lambda_1,\dots,\lambda_{2s}}\\ \cdot(x_2x_3x_1(x_1x_2x_3)^{\lambda_{2s+1}}x_1)^{\lambda_{2s+2}}.\end{aligned}$$

Fourth identity. By definition,

$$\begin{aligned}&{}^{\mathbf{Oc}}((x_1,x_2,x_3)^{\lambda_1,\dots,\lambda_{2s+2}}\\ &={}^{\mathbf{Oc}}(x_1,x_2,x_3)^{\lambda_3,\dots,\lambda_{2s+2}}({}^{\mathbf{Th}}(x_1,x_2,x_3)_3^{\lambda_3,\dots,\lambda_{2s+2}}x_3\,{}^{\mathbf{Th}}(x_1,x_2,x_3)_1^{\lambda_3,\dots,\lambda_{2s+2}}x_1\\ &\qquad\cdot{}^{\mathbf{Th}}(x_1,x_2,x_3)_2^{\lambda_3,\dots,\lambda_{2s+2}}x_2)^{\lambda_1}.\end{aligned}$$

According to the induction proposition and the first, second and third identities, this is equal to

$$\begin{aligned}&(x_3x_1x_2)^{\lambda_{2s+1}}\,{}^{\mathbf{Oc}}((x_1x_2x_3)^{\lambda_{2s+1}}x_1,(x_2x_3x_1(x_1x_2x_3)^{\lambda_{2s+1}}x_1)^{\lambda_{2s+2}}x_2,x_1x_2x_3)^{\lambda_3,\dots,\lambda_{2s}}\\ &\cdot({}^{\mathbf{Th}}((x_1x_2x_3)^{\lambda_{2s+1}}x_1,(x_2x_3x_1(x_1x_2x_3)^{\lambda_{2s+1}}x_1)^{\lambda_{2s+2}}x_2,x_1x_2x_3)_3^{\lambda_3,\dots,\lambda_{2s}}x_1x_2x_3\\ &\quad\cdot{}^{\mathbf{Th}}((x_1x_2x_3)^{\lambda_{2s+1}}x_1,(x_2x_3x_1(x_1x_2x_3)^{\lambda_{2s+1}}x_1)^{\lambda_{2s+2}}x_2,x_1x_2x_3)_1^{\lambda_3,\dots,\lambda_{2s}}\\ &\qquad\cdot(x_1x_2x_3)^{\lambda_{2s+1}}x_1\\ &\quad\cdot{}^{\mathbf{Th}}((x_1x_2x_3)^{\lambda_{2s+1}}x_1,(x_2x_3x_1(x_1x_2x_3)^{\lambda_{2s+1}}x_1)^{\lambda_{2s+2}}x_2,x_1x_2x_3)_2^{\lambda_3,\dots,\lambda_s}\\ &\qquad\cdot(x_2x_3x_1(x_1x_2x_3)^{\lambda_{2s+1}}x_1)^{\lambda_{2s+2}}x_2)^{\lambda_1}.\end{aligned}$$

This is equal by definition to

$$\begin{aligned}(x_3x_1x_2)^{\lambda_{2s+1}}\,{}^{\mathbf{Oc}}((x_1x_2x_3)^{\lambda_{2s+1}}x_1,(x_2x_3x_1(x_1x_2x_3)^{\lambda_{2s+1}}x_1)^{\lambda_{2s+2}}x_2,\\ x_1x_2x_3)^{\lambda_1,\dots,\lambda_{2s}}.\end{aligned}$$

Consider the equation

$$x_1x_2x_3x_4=x_3x_1^2x_2$$

and the sequence of joint parametric transformations (13):

$$(13.1)\qquad \begin{cases} x_1 \to (x_1x_2x_3)^{\lambda_1}x_1, \\ x_2 \to (x_2x_3x_1(x_1x_2x_3)^{\lambda_1}x_1)^{\lambda_2}x_2, \\ x_3 \to x_1x_2x_3, \\ x_4 \to x_4(x_3x_1x_2)^{\lambda_1}; \end{cases}$$

$$(13.2)\qquad \begin{cases} x_1 \to (x_1x_2x_3)^{\lambda_3}x_1, \\ x_2 \to (x_2x_3x_1(x_1x_2x_3)^{\lambda_3}x_1)^{\lambda_4}x_2, \\ x_3 \to x_1x_2x_3, \\ x_4 \to x_4(x_3x_1x_2)^{\lambda_3}; \end{cases}$$

. . . .

$$(13.s)\qquad \begin{cases} x_1 \to (x_1x_2x_3)^{\lambda_{2s-1}}x_1, \\ x_2 \to (x_2x_3x_1(x_1x_2x_3)^{\lambda_{2s-1}}x_1)^{\lambda_{2s}}x_2, \\ x_3 \to x_1x_2x_3, \\ x_4 \to x_4(x_3x_1x_2)^{\lambda_{2s-1}}; \end{cases}$$

$$(13.s+1)\qquad \begin{cases} x_1 \to (x_1x_2x_3)^{\lambda_{2s+1}}x_1, \\ x_2 \to (x_2x_3x_1(x_1x_2x_3)^{\lambda_{2s+1}}x_1)^{\lambda_{2s+2}}x_2, \\ x_3 \to x_1x_2x_3, \\ x_4 \to x_4(x_3x_1x_2)^{\lambda_{2s+1}}; \end{cases}$$

. . . .

Theorem $^{\mathbf{Th}}$2. *For every natural s the sequence of the s joint parametric transformations* (13) *can be collected by the following common transformation:*

$$(14)\qquad \begin{cases} x_1 \to {}^{\mathbf{Th}}(x_1,x_2,x_3)_1^{\lambda_1,\dots,\lambda_{2s}}x_1, \\ x_2 \to {}^{\mathbf{Th}}(x_1,x_2,x_3)_2^{\lambda_1,\dots,\lambda_{2s}}x_2, \\ x_3 \to {}^{\mathbf{Th}}(x_1,x_2,x_3)_3^{\lambda_1,\dots,\lambda_{2s}}x_3, \\ x_4 \to x_4{}^{\mathbf{Oc}}(x_1,x_2,x_3)^{\lambda_1,\dots,\lambda_{2s}}. \end{cases}$$

Proof. Suppose the sequence of the first s transformations can be collected by the common transformation (14), and let the $(s+1)$st transformation be of the form $(13.s+1)$. Then the sequence of the first $s+1$ transformations can be collected by the common transformation

$$\begin{cases} x_1 \to {}^{\mathbf{Th}}((x_1x_2x_3)^{\lambda_{2s+1}}x_1, (x_2x_3x_1(x_1x_2x_3)^{\lambda_{2s+1}}x_1)^{\lambda_{2s+2}}x_2, x_1x_2x_3)_1^{\lambda_1,\dots,\lambda_{2s}} \\ \qquad \cdot(x_1x_2x_3)^{\lambda_{2s+1}}x_1, \\ x_2 \to {}^{\mathbf{Th}}((x_1x_2x_3)^{\lambda_{2s+1}}x_1, (x_2x_3x_1(x_1x_2x_3)^{\lambda_{2s+1}}x_1)^{\lambda_{2s+2}}x_2, x_1x_2x_3)_2^{\lambda_1,\dots,\lambda_{2s}} \\ \qquad \cdot(x_2x_3x_1(x_1x_2x_3)^{\lambda_{2s+1}}x_1)^{\lambda_{2s+2}}x_2, \\ x_3 \to {}^{\mathbf{Th}}((x_1x_2x_3)^{\lambda_{2s+1}}x_1, (x_2x_3x_1(x_1x_2x_3)^{\lambda_{2s+1}}x_1)^{\lambda_{2s+2}}x_2, x_1x_2x_3)_3^{\lambda_1,\dots,\lambda_{2s}} \\ \qquad \cdot x_1x_2x_3, \\ x_4 \to x_4(x_3x_1x_2)^{\lambda_{2s+1}} \\ \qquad \cdot{}^{\mathbf{Oc}}((x_1x_2x_3)^{\lambda_{2s+1}}x_1, (x_2x_3x_1(x_1x_2x_3)^{\lambda_{2s+1}}x_1)^{\lambda_{2s+2}}x_2, x_1x_2x_3)^{\lambda_1,\dots,\lambda_{2s}}. \end{cases}$$

According to Theorem $^{\mathbf{Th}}1$ this transformation coincides with the transformation

$$\begin{cases} x_1 \to {}^{\mathbf{Th}}(x_1, x_2, x_3)_1^{\lambda_1,\ldots,\lambda_{2s+2}} x_1, \\ x_2 \to {}^{\mathbf{Th}}(x_1, x_2, x_3)_2^{\lambda_1,\ldots,\lambda_{2s+2}} x_2, \\ x_3 \to {}^{\mathbf{Th}}(x_1, x_2, x_3)_3^{\lambda_1,\ldots,\lambda_{2s+2}} x_3, \\ x_4 \to x_4{}^{\mathbf{Oc}}(x_1, x_2, x_3)^{\lambda_1,\ldots,\lambda_{2s+2}}. \end{cases}$$

Proposition 11. *The general solution of the equation*

11 $$x_2x_1x_3x_4 = x_3x_1^2(x_3x_1)^\alpha x_2$$

with $\partial(x_3) \le \partial(x_2) < \partial(x_3x_1^2(x_3x_1)^\alpha)$, where α is a natural parameter, is described by the transformations

t $$\begin{cases} x_1 \to x_1^{\eta+\theta}, \\ x_2 \to x_3x_1^\theta, \\ x_4 \to x_4(x_1^{\eta+\theta}x_3)^\alpha x_1^\theta, \end{cases}$$
$\langle \mathbf{5} \rangle$

tt $$\begin{cases} x_2 \to x_3x_1(x_1x_3)^\gamma x_2, \\ x_1 \to x_2x_1, \\ x_4 \to (x_1x_3x_2)^{\alpha-\gamma}x_1x_2(x_1x_3x_2)^\gamma, \end{cases}$$
$\langle \mathbf{2} \rangle$

ttt $$\begin{cases} x_2 \to x_3x_1(x_1x_3)^\delta x_1x_2, \\ x_3 \to x_2x_3, \\ x_4 \to (x_3x_1x_2)^{\alpha-\delta-1}x_3x_1^2x_2(x_3x_1x_2)^\delta, \end{cases}$$
$\langle \mathbf{2} \rangle$

where $\eta, \theta, \gamma, \delta$ are natural parameters with $\gamma \le \alpha$ and $\delta < \alpha$.

Proof. It is easy to see that **11** can be divided into the following collection of equations:

(j) **11** with $\partial(x_3) \le \partial(x_2) < \partial(x_3x_1)$,
(jj) **11** with $\partial(x_3x_1(x_1x_3)^\gamma) \le \partial(x_2) < \partial(x_3x_1(x_1x_3)^\gamma x_1)$,
(jjj) **11** with $\partial(x_3x_1(x_1x_3)^\delta x_1) \le \partial(x_2) < \partial(x_3x_1(x_1x_3)^{\delta+1})$,
where γ, δ are natural parameters with $\gamma \le \alpha$ and $\delta < \alpha$.

It is obvious that (j) is reduced by the parametric transformation

$$\begin{cases} x_2 \to x_3x_2, \\ x_1 \to x_2x_1, \\ x_4 \to x_4x_2(x_1x_3x_2)^\alpha \end{cases}$$

to the equation $x_2x_1x_3x_4 = x_1x_2x_1x_3$ with $\partial(x_2) < \partial(x_1)$. The last equation falls into the system

$$\begin{cases} x_2x_1 = x_1x_2, \\ x_3x_4 = x_1x_3 \end{cases}$$

with $\partial(x_2) < \partial(x_1)$. By Proposition 1 this system is reduced by the parametric transformation

$$\begin{cases} x_1 = x_1^\eta, \\ x_2 = x_1^\theta \end{cases}$$

to the equation $x_3x_4 = x_1^\eta x_3$ with $\theta > \eta$, that is, to the equation **5**.

The equation (jj) is reduced by the parametric transformation tt to the equation **2**.

The equation (jjj) is reduced by the parametric transformation ttt to the equation **2**.

Proposition 12*. *The parametric equation*

12* $$x_2x_1x_3x_4 = x_3x_1^2(x_3x_1)^{\alpha}x_2$$

with $\partial(x_2) < \partial(x_3)$*, where* α *is a natural parameter, is reduced by the transformation*

$$\begin{cases} x_3 \to x_2x_3, \\ x_4 \to x_4(x_3x_1x_2)^{\alpha} \end{cases}$$

to the parametric equation **12***.*

Proof. This is obvious.

Proposition 12. *The general solution of the equation*

12 $$x_1x_2x_3x_4 = x_3x_1^2x_2 \quad \textit{with } \partial(x_3) > 0$$

is described by the transformation

$$\mathrm{T} \quad \begin{cases} x_1 \to {}^{\mathbf{Th}}(x_1,x_2,x_3)_1^{\xi}x_1, \\ x_2 \to {}^{\mathbf{Th}}(x_1,x_2,x_3)_2^{\xi}x_2, \\ x_3 \to {}^{\mathbf{Th}}(x_1,x_2,x_3)_3^{\xi}x_3, \\ x_4 \to x_4{}^{\mathbf{Oc}}(x_1,x_2,x_3)^{\xi}, \end{cases}$$

$$\mathrm{t} \quad \begin{cases} x_1 \to (x_1x_3)^{\sigma}x_1, \\ x_2 \to (x_3x_1^2(x_3x_1)^{\sigma})^{\rho}x_2, \\ x_3 \to x_1x_3, \\ x_4 \to x_4, \end{cases}$$

$$\langle \mathbf{11} \rangle$$

where ξ *is a variable whose values are even sequences of natural parameters, and* σ, ρ *are natural parameters.*

Proof. It is easy to apply the transformation $\mathrm{Tt}\langle\mathbf{11}\rangle$ to the equation **12** and to verify, by using the definition of the functions ${}^{\mathbf{Th}}(x_1,x_2,x_3)_i^{\lambda_1,\dots,\lambda_{2s}}$, that it is a parametric solution of **12**.

According to Proposition 8 the equation **12** is reduced by the parametric transformation $x_1 \to x_3^{\sigma}x_1$, where σ is natural parameter, to the equation E_1:

$$x_1x_2x_3x_4 = x_3x_1x_3^{\sigma}x_1x_2 \quad \text{with } \partial(x_1) \le \partial(x_3).$$

According to condition $\partial(x_1) < \partial(x_3)$ the equation E_1 is reduced by the parametric transformation $x_3 \to x_1x_3$ to the equation E_2:

$$x_2x_1x_3x_4 = x_3x_1(x_1x_3)^{\sigma}x_1x_2.$$

According to Proposition 8 the equation E_2 is reduced by the parametric transformation

$$x_2 \to (x_3x_1(x_1x_3)^{\sigma}x_1)^{\rho}x_2,$$

where ρ is a natural parameter, to the equation E_3:

$$x_2x_1x_3x_4 = x_3x_1^2(x_3x_1)^{\sigma}x_2 \quad \text{with } \partial(x_2) < \partial(x_3x_1^2(x_3x_1)^{\sigma}).$$

It is easy to see that E_3 can be divided into the following collection of equations:

(j) E_3 with $\partial(x_3) \leq \partial(x_2)$,
(jj) E_3 with $\partial(x_3) > \partial(x_2)$.
The equation (j) is equation **11**.
According to Proposition 12* the equation (jj) is reduced by the transformation

$$\begin{cases} x_3 \to x_2 x_3, \\ x_4 \to x_4 (x_3 x_1 x_2)^{\sigma} \end{cases}$$

to the equation **12**.
The sequence of transformations

$$\begin{aligned} & x_1 \to x_3^{\sigma} x_1, \\ & x_3 \to x_1 x_3, \\ & x_2 \to (x_3 x_1 (x_1 x_3)^{\sigma} x_1)^{\rho} x_2, \\ & x_3 \to x_2 x_3, \\ & x_4 \to x_4 (x_3 x_1 x_2)^{\alpha} \end{aligned}$$

can be collected by the following common transformation r:

$$\begin{cases} x_1 \to (x_1 x_2 x_3)^{\sigma} x_1, \\ x_2 \to (x_2 x_3 x_1 (x_1 x_2 x_3)^{\sigma} x_1)^{\rho} x_2, \\ x_3 \to x_1 x_2 x_3, \\ x_4 \to x_4 (x_3 x_1 x_2)^{\alpha}. \end{cases}$$

Using Theorem $^{\mathbf{Th}}2$, one can see that transformation rT can be obtained from T by replacing the parameter ξ by the sequence σ, ρ, ξ.

6. Supplement

Proposition 13. *The general solution of the equation*

13 $$x_1 x_2 x_3 x_4 = x_3 x_2 (x_2 (x_3 x_2)^{\lambda+1})^{\kappa+1} x_1$$

with $\partial(x_1) \leq \partial(x_3 x_2 (x_2 (x_3 x_2)^{\lambda+1})^{\kappa+1})$, *where* λ *and* κ *are natural parameters, is described by the transformations*

$$\text{t1} \quad \begin{cases} x_3 \to x_1 x_3, \\ x_4 \to x_4 x_3 (x_2 x_1 x_3)^{\lambda} x_2 ((x_2 x_1 x_3)^{\lambda+1} x_2)^{\kappa} x_1, \end{cases}$$

$\langle \mathbf{12} \rangle$,

$$\text{t2} \quad \begin{cases} x_1 \to x_3 x_1^{\eta}, \\ x_2 \to x_1^{\eta+\theta}, \\ x_4 \to x_4 (x_1^{\eta+\theta} x_3)^{\lambda} x_1^{\eta+\theta} ((x_1^{\eta+\theta} x_3)^{\lambda+1} x_1^{\eta+\theta})^{\kappa} x_3 x_1^{\eta}, \end{cases}$$

$\langle \mathbf{5} \rangle$,

$$\text{t3} \quad \begin{cases} x_1 \to x_3 x_1 x_2 (x_1 x_2 (x_3 x_1 x_2)^{\lambda+1})^{\tau} (x_1 x_2 x_3)^{\sigma} x_1, \\ x_2 \to x_1 x_2, \\ x_4 \to x_2 (x_3 x_1 x_2)^{\lambda-\sigma} (x_1 x_2 (x_3 x_1 x_2)^{\lambda+1})^{\kappa-\tau} x_3 x_1 x_2 \\ \qquad \cdot (x_1 x_2 (x_3 x_1 x_2)^{\lambda+1})^{\tau} (x_1 x_2 x_3)^{\sigma} x_1, \end{cases}$$

$\langle \mathbf{2} \rangle$,

$$\text{t4}\quad \begin{cases} x_1 \to x_3 x_1^{\eta+\theta}((x_1^{\eta+\theta}x_3)^{\lambda+1}x_1^{\eta+\theta})^{\tau}(x_1^{\eta+\theta}x_3)^{\lambda+1}x_1^{\eta}, \\ x_2 \to x_1^{\eta+\theta}, \\ x_4 \to x_4 x_1^{\eta+\theta}(x_3 x_1^{\eta+\theta})^{\lambda}((x_1^{\eta+\theta}x_3)^{\lambda+1}x_1^{\eta+\theta})^{\kappa-\tau-1}x_3 \\ \quad \cdot x_1^{\eta+\theta}(x_1^{\eta+\theta}x_3)^{\lambda+1}x_1^{\eta+\theta})^{\tau}(x_1^{\eta+\theta}x_3)^{\lambda+1}x_1^{\eta}, \end{cases}$$

$\langle \mathbf{5} \rangle$,

$$\text{t5}\quad \begin{cases} x_1 \to x_1 x_3 (x_2 x_1)^{\beta+1} x_2(((x_2x_1)^{\beta+1}x_2x_1x_3)^{\lambda+1}(x_2x_1)^{\beta+1}x_2)^{\kappa} \\ \quad \cdot((x_2x_1)^{\beta+1}x_2x_1x_3)^{\lambda+1}x_2x_1, \\ x_2 \to (x_2x_1)^{\beta+1}x_2, \\ x_3 \to x_1x_3, \\ x_4 \to (x_2x_1)^{\beta}x_2(((x_2x_1)^{\beta+1}x_2x_1x_3)^{\lambda+1}(x_2x_1)^{\beta+1}x_2)^{\kappa} \\ \quad \cdot((x_2x_1)^{\beta+1}x_2x_1x_3)^{\lambda+1}x_2x_1, \end{cases}$$

$\langle \mathbf{2} \rangle$,

$$\text{t6}\quad \begin{cases} x_1 \to x_3(x_2x_3x_1)^{\beta+1}x_2(((x_2x_3x_1)^{\beta+1}x_2x_3)^{\lambda+1}(x_2x_3x_1)^{\beta+1}x_2)^{\kappa} \\ \quad \cdot((x_2x_3x_1)^{\beta+1}x_2x_3)^{\lambda+1}x_2x_3x_1, \\ x_2 \to (x_2x_3x_1)^{\beta+1}x_2, \\ x_4 \to (x_2x_3x_1)^{\beta}x_2(((x_2x_3x_1)^{\beta+1}x_2x_3)^{\lambda+1}(x_2x_3x_1)^{\beta+1}x_2)^{\kappa} \\ \quad \cdot((x_2x_3x_1)^{\beta+1}x_2x_3)^{\lambda+1}x_2x_3x_1, \end{cases}$$

$\langle \mathbf{2} \rangle$,

$$\text{t7}\quad \begin{cases} x_1 \to x_1x_3x_2((x_2x_1x_3)^{\lambda+1}x_2)^{\tau}(x_2x_1x_3)^{\sigma}x_2x_1, \\ x_3 \to x_1x_3, \\ x_4 \to x_3x_2(x_1x_3x_2)^{\lambda-\sigma-1}((x_2x_1x_3)^{\lambda+1}x_2)^{\kappa-\tau}x_1x_3x_2 \\ \quad \cdot((x_2x_1x_3)^{\lambda+1}x_2)^{\tau}(x_2x_1x_3)^{\sigma}x_2x_1, \end{cases}$$

$\langle \mathbf{2} \rangle$

$$\text{t8}\quad \begin{cases} x_1 \to x_1x_3x_2((x_2x_1x_3)^{\lambda+1}x_2)^{\tau}(x_2x_1x_3)^{\lambda}x_2x_1, \\ x_3 \to x_1x_3, \\ x_4 \to x_4x_3(x_2x_1x_3)^{\lambda}x_2((x_2x_1x_3)^{\lambda+1}x_2)^{\kappa-\tau-1}x_1x_3x_2 \\ \quad \cdot((x_3 \to x_1x_3x_2x_1x_3)^{\lambda+1}x_2)^{\tau}(x_2x_1x_3)^{\lambda}x_2x_1, \end{cases}$$

$\langle \mathbf{12} \rangle$,

$$\text{t9}\quad \begin{cases} x_1 \to x_1x_3x_2((x_2x_1x_3)^{\lambda+1}x_2)^{\kappa}(x_2x_1x_3)^{\lambda}x_2x_1, \\ x_3 \to x_1x_3, \\ x_4 \to x_3x_2((x_2x_1x_3)^{\lambda+1}x_2)^{\kappa}(x_2x_1x_3)^{\lambda}x_2x_1, \end{cases}$$

$\langle \mathbf{2} \rangle$,

where $\eta, \theta, \sigma, \tau, \beta$ are natural parameters.

Proof. The equation **13** can be divided into the collection of equations
(j) **13** with $\partial(x_1) < \partial(x_3)$;
(jj) **13** with $\partial(x_3) \le \partial(x_1) \le \partial(x_3x_2)$;

(jjj) **13** with

$$\begin{aligned}&\partial(x_3x_2((x_2x_3)^{\lambda+1}x_2)^\tau({}_2x_3)^\sigma) \le \partial(x_1)\\&\quad\le \partial(x_3x_2((x_2x_3)^{\lambda+1}x_2)^\tau(x_2x_3)^\sigma x_2), \qquad \tau\le\kappa,\ \sigma\le\lambda+1;\end{aligned}$$

(jjjj) **13** with

$$\begin{aligned}&\partial(x_3x_2((x_2x_3)^{\lambda+1}x_2)^\tau(x_2x_3)^\sigma x_2) \le \partial(x_1)\\&\quad< \partial(x_3x_2((x_2x_3)^{\lambda+1}x_2)^\tau(x_2x_3)^{\sigma+1}), \qquad \tau\le\kappa,\ \sigma\le\lambda.\end{aligned}$$

The equation (j) can be reduced by the transformation

$$\begin{aligned}&x_3 \to x_1x_3,\\&x_4 \to x_4x_3(x_2x_1x_3)^\lambda x_2((x_2(x_1x_3x_2)^{\lambda+1})^\kappa x_1\end{aligned}$$

to the equation $x_2x_1x_3x_4 = x_3x_2^2x_1$ with $\partial(x_3) > 0$, that is, to the equation **12**.

The equation (jj) can be reduced by the transformation

$$\begin{aligned}&x_1 \to x_3x_1,\\&x_2 \to x_1x_2,\\&\begin{cases}x_1 \to x_1^\eta,\\ x_2 \to x_1^\theta,\end{cases}\\&x_4 \to\ x_4(x_1^{\eta+\theta}x_3)^\lambda x_1^{\eta+\theta}((x_1^{\eta+\theta}x_3)^{\lambda+1}x_1^{\eta+\theta})^\kappa x_3x_1^\eta\end{aligned}$$

to the equation $x_3x_4 = x_1^\theta x_3$, that is, to the equation **5**.

The equation (jjj) can be reduced by the transformation

$$\begin{aligned}&x_1 \to x_3x_2((x_2x_3)^{\lambda+1}x_2)^\tau(x_2x_3)^\sigma x_1,\\&x_2 \to x_1x_2\end{aligned}$$

to the equation E:

$$\begin{aligned}x_1x_2x_3x_4 &= x_2(x_3x_1x_2)^{\lambda+1-\sigma}((x_1x_2x_3)^{\lambda+1}x_1x_2)^{\kappa-\tau}x_3x_1x_2\\&\quad\cdot((x_1x_2x_3)^{\lambda+1}x_1x_2)^\tau(x_1x_2x_3)^\sigma x_1.\end{aligned}$$

The equation E with $\lambda+1>\sigma$ can be reduced by the transformation

$$\begin{aligned}x_4 &\to x_2(x_3x_1x_2)^{\lambda-\sigma}((x_1x_2x_3)^{\lambda+1}x_1x_2)^{\kappa-\tau}x_3x_1x_2\\&\quad\cdot((x_1x_2x_3)^{\lambda+1}x_1x_2)^\tau(x_1x_2x_3)^\sigma x_1\end{aligned}$$

to the equation $x_1x_2x_3 = x_2x_3x_1$, that is, to the equation **2**.

The equation E with $\lambda+1=\sigma$ and $\tau<\kappa$ can be reduced by the transformation

$$\begin{aligned}&\begin{cases}x_1 \to x_1^\eta,\\ x_2 \to x_1^\theta,\end{cases}\\&x_4 \to x_4x_1^{\eta+\theta}(x_3x_1^{\eta+\theta})^\lambda((x_1^{\eta+\theta}x_3)^{\lambda+1}x_1^{\eta+\theta})^{\kappa-\tau-1}\\&\qquad\cdot x_3x_1^{\eta+\theta}(x_1^{\eta+\theta}x_3)^{\lambda+1}x_1^{\eta+\theta})^\tau(x_1^{\eta+\theta}x_3)^{\lambda+1}x_1^\eta\end{aligned}$$

to the equation $x_3x_4 = x_1^\theta x_3$, that is, to the equation **5**.

The equation E with $\lambda+1=\sigma$ and $\tau=\kappa$ can be reduced by the transformation

$$x_2 \to x_1^\beta x_2,$$
$$x_1 \to x_2 x_1,$$
$$x_4 \to (x_2x_1)^\beta x_2(((x_2x_1)^{\beta+1}x_2x_3)^{\lambda+1}(x_2x_1)^{\beta+1}x_2)^\tau)(x_2x_1)^{\beta+1}x_2x_3)^\sigma x_2x_1,$$

then both $x_1 \to x_3x_1$ and $x_3 \to x_1x_3$, to the equation **2**.

The equation (jjjj) can be reduced by the transformation

$$x_1 \to x_3x_2((x_2x_3)^{\lambda+1}x_2)^\tau(x_2x_3)^\sigma x_2x_1,$$
$$x_3 \to x_1x_3$$

to the equation F:

$$x_2x_1x_3x_4 = x_3x_2(x_1x_3x_2)^{\lambda-\sigma}((x_2x_1x_3)^{\lambda+1}x_2)^{\kappa-\tau}x_1x_3x_2 \cdot((x_2x_1x_3)^{\lambda+1}x_2)^\tau(x_2x_1x_3)^\sigma x_2x_1 \quad \text{with } \partial(x_3)>0.$$

The equation F with $\lambda>\sigma$ can be reduced by the transformation

$$x_4 = x_3x_2(x_1x_3x_2)^{\lambda-\sigma-1}((x_2x_1x_3)^{\lambda+1}x_2)^{\kappa-\tau}x_1x_3x_2 \cdot((x_2x_1x_3)^{\lambda+1}x_2)^\tau(x_2x_1x_3)^\sigma x_2x_1$$

to the equation $x_1x_2x_3 = x_3x_1x_2$, that is, to the equation **2** with $\partial(x_3)>0$. On the other hand, t7 is the parametric solution of **13**.

The equation F with $\lambda=\sigma$ and $\kappa>\tau$ can be reduced by the transformation

$$x_4 \to x_4x_3(x_2x_1x_3)^\lambda x_2((x_2x_1x_3)^{\lambda+1}x_2)^{\kappa-\tau-1}x_1x_3x_2 \cdot((x_2x_1x_3)^{\lambda+1}x_2)^\tau(x_2x_1x_3)^\lambda x_2x_1$$

to the equation $x_2x_1x_3x_4 = x_3x_2^2x_1$ with $\partial(x_3)>0$, that is, to the equation **12**.

The equation F with $\lambda=\sigma$ and $\kappa=\tau$ can be reduced by the transformation

$$x_4 \to x_3x_2((x_2x_1x_3)^{\lambda+1}x_2)^\kappa(x_2x_1x_3)^\lambda x_2x_1$$

to the equation $x_2x_1x_3 = x_3x_2x_1$ with $\partial(x_3)>0$, that is, to the equation **2** with $\partial(x_3)>0$. On the other hand, t9 is the parametric solution of **13**.

Proposition 14. *The general solution of the equation*

14 $$x_1x_2x_3x_4 = x_3x_2(x_2(x_3x_2)^{\lambda+1})^{\tau+1}x_1$$

where λ and τ are natural parameters, is described by the transformations

$$\underset{\langle \mathbf{13}\rangle}{x_1 \to (x_3x_2(x_2(x_3x_2)^{\lambda+1})^{\tau+1})^\rho x_1,} \qquad \begin{cases} x_2 \to 1, \\ x_3 \to 1, \\ x_4 \to 1, \end{cases}$$

where ρ is a natural parameter.

Proof. This follows directly from Propositions 8 and 13.

Proposition 15. *The general solution of the equation*

15 $$x_1x_2x_3x_4 = (x_3x_1)^{\tau+2}x_2,$$

where τ is a natural parameter, is described by the transformations

$$x_3 \to 1, \quad \langle \mathbf{5} \rangle, \qquad \begin{cases} x_1 \to (x_1x_3)^{\alpha}x_1, \\ x_3 \to x_1x_3, \end{cases} \quad \langle \mathbf{14} \rangle,$$

where α is a natural parameter.

Proof. The equation **15** can be divided into the collection of the equations **15** with $\partial(x_3) = 0$ and **15** with $\partial(x_3) > 0$. In the second case we use Proposition 8.

Proposition 16. *The general solution of the equation*

16 $$x_1x_2x_3x_4 = x_3x_2^{\tau+2}x_1$$

with $\partial(x_1) < \partial(x_3x_2^{\tau+2})$ and $\partial(x_3) < \partial(x_1x_2)$, where τ is a natural parameter, is described by the transformations

$$\begin{cases} x_1 \to x_3(x_1x_2)^{\nu}x_1, \\ x_2 \to x_1x_2, \\ x_4 \to x_4(x_1x_2)^{\nu}x_1, \end{cases} \qquad \begin{cases} x_1 \to x_3(x_1x_2)^{\tau+1}x_1, \\ x_2 \to x_1x_2, \\ x_4 \to (x_2x_1)^{\tau+1}, \end{cases} \quad \langle \mathbf{2} \rangle$$

$$\begin{cases} x_1 \to x_1^{\eta}, \\ x_2 \to x_1^{\theta}, \end{cases} \quad \langle \mathbf{5} \rangle$$

$$\begin{cases} x_3 \to x_1x_3, \\ x_2 \to x_3x_2, \end{cases} \quad \langle \mathbf{15} \rangle,$$

where ν, η, θ are natural parameters with $\nu \le \tau$.

Proof. The equation **16** can be divided into the collection of equations

(j) **16** with $\partial(x_3) \le \partial(x_1)$,

(jj) **16** with $\partial(x_1) \le \partial(x_3)$.

The equation (j) can be reduced by the transformation $x_1 \to x_3x_1$ to the equation E_1:

$$x_1x_2x_3x_4 = x_2^{\tau+2}x_3x_1 \quad \text{with } \partial(x_1) < \partial(x_2^{\tau+2}).$$

The equation E_1 can be reduced by the transformation

$$x_1 \to x_2^{\nu}x_1 \qquad (\nu < \tau + 2),$$
$$x_2 \to x_1x_2$$

to the equation E_2:

$$x_1x_2x_3x_4 = x_2(x_1x_2)^{\tau+1-\nu}x_3(x_1x_2)^{\tau}x_1.$$

If $\tau + 1 > \nu$, then the equation E_2 falls into the system E_3:

$$\begin{cases} x_1x_2 = x_2x_1, \\ x_3x_4 = x_2(x_1x_2)^{\tau-\nu}x_3(x_1x_2)^{\nu}x_1. \end{cases}$$

The system E_3 can be reduced by the transformation

$$x_4 \to x_4(x_1x_2)^{\tau}x_1$$

to the system E_4:

$$\begin{cases} x_1x_2 = x_2x_1, \\ x_3x_4 = x_2(x_1x_2)^{\tau-\nu}x_3. \end{cases}$$

By Proposition 1 the system E_4 is reduced by the parametric transformation

$$\begin{cases} x_1 \to x_1^{\eta}, \\ x_2 \to x_1^{\theta} \end{cases}$$

to the equation

$$x_3x_4 = x_1^{\theta+(\nu+\theta)(\tau-\nu)}x_3,$$

that is, to the equation **5**.

If $\tau + 1 = \nu$, then the equation E_2 can be reduced by the transformation $x_4 \to (x_2x_1)^{\nu}$ to the equation $x_1x_2x_3 = x_2x_3x_1$, that is, to the equation **2**.

The equation (jj) can be reduced by the transformation $x_3 \to x_1x_3$ to the equation E_5:

$$x_2x_1x_3x_4 = x_3x_2^{\tau+2}x_1 \quad \text{with } \partial(x_3) < \partial(x_2).$$

The equation E_5 can be reduced by the transformation $x_2 \to x_3x_2$ to the equation

$$x_2x_1x_3x_4 = (x_3x_2)^{\tau+2}x_1,$$

that is the equation **15**.

Proposition 17. *The general solution of the equation*

17 $$x_1x_2x_3x_4 = x_3x_2^{\tau+2}x_1,$$

where τ is a natural parameter, is described by the transformation

$$\begin{cases} x_1 \to {}^{\mathbf{Fi}}(x_1x_2, x_3x_2^{\tau+2})^{\mu}x_1, \\ x_3 \to {}^{\mathbf{Fi}}(x_3x_2^{\tau+2}, x_1x_2)^{\mu|}x_3, \end{cases}$$

where μ is a variable whose values are sequences of natural parameters, followed by one of the three transformations

$$\begin{cases} x_2 \to 1, \\ x_3 \to 1, \\ x_4 \to 1, \end{cases} \qquad \begin{cases} x_1 \to 1, \\ x_2 \to 1, \\ x_4 \to 1, \end{cases} \qquad \langle \mathbf{16} \rangle.$$

Proof. This follows directly from Propositions 9 and 16.

Proposition 18. *The general solution of the equation*

18 $$x_1x_2x_3x_4 = x_2^{\sigma+3}x_3x_1,$$

where σ is a natural parameter, is described by the transformations

$$\underset{\langle \mathbf{17} \rangle,}{x_1 \to x_2^{\sigma+3}x_1,} \qquad \underset{\langle \mathbf{2} \rangle,}{\begin{cases} x_1 \to (x_1x_2)^{\sigma+2}x_1, \\ x_2 \to x_1x_2, \\ x_4 \to (x_2x_1)^{\sigma+2}, \end{cases}} \qquad \begin{cases} x_1 \to (x_1x_2)^{\nu}x_1, \\ x_2 \to x_1x_2, \\ x_4 \to x_4(x_1x_2)^{\nu}x_1, \end{cases} \begin{cases} x_1 \to x_1^{\eta}, \\ x_2 \to x_1^{\theta}, \end{cases} \langle \mathbf{5} \rangle,$$

where ν, η, θ are natural parameters with $\nu \leq \sigma + 1$.

Proof. The equation **18** can be divided into the collection of equations
(j) **18** with $\partial(x_2^{\sigma+3}) \leq \partial(x_1)$,
(jj) **18** with $\partial(x_2^{\sigma+2}) \leq \partial(x_1) < \partial(x_2^{\sigma+3})$,
(jjj) **18** with $\partial(x_2^{\nu}) \leq \partial(x_1) < \partial(x_2^{\nu+1}),\ \nu < \sigma + 2$.

The equation (j) can be reduced by the transformation $x_1 \to x_2^{\sigma+3}x_1$ to the equation $x_1x_2x_3x_4 = x_3x_2^{\sigma+3}x_1$, that is, to **17** with $\tau > 0$.

The equation (jj) can be reduced by the transformation $x_1 \to x_2^{\sigma+2}x_1$, $x_2 \to x_1x_2$, $x_4 \to (x_2x_1)^{\sigma+2}$ to the equation $x_1x_2x_3 = x_2x_3x_1$, that is, to **2**.

The equation (jjj) can be reduced by the transformation $x_1 \to x_2^{\nu}x_1$ with $\nu < \sigma + 2$, $x_2 \to x_1x_2$, $x_4 \to x_4(x_1x_2)^{\nu}x_1$ to the equation E:

$$x_1x_2x_3x_4 = x_2x_1x_2(x_1x_2)^{\sigma+1-\nu}x_3.$$

By Proposition 1 the equation E is reduced by the parametric transformation

$$\begin{cases} x_1 \to x_1^{\eta}, \\ x_2 \to x_1^{\theta} \end{cases}$$

to the equation

$$x_3x_4 = x_1^{\theta+(\eta+\theta)(\sigma+1-\nu)}x_3,$$

that is, to **5**.

Proposition 19. *The general solution of the equation*

19 $$x_1x_2x_3x_4 = x_3x_1^{\sigma+3}x_2 \quad \text{with } \partial(x_3) < \partial(x_1x_2),$$

where σ is a natural parameter, is described by the transformations

$$\underset{\langle \mathbf{15} \rangle}{x_1 \to x_3x_1,}, \qquad \underset{\langle \mathbf{18} \rangle.}{\begin{cases} x_3 \to x_1x_3, \\ x_2 \to x_3x_2, \end{cases}}$$

Proof. The equation **19** can be divided into the collection of equations
(j) **19** with $\partial(x_3) \leq \partial(x_1)$,
(jj) **19** with $\partial(x_3) \geq \partial(x_1)$.

The equation (j) can be reduced by the transformation $x_1 \to x_3x_1$ to the equation $x_1x_2x_3x_4 = (x_3x_1)^{\sigma+3}x_2$, that is, to **15** with $\tau > 0$.

The equation (jj) can be reduced by the transformation $x_3 \to x_1x_3$, $x_2 \to x_3x_2$ to the equation $x_2x_1x_3x_4 = x_1^{\sigma+3}x_3x_2$, that is, to **18**.

Proposition 20. *The general solution of the equation*

20 $$x_1x_2x_3x_4 = x_3x_1^{\tau+3}x_2,$$

where τ is a natural parameter, is described by the transformations

$$\begin{cases} x_1 \to 1, \\ x_2 \to 1, \\ x_4 \to 1, \end{cases} \qquad \underset{\langle \mathbf{19} \rangle,}{x_3 \to (x_1x_2)^{\alpha}x_3,}$$

where α is a natural parameter.

Proof. The equation **20** can be divided into the collection of equations **20** with $\partial(x_1x_2) = 0$ and **20** with $\partial(x_1x_2) > 0$. In the second case we use Proposition 8.

7. The function ${}^{\mathbf{Ro}}(x_1, x_2, x_3)_i^{\mu_1,\dots,\mu_t}$

Consider the equation

$$x_1x_2x_3x_4 = x_2x_3^2x_1$$

and a sequence of transformations (15):

$$(15.1)\qquad \begin{cases} x_1 \to {}^{\mathbf{Fi}}(x_3x_2x_1, x_3x_2(x_1x_3)^2)^{\mu_1}x_3x_2x_1, \\ x_2 \to {}^{\mathbf{Fi}}(x_3x_2(x_1x_3)^2, x3x_2x_1)^{\mu_1|}x_3x_2, \\ x_3 \to x_1x_3, \\ x_4 \to x_4x_1, \end{cases}$$

$$(15.2)\qquad \begin{cases} x_1 \to {}^{\mathbf{Fi}}(x_3x_1(x_2x_3)^2, x_3x_1x_2)^{\mu_2|}x_3x_1, \\ x_2 \to {}^{\mathbf{Fi}}(x_3x_1x_2, x_3x_1(x_2x_3)^2)^{\mu_2}x_3x_1x_2, \\ x_3 \to x_2x_3, \\ x_4 \to x_4x_2, \end{cases}$$

. . . .

$$(15.2k+1)\qquad \begin{cases} x_1 \to {}^{\mathbf{Fi}}(x_3x_2x_1, x_3x_2(x_1x_3)^2)^{\mu_{2k+1}}x_3x_2x_1, \\ x_2 \to {}^{\mathbf{Fi}}(x_3x_2(x_1x_3)^2, x_3x_2x_1)^{\mu_{2k+1}|}x_3x_2, \\ x_3 \to x_1x_3, \\ x_4 \to x_4x_1, \end{cases}$$

$$(15.2k+2)\qquad \begin{cases} x_1 \to {}^{\mathbf{Fi}}(x_3x_1(x_2x_3)^2, x_3x_1x_2)^{\mu_{2k+2}|}x_3x_1, \\ x_2 \to {}^{\mathbf{Fi}}(x_3x_1x_2, x_3x_1(x_2x_3)^2)^{\mu_{2k+2}}x_3x_1x_2, \\ x_3 \to x_2x_3, \\ x_4 \to x_4x_2, \end{cases}$$

. . . .

Theorem ${}^{\mathbf{Ro}}$1. *For every natural t the sequence of the t joint parametric transformations* (15) *can be collected by the following common transformation:*

$$(16)\qquad \begin{cases} x_1 \to {}^{\mathbf{Ro}}(x_1, x_2, x_3)_1^{\mu_1,\dots,\mu_t}x_1, \\ x_2 \to {}^{\mathbf{Ro}}(x_1, x_2, x_3)_2^{\mu_1,\dots,\mu_t}x_2, \\ x_3 \to {}^{\mathbf{Ro}}(x_1, x_2, x_3)_3^{\mu_1,\dots,\mu_t}x_3, \\ x_4 \to x_4{}^{\mathbf{Re}}(x_1, x_2, x_3)^{\mu_1,\dots,\mu_t}. \end{cases}$$

Proof. If $t = 0$, it is obvious. Let $t = 1$. By the definition of ${}^{\mathbf{Ro}}(x_1, x_2, x_3)_i^{\mu_1,\dots,\mu_t}$ we have

$${}^{\mathbf{Ro}}(x_1, x_2, x_3)_1^{\mu_1} = {}^{\mathbf{Fi}}(x_3x_2x_1, x_3x_2(x_1x_3)^2)^{\lambda_1,\dots,\lambda_s}x_3x_2,$$

$${}^{\mathbf{Ro}}(x_1, x_2, x_3)_2^{\mu_1} = {}^{\mathbf{Fi}}(x_3x_2(x_1x_3)^2, x_3x_2x_1)^{\lambda_2,\dots,\lambda_s}x_3,$$

$${}^{\mathbf{Ro}}(x_1, x_2, x_3)_3^{\mu_1} = x_1,$$

$${}^{\mathbf{Re}}(x_1, x_2, x_3)^{\mu_1} = x_1,$$

and the transformation (16) coincides with (15.1).

Let $t > 1$. By the inductive proposition the sequence of transformations (15.2), $\dots,$ (15.t) can be collected into the transformation

$$(17)\qquad \begin{cases} x_1 \to {}^{\mathbf{Ro}}(x_2, x_1, x_3)_2^{\mu_2,\dots,\mu_t} x_1, \\ x_2 \to {}^{\mathbf{Ro}}(x_2, x_1, x_3)_1^{\mu_2,\dots,\mu_t} x_2, \\ x_3 \to {}^{\mathbf{Ro}}(x_2, x_1, x_3)_3^{\mu_2,\dots,\mu_t} x_3, \\ x_4 \to x_4{}^{\mathbf{Re}}(x_2, x_1, x_3)^{\mu_2,\dots,\mu_t}. \end{cases}$$

Substituting (17) into (15.1), we obtain (16).

Proposition 21. *The general solution of the equation*

21 $$x_1x_2x_3x_4 = x_2x_3^2x_1$$

is described by the transformations

$$\begin{cases} x_1 \to {}^{\mathbf{Fi}}(x_1, x_2x_3^2)^{\mu} x_1, \\ x_2 \to {}^{\mathbf{Fi}}(x_2x_3^2, x_1)^{\mu|} x_2, \end{cases}$$

where μ is a variable for sequences of natural parameters, followed by one of the three transformations

$$\begin{cases} x_2 \to 1, \\ x_3 \to 1, \\ x_4 \to 1, \end{cases} \qquad \begin{cases} x_1 \to 1, \\ x_4 \to x_3, \end{cases} \qquad \langle \mathbf{22} \rangle.$$

Proof. This follows directly from Propositions 9 and 22.

Proposition 22. *The general solution of the equation*

22 $$x_1x_2x_3x_4 = x_2x_3^2x_1 \quad \textit{with } \partial(x_2) < \partial(x_1) < \partial(x_2x_3^2)$$

is described by the transformations $\langle \mathbf{23} \rangle$, $\langle \mathbf{24} \rangle$, $\langle \mathbf{25} \rangle$.

Proof. The equation **22** can be divided into the collection of equations:
(j) **22** with $\partial(x_1) \le \partial(x_3)$,
(jj) **22** with $\partial(x_2x_3) \le \partial(x_1)$,
(jjj) **22** with $\partial(x_3) \le \partial(x_1) \le \partial(x_2x_3)$.
By definition (j) is **23**, (jj) is **24**, (jjj) is **25**.

Proposition 23. *The general solution of the equation*

23 $$x_1x_2x_3x_4 = x_2x_3^2x_1,$$

with $\partial(x_2) < \partial(x_1) < \partial(x_2x_3^2)$ and $\partial(x_1) \le \partial(x_3)$, is described by the transformation

$$\text{t} \quad \begin{cases} x_1 \to x_2x_1, \\ x_3 \to x_1x_2x_3, \\ x_4 \to x_3x_2x_1, \end{cases}$$
$$\langle \mathbf{2} \rangle.$$

Proof. The equation **23** can be reduced by the transformation $x_1 \to x_2x_1$ to the equation E_1:

$$x_1x_2x_3x_4 = x_3^2x_2x_1 \quad \text{with } 0 < \partial(x_1) < \partial(x_3^2), \partial(x_2x_1) \le \partial(x_3).$$

The equation E_1 can be reduced by the transformation $x_3 \to x_1x_2x_3$ to the equation E_2:

$$x_1x_2x_3x_4 = x_3x_1x_2x_3x_2x_1 \quad \text{with } \partial(x_1) > 0.$$

The equation E_2 can be reduced by the transformation $x_4 \to x_3x_2x_1$ to the equation $x_1x_2x_3 = x_3x_1x_2$ with $\partial(x_1) > 0$, that is, to the equation **2** with $\partial(x_1) > 0$.

On the other hand, the transformation t is a parametric solution of **23**.

Proposition 24. *The general solution of the equation*

24 $$x_1x_2x_3x_4 = x_2x_3^2x_1,$$

with $\partial(x_2) < \partial(x_1) < \partial(x_2x_3^2)$ and $\partial(x_2x_3) \leq \partial(x_1)$, is described by the transformation

$$\text{t} \quad \begin{cases} x_1 \to x_2x_3x_1, \\ x_3 \to x_1x_3, \\ x_4 \to x_3x_1, \end{cases}$$
$$\langle \mathbf{2} \rangle.$$

Proof. The equation **24** can be reduced by the transformation $x_1 \to x_2x_3x_1$ to the equation E_1:

$$x_1x_2x_3x_4 = x_3x_2x_3x_1 \quad \text{with } 0 < \partial(x_1) < \partial(x_3).$$

The equation E_1 can be reduced by the transformation $x_3 \to x_1x_3$ to the equation E_2:

$$x_2x_1x_3x_4 = x_3x_2x_1x_3x_1$$

with $\partial(x_1) > 0$, $\partial(x_3) > 0$. The equation E_2 can be reduced by the transformation $x_4 \to x_3x_1$ to the equation $x_2x_1x_3 = x_3x_2x_1$ with $\partial(x_1) > 0$, $\partial(x_3) > 0$, that is, to the equation **2** with $\partial(x_1) > 0$, $\partial(x_3) > 0$.

On the other hand, the transformation t is a parametric solution of **24**.

Proposition 25. *The general solution of the equation*

25 $$x_1x_2x_3x_4 = x_2x_3^2x_1,$$

with $\partial(x_2) < \partial(x_1) < \partial(x_2x_3^2)$ and $\partial(x_3) \leq \partial(x_1) \leq \partial(x_2x_3)$, is described by the transformation

$$\text{t} \quad \begin{cases} x_1 \to x_2x_1, \\ x_3 \to x_1x_3, \\ x_2 \to x_3x_2, \\ x_4 \to x_4x_1, \end{cases}$$
$$\begin{cases} x_1 \to x_2, \\ x_2 \to x_1, \end{cases}$$
$$\langle \mathbf{21} \rangle.$$

Proof. The equation **25** can be reduced by the transformation $x_1 \to x_2x_1$ to the equation E_1:

$$x_1x_2x_3x_4 = x_3^2x_2x_1$$

with $0 < \partial(x_1) < \partial(x_3^2)$ and $\partial(x_3) \le \partial(x_1 x_2)$. The equation E_1 can be reduced by the transformation $x_3 \to x_1 x_3$ to the equation E_2:

$$x_2 x_1 x_3 x_4 = x_3 x_1 x_3 x_2 x_1$$

with $\partial(x_3) \le \partial(x_2)$ and $\partial(x_1) > 0$. The equation E_2 can be reduced by the transformations $x_2 \to x_3 x_2$, $x_4 \to x_4 x_1$ to the equation E_3:

$$x_2 x_1 x_3 x_4 = x_1 x_3^2 x_2$$

with $\partial(x_1) > 0$. The equation E_3 can be reduced by the transformation

$$\begin{cases} x_1 \to x_2, \\ x_2 \to x_1 \end{cases}$$

to the equation **21** with $\partial(x_2) > 0$.

On the other hand, the transformation t is a parametric solution of **25**.

Proposition 26. *The general solution of the equation*

26 $$x_1 x_2 x_3 x_4 = x_2 x_3^2 x_1$$

is described by the transformation

$$\mathrm{t} \quad \begin{cases} x_1 \to {}^{\mathbf{Fi}}(x_1, x_2 x_3^2)^{\mu} x_1, \\ x_2 \to {}^{\mathbf{Fi}}(x_2 x_3^2, x_1)^{\mu|} x_2, \end{cases}$$

where μ is a variable for sequences of natural parameters, followed by one of the four transformations

$$\begin{cases} x_2 \to 1, \\ x_3 \to 1, \\ x_4 \to 1, \end{cases} \qquad \begin{cases} x_1 \to 1, \\ x_4 \to x_3, \end{cases} \qquad \langle \mathbf{23} \rangle, \qquad \langle \mathbf{24} \rangle,$$

and the transformation

$$\mathrm{tt} \quad \begin{cases} x_1 \to {}^{\mathbf{Fi}}(x_3 x_2 x_1, x_3 x_2 (x_1 x_3)^2)^{\mu} x_3 x_2 x_1, \\ x_2 \to {}^{\mathbf{Fi}}(x_3 x_2 (x_1 x_3)^2, x_3 x_2 x_1)^{\mu|} x_3 x_2, \\ x_3 \to x_1 x_3, \\ x_4 \to x_4 x_1, \end{cases}$$

where μ is a variable for sequences of natural parameters, followed by the transformation

$$\begin{cases} x_1 \to x_2, \\ x_2 \to x_1, \end{cases}$$
$$\langle \mathbf{21} \rangle.$$

Proof. Proposition 26 is deduced by means of Propositions 21–25.

Proposition 27. *The general solution of the equation*

27 $$x_1 x_2 x_3 x_4 = x_2 x_3^2 x_1$$

is described by the transformation

$$\mathrm{T}\quad\begin{cases} x_1 \to {}^{\mathbf{Ro}}(x_1,x_2,x_3)_1^{\nu}x_1, \\ x_2 \to {}^{\mathbf{Ro}}(x_1,x_2,x_3)_2^{\nu}x_2, \\ x_3 \to {}^{\mathbf{Ro}}(x_1,x_2,x_3)_3^{\nu}x_3, \\ x_4 \to x_4{}^{\mathbf{Re}}(x_1,x_2,x_3)^{\nu}, \end{cases}$$

where ν *is a variable whose values are even sequences of variables from the alphabet* (4)*, followed by the transformation*

$$\begin{cases} x_1 \to {}^{\mathbf{Fi}}(x_1,x_2x_3^2)^{\mu}x_1, \\ x_2 \to {}^{\mathbf{Fi}}(x_2x_3^2,x_1)^{\mu|}x_2, \end{cases}$$

where μ *is a variable whose values are finite sequences of natural parameters, followed by one of the four transformations*

$$\begin{cases} x_2 \to 1, \\ x_3 \to 1, \\ x_4 \to 1, \end{cases}\qquad \begin{cases} x_1 \to 1, \\ x_4 \to x_3, \end{cases}\qquad \langle\mathbf{23}\rangle,\qquad \langle\mathbf{24}\rangle,$$

and the transformation

$$\mathrm{TT}\quad\begin{cases} x_1 \to {}^{\mathbf{Ro}}(x_1,x_2,x_3)_1^{\kappa}x_1, \\ x_2 \to {}^{\mathbf{Ro}}(x_1,x_2,x_3)_2^{\kappa}x_2, \\ x_3 \to {}^{\mathbf{Ro}}(x_1,x_2,x_3)_3^{\kappa}x_3, \\ x_4 \to x_4{}^{\mathbf{Re}}(x_1,x_2,x_3)^{\kappa}, \end{cases}$$

where κ *is a variable whose values are odd sequences of variables from the alphabet* (4)*, followed by the transformation*

$$\begin{cases} x_1 \to {}^{\mathbf{Fi}}(x_1x_3^2,x_2)^{\mu|}x_1, \\ x_2 \to {}^{\mathbf{Fi}}(x_2,x_1x_3^2)^{\mu}x_2, \end{cases}$$

where μ *is a variable whose values are finite sequences of natural parameters, followed by one of the four transformations*

$$\begin{cases} x_1 \to 1, \\ x_3 \to 1, \\ x_4 \to 1, \end{cases}\qquad \begin{cases} x_2 \to 1, \\ x_4 \to x_3, \end{cases}\qquad \begin{cases} x_1 \to x_2, \\ x_2 \to x_1, \end{cases} \langle\mathbf{23}\rangle,\qquad \begin{cases} x_1 \to x_2, \\ x_2 \to x_1, \end{cases} \langle\mathbf{24}\rangle.$$

Proof. It is easy to apply the transformations T and TT to the equation **27** and to verify, by using the definition of the functions ${}^{\mathbf{Ro}}(x_1,x_2,x_3)_i^{\mu_1,\dots,\mu_t}$ $(i=1,2,3)$, that they are parametric solution **27**.

Let

$$\mathrm{S}\quad\begin{cases} x_1 \to X_1, \\ x_2 \to X_2, \\ x_3 \to X_3, \\ x_4 \to X_4, \end{cases}$$

where the X_i are words in the alphabet (1), be an arbitrary solution of **27**. We prove by induction on $|X_1X_2X_3X_4|$ that S is contained either in the transformation T or in TT.

The equation **27** can be divided into the following collection of equations:

(j) **27** with $\partial(x_2) \le \partial(x_1) \le \partial(x_2x_3^2)$,
(jj) **27** with $\partial(x_1) \le \partial(x_2)$,
(jjj) **27** with $\partial(x_2x_3^2) \le \partial(x_1)$.

Let S be a solution of (j). By Proposition 22, S is contained in ⟨**23**⟩, ⟨**24**⟩, ⟨**25**⟩.

If S is contained in ⟨**23**⟩ or ⟨**24**⟩, then S is contained in T with $\nu = \varnothing$ and $\mu = \varnothing$.

Let S be contained in ⟨**25**⟩. If $|X_1X_2X_3| = 0$, then S is contained in T. Let $|X_1X_2X_3| > 0$. Then $X_1 = X_2Y_1$ for some Y_1, and $Y_1X_2X_3X_4 = X_3^2X_2Y_1$. Then $X_3 = Y_1Y_3$ and $X_2Y_1Y_3X_4 = Y_3Y_1Y_3X_2Y_1$. But then $X_2 = Y_3Y_2$ and $Y_2Y_1Y_3X_4 = Y_1Y_3Y_3Y_2Y_1$. By Proposition 25 the equation **25** is reduced by the parametric transformation

$$\mathrm{t}\quad \begin{cases} x_1 \to x_3x_2x_1, \\ x_2 \to x_3x_2, \\ x_3 \to x_1x_3, \\ x_4 \to x_4x_1, \end{cases}$$

$$\mathrm{c}\quad \begin{cases} x_1 \to x_2, \\ x_2 \to x_1 \end{cases}$$

to the equation **21**, that is, to **27**. The image of the solution S via the transformation t is the coefficient transformation

$$\mathrm{S}_1\quad \begin{cases} x_1 \to Y_1, \\ x_2 \to Y_2, \\ x_3 \to Y_3, \\ x_4 \to X_4. \end{cases}$$

Since $|Y_1Y_2Y_3X_4| < |X_1X_2X_3X_4|$, one can use the inductive proposition to see that S_1 is contained in the parametric solutions cT, cTT. According to Lemma 1 the parametric solutions tcT, tcTT of **27** contain the solution tcS_1 of **27**, that is, S.

According to Theorem $^{\mathbf{Ro}}1$ the transformations tcT and tcTT can be obtained from T and TT by replacing the parameter ν by $\varnothing, \nu$.

Let S be a solution of t equation (jj). If $|X_1| = 0$, then S is contained in T and TT. Let $|X_1| > 0$. Then $X_2 = X_1Y_2$ for some Y_2, and $X_1Y_2X_3X_4 = Y_2X_3^2X_1$. The equation **27** is reduced by the transformation t: $x_2 \to x_1x_2$ to **27**. The image of the solution S via the transformation t is the coefficient transformation

$$\mathrm{S}_1\quad \begin{cases} x_1 \to X_1, \\ x_2 \to Y_2, \\ x_3 \to X_3, \\ x_4 \to X_4. \end{cases}$$

Since $|Y_2| < |X_2|$, one can use the inductive proposition to see that S_1 is contained in T and TT. According to Lemma 1 the parametric solution tT, tTT of **27** contains the solution tS_1, that is, S.

According to Theorems $^{\mathbf{Fi}}1$ and $^{\mathbf{Ro}}1$ the transformations tT and tTT can be obtained from T and TT by replacing the parameters ν, μ by ν', μ', where if $\nu = \varnothing$, then $\nu' = \varnothing$ and $\mu' = 0, 1, \mu$; if $\nu = \mu_1, \dots, \mu_s$ $(s \ge 1)$, then $\nu' = \mu_1', \dots, \mu_s$, where $\mu_1' = 0, 1, \mu_1$ and $\mu' = \mu$.

Let S be a solution of the equation (jjj). If $|X_2X_3^2| = 0$, then S is contained in T, TT. Let $|X_2X_3^2| > 0$. Then $X_1 = X_2X_3^2Y_1$ for some Y_1, and $Y_1X_2X_3X_4 =$

$X_2X_3^2Y_1$. The equation **27** is reduced by the transformation t: $x_1 \to x_2x_3^2x_1$ to the equation **27**. The image of the solution S via the transformation t is the coefficient transformation

$$\mathrm{S}_1 \quad \begin{cases} x_1 \to Y_1, \\ x_2 \to X_2, \\ x_3 \to X_3, \\ x_4 \to X_4. \end{cases}$$

Since $|Y_1| < |X_1|$, one can use the inductive proposition to see that S_1 is contained in T and TT. According to Lemma 1 a parametric solution tT, tTT of **27** contains the solution tS_1, that is, S.

According to Theorems $^{\mathbf{Fi}}1$ and $^{\mathbf{Ro}}1$ the transformations tT and tTT can be obtained from T and TT by replacing the parameters ν, μ by parameters ν', μ', where if $\nu = \varnothing$, then $\nu' = \varnothing$ and $\mu' = 1, 0, \mu$; if $\nu = \mu_1, \dots, \mu_s$ $(s \geq 1)$, then $\nu' = \mu_1', \dots, \mu_s$, where $\mu_1' = 1, 0, \mu_1$ and $\mu' = \mu$.

8. Equations and solutions

Proposition 28. *The general solution of the equation*

28 $$x_1x_2x_3x_4 = x_3(x_2x_3)^{\alpha+1}x_1 \quad \textit{with } \partial(x_1) < \partial(x_3(x_2x_3)^{\alpha+1}),$$

where λ is a natural parameter, is described by the transformations

$$\mathrm{t} \quad \begin{cases} x_1 \to (x_3x_2)^{\tau}x_1, \\ x_3 \to x_1x_3, \\ x_4 \to (x_3x_2x_1)^{\alpha-\tau}x_3x_1(x_3x_2x_1)^{\tau}, \end{cases}$$
$$\langle \mathbf{2} \rangle,$$

$$\mathrm{tt} \quad \begin{cases} x_1 \to (x_3x_2)^{\tau}x_3x_1, \\ x_2 \to x_1x_2, \\ x_4 \to x_2x_3(x_1x_2x_3)^{\alpha-\tau-1}x_3x_1(x_2x_3x_1)^{\tau}, \end{cases}$$
$$\langle \mathbf{2} \rangle,$$

$$\mathrm{ttt} \quad \begin{cases} x_1 \to x_3(x_2x_3)^{\alpha}x_1, \\ x_2 \to x_1x_2, \\ x_4 \to x_4(x_2x_3x_1)^{\alpha}, \end{cases}$$
$$\langle \mathbf{27} \rangle,$$

$$\mathrm{tttt} \quad \begin{cases} x_1 \to (x_3x_2)^{\alpha+1}x_1, \\ x_3 \to x_1x_3, \\ x_2 \to x_3x_2, \\ x_4 \to x_4(x_1x_3^2x_2)^{\alpha}x_1, \end{cases}$$
$$\langle \mathbf{27} \rangle,$$

$$\mathrm{ttttt} \quad \begin{cases} x_1 \to (x_3x_2)^{\alpha+1}x_1, \\ x_3 \to x_1x_2x_3, \\ x_4 \to x_3x_2x_1(x_2x_3x_2x_1)^{\alpha}, \end{cases}$$
$$\langle \mathbf{2} \rangle.$$

Proof. The equation **28** can be divided into the collection of equations:
(j) **28** with $\partial((x_3x_2)^\tau) \le \partial(x_1) < \partial(x_3(x_2x_3)^\tau)$, $\tau \le \alpha$,
(jj) **28** with $\partial(x_3(x_2x_3)^\tau) \le \partial(x_1) < \partial((x_3x_2)^{\tau+1})$, $\tau \le \alpha$,
(jjj) **28** with $\partial((x_3x_2)^{\alpha+1}) \le \partial(x_1) < \partial(x_3(x_2x_3)^{\alpha+1})$.

The equation (j) can be reduced by the transformation

$$x_1 \to (x_3x_2)^\tau x_1,$$
$$x_3 \to x_1x_3,$$
$$x_4 \to (x_3x_2x_1)^{\alpha-\tau}x_3x_1(x_3x_2x_1)^\tau$$

to the equation $x_2x_1x_3 = x_3x_2x_1$ with $\partial(x_3) > 0$, that is, to **2** with $\partial(x_3) > 0$. On the other hand, the transformation t is a parametric solution of **28**.

The equation (jj) can be reduced by the transformation

$$x_1 \to x_3(x_2x_3)^\tau x_1,$$
$$x_2 \to x_1x_2$$

to the equation E:

$$x_1x_2x_3x_4 = x_2x_3(x_1x_2x_3)^{\alpha-\tau}x_3x_1(x_2x_3x_1)^\tau$$

with $\partial(x_2) > 0$. If $\tau < \alpha$, then the equation E can be reduced by the transformation

$$x_4 \to x_2x_3(x_1x_2x_3)^{\alpha-\tau-1}x_3x_1(x_2x_3x_1)^\tau$$

to the equation $x_1x_2x_3 = x_2x_3x_1$ with $\partial(x_2) > 0$, that is, to **2** with $\partial(x_2) > 0$. On the other hand, the transformation tt is a parametric solution of **28**. If $\tau = \alpha$, then the equation E can be reduced by the transformation

$$x_4 \to x_4(x_2x_3x_1)^\tau$$

to the equation $x_1x_2x_3x_4 = x_2x_3^2x_1$ with $\partial(x_2) > 0$, that is, to **27** with $\partial(x_2) > 0$. On the other hand, the transformation ttt is a parametric solution of **28**.

The equation (jjj) can be reduced by the transformation

$$x_1 \to (x_3x_2)^{\alpha+1}x_1,$$
$$x_3 \to x_1x_3$$

to the equation E_1:

$$x_2x_1x_3x_4 = x_3x_1(x_3x_2x_1)^{\alpha+1} \quad \text{with } \partial(x_3) > 0.$$

The equation E_1 can be divided by means of the conditions $\partial(x_2) \ge \partial(x_3)$ and $\partial(x_3) \ge \partial(x_2)$.

The equation E_1 with $\partial(x_2) \ge \partial(x_3)$ can be reduced by the transformation

$$x_2 \to x_3x_2,$$
$$x_4 \to x_4(x_1x_3^2x_2)^\alpha x_1$$

to the equation $x_2x_1x_3x_4 = x_1x_3^2x_2$ with $\partial(x_3) > 0$, that is, to **27** with $x_1 \leftrightarrow x_2$ and $\partial(x_3) > 0$. On the other hand, the transformation tttt is a parametric solution of **28**.

The equation E_1 with $\partial(x_3) \ge \partial(x_2)$ can be reduced by the transformation

$$x_3 \to x_2x_3,$$
$$x_4 \to x_3x_2x_1(x_2x_3x_2x_1)^\alpha$$

to the equation $x_1x_2x_3 = x_3x_1x_2$, that is, to **2**.

Proposition 29. *The general solution of the equation*

29 $$x_1x_2x_3x_4 = x_3(x_2x_3)^{\alpha+1}x_1,$$

where λ is a natural parameter, is described by the transformations

$$\mathrm{t}\quad \begin{cases} x_2 \to 1, \\ x_3 \to 1, \\ x_4 \to 1, \end{cases} \qquad \begin{matrix} x_1 \to (x_3(x_2x_3)^{\alpha+1})^{\beta}x_1, \\ \langle \mathbf{28} \rangle, \end{matrix}$$

where β is a natural parameter.

Proof. The equation **29** can be divided into a collection of equations by means of the conditions $\partial(x_2x_3) = 0$ and $\partial(x_2x_3) > 0$.

The equation **29** with $\partial(x_2x_3) = 0$ can be reduced by the transformation t to the equation **1**.

The equation **29** with $\partial(x_2x_3) > 0$ by Proposition 8 can be reduced by the transformation $x_1 \to (x_3(x_2x_3)^{\alpha+1})^{\beta}x_1$, where β is a natural parameter, to **28**.

Proposition 30. *The general solution of the equation*

30 $$x_1x_2x_3x_4 = x_2x_3^{\alpha+3}x_1 \quad \textit{with } \partial(x_2) < \partial(x_1) < \partial(x_2x_3^{\alpha+3}),$$

where α is a natural parameter, is described by the transformations

$$\begin{matrix} x_1 \to x_2x_3^{\alpha+2}x_1, \\ x_3 \to x_1x_3, \\ x_4 \to (x_3x_1)^{\alpha+2}, \\ \langle \mathbf{2} \rangle, \end{matrix} \qquad \begin{matrix} x_1 \to x_2x_3^{\beta}x_1, \\ x_3 \to x_1x_3, \\ x_4 \to x_4(x_1x_3)^{\beta}x_1, \\ \langle \mathbf{29} \rangle, \end{matrix}$$

where β is a natural parameter with $\beta \le \alpha + 2$.

Proof. The equation **30** can be reduced by the transformation

$$x_1 \to x_2x_3^{\beta}x_1 \qquad (\beta \le \alpha + 2),$$
$$x_3 \to x_1x_3$$

to the equation E:

$$x_2x_1x_3x_4 = x_3(x_1x_3)^{\alpha+2-\beta}x_2(x_1x_3)^{\beta}x_1.$$

The equation E can be divided into a collection of equations by means of the conditions $\beta = \alpha + 2$ and $\beta < \alpha + 2$.

The equation E with $\beta = \alpha + 2$ can be reduced by the transformation $x_4 \to (x_3x_1)^{\alpha+2}$ to the equation $x_2x_1x_3 = x_3x_2x_1$, that is, to **2**.

The equation E with $\beta < \alpha + 2$ can be reduced by the transformation $x_4 \to x_4(x_1x_3)^{\beta}x_1$ to the equation

$$x_2x_1x_3x_4 = x_3(x_1x_3)^{\alpha+2-\beta}x_2,$$

that is, to **29**.

Proposition 31. *The general solution of the equation*

31 $$x_1x_2x_3x_4 = x_2x_3^{\alpha+3}x_1,$$

where α is a natural parameter, is described by the transformation

$$\begin{cases} x_1 \to {}^{\mathbf{Fi}}(x_1, x_2x_3^{\alpha+3})^{\mu}x_1, \\ x_2 \to {}^{\mathbf{Fi}}(x_2x_3^{\alpha+3}, x_1)^{\mu|}x_2, \end{cases}$$

where μ is a variable for sequences of natural parameters, followed by one of the three transformations

$$\begin{cases} x_2 \to 1, \\ x_3 \to 1, \\ x_4 \to 1, \end{cases} \qquad \begin{cases} x_1 \to 1, \\ x_4 \to x_3^{\alpha+2}, \end{cases} \qquad \langle \mathbf{30} \rangle.$$

Proof. This follows directly from Propositions 9 and 30.

Proposition 32. *The general solution of the equation*

32 $$x_1x_2x_3x_4 = x_2x_3^{\alpha}x_1x_5,$$

where λ is a natural parameter, is described by the transformations

$$\text{t} \quad \begin{cases} \alpha \to 0, \\ x_5 \to x_3x_4, \end{cases} \quad \langle \mathbf{1} \rangle, \qquad \text{tt} \quad \begin{cases} \alpha \to 1, \\ x_5 \to x_4, \end{cases} \quad \langle \mathbf{2} \rangle,$$

$$\text{ttt} \quad \begin{cases} \alpha \to 2, \\ x_4 \to x_4x_5, \end{cases} \quad \langle \mathbf{27} \rangle, \qquad \text{tttt} \quad \begin{cases} \alpha \to \gamma+3, \\ x_4 \to x_4x_5, \end{cases} \quad \langle \mathbf{31} \rangle,$$

where γ is a natural parameter.

Proof. The equation **32** is divided into a collection of equations by means of the conditions $\alpha = 0$, $\alpha = 1$, $\alpha = 2$, $\alpha = \gamma + 3$, where γ is a natural parameter.

The equation **32** with $\alpha = 0$ can be reduced by the transformation t to the equation $x_1x_2 = x_2x_1$, that is, to **1**.

The equation **32** with $\alpha = 1$ can be reduced by the transformation tt to the equation $x_1x_2x_3 = x_2x_3x_1$, that is, to **2**.

The equation **32** with $\alpha = 2$ can be reduced by the transformation ttt to the equation $x_1x_2x_3x_4 = x_2x_3^2x_1$, that is, to **27**.

The equation **32** with $\alpha = \gamma + 3$ can be reduced by the transformation tttt to the equation $x_1x_2x_3x_4 = x_2x_3^{\gamma+3}x_1$, that is, to **31**.

Proposition 33. *The general solution the equation*

33 $$x_1x_2 = x_3^{\alpha}x_4,$$

where α is a natural parameter, is described by the transformations

$$\begin{cases} x_1 \to x_3^{\alpha}x_4, \\ x_4 \to x_4x_2, \end{cases} \qquad \begin{cases} x_1 \to (x_1x_3)^{\beta}x_1, \\ x_2 \to x_3(x_1x_3)^{\gamma-\beta}x_4, \\ x_3 \to x_1x_3, \\ \alpha \to \gamma+1, \end{cases} \qquad \begin{cases} x_1 \to 1, \\ x_2 \to x_4, \\ \alpha \to 0, \end{cases}$$

where β, γ are natural parameters.

Proof. The equation **33** can be divided by means of the conditions $\partial(x_2) \le \partial(x_4)$ and $\partial(x_2) \ge \partial(x_4)$.

The equation **33** with $\partial(x_2) \le \partial(x_4)$ can be reduced by the transformation $x_4 \to x_4x_2$ to the equation $x_1 = x_3^{\alpha}x_4$.

The equation **33** with $\partial(x_2) \geq \partial(x_4)$ and $\alpha > 0$ can be reduced by the transformation

$$\begin{cases} x_3 \to x_1x_3, \\ x_1 \to (x_1x_3)^\beta x_1, \\ x_2 \to x_3(x_1x_3)^{\alpha-\beta-1}x_4 \end{cases}$$

to the equation **1**.

The equation **33** with $\partial(x_2) \geq \partial(x_4)$ and $\alpha = 0$ can be reduced by the transformation

$$\begin{cases} x_1 \to 1, \\ x_2 \to x_4, \\ \alpha \to 0 \end{cases}$$

to the equation **1**.

Proposition 34. *The general solution of the equation*

34 $$x_1x_2x_3x_4 = x_2^{\alpha+1}x_5,$$

where α is a natural parameter, is described by the transformations

$$\begin{cases} x_1 \to x_2^{\alpha+1}x_1, \\ x_5 \to x_1x_2x_3x_4, \end{cases} \qquad \begin{cases} x_1 \to x_1^{(\eta+\theta)\beta+\eta}, \\ x_2 \to x_1^{\eta+\theta}, \\ \alpha \to \gamma+1, \end{cases} \qquad \begin{cases} x_1 \to (x_1x_2)^\alpha x_1, \\ x_2 \to x_1x_2, \\ x_5 \to x_5x_3x_4, \end{cases}$$

$$\langle \mathbf{33} \rangle, \qquad \langle \mathbf{4} \rangle,$$

where $\beta, \gamma, \eta, \theta$ are natural parameters with $\beta \leq \gamma$.

Proof. The equation **34** can be divided by means of the conditions $\partial(x_1) \geq \partial(x_2^{\alpha+1})$ and $\partial(x_1) < \partial(x_2^{\alpha+1})$.

The equation **33** with $\partial(x_1) \geq \partial(x_2^{\alpha+1})$ can be reduced by the transformation $x_1 \to x_2^{\alpha+1}x_1$ to the equation $x_1x_2x_3x_4 = x_5$.

The equation **34** with $\partial(x_1) < \partial(x_2^{\alpha+1})$ can be reduced by the transformation

$$x_1 \to x_2^\beta x_1 \qquad (\beta < \alpha+1),$$
$$x_2 \to x_1x_2$$

to the equation E:

$$x_1x_2x_3x_4 = x_2(x_1x_2)^{\alpha-\beta}x_5 \quad \text{with } \beta \leq \alpha,\ \partial(x_2) > 0.$$

If $\beta < \alpha$, then by Proposition 1 the equation E is reduced by the parametric transformation

$$\begin{cases} x_1 \to x_1^\eta, \\ x_2 \to x_1^\theta, \end{cases}$$

where η, θ are natural parameters, to the equation

$$x_3x_4 = x_1^{\theta+(\eta+\theta)(\alpha-\beta-1)}x_5$$

with $\partial(x_1^\theta) > 0$, that is, to **33**.

If $\beta = \alpha$, then the equation E is reduced by the parametric transformation $x_5 \to x_5x_3x_4$ to the equation $x_1x_2 = x_2x_5$, that is, to **4**.

Proposition 35. *The general solution of the equation*

35 $$x_1x_2x_3x_4 = x_3(x_2x_3)^{\alpha+1}x_5,$$

where α is a natural parameter, is described by the transformations

$$\text{t}\quad \begin{cases} x_1 \to (x_1x_3x_2)^{\tau}x_1, \\ x_3 \to x_1x_3, \\ x_4 \to x_3(x_2x_1x_3)^{\alpha-\tau}x_5, \end{cases}$$
$$\langle \mathbf{2} \rangle,$$
$$\text{tt}\quad \begin{cases} x_1 \to (x_1x_3x_2)^{\alpha+1}x_1, \\ x_3 \to x_1x_3, \\ x_5 \to x_5x_4, \end{cases}$$
$$\langle \mathbf{6} \rangle,$$
$$\text{ttt}\quad \begin{cases} x_1 \to x_3(x_1x_2x_3)^{\tau}x_1, \\ x_2 \to x_1x_2, \\ x_4 \to x_2x_3(x_1x_2x_3)^{\alpha-\tau-1}x_5, \end{cases}$$
$$\langle \mathbf{2} \rangle,$$
$$\text{tttt}\quad \begin{cases} x_1 \to x_3(x_1x_2x_3)^{\alpha}x_1, \\ x_2 \to x_1x_2, \\ x_5 \to x_5x_4, \end{cases}$$
$$\langle \mathbf{7} \rangle,$$
$$\text{ttttt}\quad \begin{cases} x_1 \to x_3(x_2x_3)^{\alpha+1}x_1, \\ x_5 \to x_1x_2x_3x_4, \end{cases}$$

where τ is a natural parameter.

Proof. The equation **35** can be divided into the collection of equations
(j) **35** with $\partial((x_2x_3)^{\tau}) \le \partial(x_1) \le \partial(x_3(x_2x_3)^{\tau})$, $\tau \le \alpha+1$,
(jj) **35** with $\partial(x_3(x_2x_3)^{\tau}) \le \partial(x_1) \le \partial((x_2x_3)^{\tau+1})$, $\tau \le \alpha$,
(jjj) **35** with $\partial(x_3(x_2x_3)^{\alpha+1}) \le \partial(x_1)$.

The equation (j) can be reduced by the transformation

$$x_1 \to (x_3x_2)^{\tau}x_1,$$
$$x_3 \to x_1x_3$$

to the equation E:

$$x_2x_1x_3x_4 = x_3(x_2x_1x_3)^{\alpha+1-\tau}x_5.$$

If $\tau < \alpha+1$, then the equation E can be reduced by the transformation $x_4 \to x_3(x_2x_1x_3)^{\alpha-\tau}x_5$ to the equation $x_2x_1x_3 = x_3x_2x_1$, that is, to **2**.

If $\tau = \alpha+1$, then the equation E can be reduced by the transformation $x_5 \to x_5x_4$ to the equation $x_2x_1x_3 = x_3x_5$, that is, to **6**.

The equation (jj) can be reduced by the transformation

$$x_1 \to x_3(x_2x_3)^{\tau}x_1,$$
$$x_2 \to x_1x_2$$

to the equation E:

$$x_1x_2x_3x_4 = x_2x_3(x_1x_2x_3)^{\alpha-\tau}x_5.$$

If $\tau < \alpha$, then the equation E can be reduced by the transformation $x_4 \to x_2x_3(x_1x_2x_3)^{\alpha-\tau-1}x_5$ to the equation $x_1x_2x_3 = x_2x_3x_1$, that is, to **2**.

If $\tau = \alpha$, then the equation E can be reduced by the transformation $x_5 \to x_5x_4$ to the equation $x_1x_2x_3 = x_2x_3x_5$, that is, to **7**.

The equation (jjj) can be reduced by the transformation

$$x_1 \to x_3(x_2x_3)^{\alpha+1}x_1$$

to the equation $x_1x_2x_3x_4 = x_5$.

Proposition 36. *The general solution of the equation*

$$x_1x_2x_3x_4 = x_3^{\alpha+1}x_5,$$

where α is a natural parameter, is described by the transformations

$$\text{t}\quad \begin{cases} x_1 \to x_3^{\alpha+1}x_1, \\ x_5 \to x_1x_2x_3x_4, \end{cases}$$

$$\text{tt}\quad \begin{cases} x_1 \to (x_1x_3)^{\beta}x_1, \\ x_3 \to x_1x_3, \\ \quad\langle \mathbf{35} \rangle, \end{cases}$$

$$\text{ttt}\quad \begin{cases} x_1 \to (x_1x_3)^{\alpha}x_1, \\ x_3 \to x_1x_3, \\ x_5 \to x_5x_4, \\ \quad\langle \mathbf{6} \rangle, \end{cases}$$

where β is a natural parameter with $\beta < \alpha$.

Proof. The equation **36** can be divided by means of the conditions $\partial(x_1) \geq \partial(x_3^{\alpha+1})$ and $\partial(x_1) \leq \partial(x_3^{\alpha+1})$.

The equation **36** with $\partial(x_1) \geq \partial(x_3^{\alpha+1})$ can be reduced by the transformation $x_1 \to x_3^{\alpha+1}x_1$ to the equation $x_1x_2x_3x_4 = x_5$.

The equation **36** with $\partial(x_1) \leq \partial(x_3^{\alpha+1})$ can be reduced by the transformation

$$\begin{aligned} &x_1 \to x_3^{\beta}x_1 \qquad (\beta < \alpha + 1), \\ &x_3 \to x_1x_3 \end{aligned}$$

to the equation E:

$$x_2x_1x_3x_4 = x_3(x_1x_3)^{\alpha-\beta}x_5 \quad \text{with } \beta \leq \alpha.$$

If $\beta < \alpha$, then E is **35**.

If $\beta = \alpha$, then E can be reduced by the transformation $x_5 \to x_5x_4$ to the equation $x_2x_1x_3 = x_3x_5$, that is, to **6**.

Proposition 37. *The general solution of the equation*

37 $$x_1x_2x_3x_4 = x_2x_3^{\alpha+1}x_5,$$

where α is a natural parameter, is described by the transformations

$$\text{t}\quad \begin{cases} x_1 \to 1, \\ x_4 \to x_3^{\alpha}x_5, \end{cases} \qquad\qquad \text{tt}\quad \begin{cases} x_1 \to x_2x_1, \\ x_2 \to (x_2x_1)^{\lambda}x_2, \\ \quad\langle \mathbf{36} \rangle, \end{cases}$$

where λ is a natural parameter.

Proof. The equation **37** can be divided by means of the conditions $\partial(x_1) = 0$ and $\partial(x_1) > 0$.

The equation **37** with $\partial(x_1) = 0$ can be reduced by the transformation $x_1 \to 1$ to the equation $x_4 = x_3^{\alpha} x_5$.

The equation **37** with $\partial(x_1) > 0$ by Proposition 8 can be reduced by the transformation $x_2 \to x_1^{\lambda} x_2$ to the equation **37** with $\partial(x_1) > \partial(x_2)$, which in turn can be reduced by the transformation $x_1 \to x_2 x_1$ to the equation

$$x_1 x_2 x_3 x_4 = x_3^{\alpha+1} x_5 \quad \text{with } \partial(x_1) > 0,$$

that is, to **36** with $\partial(x_1) > 0$. On the other hand, the transformation tt, $\langle \mathbf{36} \rangle$ is a parametric solution of **37**.

Proposition 38. *The general solution of the equation*

38 $$x_2 x_1 x_3 x_4 = x_3 x_1 ((x_1 x_3)^{\lambda+1} x_1)^{\kappa+1} x_5,$$

where λ, κ are natural parameters, is described by the transformations

$$\text{t1} \quad \begin{cases} x_3 \to x_2 x_3, \\ x_4 \to x_4 x_3 (x_1 x_2 x_3)^{\lambda} x_1 ((x_1 x_2 x_3)^{\lambda+1} x_1)^{\kappa} x_5, \end{cases}$$
$$\langle \mathbf{12} \rangle,$$

$$\text{t2} \quad \begin{cases} x_1 \to x_1^{\eta+\theta}, \\ x_2 \to x_3 x_1^{\theta}, \\ x_4 \to x_4 (x_1^{\eta+\theta} x_3)^{\lambda} x_1^{\eta+\theta} ((x_1^{\eta+\theta} x_3)^{\lambda+1} x_1^{\eta+\theta})^{\kappa} x_5, \end{cases}$$
$$\langle \mathbf{5} \rangle$$

$$\text{t3} \quad \begin{cases} x_1 \to x_2 x_1, \\ x_2 \to x_3 x_2 x_1 ((x_1 x_2 x_3)^{\lambda+1} x_2 x_1)^{\tau} (x_2 x_1 x_3)^{\sigma} x_2, \\ x_4 \to x_1 (x_2 x_3 x_1)^{\lambda-\sigma} ((x_2 x_1 x_3)^{\lambda+1} x_2 x_1)^{\kappa-\tau} x_5, \end{cases}$$
$$\langle \mathbf{2} \rangle,$$

$$\text{t4} \quad \begin{cases} x_1 \to x_1^{\eta+\theta}, \\ x_2 \to x_3 x_1^{\eta+\theta} ((x_1^{\eta+\theta} x_3)^{\lambda+1} x_1^{\eta+\theta})^{\tau} (x_1^{\eta+\theta} x_3)^{\lambda+1} x_1^{\theta}, \\ x_4 \to x_4 x_1^{\eta+\theta} (x_3 x_1^{\eta+\theta})^{\lambda} ((x_1^{\eta+\theta} x_3)^{\lambda+1} x_1^{\eta+\theta})^{\kappa-\tau-1} x_5, \end{cases}$$
$$\langle \mathbf{5} \rangle,$$

$$\text{t5} \quad \begin{cases} x_1 \to (x_1 x_2)^{\beta+1} x_1, \\ x_2 \to x_3 (x_1 x_2)^{\beta} x_1 (((x_1 x_2)^{\beta} x_1 x_3)^{\lambda+1} (x_1 x_2)^{\beta} x_1)^{\kappa} \\ \qquad \cdot ((x_1 x_2)^{\beta} x_1 x_3)^{\lambda+1} x_1 x_2, \\ x_5 \to x_2 x_1 x_3 x_4, \end{cases}$$

$$\text{t6} \quad \begin{cases} x_2 \to x_2 x_3 x_1 ((x_1 x_2 x_3)^{\lambda+1} x_1)^{\tau} (x_1 x_2 x_3)^{\sigma} x_1 x_2, \\ x_3 \to x_2 x_3, \\ x_4 \to x_3 x_1 (x_2 x_3 x_1)^{\lambda-\sigma-1} ((x_1 x_2 x_3)^{\lambda+1} x_1)^{\kappa-\tau} x_5, \end{cases}$$
$$\langle \mathbf{2} \rangle,$$

$$\text{t7}\quad \begin{cases} x_2 \to x_2x_3x_1((x_1x_2x_3)^{\lambda+1}x_1)^{\tau}(x_1x_2x_3)^{\lambda}x_1x_2, \\ x_3 \to x_2x_3, \\ x_4 \to x_4x_3(x_1x_2x_3)^{\lambda}x_1((x_1x_2x_3)^{\lambda+1}x_1)^{\kappa-\tau-1}x_5, \end{cases}$$
$$\langle \mathbf{12} \rangle,$$

$$\text{t8}\quad \begin{cases} x_2 \to x_2x_3x_1((x_1x_2x_3)^{\lambda+1}x_1)^{\kappa}(x_1x_2x_3)^{\lambda}x_1x_2, \\ x_3 \to x_2x_3, \\ x_5 \to x_5x_4, \end{cases}$$
$$\langle \mathbf{10} \rangle,$$

$$\text{t9}\quad \begin{cases} x_2 \to x_3x_1((x_1x_3)^{\lambda+1}x_1)^{\kappa+1}x_2, \\ x_5 \to x_2x_1x_3x_4, \end{cases}$$

where $\eta, \theta, \tau, \sigma, \beta$ are natural parameters.

Proof. The equation **38** can be divided into the collection of equations:
(j) **38** with $\partial(x_2) < \partial(x_3)$,
(jj) **38** with $\partial(x_3) \le \partial(x_2) \le \partial(x_3x_1)$,
(jjj) **38** with

$$\partial(x_3x_1((x_1x_3)^{\lambda+1}x_1)^{\tau}(x_1x_3)^{\sigma} \le \partial(x_2)$$
$$\le \partial(x_3x_1((x_1x_3)^{\lambda+1}x_1)^{\tau}(x_1x_3)^{\sigma}x_1), \qquad \tau < \kappa+1, \sigma \le \lambda+1,$$

(jjjj) **38** with

$$\partial(x_3x_1((x_1x_3)^{\lambda+1}x_1)^{\tau}(x_1x_3)^{\sigma}x_1) \le \partial(x_2)$$
$$< \partial(x_3x_1((x_1x_3)^{\lambda+1}x_1)^{\tau}(x_1x_3)^{\sigma+1}), \qquad \tau < \kappa+1, \sigma < \lambda+1,$$

(jjjjj) **38** with $\partial(x_3x_1((x_1x3)^{\lambda+1}x_1)^{\kappa+1}) \le \partial(x_2)$.

The equation (j) can be reduced by the transformation

$$x_3 \to x_2x_3,$$
$$x_4 \to x_4x_3(x_1x_2x_3)^{\lambda}x_1((x_1x_2x_3)^{\lambda+1}x_1)^{\kappa}x_5$$

to the equation $x_1x_2x_3x_4 = x_3x_1^2x_2$ with $\partial(x_3) > 0$, that is, to **12**.

The equation (jj) can be reduced by the transformation

$$x_2 \to x_3x_2,$$
$$x_1 \to x_2x_1,$$
$$\begin{cases} x_1 \to x_1^{\eta}, \\ x_2 \to x_1^{\theta}, \end{cases}$$
$$x_4 \to x_4(x_1^{\eta+\theta}x_3)^{\lambda}x_1^{\eta+\theta}((x_1^{\eta+\theta}x_3)^{\lambda+1}x_1^{\eta+\theta})^{\kappa}x_5$$

to the equation $x_3x_4 = x_1^{\eta}x_3$, that is, $(x_2x_3x_1)^{\beta+1}x_2)^{\kappa}$ to **5**.

The equation (jjj) can be reduced by the transformation

$$x_2 \to x_3x_1((x_1x_3)^{\lambda+1}x_1)^{\tau}(x_1x_3)^{\sigma}x_2,$$
$$x_1 \to x_2x_1$$

to the equation E:

$$x_2x_1x_3x_4 = x_1(x_3x_2x_1)^{\lambda+1-\sigma}((x_2x_1x_3)^{\lambda+1}x_2x_1)^{\kappa-\tau}x_5.$$

The equation E with $\lambda+1>\sigma$ can be reduced by the transformation

$$x_4 \to x_1(x_2x_3x_1)^{\lambda-\sigma}((x_2x_1x_3)^{\lambda+1}x_2x_1)^{\kappa-\tau}x_5$$

to the equation $x_2x_1x_3 = x_1x_3x_2$, that is, to **2**.

The equation E with $\lambda+1=\sigma$ and $\tau<\kappa$ can be reduced by the transformation

$$\begin{cases} x_1 \to x_1^{\eta}, \\ x_2 \to x_1^{\theta}, \end{cases}$$
$$x_4 \to x_4x_1^{\eta+\theta}(x_3x_1^{\eta+\theta})^{\lambda}((x_1^{\eta+\theta}x_3)^{\lambda+1}x_1^{\eta+\theta})^{\kappa-\tau-1}x_5$$

to the equation $x_3x_4 = x_1^{\eta}x_3$, that is, **5**.

The equation E with $\lambda+1=\sigma$ and $\tau=\kappa$ can be reduced by the transformation

$$x_1 \to x_2^{\beta}x_1,$$
$$x_2 \to x_1x_2,$$
$$x_5 \to x_2x_1x_3x_4$$

to **1**.

The equation (jjjj) can be reduced by the transformation

$$x_2 \to x_3x_1((x_1x_3)^{\lambda+1}x_1)^{\tau}(x_1x_3)^{\sigma}x_1x_2,$$
$$x_3 \to x_2x_3$$

to the equation E:

$$x_1x_2x_3x_4 = x_3x_1(x_2x_3x_1)^{\lambda-\sigma}((x_1x_2x_3)^{\lambda+1}x_1)^{\kappa-\tau}x_5$$

with $\partial(x_3)>0$.

The equation E with $\lambda>\sigma$ can be reduced by the transformation

$$x_4 \to x_3x_1(x_2x_3x_1)^{\lambda-\sigma-1}((x_1x_2x_3)^{\lambda+1}x_1)^{\kappa-\tau}x_5$$

to the equation $x_2x_1x_3 = x_1x_3x_2$, that is, to **2** with $\partial(x_3)>0$. On the other hand, t6 is a parametric solution of **38**.

The equation E with $\lambda=\sigma$ and $\kappa>\tau$ can be reduced by the transformation

$$x_4 \to x_4x_3(x_1x_2x_3)^{\lambda}x_1((x_1x_2x_3)^{\lambda+1}x_1)^{\kappa-\tau-1}x_5$$

to the equation $x_1x_2x_3x_4 = x_3x_1^2x_2$ with $\partial(x_3)>0$, that is, to **12**.

The equation E with $\lambda=\sigma$ and $\kappa=\tau$ can be reduced by the transformation $x_5 \to x_5x_4$ to the equation $x_1x_2x_3 = x_3x_1x_5$ with $\partial(x_3)>0$, that is, to **10** with $\partial(x_3)>0$. On the other hand, t8 is a parametric solution of **38**.

The equation (jjjjj) can be reduced by the transformation

$$x_2 \to x_3x_1((x_1x_3)^{\lambda+1}x_1)^{\kappa+1}x_2,$$
$$x_5 \to x_2x_1x_3x_4$$

to the equation **1**.

Proposition 39. *The general solution of the equation*

39 $$x_1x_2x_3x_4 = x_3x_1^{\alpha+1}x_5,$$

where α is a natural parameter, is described by the transformations

$$\text{t1} \quad \begin{cases} x_1 \to 1, \\ x_2 \to 1, \\ x_4 \to x_5, \end{cases} \qquad \text{t2} \quad \begin{cases} x_2 \to x_3x_2, \\ x_3 \to (x_1x_3x_2)^\lambda x_1x_3, \end{cases} \langle \mathbf{34} \rangle,$$

$$\text{t3} \quad \begin{cases} x_1 \to (x_1x_3)^{\gamma+1}x_1, \\ x_3 \to ((x_1x_3)^{\gamma+1}x_1x_2)^\lambda x_1x_3, \\ \alpha \to \kappa+1, \end{cases} \langle \mathbf{38} \rangle, \qquad \text{t4} \quad \begin{cases} x_1 \to (x_1x_3)^{\gamma+1}x_1, \\ x_2 \to x_3x_2, \\ x_3 \to ((x_1x_3)^{\gamma+2}x_2)^\lambda x_1x_3, \\ x_5 \to x_5x_3x_4, \\ \alpha \to 0, \end{cases} \langle \mathbf{4} \rangle,$$

$$\text{t5} \quad \begin{cases} x_1 \to (x_1x_2x_3)^{\gamma+1}x_1, \\ x_3 \to ((x_1x_2x_3)^{\gamma+1}x_1x_2)^\lambda x_1x_2x_3, \\ x_5 \to x_5x_4, \\ \alpha \to 0, \end{cases} \langle \mathbf{10} \rangle, \qquad \text{t6} \quad \begin{cases} x_2 \to x_1^\alpha x_5, \\ x_3 \to (x_1^{\alpha+1}x_5)^\lambda, \\ x_5 \to x_5x_4, \end{cases}$$

$$\text{t7} \quad \begin{cases} x_2 \to 1, \\ x_3 \to x_1^\lambda, \\ x_4 \to x_5, \\ \alpha \to 0, \end{cases} \qquad \text{t8} \quad \begin{cases} x_1 \to x_2x_1, \\ x_2 \to (x_2x_1)^\gamma x_2, \\ x_3 \to ((x_2x_1)^{\gamma+1}x_2)^\lambda, \\ x_4 \to (x_1x_2)^{\beta-\gamma}x_1x_5, \\ \alpha \to \beta+1, \end{cases}$$

where $\lambda, \gamma, \kappa, \beta$ are natural parameters.

Proof. The equation **39** can be divided into the collection of equations:

(j) **39** with $\partial(x_1x_2) = 0$,

(jj) **39** with $\partial(x_1x_2) > 0$.

The equation (j) can be reduced by the transformation $x_1 \to 1$, $x_2 \to 1$ to the equation $x_4 = x_5$.

The equation (jj) by Proposition 8 can be reduced by the transformation

$$x_3 \to (x_1x_2)^\lambda x_3,$$

where λ is a natural parameter, to the equation E:

$$x_1x_2x_3x_4 = x_3x_1^{\alpha+1}x_5 \quad \text{with } \partial(x_3) < \partial(x_1x_2).$$

The equation E can be divided by mean of the conditions $\partial(x_3) \geq \partial(x_1)$ and $\partial(x_1) \geq \partial(x_3)$.

The equation E with $\partial(x_3) \geq \partial(x_1)$ can be reduced by the transformation $x_3 \to x_1x_3$, $x_2 \to x_3x_2$ to the equation $x_2x_1x_3x_4 = x_1^{\alpha+1}x_5$ with $\partial(x_2) > 0$, that is, to **34** with $\partial(x_1) > 0$. On the other hand, t2 is a parametric solution of **39**.

The equation E with $\partial(x_1) \geq \partial(x_3)$ can be reduced by the transformation $x_1 \to x_3x_1$ to the equation E_1:

$$x_1x_2x_3x_4 = (x_3x_1)^{\alpha+1}x_5 \quad \text{with } \partial(x_1x_2) > 0.$$

The equation E_1 with $\partial(x_3) > 0$ by Proposition 8 can be reduced by the transformation $x_1 \to x_3^\gamma x_1$, $x_3 \to x_1x_3$ to the equation E_2:

$$x_2x_1x_3x_4 = x_3x_1((x_1x_3)^{\gamma+1}x_1)^\alpha x_5 \quad \text{with } \partial(x_3) > 0.$$

The equation E_2 with $\alpha > 0$ is **38** with $\partial(x_3) > 0$. On the other hand, t3 is a parametric solution of **39**.

The equation E_2 with $\alpha = 0$ and $\partial(x_2) \geq \partial(x_3)$ can be reduced by the transformation $x_2 \to x_3x_2$, $x_5 \to x_5x_3x_4$ to the equation $x_2x_1 = x_1x_5$, that is, to **4**.

The equation E_2 with $\alpha = 0$ and $\partial(x_2) \leq \partial(x_3)$ can be reduced by the transformation $x_3 \to x_2x_3$ to the equation $x_1x_2x_3 = x_3x_1x_5$, that is, to **10**.

The equation E_1 with $\partial(x_3) = 0$ can be reduced by the transformation $x_3 \to 1$ the equation E_3:

$$x_2x_4 = x_1^\alpha x_5.$$

The equation E_3 with $\partial(x_4) \leq \partial(x_5)$ can be reduced by the transformation $x_5 \to x_5x_4$ to the equation $x_2 = x_1^\alpha x_5$.

The equation E_3 with $\partial(x_4) \geq \partial(x_5)$ and $\alpha = 0$ can be reduced by the transformation $x_4 = x_4x_5$, $\alpha \to 0$ to the equation $x_2x_4 = 1$.

The equation E_3 with $\partial(x_4) \geq \partial(x_5)$ and $\alpha = \beta + 1$ can be reduced by the transformation

$$\begin{aligned} \alpha &\to \beta + 1, \\ x_1 &\to x_2x_1, \\ x_2 &\to (x_2x_1)^\gamma x_2, \\ x_4 &\to (x_1x_2)^{\beta-\gamma}x_1x_5 \end{aligned}$$

to the equation **1**.

9. More equations and solutions

In this section we will denote the words in the alphabet of primitive parametric words $P_1, \dots, P_k$ by $\zeta(P_1, \dots, P_k)$, $\upsilon(P_1, \dots, P_k)$, $\varphi(P_1, \dots, P_k)$. The natural numbers will be denoted by q, r, s, the natural parameters by $\alpha, \beta, \lambda, \tau$, and the primitive parametric word by P.

The proofs of Propositions 40–55 are simple and very similar to proofs of the previous propositions, so we shall omit them.

Proposition 40. *The general solution of the equation*

40 $$x_1x_2x_3x_4 = x_2\zeta(x_1x_2, x_3)x_1 \quad \textit{with } \partial(x_2) > 0,$$

where x_3 occurs in $\zeta(x_1x_2, x_3)$, is described by the transformations

$$\begin{cases} x_1 \to x_1^\alpha, \\ x_2 \to x_1^\beta, \\ x_4 \to x_4\upsilon(x_1^{\alpha+\beta}, x_3)x_1^\alpha, \end{cases}$$
$$\langle \mathbf{5} \rangle,$$

where $\zeta(x_1x_2, x_3)$ is $(x_1x_2)^{q+1}x_3\upsilon(x_1x_2, x_3)$;

$$x_4 \to x_4x_2\upsilon(x_1x_2, x_3)x_1,$$
$$\langle \mathbf{32} \rangle,$$

where $\zeta(x_1x_2, x_3)$ is $x_3^{q+1}x_1x_2\upsilon(x_1x_2, x_3)$; and

$$\langle \mathbf{32} \rangle,$$

where $\zeta(x_1x_2, x_3)$ is x_3^{q+1}.

Proposition 41. *The general solution of the equation*

41 $$x_2x_1x_3x_4 = x_3\zeta(x_1x_3, x_2)x_1 \quad \textit{with } \partial(x_3) > 0,$$

where x_2 *occurs in* $\zeta(x_1x_3, x_2)$*, is described by the transformations*

$$x_4 \to x_4\upsilon(x_1x_3, x_2)x_1, \quad \langle \mathbf{29} \rangle,$$

where $\zeta(x_1x_3, x_2)$ *is* $(x_1x_3)^{q+1}x_2\upsilon(x_1x_3, x_2)$*;*

$$x_4 \to x_4x_3\upsilon(x_1x_3, x_2), \quad \langle \mathbf{20} \rangle,$$

where $\zeta(x_1x_3, x_2)$ *is* $x_2^{r+1}x_1x_3\upsilon(x_1x_3, x_2)$*;*

$$x_4 \to x_4x_3\upsilon(x_1x_3, x_2), \quad \langle \mathbf{12} \rangle,$$

where $\zeta(x_1x_3, x_2)$ *is* $x_2^2x_1x_3\upsilon(x_1x_3, x_2)$*;*

$$x_4 \to x_4x_3\upsilon(x_1x_3, x_2), \quad \langle \mathbf{2} \rangle,$$

where $\zeta(x_1x_3, x_2)$ *is* $x_2x_1x_3\upsilon(x_1x_3, x_2)$*;*

$$\langle \mathbf{20} \rangle,$$

where $\zeta(x_1x_3, x_2)$ *is* x_2^{r+3}*;*

$$\langle \mathbf{12} \rangle,$$

where $\zeta(x_1x_3, x_2)$ *is* x_2^2*; and*

$$\langle \mathbf{2} \rangle,$$

where $\zeta(x_1x_3, x_2)$ *is* x_2*.*

Proposition 42. *The general solution of the equation*

42 $$x_1x_2x_3x_4 = \zeta(x_2, x_3)x_1$$

where x_2 *and* x_3 *occur in* $\zeta(x_2, x_3)$*, is described by the transformations*

$$\begin{cases} x_2 \to 1, \\ x_3 \to 1, \\ x_4 \to 1; \end{cases}$$

$$x_1 \to (\varphi(x_2, x_3)x_2\upsilon(x_2, x_3))^{\alpha}\varphi(x_2, x_3)x_1, \quad x_2 \to x_1x_2, \quad \langle \mathbf{40} \rangle,$$

where $\zeta(x_2, x_3)$ *is* $\varphi(x_2, x_3)x_2\upsilon(x_2, x_3)$*; and*

$$x_1 \to (\varphi(x_2, x_3)x_3\upsilon(x_2, x_3))^{\alpha}\varphi(x_2, x_3)x_1, \quad x_3 \to x_1x_3, \quad \langle \mathbf{41} \rangle,$$

where $\zeta(x_2, x_3)$ *is* $\varphi(x_2, x_3)x_3\upsilon(x_2, x_3)$*.*

Proposition 43. *The general solution of the equation*

43 $$x_1x_2x_3x_4 = x_2\zeta(x_1x_2, x_3)x_5 \quad \text{with } \partial(x_2) > 0$$

is described by the transformations

$$\langle \mathbf{34} \rangle$$

where $\zeta(x_1x_2, x_3)$ *is* 1;

$$\begin{cases} x_1 \to x_1^{\alpha}, \\ x_2 \to x_1^{\beta}, \end{cases}$$
$$\langle \mathbf{33} \rangle,$$

where $\zeta(x_1x_2, x_3)$ *is* $(x_1x_2)^{q+1}$;

$$x_4 \to x_4x_2\upsilon(x_1x_2, x_3)x_5,$$
$$\langle \mathbf{40} \rangle,$$

where $\zeta(x_1x_2, x_3)$ *is* $(x_1x_2)^q x_3^{r+1}x_1x_2\upsilon(x_1x_2, x_3)x_5$;

$$\langle \mathbf{37} \rangle,$$

where $\zeta(x_1x_2, x_3)$ *is* x_3^{r+1}; *and*

$$\begin{cases} x_1 \to x_1^{\alpha}, \\ x_2 \to x_1^{\beta}, \end{cases}$$
$$x_4 \to x_4x_3^r x_5,$$
$$\langle \mathbf{5} \rangle,$$

where $\zeta(x_1, x_2)$ *is* $(x_1x_2)^{q+1}x_3^{r+1}$.

Proposition 44. *The general solution of the equation*

44 $$x_2x_1x_3x_4 = x_3\zeta(x_1x_3, x_2)x_5 \quad \text{with } \partial(x_3) > 0$$

is described by the transformations

$$\langle \mathbf{36} \rangle,$$

where $\zeta(x_1x_3, x_2)$ *is* 1;

$$\langle \mathbf{35} \rangle,$$

where $\zeta(x_1x_3, x_2)$ *is* $(x_1x_3)^{q+1}$;

$$\langle \mathbf{39} \rangle,$$

where $\zeta(x_1x_3, x_2)$ *is* x_2^{r+1};

$$x_4 \to x_4\upsilon(x_1x_3, x_2)x_5,$$
$$\langle \mathbf{29} \rangle,$$

where $\zeta(x_1x_3, x_2)$ *is* $(x_1x_3)^{q+1}x_2\upsilon(x_1x_3, x_2)$;

$$x_4 \to x_4x_3\upsilon(x_1x_3, x_2)x_5,$$
$$\langle \mathbf{2} \rangle,$$

where $\zeta(x_1x_3, x_2)$ *is* $x_2x_1x_3\upsilon(x_1x_3, x_2)x_5$;

$$\langle \mathbf{12} \rangle,$$

where $\zeta(x_1x_3, x_2)$ *is* $x_2^2x_1x_3\upsilon(x_1x_3, x_2)x_5$; *and*

$$\langle \mathbf{20} \rangle,$$

where $\zeta(x_1x_3, x_2)$ *is* $x_2^{s+3}x_1x_3\upsilon(x_1x_3, x_2)x_5$.

Proposition 45. *The general solution of the equation*

45 $$x_1x_2x_3x_4 = \zeta(x_2, x_3)x_5$$

where x_2, x_3 *occur in* $\zeta(x_2, x_3)$ *is described by the transformations*

$$\begin{cases} x_1 \to \zeta(x_2, x_3)x_1, \\ x_5 \to x_1x_2x_3x_4; \end{cases}$$
$$x_1 \to \varphi(x_2, x_3)x_1,$$
$$x_2 \to x_1x_2,$$
$$\langle \mathbf{43} \rangle,$$

where $\zeta(x_2, x_3)$ *is* $\varphi(x_2, x_3)x_2\upsilon(x_2, x_3)$; *and*

$$x_1 \to \varphi(x_2, x_3)x_1,$$
$$x_3 \to x_1x_3,$$
$$\langle \mathbf{44} \rangle,$$

where $\zeta(x_2, x_3)$ *is* $\varphi(x_2, x_3)x_3\upsilon(x_2, x_3)$.

Proposition 46. *The general solution of the equation*

46 $$x_1x_2x_3x_4 = \zeta(x_1, x_2)x_3,$$

where x_1, x_2 *occur in* $\zeta(x_1, x_2)$, *is described by the transformation*

$$\begin{cases} x_1 \to x_1^{\alpha}, \\ x_2 \to x_1^{\beta}, \end{cases}$$
$$\langle \mathbf{5} \rangle.$$

Proposition 47. *The general solution of the equation*

47 $$x_1x_2x_3x_4 = \zeta(x_1, x2)x_5,$$

where x_1, x_2 *occur in* $\zeta(x_1, x_2)$, *is described by the transformation*

$$\begin{cases} x_1 \to x_1^{\alpha}, \\ x_2 \to x_1^{\beta}, \end{cases}$$
$$\langle \mathbf{33} \rangle.$$

Proposition 48. *The general solution of the equation*

48 $$x_2x_1x_3x_4 = (x_3x_1)^{r+1}\zeta(x_1x_3, (x_1x_3)^{\lambda}x_1)x_2$$

with $\partial(x_2) \leq \partial(x_3x_1)$ *is described by the transformations*

$$x_3 \to x_2x_3,$$
$$x_4 \to (x_3x_1x_2)^r,$$
$$\langle \mathbf{2} \rangle,$$

where $\zeta(x_1x_3, (x_1x_3)^{\lambda}x_1)$ *is* 1;

$$x_3 \to x_2x_3,$$
$$x_4 \to x_4P,$$
$$\langle \mathbf{12} \rangle,$$

where $\zeta(x_1x_2x_3, (x_1x_2x_3)^\lambda x_1)x_2$ *can be* x_1x_2P*;*

$$\begin{gathered} x_3 \to x_2x_3, \\ x_4 \to x_4P, \\ \langle \mathbf{20} \rangle, \end{gathered}$$

where $\zeta(x_1x_2x_3, (x_1x_2x_3)^\lambda x_1)x_2$ *can be* $x_1^{q+2}x_2P$*;*

$$\begin{gathered} x_2 \to x_3x_2, \\ x_1 \to x_2x_1, \\ x_4 \to (x_1x_3x_2)^r, \\ \langle \mathbf{2} \rangle, \end{gathered}$$

where $\zeta(x_1x_3, (x_1x_3)^\lambda x_1)$ *is* 1*;*

$$\begin{gathered} x_2 \to x_3x_2, \\ x_1 \to x_2x_1, \\ x_4 \to x_1(x_3x_2x_1)^{r-1}\zeta(x_2x_1x_3, (x_2x_1x_3)^\lambda x_2x_1)x_3x_2, \\ \langle \mathbf{2} \rangle, \end{gathered}$$

where $r > 0$*; and*

$$\begin{gathered} x_2 \to x_3x_2, \\ x_1 \to x_2x_1, \\ x_4 \to x_4P, \\ \begin{cases} x_1 \to x_1^\alpha, \\ x_2 \to x_1^\beta, \end{cases} \\ \langle \mathbf{5} \rangle, \end{gathered}$$

where $r = 0$*, and* $x_1\zeta(x_2x_1x_3, (x_2x_1x_3)^\lambda x_2x_1)x_3x_2$ *can be* $(x_2x_1)^{s+1}x_3P$*.*

Proposition 49. *The general solution of the equation*

49 $$x_2x_1x_3x_4 = x_1x_3\zeta(x_1x_3, (x_1x_3)^\lambda x_1)x_2$$

with $\partial(x_2) \le \partial(x_1x_3)$ *is described by the transformations*

$$\begin{gathered} x_1 \to x_2x_1, \\ x_4 \to P, \\ \langle \mathbf{2} \rangle, \end{gathered}$$

where $\zeta(x_2x_1x_3, (x_2x_1x_3)^\lambda x_2x_1)x_2$ *can be* x_2P*;*

$$\begin{gathered} x_2 \to x_1x_2, \\ x_3 \to x_2x_3, \\ x_4 \to P, \\ \langle \mathbf{2} \rangle, \end{gathered}$$

where $\zeta(x_1x_2x_3, (x_1x_2x_3)^\lambda x_1)x_1x_2$ *can be* x_1x_2P*;*

$$\begin{gathered} x_2 \to x_1x_2, \\ x_3 \to x_2x_3, \\ x_4 \to x_4P, \\ \langle \mathbf{12} \rangle, \end{gathered}$$

where $\zeta(x_1x_2x_3,(x_1x_2x_3)^\lambda x_1)x_1x_2$ *can be* $x_1^2x_2P$*;*

$$\begin{gathered} x_2 \to x_1x_2, \\ x_3 \to x_2x_3, \\ x_4 \to x_4P, \\ \langle \mathbf{20} \rangle, \end{gathered}$$

where $\zeta(x_1x_2x_3,(x_1x_2x_3)^\lambda x_1)x_1x_2$ *can be* $x_1^{q+3}x_2P$*; and*

$$\begin{gathered} x_2 \to x_1x_2, \\ x_3 \to x_2x_3, \\ x_4 \to x_4P, \\ \langle \mathbf{29} \rangle, \end{gathered}$$

where $\zeta(x_1x_2x_3,(x_1x_2x_3)^\lambda x_1)x_1x_2$ *is* $x_2x_3x_1P$*.*

Proposition 50. *The general solution of the equation*

50 $$x_2x_1x_3x_4 = (x_1x_3)^\lambda x_1\zeta(x_1x_3, x_3x_1, (x_1x_3)^\lambda x_1)x_2$$

with $\partial(x_2) \le \partial((x_1x_3)^\lambda x_1)$ *is described by the transformations*

$$\begin{gathered} x_2 \to (x_1x_3)^\tau x_2, \\ x_1 \to x_2x_1, \\ x_4 \to P, \\ \langle \mathbf{2} \rangle, \end{gathered}$$

where $(x_2x_1x_3)^{\lambda-\tau}\zeta(x_2x_1x_3, x_3x_2x_1, (x_2x_1x_3)^\lambda x_2x_1)(x_2x_1x_3)^\tau x_2$ *can be* x_3x_2P*;*

$$\begin{gathered} x_2 \to (x_1x_3)^\tau x_2, \\ x_1 \to x_2x_1, \\ x_3 \to 1, \\ x_4 \to 1, \\ \langle \mathbf{1} \rangle, \end{gathered}$$

where $(x_2x_1x_3)^{\lambda-\tau}\zeta(x_2x_1x_3, x_3x_2x_1, (x_2x_1x_3)^\lambda x_2x_1)(x_2x_1x_3)^\tau x_2$ *can be* x_2*;*

$$\begin{gathered} x_2 \to (x_1x_3)^\tau x_2, \\ x_1 \to x_2x_1, \\ x_4 \to x_4P, \\ \begin{cases} x_1 \to x_1^\alpha, \\ x_2 \to x_1^\beta, \end{cases} \\ \langle \mathbf{5} \rangle, \end{gathered}$$

where $(x_2x_1x_3)^{\lambda-\tau}\zeta(x_2x_1x_3, x_3x_2x_1, (x_2x_1x_3)^\lambda x_2x_1)(x_2x_1x_3)^\tau x_2$ *can be* $x_2x_1x_3P$*;*

$$\begin{gathered} x_2 \to (x_1x_3)^\tau x_1x_2, \\ x_3 \to x_2x_3, \\ x_4 \to P, \\ \langle \mathbf{2} \rangle, \end{gathered}$$

where $(x_2x_3x_1)^{\lambda-\tau-1}\zeta(x_1x_2x_3, x_2x_3x_1, (x_1x_2x_3)^\lambda x_1)(x_1x_2x_3)^\tau x_1x_2$ *is* x_1x_2P*; and*

$$\begin{gathered} x_2 \to (x_1x_3)^\tau x_1x_2, \\ x_3 \to x_2x_3, \\ x_4 \to x_4P, \\ \langle \mathbf{29} \rangle, \end{gathered}$$

where $(x_2x_3x_1)^{\lambda-\tau-1}\zeta(x_1x_2x_3, x_2x_3x_1, (x_1x_2x_3)^\lambda x_1)(x_1x_2x_3)^\tau x_1x_2 = x_2x_3x_1P$.

Proposition 51. *The general solution of the equation*

51 $$x_2x_1x_3x_4 = (x_3x_1)^{r+1}\zeta(x_2x_3, (x_1x_3)^\lambda x_1)x_5$$

with $\partial(x_2) \le \partial(x_3x_1)$ *is described by the transformations*

$$\begin{gathered} x_3 \to x_2x_3,\\ x_5 \to x_5x_4,\\ \langle \mathbf{10} \rangle, \end{gathered}$$

where $r = 0$ *and* $\zeta(x_1x_3, (x_1x_3)^\lambda x_1)$ *is* 1;

$$\begin{gathered} x_3 \to x_2x_3,\\ x_4 \to P,\\ \langle \mathbf{2} \rangle \end{gathered}$$

where $(x_2x_3x_1)^r\zeta(x_1x_2x_3, (x_1x_2x_3)^\lambda x_1)x_5$ *is* x_2P;

$$\begin{gathered} x_3 \to x_2x_3,\\ x_4 \to x_4P,\\ \langle \mathbf{12} \rangle, \end{gathered}$$

where $(x_2x_3x_1)^r\zeta(x_1x_2x_3, (x_1x_2x_3)^\lambda x_1)x_5$ *can be* x_1x_2P;

$$\begin{gathered} x_3 \to x_2x_3,\\ x_4 \to x_4P,\\ \langle \mathbf{20} \rangle, \end{gathered}$$

where $(x_2x_3x_1)^r\zeta(x_1x_2x_3, (x_1x_2x_3)^\lambda x_1)x_5$ *can be* $x_1^{q+2}x_2P$;

$$\begin{gathered} x_2 \to x_3x_2,\\ x_1 \to x_2x_1,\\ x_5 \to x_5x_3x_4,\\ \langle \mathbf{4} \rangle, \end{gathered}$$

where $r = 0$, $\zeta(x_1x_3, (x_1x_3)^\lambda x_1) = 1$;

$$\begin{gathered} x_2 \to x_3x_2,\\ x_1 \to x_2x_1,\\ x_4 \to x_1(x_3x_2x_1)^{r-1}\zeta(x_2x_1x_3, (x_2x_1x_3)^\lambda x_2x_1)x - 5,\\ \langle \mathbf{2} \rangle, \end{gathered}$$

where $r > 0$;

$$\begin{gathered} x_2 \to x_3x_2,\\ x_1 \to x_2x_1,\\ x_4 \to x_4P,\\ \begin{cases} x_1 \to x_1^\alpha,\\ x_2 \to x_1^\beta, \end{cases}\\ \langle \mathbf{5} \rangle, \end{gathered}$$

where $r = 0$ *and* $\zeta(x_2x_1x_3, (x_2x_1x_3)^\lambda x_2x_1)x_5$ *can be* $(x_2x_1)^{q+1}x_3P$; *and*

$$\begin{gathered} x_2 \to x_3x_2, \\ x_1 \to x_2x_1, \\ \begin{cases} x_1 \to x_1^\alpha, \\ x_2 \to x_1^\beta, \end{cases} \\ \langle \mathbf{33} \rangle, \end{gathered}$$

where $r = 0$ *and* $\zeta(x_2x_1x_3, (x_2x_1x_3)^\lambda x_2x_1)$ *can be* $(x_2x_1)^{q+1}$.

Proposition 52. *The general solution of the equation*

52 $$x_2x_1x_3x_4 = x_1x_3\zeta(x_1x_3, (x_1x_3)^\lambda x_1)x_5$$

with $\partial(x_2) \le \partial(x_1x_3)$ *is described by the transformations*

$$\begin{gathered} x_1 \to x_2x_1, \\ x_5 \to x_5x_4, \\ \langle \mathbf{7} \rangle, \end{gathered}$$

where $\zeta(x_1x_3, (x_1x_3)^\lambda x_1)$ *is* 1;

$$\begin{gathered} x_1 \to x_2x_1, \\ x_4 \to Px_5, \\ \langle \mathbf{2} \rangle, \end{gathered}$$

where $\zeta(x_2x_1x_3, (x_2x_1x_3)^\lambda x_2x_1)$ *can be* x_2P;

$$\begin{gathered} x_2 \to x_1x_2, \\ x_3 \to x_2x_3, \\ x_5 \to x_5x_4, \\ \langle \mathbf{6} \rangle, \end{gathered}$$

where $\zeta(x_1x_3, (x_1x_3)^\lambda x_1)$ *is* 1;

$$\begin{gathered} x_2 \to x_1x_2, \\ x_3 \to x_2x_3, \\ \langle \mathbf{39} \rangle, \end{gathered}$$

where $\zeta(x_1x_2x_3, (x_1x_2x_3)^\lambda x_1)$ *can be* x_1^{q+1};

$$\begin{gathered} x_2 \to x_1x_2, \\ x_3 \to x_2x_3, \\ x_4 \to P \\ \langle \mathbf{2} \rangle, \end{gathered}$$

where $\zeta(x_1x_2x_3, (x_1x_2x_3)^\lambda x_1)$ *can be* x_1x_2P;

$$\begin{gathered} x_2 \to x_1x_2, \\ x_3 \to x_2x_3, \\ x_4 \to x_4P, \\ \langle \mathbf{12} \rangle, \end{gathered}$$

where $\zeta(x_1x_2x_3, (x_1x_2x_3)^\lambda x_1)$ *can be* $x_1^2x_2P$; *and*

$$\begin{gathered} x_2 \to x_1x_2, \\ x_3 \to x_2x_3, \\ x_4 \to x_4P, \\ \langle \mathbf{20} \rangle, \end{gathered}$$

where $\zeta(x_1x_2x_3,(x_1x_2x_3)^\lambda x_1)$ *can be* $x_1^{s+3}x_2P$.

Proposition 53. *The general solution of the equation*

53 $$x_2x_1x_3x_4=(x_1x_3)^\lambda x_1\zeta(x_1x_3,(x_1x_3)^\lambda x_1)x_5$$

with $\partial(x_2)\le\partial((x_1x_3)^\lambda x_1)$ *is described by the transformations*

$$\begin{gathered}x_2\to(x_1x_3)^\lambda x_2,\\ x_1\to x_2x_1,\\ x_5\to x_5x_3x_4,\\ \langle\mathbf{4}\rangle,\end{gathered}$$

where $\zeta(x_1x_3,(x_1x_3)^\lambda x_1)$ *is* 1;

$$\begin{gathered}x_2\to(x_1x_3)^\tau x_2,\\ x_1\to x_2x_1,\\ x_4\to x_4P,\\ \begin{cases}x_1\to x_1^\alpha,\\ x_2\to x_1^\beta,\end{cases}\\ \langle\mathbf{5}\rangle,\end{gathered}$$

where $(x_2x_1x_3)^{\lambda-\tau}\zeta(x_2x_1x_3,(x_2x_1x_3)^\lambda x_2x_1)x_5$ *can be* $(x_2x_1)^{q+1}x_3P$*;*

$$\begin{gathered}x_2\to(x_1x_3)^\tau x_2,\\ x_1\to x_2x_1,\\ \begin{cases}x_1\to x_1^\alpha,\\ x_2\to x_1^\beta,\end{cases}\\ \langle\mathbf{33}\rangle,\end{gathered}$$

where $(x_2x_1x_3)^{\lambda-\tau}\zeta(x_2x_1x_3,(x_2x_1x_3)^\lambda x_2x_1)$ *can be* $(x_2x_1)^{q+1}$;

$$\begin{gathered}x_2\to(x_1x_3)^{\lambda-1}x_1x_2,\\ x_3\to x_2x_3,\\ x_5\to x_5x_4,\\ \langle\mathbf{10}\rangle,\end{gathered}$$

where $\zeta(x_1x_3,(x_1x_3^\lambda x_1)$ *is* 1;

$$\begin{gathered}x_2\to(x_1x_3)^{\lambda-1}x_1x_2,\\ x_3\to x_2x_3,\\ \langle\mathbf{39}\rangle,\end{gathered}$$

where $\zeta(x_1x_2x_3,(x_1x_2x_3)^\lambda x_1)$ *can be* x_1^{s+1};

$$\begin{gathered}x_2\to(x_1x_3)^{\lambda-1},\\ x_3\to x_2x_3,\\ x_4\to P,\\ \langle\mathbf{2}\rangle,\end{gathered}$$

where $\zeta(x_1x_2x_3,(x_1x_2x_3)^\lambda x_1)$ *can be* x_1x_2P;

$$\begin{gathered}x_2\to(x_1x_3)^{\lambda-1},\\ x_3\to x_2x_3,\\ x_4\to x_4P,\\ \langle\mathbf{12}\rangle,\end{gathered}$$

where $\zeta(x_1x_2x_3, (x_1x_2x_3)^\lambda x_1)$ *can be* $x_1^2x_2P$*; and*

$$\begin{gathered} x_2 \to (x_1x_3)^{\lambda-1}, \\ x_3 \to x_2x_3, \\ x_4 \to x_4P, \\ \langle \mathbf{20} \rangle, \end{gathered}$$

where $\zeta(x_1x_2x_3, (x_1x_2x_3)^\lambda x_1)$ *can be* $x_1^{s+3}x_2$.

Proposition 54. *The general solution of the equation*

54 $$x_1x_2x_3x_4 = x_3^{r+1}x_1\zeta(x_1, x_3)x_2,$$

where r *is a natural number, is described by the transformations*

$$\begin{cases} x_1 \to 1, \\ x_3 \to 1, \\ x_4 \to 1; \end{cases}$$

$$\begin{cases} x_1 \to (x_1x_3)^\lambda x_1, \\ x_3 \to x_1x_3, \\ x_2 \to ((x_3x_1)^{r+1}\zeta((x_1x_3)^\lambda x_1, x_1x_3))^\alpha \upsilon((x_1x_3)^\lambda x_1, x_3x_1, x_1x_3)x_2, \end{cases}$$

where $x_5^{r+1}\zeta(x_1, x_3)$ *is* $\upsilon(x_1, x_5, x_3)\varphi(x_1, x_5, x_3)$*, followed by one of the three transformations* $\langle \mathbf{48} \rangle$, $\langle \mathbf{49} \rangle$, $\langle \mathbf{50} \rangle$.

Proposition 55. *The general solution of the equation*

55 $$x_1x_2x_3x_4 = x_3^{r+1}x_1\zeta(x_1, x_3)x_5,$$

where r *is a natural number, is described by the transformations*

$$\begin{cases} x_1 \to (x_1x_3)^\lambda x_1, \\ x_3 \to x_1x_3, \\ x_2 \to \varphi((x_1x_3)^\lambda x_1, x_3x_1, x_1x_3)x_2, \end{cases}$$

where $x_6^{r+1}\zeta(x_1, x_3)$ *is* $\varphi(x_1, x_6, x_3)\upsilon(x_1, x_6, x_3)$*, followed by one of the four transformations* $\langle \mathbf{51} \rangle$, $\langle \mathbf{52} \rangle$, $\langle \mathbf{53} \rangle$*, or* $x_5 \to x_2x_1x_3x_4$*, where* $\upsilon(x_1, x_6, x_3)$ *is* 1.

Proposition 56. *The general solution of the equation*

56 $$x_1x_2x_3x_4 = \zeta(x_1, x_2, x_3)x_5$$

is described by the transformations $\langle \mathbf{42} \rangle$; $\langle \mathbf{46} \rangle$; $\langle \mathbf{54} \rangle$; $\langle \mathbf{45} \rangle$; $\langle \mathbf{47} \rangle$; $\langle \mathbf{55} \rangle$; $\langle \mathbf{34} \rangle$; $\langle \mathbf{36} \rangle$; $x_5 \to x_1x_2x_3x_4$*, where* $\zeta(x_1, x_2, x_3) = 1$.

Proof. If x_1, x_2, x_3 occur in $\zeta(x_1, x_2, x_3)$, then the equation **56** has one of the following three forms:

$$E_1\colon \quad x_1x_2x_3x_4 = \upsilon(x_2, x_3)x_1\varphi(x_1, x_2, x_3)x_5,$$

where x_2 and x_3 occur in $\upsilon(x_2, x_3)$;

$$E_2\colon \quad x_1x_2x_3x_4 = \upsilon(x_1, x_2)x_3\varphi(x_1, x_2, x_3)x_5,$$

where x_1 and x_2 occur in $\upsilon(x_1, x_2)$;

$$E_3\colon \quad x_1x_2x_3x_4 = \upsilon(x_1, x_3)x_2\varphi(x_1, x_2, x_3)x_5,$$

where x_1 and x_3 occur in $\upsilon(x_1, x_3)$.

The equation E_1 can be reduced by the transformation

$$x_4 \to x_4 \varphi(x_1, x_2, x_3) x_5$$

to the equation **42**.

The equation E_2 can be reduced by the transformation

$$x_4 \to x_4 \varphi(x_1, x_2, x_3) x_5$$

to the equation **46**.

The equation E_3 can be reduced by the transformation

$$x_4 \to x_4 \varphi(x_1, x_2, x_3) x_5$$

to the equation **54**.

If only x_2, x_3 occur in $\zeta(x_1, x_2, x_3)$, then **56** is **45**.

If only x_1, x_2 occur in $\zeta(x_1, x_2, x_3)$, then **56** is **47**.

If only x_1, x_3 occur in $\zeta(x_1, x_2, x_3)$, then **56** is **55**.

If only x_2 occurs in $\zeta(x_1, x_2, x_3)$, then **56** is **34**.

If only x_3 occurs in $\zeta(x_1, x_2, x_3)$, then **56** is **36**.

If $\zeta(x_1, x_2, x_3)$ is the empty word, then **56** has the form $x_1 x_2 x_3 x_4 = x_5$.

References

[1] R. C. Lyndon, *Equations in free groups*, Trans. Amer. Math. Soc. **96** (1960), 445–457. MR **27:**1488

[2] ———, *Groups with parametric exponents*, Trans. Amer. Math. Soc. **96** (1960), 518–533. MR **27:**1487

[3] Yu. I. Khmelevskiĭ, *Solution of word equations in three unknowns*, Dokl. Akad. Nauk SSSR **177** (1967), 1023–1025; English transl., Soviet Math. Dokl. **8** (1967), 1554–1556. MR **36:**3899

[4] ———, *Equation in free semigroups*, Trudy Mat. Inst. Steklov. **107** (1971); English transl., Proc. Steklov Inst. Math. **107** (1971) (1976). MR **51:**5808; MR **52:**14094

[5] G. S. Makanin and P. Goralcik, *Structure of the solutions of one parametric equation*, Report of research, LIR, INSA of Rouen, April 1996.

[6] G. S. Makanin and H. Abdulrab, *Functions for the general solution of parametric word equations*, Report of research, LIR, INSA of Rouen, June 1996.

[7] G. S. Makanin, *Systems of standard equations in words of an n-layer alphabet with unknowns*, Sibirsk. Mat. Zh. **19** (1978), 637–645; English transl., Siberian Math. J. **19** (1978), 448–454. MR **80a:**20039

[8] ———, *Recognition of the rank of equations in a free semigroup*, Izv. Akad. Nauk SSSR Ser. Mat. **43** (1979), 547–602; English transl., Math. USSR Izv. **14** (1980), 499–545. MR **80h:**20084

[9] M. Lothaire, *Combinatorics on words*, Addison-Wesley, 1983. MR **84g:**05002

Steklov Mathematical Institute, Vavilova 42, 117 966, Moscow GSP-1, Russia