

THE NOETHERIAN PROPERTY IN SOME QUADRATIC ALGEBRAS

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ABSTRACT. We introduce a new class of noncommutative rings called *pseudopolynomial rings* and give sufficient conditions for such a ring to be Noetherian. Pseudopolynomial rings are standard finitely presented algebras over a field with some additional restrictions on their defining relations—namely that the polynomials in a Gröbner basis for the ideal of relations must be homogeneous of degree 2—and on the Ufnarovskii graph $\Gamma(A)$. The class of pseudopolynomial rings properly includes the generalized skew polynomial rings introduced by M. Artin and W. Schelter. We use the graph $\Gamma(A)$ to define a weaker notion of almost commutative, which we call *almost commutative on cycles*. We show as our main result that a pseudopolynomial ring which is almost commutative on cycles is Noetherian. A counterexample shows that a Noetherian pseudopolynomial ring need not be almost commutative on cycles.

1. INTRODUCTION

Determining whether or not certain classes of noncommutative rings are Noetherian has been the focus of much previous research, and no doubt this topic will continue to be of interest since it is one of the most basic questions one might ask about any new class of rings. In this paper, we will be concerned with *quadratic algebras*. Let K be a field, X a finite set, $K\langle X \rangle$ the free associative algebra generated by X over K , and $I \subset K\langle X \rangle$ an ideal. We say that the algebra $A = K\langle X \rangle/I$ is *quadratic* if I is generated by quadratic polynomials in $K\langle X \rangle$. The Noetherian property has been studied for several important classes of quadratic algebras, including skew polynomial rings, algebras of solvable type, and Sklyanin algebras. Algebras of solvable type are a broad class which includes commutative polynomial rings, iterated Ore extensions of K , and universal enveloping algebras of finite dimensional Lie algebras over K . We give a survey of these results in Section 5. The object of study in this paper will be a new class of quadratic algebras called *pseudopolynomial rings* which will be formally defined in Section 6 (we give a sketch below). We will see that the class of pseudopolynomial rings properly includes the generalized skew polynomial rings introduced by M. Artin and W. Schelter in [AS]. Our major tools will be the theory of Gröbner bases in free associative algebras

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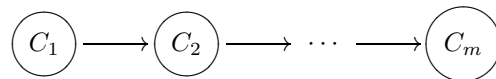
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and a graph, introduced by V. Ufnarovskii [U1], which may be associated to a quotient of a free algebra (given a sufficiently nice presentation) and used to study the quotient algebra.

Gröbner bases were first introduced for ideals in commutative polynomial rings by B. Buchberger [Bu], who also gave an algorithm for computing them. These bases have proved to be powerful computational tools in commutative algebra and algebraic geometry. That Buchberger's algorithm will indeed halt is guaranteed by the fact that commutative polynomial rings are Noetherian. One of the major difficulties in extending the theory of Gröbner bases and Buchberger's algorithm to noncommutative rings is that such rings are generally not Noetherian. Nonetheless, there is a theory of Gröbner bases for free associative algebras, due to G. Bergman [B] and F. Mora [M1], [M4]. Theories of Gröbner bases for a number of other noncommutative settings are a subject of current research; in this context, being able to determine whether an algebra is Noetherian is helpful for developing a Gröbner basis theory for the algebra. The great power of a Gröbner basis G of an ideal $I \subset K\langle X \rangle$ is that it gives us a means of determining a canonical representative in $K\langle X \rangle$ of an element in the quotient $K\langle X \rangle/I$ and of solving the ideal membership problem for I . That is, given an element $f \in K\langle X \rangle$, we can determine whether or not $f \in I$.

To give an idea of what a pseudopolynomial ring is, suppose that we have a finite set $G \subset K\langle X \rangle$, and we let I be the ideal generated by G in $K\langle X \rangle$. If G is a Gröbner basis for I , then the quotient algebra $A = K\langle X \rangle/I$ is called *standard finitely presented*. To such an algebra A we may associate a finite directed graph $\Gamma(A)$ which will reflect such properties of A as the growth of A . Pseudopolynomial rings are standard finitely presented algebras with additional restrictions on relations in I —namely that the polynomials in G must be homogeneous of degree 2—and on the graph $\Gamma(A)$. So pseudopolynomial rings are examples of quadratic algebras. For a pseudopolynomial ring, the vertices of $\Gamma(A)$ are exactly the elements of X , and the collection of all paths in $\Gamma(A)$ (including the empty path) will give a vector space basis for A . We also require that there be no intersecting cycles in $\Gamma(A)$, and that the graph $\Gamma(A)$ looks like a string of m cycles:



We allow multiple edges between cycles, and there may be edges from a cycle C_i to a cycle C_j as long as $i < j$; however, there may not be an edge from C_j to C_i . The idea of pseudopolynomial rings is that we wish to view A as being generated as an algebra by the cycles in $\Gamma(A)$, rather than the elements of X , as much as possible.

Recall that a finitely generated K -algebra R is *almost commutative* if it has a filtration $\{R_n\}$ such that $R_1^n = R_n$ for each n , $R_0 = K$, and $\dim_K R_n < \infty$, and the associated graded ring

$$\text{gr}(R) = \bigoplus_{n \geq 0} R_n/R_{n-1}$$

is commutative. (Note that $R_{-1} = 0$.) Almost commutative algebras are Noetherian, and are an interesting class in their own right, as an algebra is almost commutative if and only if it is a homomorphic image of the enveloping algebra of some finite dimensional Lie algebra over K [MR, pp. 299–300]. If $\langle X \rangle$ is the free associative monoid generated by X , we may put a *term ordering* on $\langle X \rangle$. Using

this term ordering, we may determine the set of *leading monomials* of I , denoted $\text{LM}(I)$. Then the set $N = \langle X \rangle \setminus \text{LM}(I)$ is a K -basis for A . For any $f \in K\langle X \rangle$, let $\text{Nor}(f)$ be the image of f in A . We may assume $\text{Nor}(f) \in \text{Span } N$, and denote the leading monomial of $\text{Nor}(f)$ by \bar{f} . We say A is almost commutative if for $u, v \in N$, we have $\overline{uv} = \overline{vu}$ [GI1]. If we consider the N -filtration $\{A_u : u \in N\}$ on A , where $A_u = \text{Span}\{v \in N : v \leq u\}$, we may associate to A an N -graded structure, which is commutative [GI1], [M3]. So this definition of almost commutative is a natural extension of the usual one. We will further extend this idea to a definition of *almost commutative on cycles* for pseudopolynomial rings A : if $u, v \in N$ correspond to paths in different cycles in $\Gamma(A)$, then $\overline{uv} = \overline{vu}$. Our main result, Theorem 6.15, shows that if a pseudopolynomial ring is almost commutative on cycles, then it is Noetherian. The converse is false, however.

Finally, we note that given a standard finitely presented algebra $A = K\langle X \rangle/I$, one can effectively determine, by means of the graph $\Gamma(A)$, whether or not A is indeed a pseudopolynomial ring. The Gröbner basis G of I allows us to draw the graph, and once we have $\Gamma(A)$, it is easy to determine the cycles of $\Gamma(A)$ and thus to check whether A satisfies the conditions of a pseudopolynomial ring. From the graph, we can read off exactly what relations are needed for A to be almost commutative on cycles. We then see if these relations are in the ideal I . If so, we know A is almost commutative on cycles, and hence, Noetherian.

2. GRÖBNER BASES

In this section, we establish notation and briefly review the theory of Gröbner bases in free associative algebras. A detailed treatment may be found in [M1] or [M4].

Let K be a field, X a finite set, and $\langle X \rangle$ the free associative monoid generated by X . The elements of X are called *letters*, and those of $\langle X \rangle$ are called *words*. The identity element 1 in $\langle X \rangle$ is the empty word (i.e., the word composed of no letters). For a subset $U \subset \langle X \rangle$, we write $\langle U \rangle$ for the monoid ideal generated by U in $\langle X \rangle$. Let $K\langle X \rangle$ be the free associative algebra generated by $\langle X \rangle$ over K . For a subset $F \subset K\langle X \rangle$, the ideal generated by F will be written $\langle F \rangle$.

For any nonzero element $f \in K\langle X \rangle$, we define the *leading monomial* of f , denoted $\text{LM}(f)$, to be the largest monomial (with respect to the ordering $<$) to appear in the representation of f in the K -basis $\langle X \rangle$. The *leading coefficient* $\text{LC}(f)$ of f is the coefficient of $\text{LM}(f)$, and the *leading term* $\text{LT}(f)$ of f is

$$\text{LT}(f) = \text{LC}(f)\text{LM}(f).$$

For $F \subseteq K\langle X \rangle$, we define

$$\text{LM}(F) = \{u \in \langle X \rangle : u = \text{LM}(f) \text{ for some } f \in F\}.$$

For any (left, right, or two-sided) ideal $I \subset K\langle X \rangle$, we may determine the set

$$\text{LM}(I) = \{\text{LM}(f) \in \langle X \rangle : f \in I\},$$

which is a (left, right, two-sided) ideal of $\langle X \rangle$. We give the definition of a Gröbner basis for such an ideal I .

Definition 2.1. Let $I \subset K\langle X \rangle$ be an ideal. A set $F \subset I$ is a *Gröbner basis* for I if the monoid ideal generated by $\text{LM}(F)$ in $\langle X \rangle$ equals $\text{LM}(I)$. That is,

$$\langle \text{LM}(F) \rangle = \text{LM}(I);$$

or, equivalently,

$$(\text{LM}(F)) = (\text{LM}(I))$$

as ideals in $K\langle X \rangle$.

A Gröbner basis F is a generating set for the ideal I . Moreover, having this Gröbner basis allows us to find canonical representatives in $K\langle X \rangle$ of elements in the quotient $K\langle X \rangle/I$. Let $N(I) = \langle X \rangle \setminus \text{LM}(I)$. (We will usually write $N(I)$ as just N .) Then

$$K\langle X \rangle = I \oplus \text{Span } N$$

as vector spaces. So for all $f \in K\langle X \rangle$ [M1], [M4], there is a unique $g \in \text{Span } N$ such that $f - g \in I$. We denote this g by $\text{Nor}(f)$ and call it the *normal form* of f . The element $\text{Nor}(f) \in K\langle X \rangle$ is the canonical representation of the image of f in the quotient $K\langle X \rangle/I$. We now note two simple, but important, facts:

- (1) $f \in I$ if and only if $\text{Nor}(f) = 0$.
- (2) For $u \in \langle X \rangle$, $\text{Nor}(u) = u$ if and only if $u \in N$. Hence, the elements of N are called the *normal monomials* of $K\langle X \rangle/I$.

Finally, we note that some Gröbner bases are better than others.

Definition 2.2. A Gröbner basis G of an ideal $I \subset K\langle X \rangle$ is a *reduced Gröbner basis* if

1. $\{\text{LM}(g) : g \in G\}$ is an irredundant basis of $\text{LM}(I)$,
2. $\text{LC}(g) = 1$ for each $g \in G$, and
3. $g = \text{LM}(g) - \text{Nor}(\text{LM}(g))$ for each $g \in G$.

Any ideal $I \subset K\langle X \rangle$ has a unique reduced Gröbner basis [M4, Proposition 1.5]. If we have a finite Gröbner basis for I , it is then easy to find the reduced Gröbner basis for I .

3. THE UFNAROVSKII GRAPH

Let K be a field, X a finite set, and let $\langle X \rangle$ be the free monoid generated by X . Let $K\langle X \rangle$ be the free associative algebra generated by $\langle X \rangle$ over K . For a finite set $G \subset K\langle X \rangle$, let $R = (G)$, the (two-sided) ideal generated by G in $K\langle X \rangle$. Then the quotient, $A = K\langle X \rangle/R$, is a finitely presented algebra. If, in addition, the set G forms a Gröbner basis for R , we say that A is *standard finitely presented* (abbreviated s.f.p.). We will assume without loss of generality that G is a reduced Gröbner basis.

As noted in Section 2, the set $N = \langle X \rangle \setminus \text{LM}(R)$ forms a K -basis for A . Words in N are called *normal*. Every subword of a normal word is normal. A word that is not in N (so it is in $\text{LM}(R)$), but whose proper subwords are normal, is called an *obstruction*. So the set of obstructions F is just $\text{LM}(G)$ (since G is reduced). In [U1], Ufnarovskii shows how we may associate a finite directed graph $\Gamma(A)$ (which we will call the *Ufnarovskii graph*) to the algebra A in such a way that the elements of N are usefully reflected in the graph as paths.

A *graph* Γ is a finite set of vertices V joined by a finite set of edges E . We will assume that all graphs are directed. By a path of length n in Γ , we mean an alternating sequence of vertices and edges $v_0, e_1, v_1, e_2, v_2, \dots, v_{n-1}, e_n, v_n$ in which every edge e_i begins at vertex v_{i-1} and ends at vertex v_i . If $v_0 = v_n$, the path is called *cyclic*; and the subgraph of Γ consisting of the vertices v_i , $0 \leq i < n$, and

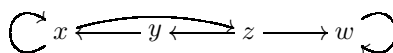
edges e_j , $1 \leq j \leq n$, is called a *cycle*. If a vertex in Γ is contained in some cycle, the vertex is *cyclic*; if the vertex is contained in no cycle, it is *acyclic*. A vertex which is contained in two distinct cycles is *doubly cyclic*.

In giving the construction of the graph $\Gamma(A)$, we will restrict our attention to the case in which G consists only of homogeneous polynomials of degree 2 since this case includes all pseudopolynomial rings. (We will define pseudopolynomial rings in Section 6.) We construct the graph $\Gamma(A)$ in the following way: Let the set of vertices $V = X$. We draw an edge $e = x_i x_j$ from a vertex x_i to a vertex x_j if and only if $x_i x_j \notin (F)$. Thus there is a one-to-one correspondence between the edges in the graph $\Gamma(A)$ and the normal words of length 2. (Since G is a Gröbner basis for R , we know that $(\text{LM}(G)) = (\text{LM}(R))$ as ideals in $K\langle X \rangle$. So words not in $(F) = (\text{LM}(G))$ are exactly the words in N .)

Example 3.1. Let $X = \{x, y, z, w\}$, and suppose

$$F = \{xy, xw, yy, yz, yw, zx, zz, wx, wy, wz\}.$$

So $\Gamma(A)$ is



For instance, there is an edge from x to z since $xz \notin (F)$. There is no edge from x to y since $xy \in (F)$.

We may extend the one-to-one correspondence between edges and normal words of length 2 to a one-to-one correspondence between paths in the graph and normal words of length greater than or equal to 2. Let $u \in N$ be such that the length of $u = m > 1$. Write

$$u = x_1 x_2 \cdots x_m$$

where each $x_i \in X$. Then we may uniquely associate u with the path of length $m - 1$ in $\Gamma(A)$ through the successive edges

$$x_1 x_2, x_2 x_3, \dots, x_{m-1} x_m.$$

Example 3.2. The word $yxzxw$ is normal in the algebra of Example 3.1. Since the length of $yxzxw$ is 5, this word corresponds to a path of length 4 in Γ . This path passes through the edges yx, xx, xz, zw ; that is, it is the path

$$\begin{aligned} v_0 = y, e_1 = yx, v_1 = x, e_2 = xx, v_2 = x, \\ e_3 = xz, v_3 = z, e_4 = zw, v_4 = w. \end{aligned}$$

4. THE GROWTH OF ALGEBRAS

We first recall the notion of growth for monotone increasing functions. For monotone increasing functions $f, g : \mathbb{N} \rightarrow \mathbb{R}^+$, we say that $f \leq g$ if and only if there exist natural numbers $m, c > 0$ such that $f(n) \leq cg(mn)$ for all $n \in \mathbb{N}$. Then we define an equivalence relation \sim on such functions by

$$f \sim g \text{ if and only if } f \leq g \text{ and } g \leq f.$$

The equivalence class of a function f is denoted $[f]$ and is called the *growth* of f . Growth $[2^n]$ is called *exponential*, and growth $[n^d]$ (where $d \in \mathbb{N}$) is called *polynomial of degree d* .

To define the growth of an algebra, suppose V is a finite dimensional vector space over K which generates A as an algebra. We define the function

$$d_V(n) = \dim_K(K + V + V^2 + \dots + V^n)$$

and take the growth of A to be the equivalence class $[d_V(n)]$. The growth of A is independent of the choice of V . For an extensive discussion of notion of growth in algebras, see [KL].

In [U1], Ufnarovskii shows that the growth of an s.f.p. algebra is either polynomial or exponential, and gives the following criterion in terms of the graph described in the previous section.

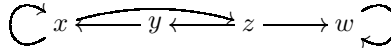
Theorem 4.1 ([U1]). *Suppose A is an s.f.p. algebra. Then the growth of A is either polynomial or exponential. Furthermore:*

- (1) *the growth of A is exponential if and only if $\Gamma(A)$ has a doubly cyclic vertex (i.e., there are two intersecting cycles);*
- (2) *the growth of A is polynomial if and only if $\Gamma(A)$ is cyclically simple (i.e., there are no intersecting cycles). The polynomial degree d is given by the maximal number of different cycles through which one path can pass.*

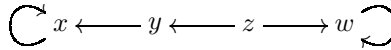
Example 4.2. If $A = K\langle X \rangle / (F)$ where $X = \{x, y, z, w\}$, and

$$F = \{xy, xw, yy, yz, yw, zx, zz, wx, wy, wz\}$$

as in Example 3.1, then we know that $\Gamma(A)$ is



Since x is a doubly cyclic vertex, the growth of A is exponential. If we were to add the word xz to F , then we would have



The growth would be polynomial of degree 1. If we were to add xx, xz , and w to F , the graph would be

$$x \longleftarrow y \longleftarrow z \longrightarrow w$$

and the growth of A would be polynomial of degree 0, i.e., constant. So A would be finite dimensional.

5. THE NOETHERIAN PROPERTY IN QUADRATIC ALGEBRAS

Let $A = K\langle X \rangle / I$ be a finitely presented algebra; if the defining relations in I are all quadratic, we will call the algebra A *quadratic* as well. In this section, we look at a variety of quadratic algebras and summarize results regarding the Noetherian property in these algebras. In the next section, we introduce a new class of quadratic algebras, which we call *pseudopolynomial rings*. Pseudopolynomial rings do not fit directly into any of the classes discussed below; however, we will see that the class of pseudopolynomial rings properly contains the generalized skew polynomial rings discussed by Artin and Schelter [AS].

An algebra $A = K\langle X \rangle / I$ for which I is a monomial ideal (i.e., the defining relations in I are all monomials) is called a *monomial algebra*. J. Okniński [O] gives necessary and sufficient conditions for a monomial (not necessarily quadratic) algebra to be Noetherian. Let $M = \langle \text{LM}(I) \rangle \subset \langle X \rangle$, and define the *right multiplication graph* of $\langle X \rangle / M$ to be the directed graph with the set of vertices $V = \langle X \rangle / M$ and

an edge from a vertex v to a vertex w if and only if $vx = w$ for a letter $x \in X$. This graph is clearly a tree. Then we have the following result.

Proposition 5.1 ([O]). *The algebra $K\langle X\rangle/I$ is right Noetherian if and only if the right multiplication graph of $\langle X\rangle/M$ has finitely many branchings.*

Ufnarovskii shows in [U2] that a monomial algebra is right Noetherian if and only if there are no edges coming out of any cycle of its graph $\Gamma(A)$. (Though the graphs used in [U2] are somewhat different from those discussed in Sections 3 and 4, the two types of graphs are equivalent with respect to this property.)

Regular algebras were introduced by Artin and Schelter in [AS]. Let $A = K \oplus A_1 \oplus A_2 \oplus \dots$ be a finitely presented graded algebra over a field K . The algebra A is called *regular* if it has the following properties:

- (1) A has *finite global dimension* d : every graded A -module has projective dimension $\leq d$.
- (2) A has polynomial growth.
- (3) A is *Gorenstein*; that is, $\text{Ext}_A^q(k, A) = 0$ for positive $q \neq d$, and $\text{Ext}_A^d(k, A) \cong k$.

Regular algebras of global dimension 3 are classified in [AS], where it is shown that if A is regular of global dimension 3, then either A is generated by two elements satisfying two cubic relations, or by three elements satisfying three quadratic relations. Artin, J. Tate and M. Van Den Bergh have shown that regular algebras of global dimension 3 are Noetherian [ATV]. A generalized notion of skew polynomial rings is also introduced in [AS]. Let $X = \{x_1, x_2, \dots, x_n\}$. Set a degree-lexicographic (hereafter abbreviated deg-lex) term ordering with $\text{deg } x_i = d_i \geq 1$ for $1 \leq i \leq n$, and $x_1 < x_2 < \dots < x_n$. An *ordered monomial* is one of the form $x_1^{e_1} \dots x_n^{e_n}$; let T denote the set of all ordered monomials in $\langle X \rangle$.

Definition 5.2. An s.f.p. algebra $A = K\langle X\rangle/I$ is a skew polynomial ring if the reduced Gröbner basis G of I is of the form

$$G = \{x_j x_i - \phi_{ji} : 1 \leq i < j \leq n\}$$

where $\phi_{ji} \in \text{Span } T$, the polynomial ϕ_{ji} is homogeneous of degree $d_i + d_j$, and $\text{LM}(\phi_{ji}) < x_j x_i$.

Artin and Schelter note that skew polynomial rings may not be regular, and give a criterion for a regular algebra of global dimension 3 to be a skew polynomial ring [AS].

A. Kandri-Rody and V. Weispfenning study *algebras of solvable type* and show that such algebras are Noetherian [KRW]. They define these algebras for a large class of term orderings on $\langle X \rangle$, but we will consider only the case where $<$ is a deg-lex ordering. Then we may make the following definition.

Definition 5.3. An s.f.p. algebra $A = K\langle X\rangle/I$ is an *algebra of solvable type* if the reduced Gröbner basis G of I is of the form

$$G = \{x_j x_i - \alpha_{ij} x_i x_j - p_{ij} : 1 \leq i < j \leq n\}$$

where $\alpha_{ij} \in K \setminus \{0\}$, $p_{ij} \in \text{Span } T$, and $\text{LM}(p_{ij}) < x_i x_j$.

If p_{ij} is homogeneous of degree $d_i + d_j$, then clearly A is also a skew polynomial ring.

J. Apel [A] studies Gröbner bases in algebras A to which graded rings G with certain desirable properties may be associated. The algebras A are termed *G-algebras*, and are defined as follows: let A be a K -algebra and $(\Lambda, <)$ a well-ordered commutative Noetherian monoid. (We will indicate multiplication in Λ by \circ .) For technical reasons, we introduce an element $-\infty$ which satisfies $-\infty \circ \lambda = -\infty$ and $-\infty < \lambda$ for all $\lambda \in \Lambda$ and denote $\Lambda \cup \{-\infty\}$ by Λ^∞ . A function $\varphi : A \rightarrow \Lambda^\infty$ is a *pseudovaluation* of A if it satisfies the following conditions:

- (1) $(a + b)^\varphi \leq \max\{a^\varphi, b^\varphi\}$ for all $a, b \in A$;
- (2) $(ab)^\varphi \leq a^\varphi \circ b^\varphi$ for all $a, b \in A$;
- (3) $(\alpha a)^\varphi = a^\varphi$ for $\alpha \in K \setminus \{0\}$ and $a \in A$.

We can use such a pseudovaluation to define a filtration F of A . For each $\lambda \in \Lambda^\infty$, let $F_\lambda = \{a \in A : a^\varphi \leq \lambda\}$ and $\hat{F}_\lambda = \bigcup_{\lambda' < \lambda} F_{\lambda'}$. Then let $G_\lambda = F_\lambda / \hat{F}_\lambda$, and define the algebra

$$G = \bigoplus_{\lambda \in \Lambda} G_\lambda,$$

which is called the associated graded algebra of A . If an element $g \in G$ belongs to a single G_λ , then g is called homogeneous of grade λ . For an arbitrary $g \in G$ such that $g \neq 0$, the grade of g is the grade of its highest (with respect to $<$) homogeneous part. The grade of the zero element is $-\infty$. If A satisfies the conditions:

- (1) $\dim_K G_\lambda \leq 1$ for $\lambda \in \Lambda$ and
- (2) $G_\lambda G_{\lambda'} = G_{\lambda \circ \lambda'}$ for $\lambda, \lambda' \in \Lambda$,

then the quadruple $(A, \Lambda, <, \varphi)$ is called a *G-algebra*.

Apel shows that *G-algebras* are Noetherian. He notes that a commutative polynomial ring $K[x_1, \dots, x_n]$ may be regarded as a *G-algebra* by setting $\Lambda = \mathbb{N}^n$, $<$ any ordering on \mathbb{N}^n which makes it a well-ordered monoid, and φ the function which assigns to any nonzero polynomial the exponent vector of the leading monomial. Algebras of solvable type may be viewed as *G-algebras* in a similar way.

Mora mentions a result for another class of algebras in [M2]. These are finitely presented algebras $A = K\langle X \rangle / I$ in which the defining relations in I are

$$G = \{x_j x_i - \alpha_{ij} x_i x_j : 1 \leq i < j \leq n\}$$

where $\alpha_{ij} \in K$. (We may have $\alpha_{ij} = 0$.) Mora shows that the set G is a Gröbner basis for I , and determines when A will be Noetherian. Let

$$V_1 = \{x_j : \alpha_{ij} \neq 0 \text{ for each } i < j\}$$

and

$$V_2 = \{x_j : \alpha_{ij} = 0 \text{ for some } i < j\}.$$

We may assume without loss of generality that $V_1 = \{x_1, \dots, x_s\}$ and $V_2 = \{x_{s+1}, \dots, x_n\}$, by rearranging the variables if necessary.

Proposition 5.4 ([M2]). *The algebra $K\langle X \rangle / (G)$ is Noetherian if and only if for each $x_k \in V_2$, for each $j > k$, we have $\alpha_{kj} \neq 0$.*

Mora then uses this result to find finite Gröbner bases for certain ideals in the free algebra $K\langle X \rangle$. The ideals are those which contain, for each i, j , $i < j$, the relations

$$x_j x_i - \alpha_{ij} x_i x_j - p_{ij}$$

where $\alpha_{ij} \in K$, $p_{ij} \in \text{Span } T$, and $\text{LM}(p_{ij}) < x_i x_j$.

Strictly ordered algebras are studied by T. Gateva-Ivanova in [GI1] and [GI2]. Before giving the definition, we establish some notation that will also be used in the next section. Let $X = \{x_1, x_2, \dots, x_n\}$. First, we fix the notation $<_0$ to be the deg-lex term ordering on X , where $\text{deg } x_i = 1$ for $1 \leq i \leq n$, and

$$x_1 <_0 x_2 <_0 \dots <_0 x_n.$$

Let $I \subset K\langle X \rangle$ be an ideal. Then for $f \in K\langle X \rangle$, we will denote the largest monomial in the normal form of f by \overline{f} ; i.e.,

$$\overline{f} = \text{LM}(\text{Nor}(f)).$$

Now for $f, g \in K\langle X \rangle$, we define

$$f \sim g \text{ if and only if } \overline{f} = \overline{g}.$$

(If $f \in I$, then we write $\overline{f} = 0$ and $f \sim 0$.) Clearly \sim gives an equivalence relation on $K\langle X \rangle$. We also define a partial order on $K\langle X \rangle$ as follows:

$$f < g \text{ if and only if } \overline{f} <_0 \overline{g}$$

for $\overline{f}, \overline{g} \in K\langle X \rangle \setminus \{0\}$, and $f > 0$ for $\overline{f} \in K\langle X \rangle \setminus \{0\}$.

Definition 5.5 ([GI1]). An algebra $A = K\langle X \rangle/I$ is *strictly ordered* if

$$\text{deg}(\overline{x_i x_j}) = 2 \text{ for } 1 \leq i, j \leq n,$$

and if for $u, v, w \in N(I)$ such that $v < w$, we have $uv < uw$ and $vu < wu$.

Note that the second condition implies that strictly ordered algebras have no zero divisors.

Definition 5.6. An algebra $A = K\langle X \rangle/I$ is *almost commutative* if for any $u, v \in N(I)$, we have $uv \sim vu$.

Gateva-Ivanova proves the following theorem in [GI1], [GI2].

Theorem 5.7. *Let $I \subset K\langle X \rangle$ be an ideal generated by homogeneous elements. If $A = K\langle X \rangle/I$ is a strictly ordered s.f.p. algebra, then the following conditions are equivalent:*

- (1) *A is almost commutative;*
- (2) *A has polynomial growth;*
- (3) *A is right Noetherian;*
- (4) *A is left Noetherian;*
- (5) *The reduced Gröbner basis G of I has the following property: for any pair of integers i, j , $1 \leq i \leq j \leq n$, either*
 - (a) *there is a polynomial $f_{ji} \in G$ such that*

$$f_{ji} = x_j x_i - \alpha_{ji} x_i x_j - p_{ji}$$

where $\alpha_{ji} \in K \setminus \{0\}$, $p_{ji} \in \text{Span } N$, and (if $p_{ji} \neq 0$) $\text{LM}(p_{ji}) <_0 x_i x_j$; or

- (b) *there are two polynomials $f_{ji}, f_{ij} \in F$ such that*

$$f_{ji} = x_j x_i - \alpha_{ji} x_s x_t - p_{ji},$$

$$f_{ij} = x_i x_j - \alpha_{ij} x_s x_t - p_{ij}$$

where $s < i$, $t > j$, $\alpha_{ji}, \alpha_{ij} \in K \setminus \{0\}$, $p_{ji}, p_{ij} \in \text{Span } N$, and $\text{LM}(p_{ji}) <_0 x_s x_t$, $\text{LM}(p_{ij}) <_0 x_s x_t$.

In [GI3], Gateva-Ivanova considers *binomial* skew polynomial rings, which are skew polynomial rings (as defined above) in which the reduced Gröbner basis G is of the form

$$G = \{x_j x_i - \alpha_{ij} x_{i'} x_{j'} : 1 \leq i < j \leq n\}$$

where $\alpha_{ij} \in K \setminus \{0\}$, $1 \leq i' \leq j' \leq n$, $x_{i'} x_{j'} <_0 x_j x_i$.

Under the further assumption that $i' < j'$, thus excluding the case $i' = j'$, the following theorem is obtained.

Theorem 5.8. *Let $A = K\langle x_1, \dots, x_n \rangle / (G)$ be a binomial skew polynomial ring without zero divisors, with reduced Gröbner basis*

$$G = \{x_j x_i - \alpha_{ij} x_{i'} x_{j'} : 1 \leq i < j \leq n\},$$

where for $1 \leq i < j \leq n$ one has $1 \leq i' < j' \leq n$, and $x_{i'} x_{j'} <_0 x_j x_i$. Then the following conditions are equivalent:

- (1) The set $\{x_{i'} x_{j'} : 1 \leq i < j \leq n\}$ contains precisely $\binom{n}{2}$ elements;
- (2) A is left Noetherian;
- (3) A is right Noetherian.

In [GI4], the restriction that A have no zero divisors is removed, and finite Gröbner bases are found for any left or right ideal of A .

The *Sklyanin algebra* S studied by S. Smith and J. Stafford [SmSt] is another example of a quadratic algebra, and may be defined as follows: Let $\alpha, \beta, \gamma \in K$ satisfy

$$\alpha + \beta + \gamma + \alpha\beta\gamma = 0.$$

Then the algebra S is given by $K\langle x_1, x_2, x_3, x_4 \rangle / (F)$, where

$$F = \{f_1, f_2, f_3, f_4, f_5, f_6\},$$

and

$$\begin{aligned} f_1 &= x_1 x_2 - x_2 x_1 - \alpha(x_3 x_4 + x_4 x_3), \\ f_2 &= x_1 x_2 + x_2 x_1 - (x_3 x_4 - x_4 x_3), \\ f_3 &= x_1 x_3 - x_3 x_1 - \beta(x_4 x_2 + x_2 x_4), \\ f_4 &= x_1 x_3 + x_3 x_1 - (x_4 x_2 - x_2 x_4), \\ f_5 &= x_1 x_4 - x_4 x_1 - \gamma(x_2 x_3 + x_3 x_2), \\ f_6 &= x_1 x_4 + x_4 x_1 - (x_2 x_3 - x_3 x_2). \end{aligned}$$

It is shown in [SmSt] that with some restrictions on α, β, γ , these algebras are regular (in the sense of [AS]) and are Noetherian domains.

6. PSEUDOPOLYNOMIAL RINGS

Let $X = \{x_1, x_2, \dots, x_n\}$. As in the previous section, we will use the notation $<_0$ for the deg-lex term ordering such that $\deg x_i = 1$ for $1 \leq i \leq n$, and

$$x_1 <_0 x_2 <_0 \cdots <_0 x_n.$$

This term ordering will be the only one used for the rest of the paper. Recall that given an ideal I of $K\langle X \rangle$, we define

$$\bar{f} = \text{LM}(\text{Nor}(f)),$$

for $f \in K\langle X \rangle$, and the relations

$$f \sim g \text{ if and only if } \overline{f} = \overline{g},$$

$$f < g \text{ if and only if } \overline{f} <_0 \overline{g}$$

for $f, g \in K\langle X \rangle$.

In the discussion that follows, we consider s.f.p. algebras $A = K\langle X \rangle/R$ with polynomial growth. For such an algebra A , the Ufnarovskii graph $\Gamma(A)$ must be cyclically simple; so we define a partial order on cycles in Γ as follows: $C_1 \leq C_2$ if there is a path in Γ which originates at a vertex in C_1 and terminates at a vertex in C_2 . If there are any acyclic vertices in Γ , we regard them as cycles of length 0. We saw in Section 3 that for any $u \in N$ (where the length of u is greater than or equal to 2), we may associate to u a unique path in $\Gamma(A)$. We say that the monomial u corresponds to a path in the cycle C if all vertices and edges of the path we associate to u lie in a single cycle C in $\Gamma(A)$.

We further restrict the algebras we consider to those that satisfy the following definition.

Definition 6.1. An s.f.p. algebra $A = K\langle X \rangle/R$ with polynomial growth is called a *pseudopolynomial ring* if it satisfies the following conditions:

- (1) The defining relations in R are homogeneous of degree 2.
- (2) The partial order on cycles in Γ is a total order which is compatible with the term ordering $<_0$; i.e., for two distinct cycles C_i, C_j , if $C_i \leq C_j$, then for any monomial u corresponding to a path in C_i and monomial v corresponding to a path in C_j such that $\deg u = \deg v$, we have $u <_0 v$.
- (3) If x_i, x_j are vertices in distinct cycles such that $i < j$, then $x_i x_j \in N$.
- (4) Suppose $v, w \in N$ correspond to paths in the same cycle C in Γ and $vw \notin N$. Then $\overline{vw} = 0$.

Suppose $A = K\langle X \rangle/R$ is a pseudopolynomial ring. Since the defining relations in R are quadratic (by condition (1)), the vertices of the Ufnarovskii graph of A are exactly the elements of X ; i.e., $V = X$. So for any path $v_0, e_1, v_1, \dots, v_{t-1}, e_t, v_t$ in $\Gamma(A)$, the vertices v_i are just letters in X , and the path corresponds to the monomial $v_0 v_1 \cdots v_t$ in N . From now on, we will use the notation

$$v_0 \rightarrow v_1 \rightarrow \cdots \rightarrow v_t$$

to indicate this path. Similarly, if we have a cycle in $\Gamma(A)$ consisting of the vertices v_0, v_1, \dots, v_t and the edges e_1, e_2, \dots, e_t , then we will use the notation

$$(v_0 \rightarrow v_1 \rightarrow \cdots \rightarrow v_t \rightarrow v_0)$$

to indicate this cycle (which is a subgraph of Γ).

Since the defining relations in R are also homogeneous, if we have $u, v \in N$ such that $\overline{uv} \neq 0$, then $\text{Nor}(uv)$ is homogeneous of degree $(\deg u + \deg v)$. So the algebra A is graded. Let $U_i = \text{Span}\{u \in N : \deg u \leq i\}$; then

$$U_0 \subset U_1 \subset \cdots \subset U_i \subset \cdots$$

gives a filtration. So

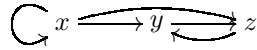
$$A = \bigoplus_{i \geq 0} A_i$$

where $A_0 = K$ and

$$A_i = U_i/U_{i-1} \text{ for } i > 0.$$

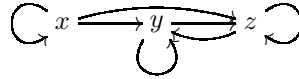
Since by condition (2), we have a total order on cycles, let us assume that the cycles are numbered in such a way that if $i < j$, we have $C_i < C_j$. Any letter $x_i \in X$ may be viewed as a monomial corresponding to the path consisting of the single vertex x_i in a cycle C_i of Γ . Thus, if C_i, C_j are cycles in Γ such that $i < j$, then for any vertex y of C_i and z of C_j , we must have $y <_0 z$. Similarly, if y, z are vertices in the same cycle, condition (4) says that either $yz \in N$ or $\overline{yz} = 0$.

Example 6.2. Let $X = \{x, y, z\}$, and let $G = \{z^2, y^2, zx - xy, yx - xz\}$. Then the algebra $A = K\langle X \rangle / (G)$ is a pseudopolynomial ring. It is easily checked that G is a Gröbner basis for (G) . (We assume $x <_0 y <_0 z$.) So the set of obstructions is $\text{LM}(G) = \{z^2, y^2, zx, yx\}$, and $\Gamma(A)$ is



Here the cycles are $C_1 = (x \rightarrow x)$ and $C_2 = (y \rightarrow z \rightarrow y)$.

If we let $F = \{zx - xy, yx - xz\}$, then F is a Gröbner basis for (F) , but the algebra $B = K\langle X \rangle / (F)$ is not a pseudopolynomial ring. Its graph $\Gamma(B)$ is



We see that $\Gamma(B)$ has two doubly cyclic vertices: y and z . Hence, B has exponential growth and cannot be a pseudopolynomial ring.

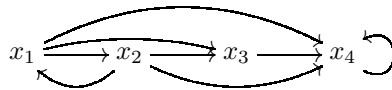
Example 6.3. Let $X = \{x_1, x_2, x_3, x_4\}$, and let

$$G = \{x_1^2, x_2^2, x_3^2, x_3x_1 - x_2x_3, x_3x_2 - x_1x_3, x_4x_1 - x_2x_4, x_4x_2 - x_1x_4, x_4x_3 - x_3x_4\}.$$

Then the algebra $A = K\langle X \rangle / (G)$ is a pseudopolynomial ring, and G is a Gröbner basis for R . The set of obstructions is

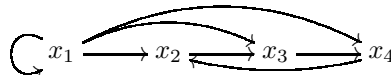
$$\text{LM}(G) = \{x_1^2, x_2^2, x_3^2, x_3x_1, x_3x_2, x_4x_1, x_4x_2, x_4x_3\}.$$

So $\Gamma(A)$ is



The cycles are $C_1 = (x_1 \rightarrow x_2 \rightarrow x_1)$, $C_2 = (x_3)$ (a cycle of length 0), and $C_3 = (x_4 \rightarrow x_4)$.

It is fairly easy to create examples of pseudopolynomial rings. Start with a graph $\Gamma(A)$ that satisfies (2) and (3): say let $\Gamma(A)$ be



So the cycles are $C_1 = (x_1 \rightarrow x_1)$ and $C_2 = (x_2 \rightarrow x_3 \rightarrow x_4 \rightarrow x_2)$. Since there are four vertices in the graph, we put $X = \{x_1, x_2, x_3, x_4\}$. We now choose the set G of generating relations for R in such a way that G forms a Gröbner basis for R and conditions (1) and (4) are satisfied. Condition (4) requires that we put

the monomials $x^2, x^3, x^4, x_3x_2, x_4x_3,$ and x_4x_2 into G . Now we must choose homogeneous polynomials with leading terms $x_2x_1, x_3x_1,$ and x_4x_1 . An easy way is just to add these monomials to G . Otherwise, we must be careful that our choice results in G being a Gröbner basis for R . The polynomials $x_2x_1 - x_1x_3, x_3x_1 - x_1x_4,$ and $x_4x_1 - x_1x_2$ will work.

Skew polynomial rings are examples of pseudopolynomial rings. For a skew polynomial ring $A = K\langle X \rangle / R$, we know that the reduced Gröbner basis G of R is of the form

$$G = \{x_jx_i - \phi_{ji} : 1 \leq i < j \leq n\}$$

where $\phi_{ji} \in \text{Span } T$, the polynomial ϕ_{ji} is homogeneous of degree 2 (since we are assuming the term ordering $<_0$), and $\text{LM}(\phi_{ji}) < x_jx_i$. The Ufnarovskii graph $\Gamma(A)$ is most easily expressed in terms of an *adjacency matrix* $M(\Gamma) = (a_{ij})$, where $a_{ij} = 1$ if and only if there is an edge from the vertex x_i to the vertex x_j . Otherwise, $a_{ji} = 0$. Since $\text{LM}(G) = \{x_jx_i : 1 \leq i < j \leq n\}$, we have

$$M(\Gamma) = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 0 & 1 & \dots & 1 \\ \vdots & 0 & 1 & \dots & \vdots \\ & \vdots & & \ddots & \\ 0 & 0 & \dots & & 1 \end{pmatrix}.$$

That is, $M(\Gamma)$ has 1's on and above the main diagonal and 0's below the main diagonal. The 1's on the diagonal mean that we have n cycles of length 1: $C_i = (x_i \rightarrow x_i)$ for each $i, 1 \leq i \leq n$. The graph Γ is cyclically simple since the matrix entry $a_{ij} = 0$ for $i > j$. If u, v correspond to paths in cycles C_i, C_j , respectively, with $i < j$ and $\text{deg } u = \text{deg } v$, then $u = x_i^k$ and $v = x_j^k$ for some $k > 0$. Clearly $u <_0 v$; so condition (2) is satisfied. Since $a_{ij} = 1$ for $i < j$, condition (3) is satisfied. If $u, v \in N$ correspond to paths in the same cycle C_i , the $u = x_i^k$ and $v = x_i^l$ for some $k, l > 0$. So $uv = x_i^{k+l} \in N$, and we see that (4) is satisfied. Thus, skew polynomial rings are examples of pseudopolynomial rings. Pseudopolynomial rings which are not skew polynomial rings are shown in Examples 6.2 and 6.3.

Remark 6.4. If A is a pseudopolynomial ring with monomials $u, v, w \neq 1$ such that $uv, vw \in N$, then $uvw \in N$. Since none of u, v, w is the empty word, we have $\text{deg } u, \text{deg } v, \text{deg } w \geq 1$, which implies that $\text{deg } uv, \text{deg } vw \geq 2$. Now suppose $uvw \notin N$, that is, uvw contains some obstruction f . By 1, we know that $\text{deg } f = 2$; so either f is a subword of uv or f is a subword of vw . But both $uv, vw \in N$, are contradictions. So we must have $uvw \in N$.

Definition 6.5. A pseudopolynomial ring $A = K\langle x_1, \dots, x_n \rangle / R$ is *almost commutative on cycles* if for vertices x_i, x_j from distinct cycles in $\Gamma(A)$, we have

$$x_jx_i \sim x_ix_j.$$

Remark 6.6. If x_i, x_j are vertices in distinct cycles C_i, C_j , respectively, of $\Gamma(A)$ for some pseudopolynomial ring A with $i < j$, then condition (2) implies that $x_jx_i \notin N$. Since $i < j$ implies that $C_i < C_j$ (since we have a total ordering on cycles) and Γ is cyclically simple, there cannot be a path from C_j to C_i . If x_jx_i were normal, the edge from x_j to x_i would be such a path. Hence, $x_jx_i \notin N$. But $x_ix_j \in N$ by

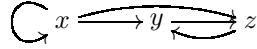
condition (3); so if A is almost commutative on cycles, we have

$$\overline{x_j x_i} = x_i x_j.$$

Example 6.7. Let $X = \{x, y, z\}$ (assume $x <_0 y <_0 z$), and let

$$G = \{z^2, y^2, yx - xy, zx - xz\}.$$

Then if $A = K\langle x, y, z \rangle / R$ where $R = (G)$, it is easily checked that G is a Gröbner basis for R . The graph $\Gamma(A)$ is



So, as in Example 6.2, the cycles are $C_1 = (x \rightarrow x)$ and $C_2 = (y \rightarrow z \rightarrow y)$, and A is clearly a pseudopolynomial ring but not a skew polynomial ring. For A to be almost commutative on cycles, we need the vertex x of C_1 to almost commute with both vertices, y, z , of C_2 ; i.e., we need

$$\begin{aligned} \overline{y x} &= xy, \\ \overline{z x} &= xz. \end{aligned}$$

These conditions are clearly satisfied since the relations $yx - xy$ and $zx - xz$ are in R . Notice that A is *not* almost commutative since $yz \not\sim zy$. Also, it is easy to see that neither the algebra of Example 6.2 nor that of Example 6.3 are almost commutative on cycles.

If an algebra $A = K\langle x_1, \dots, x_n \rangle / R$ is a skew polynomial ring, the cycles in $\Gamma(A)$ are exactly $C_i = (x_i \rightarrow x_i)$ for each i , $1 \leq i \leq n$. So a skew polynomial ring is almost commutative on cycles if and only if it is almost commutative.

Lemma 6.8 is a technical lemma needed to prove Lemma 6.9, which tells us the sense in which we may think of cycles in the Ufnarovskii graph as almost commuting.

Lemma 6.8. *Let $A = K\langle x_1, x_2, \dots, x_n \rangle / R$ be a pseudopolynomial ring which is almost commutative on cycles. Suppose $u \in N$ corresponds to a path in a cycle C_i in Γ and $x_l \in \{x_1, x_2, \dots, x_n\}$ is a vertex in a cycle C_j where $i < j$. Then $\overline{x_l u} = ux_l \in N$.*

Proof. We proceed by induction on $\deg u$.

If $\deg u = 1$, then $u = x_k$, where $x_k \in \{x_1, \dots, x_n\}$. Since $i < j$, we have $x_k < x_l$; so $\overline{x_l u} = \overline{x_l x_k} = x_k x_l \in N$ by Remark 6.6.

If $\deg u = m > 1$, then write $u = x_{k_1} \cdots x_{k_m}$, where each $x_{k_p} \in \{x_1, \dots, x_n\}$. Since $i < j$, we have $x_{k_p} <_0 x_l$ for $1 \leq p \leq m$. Then

$$\begin{aligned} \overline{x_l u} &= \overline{x_l (x_{k_1} \cdots x_{k_m})} \\ &= \overline{(x_l x_{k_1}) (x_{k_2} \cdots x_{k_m})} \\ &= \overline{(x_{k_1} x_l) (x_{k_2} \cdots x_{k_m})} \text{ by Remark 6.6} \\ &= \overline{x_{k_1} (x_l (x_{k_2} \cdots x_{k_m}))} \\ &= \overline{x_{k_1} ((x_{k_2} \cdots x_{k_m}) x_l)} \text{ by the induction hypothesis.} \end{aligned}$$

We also have that $x_{k_2} \cdots x_{k_m} x_l \in N$ by the induction hypothesis, and we know that $x_{k_1} \cdots x_{k_m} = u \in N$. So $x_{k_1} \cdots x_{k_m} x_l \in N$ by Remark 6.4; thus, $\overline{x_l u} = x_{k_1} \cdots x_{k_m} x_l = ux_l \in N$. \square

Lemma 6.9. *Let $A = K\langle x_1, \dots, x_n \rangle / R$ be a pseudopolynomial ring which is almost commutative on cycles. Suppose $u \in N$ corresponds to a path in a cycle C_i in $\Gamma(A)$ and $v \in N$ corresponds to a path in a cycle C_j where $i < j$. Then $\overline{vu} = uv \in N$.*

Proof. We proceed by induction on $\deg v$.

If $\deg v = 1$, then $v = x_l$ where $x_l \in \{x_1, \dots, x_n\}$. We have shown that $\overline{vu} = \overline{x_l u} = ux_l \in N$ in Lemma 6.8.

If $\deg v = m > 1$, then write $v = x_{l_1} \cdots x_{l_m}$, where each $x_{l_p} \in \{x_1, \dots, x_n\}$. Note that each x_{l_p} is a vertex in C_j by assumption. Then

$$\begin{aligned} \overline{vu} &= \overline{(x_{l_1} \cdots x_{l_m})u} \\ &= \overline{(x_{l_1} \cdots x_{l_{(m-1)}})\overline{x_{l_m}u}} \\ &= \overline{(x_{l_1} \cdots x_{l_{(m-1)}})ux_{l_m}} \text{ by Lemma 6.8} \\ &= \overline{((x_{l_1} \cdots x_{l_{(m-1)}})u)x_{l_m}} \\ &= \overline{u(x_{l_1} \cdots x_{l_{(m-1)}})x_{l_m}} \text{ by the induction hypothesis.} \end{aligned}$$

Note that the induction hypothesis also tells us that $ux_{l_1} \cdots x_{l_{(m-1)}} \in N$, and we already know that $x_{l_1} \cdots x_{l_m} = v \in N$. So $ux_{l_1} \cdots x_{l_m} \in N$, and we have $\overline{vu} = ux_{l_1} \cdots x_{l_m} = uv \in N$. □

Lemma 6.10 tells us to what extent the relations in a pseudopolynomial ring $K\langle X \rangle / R$ which is almost commutative on cycles are compatible with the term ordering on $\langle X \rangle$.

Lemma 6.10. *Let $A = K\langle x_1, \dots, x_n \rangle / R$ be a pseudopolynomial ring. Then if A is almost commutative on cycles, A satisfies the following condition: If $u, v, w \in N$ such that $v < w$, then*

- (1) $\overline{uw} \neq 0$ implies $uv < uw$, and
- (2) $\overline{wu} \neq 0$ implies $vu < wu$.

Proof. (1) Suppose $\overline{uw} \neq 0$. If $\overline{uv} = 0$, then clearly $uv < uw$; so we may assume that $\overline{uv} \neq 0$. If $\deg v < \deg w$, then since A is graded, we have

$$\deg(\overline{uv}) = \deg u + \deg v < \deg u + \deg w = \deg(\overline{uw}).$$

So $uv < uw$.

Suppose $\deg v = \deg w$. We know that $v <_0 w$; the idea here is to look at v and w as products of cycles in the Ufnarovskii graph $\Gamma(A)$ and compare them cycle by cycle. Say $\Gamma(A)$ has m distinct cycles and the path corresponding to w originates in C_j . Then we may write

$$w = w_j w_{j+1} \cdots w_m$$

where each w_k , for $j \leq k \leq m$, corresponds to a path in C_k or $w_k = 1$, the empty word, if w_k does not pass through C_k . Note that since the path corresponding to w originates in C_j , we must have $w_j \neq 1$. We write v in a similar way: if the path corresponding to v originates in C_i , then

$$v = v_i v_{i+1} \cdots v_m$$

where each v_l , for $i \leq l \leq m$, corresponds to a path in C_l or $v_l = 1$. Again note that $v_i \neq 1$. Since $v < w$, we must have $i \leq j$. We must consider case by case how v_i and w_j relate given that $v <_0 w$. If we have $v_i = w_j$, we then move on to

compare v_{i+1} and w_{j+1} (hence our use of induction). We proceed by induction on $m - j$.

If $m - j = 0$, that is, $j = m$, then $w = w_m$. Write $u = u_1 \cdots u_m$ where each u_p , for $1 \leq p \leq m$ corresponds to a path in C_p or $u_p = 1$.

If $i < j$, observe that

$$\begin{aligned} \overline{uv} &= \overline{(u_1 \cdots u_m)(v_i \cdots v_m)} \\ &= \overline{u_1 \cdots u_i v_i u_{i+1} v_{i+1} \cdots u_m v_m} \end{aligned}$$

by Lemma 6.9; and

$$u_1 \cdots u_i v_i u_{i+1} v_{i+1} \cdots u_m v_m \in N$$

by repeated applications of Remark 6.4. So

$$\overline{uv} = u_1 \cdots u_i v_i u_{i+1} v_{i+1} \cdots u_m v_m \in N.$$

Similarly, we get

$$\begin{aligned} \overline{uw} &= \overline{(u_1 \cdots u_m)w_m} \\ &= u_1 \cdots u_i u_{i+1} \cdots u_m w_m \in N. \end{aligned}$$

We know that $v_i \neq 1$ and v_i corresponds to a path in a cycle C_i . If for some p , $i + 1 \leq p \leq m$, we have $u_p \neq 1$, then u_p corresponds to a path in a cycle C_p where $p > i$. So if $v_i = yv'_i$, where $y \in \{x_1, x_2, \dots, x_n\}$, and $u_p = zu'_p$, where $z \in \{x_1, x_2, \dots, x_n\}$, then we must have $y <_0 z$. If $u_p = 1$ for all p , $i + 1 \leq p \leq m$, we know that $w_m \neq 1$; so we may write $w_m = aw'_m$, where $a \in \{x_1, x_2, \dots, x_n\}$. Again we must have $y <_0 a$. So

$$\overline{uv} = u_1 \cdots u_i v_i u_{i+1} v_{i+1} \cdots u_m w_m <_0 u_1 \cdots u_i \cdots u_m w_m = \overline{uw}.$$

That is, we have $uv < uw$.

If $i = j$, then $v = v_m$; and we must have $v_m <_0 w_m$. So $\overline{uv} = u_1 \cdots u_m v_m \in N$, and $\overline{uw} = u_1 \cdots u_m w_m \in N$. Since

$$u_1 \cdots u_m v_m <_0 u_1 \cdots u_m w_m,$$

we have $uv < uw$.

Now suppose $m - j > 0$; that is, $j < m$. So we may write $w = w_j \cdots w_m$, where each w_k , for $j \leq k \leq m$, corresponds to a path in the cycle C_k or $w_k = 1$. Since we assume that the path in $\Gamma(A)$ corresponding to w originates in C_j , we know that $w_j \neq 1$. Again, we write v as $v = v_i \cdots v_m$, where each v_l , for $i \leq l \leq m$, corresponds to a path in C_l or $v_l = 1$. Since we assume that the path in $\Gamma(A)$ corresponding to v originates in C_i , we have that $v_i \neq 1$. Since $v < w$, we must have $i \leq j$.

If $i < j$, we have

$$\begin{aligned} \overline{uv} &= \overline{(u_1 \cdots u_m)(v_i \cdots v_m)} \\ &= u_1 \cdots u_i v_i u_{i+1} v_{i+1} \cdots u_m v_m \in N, \end{aligned}$$

and

$$\begin{aligned} \overline{uw} &= \overline{(u_1 \cdots u_m)(w_j \cdots w_m)} \\ &= u_1 \cdots u_i \cdots u_j w_j u_{j+1} w_{j+1} \cdots u_m w_m \in N. \end{aligned}$$

If $u_p \neq 1$ for some p , $i + 1 \leq p \leq j$, then we may write $v_i = yv'_i$, where $y \in \{x_1, \dots, x_n\}$, and $u_p = zu'_p$, where $z \in \{x_1, \dots, x_n\}$. Since $p > i$, we must have $y <_0 z$. So

$$u_1 \cdots u_i v_i u_{i+1} v_{i+1} \cdots u_m v_m <_0 u_1 \cdots u_i u_{i+1} \cdots u_j w_j \cdots u_m w_m;$$

that is, $uv < uw$.

If $u_p = 1$ for all p , $i + 1 \leq p \leq j$, then since we know that $w_j \neq 1$, we may write $w_j = aw'_j$, where $a \in \{x_1, \dots, x_n\}$. Since $j > i$, we must have $y <_0 a$. So again we have

$$u_1 \cdots u_i v_i u_{i+1} v_{i+1} \cdots u_m v_m <_0 u_1 \cdots u_i u_{i+1} \cdots u_j w_j u_{j+1} w_{j+1} \cdots u_m w_m;$$

that is, $uv < uw$.

If $i = j$, then $v = v_j \cdots v_m$ and $w = w_j \cdots w_m$, where each v_k, w_k corresponds to a path in a cycle C_k in $\Gamma(A)$, or v_k or w_k may equal 1 (the empty word). We have

$$\begin{aligned} \overline{uv} &= \overline{(u_1 \cdots u_m)(v_j \cdots v_m)} \\ &= u_1 \cdots u_j v_j u_{j+1} v_{j+1} \cdots u_m v_m \in N, \end{aligned}$$

and

$$\begin{aligned} \overline{uw} &= \overline{(u_1 \cdots u_m)(w_j \cdots w_m)} \\ &= u_1 \cdots u_j w_j u_{j+1} w_{j+1} \cdots u_m w_m \in N. \end{aligned}$$

From the assumption that the paths corresponding to v and w both originate in C_j , we know that $v_j \neq 1$ and $w_j \neq 1$. So we may write $v_j = y_1 \cdots y_s$ and $w_j = z_1 \cdots z_t$, where $y_q, z_{q'} \in \{x_1, \dots, x_n\}$ for each q , $1 \leq q \leq s$, and for each q' , $1 \leq q' \leq t$.

Suppose $s > t$. If there is a q_0 , $1 \leq q_0 \leq t$, such that

$$y_1 = z_1, \dots, y_{q_0-1} = z_{q_0-1}, \text{ and } y_{q_0} <_0 z_{q_0},$$

then

$$u_1 \cdots u_j v_j u_{j+1} v_{j+1} \cdots u_m v_m <_0 u_1 \cdots u_j w_j u_{j+1} w_{j+1} \cdots u_m w_m;$$

that is, $uv < uw$. But if $y_1 = z_1, \dots, y_t = z_t$, then $w_k \neq 1$ for some k , $j < k \leq m$. Pick the smallest such k and write $w_k = aw'_k$, where $a \in \{x_1, \dots, x_n\}$. Then the first $t + 1$ letters of w are $y_1 \cdots y_t a$, and as $k > j$, we must have $y_{t+1} <_0 a$. Thus,

$$u_1 \cdots u_j v_j u_{j+1} v_{j+1} \cdots u_m v_m <_0 u_1 \cdots u_j w_j u_{j+1} w_{j+1} \cdots u_m w_m;$$

that is, $uv < uw$.

Now suppose that $s \leq t$. There are two possibilities:

- (1) either there is some q_0 , $1 \leq q_0 \leq s$ such that

$$y_1 = z_1, \dots, y_{q_0-1} = z_{q_0-1} \text{ and } y_{q_0} <_0 z_{q_0},$$

or

- (2) $s = t$ and $y_q = z_q$ for all $1 \leq q \leq s$, i.e., $v_j = w_j$.

If (1) is the case, we proceed as we did when we had $s > t$. If we have (2), then let

$$\begin{aligned} v' &= v_{j+1} \cdots v_m, \\ w' &= w_{j+1} \cdots w_m. \end{aligned}$$

Let

$$\begin{aligned} u' &= \overline{uv_j} (= \overline{uw_j}) \\ &= \overline{(u_1 \cdots u_m)v_j} \\ &= u_1 \cdots u_j v_j u_{j+1} \cdots u_m \in N. \end{aligned}$$

Since $v, w \in N$ and $v < w$, we have $v <_0 w$. Since $v_j = w_j$, and $v = v_j v'$ and $w = w_j w'$, we must have $v' <_0 w'$. So $v' < w'$, and

$$\begin{aligned} \overline{u'w'} &= \overline{(uw_j)(w_{j+1} \cdots w_m)} \\ &= \overline{u(w_j w_{j+1} \cdots w_m)} \\ &= \overline{uw} \neq 0. \end{aligned}$$

Thus, by the induction hypothesis, $u'v' < u'w'$, or, equivalently, $\overline{u'v'} <_0 \overline{u'w'}$. But we have shown above that $\overline{u'w'} = \overline{uw}$; a similar argument shows that $\overline{u'v'} = \overline{uv}$. So, finally, we have that $\overline{uv} <_0 \overline{uw}$, or $uv < uw$.

The proof of (2), that $\overline{uv} \neq 0$ implies $vu < uv$, is similar. \square

Remark 6.11. Let $A = K\langle X \rangle / R$ be a pseudopolynomial ring which is almost commutative on cycles, and suppose $f \in A$. Then we may write

$$f = \sum_{i=1}^s \alpha_i u_i$$

where $\alpha_i \in K \setminus \{0\}$ and $u_i \in N$ such that

$$u_s <_0 u_{s-1} <_0 \cdots <_0 u_1.$$

So $\text{LM}(f) = u_1$. Now let $a \in N$ such that $\overline{au_1} \neq 0$. Then $\overline{af} = \overline{au_1}$ since $u_1 >_0 u_i$ for $1 < i \leq s$ implies $\overline{au_1} >_0 \overline{au_i}$. We will need this fact in the proof of Theorem 6.15.

Recall that $T \subset \langle X \rangle$ denotes the set of ordered monomials, i.e., words of the form $x_1^{e_1} x_2^{e_2} \cdots x_n^{e_n}$. The following lemma was proved by Dickson [D] in the context of commutative polynomial rings.

Dickson's Lemma. *For every infinite sequence of ordered monomials $\{t_i \in T : i = 1, 2, \dots\}$, there is an M such that for any $k > M$, there exists $j \leq M$ such that t_j divides t_k .*

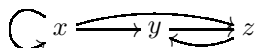
Dickson's lemma carries over to algebras $A = K\langle X \rangle / R$ where $N(R) \subseteq T$ and the relations in the algebra are such that it makes sense to say that $u = x_1^{p_1} \cdots x_n^{p_n}$ divides $v = x_1^{q_1} \cdots x_n^{q_n}$ if $p_i \leq q_i$ for $1 \leq i \leq n$. (That is, there is $a \in N$ such that $\overline{au} = v$.) Algebras of solvable type fall into this class [KRW], and so do s.f.p. algebras which are strictly ordered and almost commutative, as shown in [GI1]. But for many pseudopolynomial rings, it is not true that all normal monomials are also ordered monomials; see Examples 6.2, 6.3, and 6.7. So we give a new definition of "divides":

Definition 6.12. Let $u, v \in N$ be such that $u = u_1 u_2 \cdots u_m$ and $v = v_1 v_2 \cdots v_m$ where u_i, v_i correspond to paths in a cycle C_i in $\Gamma(A)$ for $1 \leq i \leq m$. If for each i , $1 \leq i \leq m$, the paths corresponding to u_i and v_i begin at the same vertex y_i and end at the same vertex z_i and $\deg(u_i) \leq \deg(v_i)$, then we say that u divides v .

Example 6.13. We look at the algebra of Example 6.7 again: $A = K\langle x, y, z \rangle / (G)$ where

$$G = \{z^2, y^2, yx - xy, zx - xz\}.$$

We know that $\text{LM}(G)$ is $\{z^2, y^2, zx, yx\}$, and $\Gamma(A)$ is



with cycles $C_1 = (x \rightarrow x)$ and $C_2 = (y \rightarrow z \rightarrow y)$. The idea of the definition of “divides” is to split the words in N into (in this case) nine different piles according to how many cycles the corresponding paths in Γ pass through, and where the paths enter and exit the cycles. For A , we have monomials which correspond to a path in a single cycle: $x^i, (yz)^i, (yz)^i y, (zy)^i, (zy)^i z, i > 0$, and monomials which correspond to a path through both cycles: $x^i (yz)^j, x^i (yz)^j y, x^i (zy)^j, x^i (zy)^j z, i, j > 0$. Say we have two monomials in the same pile, e.g., $x^i (yz)^j z$ and $x^k (yz)^l z$ for some $i, j, k, l > 0$. Then if $i \leq k$ and $j \leq l$, we wish to say that $x^i (yz)^j z$ divides $x^k (yz)^l z$ since (using the relations in R)

$$\overline{(x^{k-i} (yz)^{l-j} (x^i (yz)^j z))} = x^k (yz)^l z.$$

Using this new definition of “divides,” we get a generalized version of Dickson’s lemma:

Lemma 6.14. *Let $S = \{u_k \in N : k = 1, 2, \dots\}$ be an infinite sequence of monomials such that each u_k may be written as*

$$u_k = u_{k1} u_{k2} \cdots u_{km}$$

where each u_{ki} corresponds to a path in a cycle C_i in $\Gamma(A)$ for $1 \leq i \leq m$, and for a given i , all u_{ki} (for $k = 1, 2, \dots$) begin at the same vertex y_i and end at the same vertex z_i in C_i . Then there exists an M such that for any $k > M$ there is $j \leq M$ such that u_j divides u_k .

Proof. We proceed by induction on m .

If $m = 1$, then $u_k = u_{k1}$ for each k ; i.e., each u_k corresponds to a path in a cycle $C_1 = (x_{i_1} \rightarrow x_{i_2} \rightarrow \cdots \rightarrow x_{i_t} \rightarrow x_{i_1})$ and each begins at the same vertex x_{i_1} and ends at the same vertex x_{i_e} , where $1 \leq e \leq t$. So each u_k is of the form $u_k = \gamma^i w$, for some $i \geq 0$, where γ corresponds to the path $x_{i_1} \rightarrow x_{i_2} \rightarrow \cdots \rightarrow x_{i_t}$ and w corresponds to the path $x_{i_1} \rightarrow x_{i_2} \rightarrow \cdots \rightarrow x_{i_e}$ (or if $x_{i_1} = x_{i_e}$, then $w = 1$).

Let $p = \min\{i : \gamma^i w = u_k \text{ for some } k\}$. Then there is an M such that $u_M = \gamma^p w$. Clearly u_M divides u_k for any $k \geq M$.

Now let $m > 1$. Each u_k may be written as

$$u_k = u_{k1} u_{k2} \cdots u_{km}.$$

Let

$$v_k = u_{k1} u_{k2} \cdots u_{k(m-1)},$$

and look at the sequence $T = \{v_k \in N : k = 1, 2, \dots\}$. By the induction hypothesis, we can find k_0 such that for any $k > k_0$, there is $j \leq k_0$ such that v_j divides v_k .

For each $k, 1 \leq k \leq k_0$, there is a u_{km} such that $v_k u_{km} = u_k$. If $C_m = (x_{i_1} \rightarrow x_{i_2} \rightarrow \cdots \rightarrow x_{i_t})$, we note that each u_{km} is of the form $u_{km} = \gamma^i w$ for some $i \geq 0$, where γ corresponds to the path $x_{i_1} \rightarrow x_{i_2} \rightarrow \cdots \rightarrow x_{i_t}$ and w corresponds to the path $x_{i_1} \rightarrow x_{i_2} \rightarrow \cdots \rightarrow x_{i_e}$ if $2 \leq e \leq t$. If $e = 1$, then $w = 1$. So for each $k, 1 \leq k \leq k_0$, we have $u_k = v_k \gamma^{i(k)} w$ for some $i(k) \geq 0$.

Let $L = \max\{i(k) : 1 \leq k \leq k_0\}$.

Consider the following subsequences of T :

$$\begin{aligned} V_0 &= \{v_{k_j} \in T : u_{k_j} = v_{k_j} w \in S, j = 1, 2, \dots\} \\ V_1 &= \{v_{k_j} \in T : u_{k_j} = v_{k_j} \gamma w \in S, j = 1, 2, \dots\} \\ &\vdots \\ V_i &= \{v_{k_j} \in T : u_{k_j} = v_{k_j} \gamma^i w \in S, j = 1, 2, \dots\} \\ &\vdots \\ V_{L-1} &= \{v_{k_j} \in T : u_{k_j} = v_{k_j} \gamma^{L-1} w \in S, j = 1, 2, \dots\}. \end{aligned}$$

By the induction hypothesis, there exist j_0, j_1, \dots, j_{L-1} such that for any $i, 0 \leq i \leq L - 1$, for any $j > j_i$, for $v_{k_j} \in V_i$, there is a $p \leq j_i$ such that v_{k_p} divides v_{k_j} .

Let $M = \max\{k_0, k_{j_0}, \dots, k_{j_{(L-1)}}\}$.

We now show that for any $k > M$, there is a $j \leq M$ such that u_j divides u_k . Write u_k as $u_k = v_k \gamma^i w$. If $i \geq L$, then there is a $j \leq k_0 \leq M$ such that v_j divides v_k . Then $u_j = v_j \gamma^{i(j)} w$ divides u_k since $i(j) \leq L \leq i$.

If $i < L$, then $v_k \in V_i$. So there is a $j \leq j_i \leq M$ such that v_{k_j} divides v_k . So $u_{k_j} = v_{k_j} \gamma^i w$ divides $v_k \gamma^i w = u_k$. □

We now come to the main result.

Theorem 6.15. *Suppose A is a pseudopolynomial ring. Then, if A is almost commutative on cycles, A is Noetherian.*

Proof. Let J be a left ideal in A . The idea is that we wish to look at the ideal generated by $\text{LM}(J)$, which is a monomial ideal, and find a finite generating set using Lemma 6.14. To use the lemma, we will need to use the graph $\Gamma(A)$ to split the monomials of $\text{LM}(J)$ into several different piles using the definition of “divides” introduced above. Then we use Lemma 6.14 on each pile separately, finally gathering the generators we find in this way into one big pile which generates all of $(\text{LM}(J))$. We find a set of polynomials in J which has the generators of $(\text{LM}(J))$ as leading monomials. We then show that these polynomials generate J .

We may assume that all elements of J are in normal form; i.e., $J \subseteq \text{Span } N$. So

$$\text{LM}(J) = \{\bar{g} : g \in J\} \subseteq N;$$

that is, each $u \in \text{LM}(J)$ corresponds to a path in $\Gamma(A)$. Clearly $\text{LM}(J)$ is countable. Let the sequence

$$U = \{u_1, u_2, u_3, \dots\}$$

be an enumeration of $\text{LM}(J)$ where $u_i < u_j$ for $1 \leq i < j$.

Since $\Gamma(A)$ is a finite graph, we may partition U into a finite number of subsequences U_1, U_2, \dots, U_r such that for any $v, w \in U_i, 1 \leq i \leq r$, we may write

$$v = v_1 v_2 \cdots v_m \text{ and } w = w_1 w_2 \cdots w_m,$$

where v_j, w_j correspond to paths in a cycle C_j in $\Gamma(A)$ for $1 \leq j \leq m$ and for each $j, 1 \leq j \leq m$, the paths corresponding to v_j and w_j begin at the same vertex y_j

and end at the same vertex z_j . Note that

$$U = \bigcup_{i=1}^r U_i.$$

For each i , $1 \leq i \leq r$, write

$$U_i = \{u_{i_k} : k = 1, 2, \dots\}.$$

By Lemma 6.14, there exists $k(i)$ such that for any $k > k(i)$, there is a $j \leq k(i)$ such that u_{i_j} divides u_{i_k} . Since $u_{i_{k(i)}} \in U_i$, we have $u_{i_{k(i)}} = u_{p(i)} \in U$ for some $p(i)$. Let $M = \max\{p(i) : 1 \leq i \leq r\}$. Then for any $k > M$, there is a $j \leq M$ such that u_j divides u_k . Clearly, there exist elements f_1, f_2, \dots, f_M of J such that

$$f_i = u_i - \varphi_i$$

where $\varphi_i \in \text{Span } N$ and (if $\varphi_i \neq 0$) $\text{LM}(\varphi_i) < u_i$, $1 \leq i \leq M$. Let $F = \{f_1, f_2, \dots, f_M\}$.

Let $g \in J$, and let $u = \bar{g}$. Since $u \in \text{LM}(J)$, by choice of M , there exists $j \leq M$ such that u_j divides u ; i.e., $u = v_1 \cdots v_m$ and $u_j = v_{j_1} \cdots v_{j_m}$ where v_i, v_{j_i} correspond to paths in C_i in $\Gamma(A)$ for $1 \leq i \leq m$. For each i , the paths corresponding to v_i and v_{j_i} begin at the same vertex y_i and end at the same vertex z_i , and $\deg(v_{j_i}) \leq \deg(v_i)$. That is, for each i , we have

$$v_i = \gamma_i^{p(i)} w_i \text{ and } v_{j_i} = \gamma_i^{q(i)} w_i$$

where γ_i corresponds to the path once around the cycle $C_i = (x_{i_1} \rightarrow x_{i_2} \rightarrow \dots \rightarrow x_{i_{t(i)}} \rightarrow x_{i_1})$ and w_i corresponds to the path $x_{i_1} \rightarrow \dots \rightarrow x_{i_{e(i)}}$, where $2 \leq e(i) \leq t(i)$ (or if $y_i = z_i$, then $w_i = 1$). Since $\deg(v_{j_i}) \leq \deg(v_i)$, we have $q(i) \leq p(i)$. Let $s(i) = p(i) - q(i)$, and let $a = \gamma_1^{s(1)} \cdots \gamma_m^{s(m)}$. (If $s(i) = 0$ for some i , then we let $\gamma_i^{s(i)} = 1$.) The monomial $a \in N$ since condition (3) on pseudopolynomial rings tells us that

$$\gamma_i \gamma_{i+1} \in N \text{ for each } 1 \leq i \leq n - 1;$$

so we may apply Remark 6.4. Then, since A is almost commutative on cycles, we have

$$\begin{aligned} \overline{au_j} &= \overline{(\gamma_1^{s(1)} \cdots \gamma_m^{s(m)})(\gamma_1^{q(1)} w_1 \cdots \gamma_m^{q(m)} w_m)} \\ &= \gamma_1^{s(1)} \gamma_1^{q(1)} w_1 \cdots \gamma_m^{s(m)} \gamma_m^{q(m)} w_m \\ &= u. \end{aligned}$$

We now show that F generates J as a left ideal. Suppose $g \in J$ and $\text{LM}(g) = u$. Then there is a $j_1 \leq M$ such that u_{j_1} divides u . Let $a_1 \in N$ be such that $\overline{a_1 u_{j_1}} = u$. Then by Remark 6.11, we know that $\overline{a_1 f_{j_1}} = u$. For notational convenience, let $g = g_0$. Then let

$$g_1 = g_0 - \text{LC}(g_0) \text{Nor}(a_1 f_{j_1})$$

and let

$$h_1 = \text{LC}(g_0) \text{Nor}(a_1 f_{j_1}).$$

Note that $\text{LM}(g_1) < \text{LM}(g)$ and $g = g_1 + h_1$. Now for $k \geq 1$, if $\text{LM}(g_k) \neq 0$, then there exists $j_k \leq M$ such that u_{j_k} divides $\text{LM}(g_k)$. So there is an $f_{j_k} \in F$ such that $\text{LT}(f_{j_k}) = u_{j_k}$. Let $a_k \in N$ be such that $\overline{a_k u_{j_k}} = \text{LM}(g_k)$. Then let

$$g_{k+1} = g_k - \text{LC}(g_k)\text{Nor}(a_k f_{j_k})$$

and

$$h_{k+1} = h_k + \text{LC}(g_k)\text{Nor}(a_k f_{j_k}).$$

For each $k \geq 1$, we have $\text{LM}(g_{k+1}) < \text{LM}(g_k)$ and $g = g_k + h_k$. Since we have a well-founded term ordering, we must have $g_s = 0$ for some s . Then

$$g = h_s = \sum_{l=1}^s \text{LC}(g_{l-1})\text{Nor}(a_l f_{j_l}).$$

That is, F is a left basis for J .

The proof that A is right Noetherian is similar. □

There is a lot of redundancy built into our new notion of “divides” and thus in the proofs of Lemma 6.14 and Theorem 6.15. Consider again the algebra of Examples 6.7 and 6.13: $A = K\langle x, y, z \rangle / (G)$ where

$$G = \{z^2, y^2, yx - xy, zx - xz\}.$$

Suppose I is a left ideal of A . To find a left basis for I , we first find a left basis for $(\text{LM}(I))$, which is a monomial ideal in A . Considering monomials of the form $(yz)^j$ and $x^i(yz)^j$ separately is somewhat inefficient; after all, if we have a monomial $(yz)^j$, we can left multiply by x^i to obtain $x^i(yz)^j$. We could minimize this redundancy by introducing a notion of “left divides” (or “divides with respect to left multiplication”), but this approach unnecessarily complicates the proofs of Lemma 6.14 and Theorem 6.15. This tactic *will* be useful in Example 6.16, however, since we will need two separate arguments to show that the algebra considered there is both right and left Noetherian.

The converse to Theorem 6.15 is false in general. For a skew polynomial ring $A = K\langle x_1, x_2, \dots, x_n \rangle / R$, the cycles in $\Gamma(A)$ are all of the form $(x_i \rightarrow x_i)$ for $1 \leq i \leq n$; so a skew polynomial ring is almost commutative on cycles if and only if it is almost commutative. But Theorem 5.8 implies that there are many examples of Noetherian binomial skew polynomial rings that are not almost commutative. For instance, suppose $A = K\langle x_1, x_2, x_3 \rangle / (G)$ where

$$G = \{x_3x_2 - x_1x_3, x_3x_1 - x_2x_3, x_2x_1 - x_1x_2\}.$$

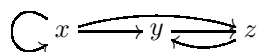
One can easily check that G is a Gröbner basis; so A is indeed a binomial skew polynomial ring. The set $\{x_i x_{j'} : 1 \leq i < j \leq n\}$ of Theorem 5.8 is $\{x_1x_3, x_2x_3, x_1x_2\}$ which does indeed have $\binom{3}{2} = 3$ elements. So A is Noetherian, but since $x_3x_2 \not\sim x_2x_3$, it is not almost commutative.

We also give an example of a pseudopolynomial ring which is not a skew polynomial ring and is not almost commutative on cycles, but is Noetherian.

Example 6.16. Consider again $A = K\langle x, y, z \rangle / R$ where

$$R = (z^2, y^2, zx - xy, yx - xz),$$

the algebra of Examples 6.7 and 6.13. The Ufnarovskii graph $\Gamma(A)$ is



Clearly, the algebra A is not almost commutative on cycles since $yx \not\sim xy$, but A is Noetherian.

To show that A is left Noetherian, we first show that all monomial ideals of A are finitely generated. So let I be a monomial left ideal in A , and let $G = \{m_p \in N : p \in P\}$ be a (possibly infinite) set of generators for I ; so $I = \langle G \rangle$. Let

$$\begin{aligned} J &= (x^i : i \geq 0, x^i \in I), \\ Y_0 &= (x^i : i \geq 0, x^i(yz)^j \in I \text{ for some } j > 0), \\ Y_1 &= (x^i : i \geq 0, x^i(yz)^j y \in I \text{ for some } j \geq 0), \\ Z_0 &= (x^i : i \geq 0, x^i(z y)^j \in I \text{ for some } j > 0), \\ Z_1 &= (x^i : i \geq 0, x^i(z y)^j z \in I \text{ for some } j \geq 0). \end{aligned}$$

(Notice that if there are any monomials in I of the form $(yz)^j$, then $1 \in Y_0$. Similarly for Y_1, Z_0 , and Z_1 .) Then J (if $J \neq \emptyset$) is generated by some x^α . Similarly, Y_0, Y_1, Z_0, Z_1 are generated by $x^\beta, x^\gamma, x^\delta, x^\epsilon$, respectively. So there is an $m > 0$ such that $x^\beta(yz)^m \in I$. There could be monomials in I of the form $x^i(yz)^j$ where $j < m$ (and of course $i > \beta$ by the way β was chosen); so to avoid missing any of these monomials we define the following ideals. Let

$$\begin{aligned} S_1 &= (x^i : i \geq 0, x^i(yz) \in I) \\ S_2 &= (x^i : i \geq 0, x^i(yz)^2 \in I) \\ &\vdots \\ S_k &= (x^i : i \geq 0, x^i(yz)^k \in I) \\ &\vdots \\ S_{m-1} &= (x^i : i \geq 0, x^i(yz)^{m-1} \in I). \end{aligned}$$

Each nonempty S_k is generated by some $x^{\beta(k)}$ for $1 \leq k \leq m - 1$. (Note: Throughout this proof, the letter k will be used as a generic index.)

We now proceed in the same way with the generators of Y_0, Y_1, Z_0 , and Z_1 . There is an $n \geq 0$ such that $x^\gamma(yz)^n y \in I$. Let

$$\begin{aligned} T_0 &= (x^i : i \geq 0, x^i y \in I) \\ T_1 &= (x^i : i \geq 0, x^i(yz)y \in I) \\ &\vdots \\ T_k &= (x^i : i \geq 0, x^i(yz)^k y \in I) \\ &\vdots \\ T_{n-1} &= (x^i : i \geq 0, x^i(yz)^{n-1} y \in I). \end{aligned}$$

Each nonempty T_k is generated by some $x^{\gamma(k)}$ for $1 \leq k \leq n - 1$.

There is an $l > 0$ such that $x^\delta(zy)^l \in I$. Let

$$\begin{aligned} U_1 &= (x^i : i \geq 0, x^i(zy) \in I) \\ &\vdots \\ U_k &= (x^i : i \geq 0, x^i(zy)^k \in I) \\ &\vdots \\ U_{l-1} &= (x^i : i \geq 0, x^i(zy)^{l-1} \in I). \end{aligned}$$

Each nonempty U_k is generated by some $x^{\delta(k)}$ for $1 \leq k \leq l - 1$.

There is a $q \geq 0$ such that $x^\epsilon(zy)^q z \in I$. Let

$$\begin{aligned} V_0 &= (x^i : i \geq 0, x^i z \in I) \\ V_1 &= (x^i : i \geq 0, x^i(zy)z \in I) \\ &\vdots \\ V_k &= (x^i : i \geq 0, x^i(zy)^k z \in I) \\ &\vdots \\ V_{q-1} &= (x^i : i \geq 0, x^i(zy)^{q-1} z \in I). \end{aligned}$$

Each nonempty V_k is generated by some $x^{\epsilon(k)}$ for $1 \leq k \leq q - 1$.

Finally, we put all the generators we have found into one big pile. Let

$$\begin{aligned} G' = \{ &x^\alpha, x^\beta(yz)^m, x^{\beta(k)}(yz)^k \text{ for } 1 \leq k \leq m - 1, \\ &x^\gamma(yz)^n y, x^{\gamma(k)}(yz)^k y \text{ for } 1 \leq k \leq n - 1, \\ &x^\delta(zy)^l, x^{\delta(k)}(zy)^k \text{ for } 1 \leq k \leq l - 1, \\ &x^\epsilon(zy)^q z, x^{\epsilon(k)}(zy)^k z \text{ for } 1 \leq k \leq q - 1\}. \end{aligned}$$

We show that G' generates I as a left ideal.

Let $u \in G$.

Case 1. Suppose $u = x^i$ for some $i \geq 0$. Then $u \in J$; so $i \geq \alpha$ and $x^{i-\alpha}(x^\alpha) = x^i$.

Note that $x^\alpha \in G'$.

Case 2. Suppose $u = x^i(yz)^j$ for some $i \geq 0, j > 0$.

Subcase (a). Suppose $j < m$. Then $x^i \in S_j$; so $i \geq \beta(j)$ and

$$x^{i-\beta(j)}(x^{\beta(j)}(yz)^j) = x^i(yz)^j.$$

Note that $x^{\beta(j)}(yz)^j \in G'$.

Subcase (b). Suppose $j \geq m$. Then $x^i \in Y_0, i \geq \beta$, and $x^{i-\beta}(x^\beta) = x^i$. If β is even, then

$$\overline{(x^{i-\beta}(yz)^{j-m})(x^\beta(yz)^m)} = x^i(yz)^j.$$

Actually, for this algebra, we really have

$$\text{Nor}(x^{i-\beta}(yz)^{j-m})(x^\beta(yz)^m) = x^i(yz)^j.$$

If β is odd, then

$$\overline{(x^{i-\beta}(zy)^{j-m})(x^\beta(yz)^m)} = x^i(yz)^j.$$

The monomial $x^\beta(yz)^m \in G'$.

Case 3. Suppose $u = x^i(yz)^jy$ for some $i \geq 0, j \geq 0$.

Subcase (a). Suppose $j < n$. Then $x^i \in T_j$; so $i \geq \gamma(j)$ and

$$x^{i-\gamma(j)}(x^{\gamma(j)}(yz)^jy) = x^i(yz)^jy,$$

where $x^{\gamma(j)}(yz)^jy \in G'$.

Subcase (b). Suppose $j \geq n$. Then $x^i \in Y_1, i \geq \gamma$, and $x^{i-\gamma}(x^\gamma) = x^i$. If γ is even, then

$$\overline{(x^{i-\gamma}(yz)^{j-n})(x^\gamma(yz)^ny)} = x^i(yz)^jy.$$

If γ is odd, then

$$\overline{(x^{i-\gamma}(zy)^{j-n})(x^\gamma(yz)^ny)} = x^i(yz)^jy.$$

Case 4. Suppose $u = x^i(zy)^j$ for some $i \geq 0, j > 0$. This case is similar to Case 2.

Case 5. Suppose $u = x^i(zy)^jz$ for some $i \geq 0, j \geq 0$. This case is similar to Case 3.

We have shown that any monomial left ideal in A is finitely generated. Now let I be an arbitrary ideal of A . The ideal $(LM(I))$ must be finitely generated, say by u_1, u_2, \dots, u_s . For each u_i , there is an $f_i \in I$ such that $LM(f_i) = u_i$. Let $F = \{f_i : 1 \leq i \leq s\}$. For this algebra, it is easily seen that for $u, v, w \in N$ such that $v < w$ and $\overline{uv} \neq 0$, we have $uv < uw$. Thus, we may use the procedure given in the proof of Theorem 6.15 to show that F generates I as a left ideal.

To see that A is right Noetherian, we will need a somewhat different argument, since it is not true that for $u, v, w \in N$ such that $v < w$ and $\overline{vu} \neq 0$, we always have $vu < wu$. For instance, $y < z$, but

$$\overline{yx} = xz \not<_0 \overline{zx} = xy;$$

i.e., $yx \not< zx$. However, if we know that u is of the form $x^i u'$ where i is even and u' corresponds to a path in the second cycle, then we may conclude that $vu < wu$. So $y < z$, and

$$\overline{yx^2} = x^2y < \overline{zx^2} = x^2z.$$

The point is that as long as we multiply by even powers of x , we are safe, and being able to multiply by these even powers turns out to be good enough. We again start with a monomial ideal: let I be a monomial right ideal in A , and let $G = \{m_p \in N : p \in P\}$ be a (possibly infinite) set of generators for I ; so $I = (G)$.

Let

$$Y = \{(yz)^j y^\epsilon : j \geq 0, \epsilon = 0 \text{ or } 1, x^i(yz)^j y^\epsilon \in I \text{ for some } i \geq 0\},$$

and let

$$Z = \{(zy)^j z^\epsilon : j \geq 0, \epsilon = 0 \text{ or } 1, x^i(zy)^j z^\epsilon \in I \text{ for some } i \geq 0\}.$$

Let u be the element of Y of least degree. Then for any $m \in Y$, there exists $a \in K\langle x, y, z \rangle$ such that $ua = m$. Similarly, let v be the element of Z of least degree.

There is an α such that $x^\alpha u \in I$ and a β such that $x^\beta v \in I$.

Let

$$\begin{aligned} Y_0 &= \{(yz)^j y^\epsilon : j \geq 0, \epsilon = 0 \text{ or } 1, (yz)^j y^\epsilon \in I\} \\ Y_1 &= \{(yz)^j y^\epsilon : j \geq 0, \epsilon = 0 \text{ or } 1, x(yz)^j y^\epsilon \in I\} \\ &\vdots \\ Y_k &= \{(yz)^j y^\epsilon : j \geq 0, \epsilon = 0 \text{ or } 1, x^k(yz)^j y^\epsilon \in I\} \\ &\vdots \\ Y_{\alpha-1} &= \{(yz)^j y^\epsilon : j \geq 0, \epsilon = 0 \text{ or } 1, x^{\alpha-1}(yz)^j y^\epsilon \in I\}. \end{aligned}$$

Let u_k be the element of Y_k of minimal degree for $0 \leq k \leq \alpha - 1$.

Let

$$\begin{aligned} Z_0 &= \{(zy)^j z^\epsilon : j \geq 0, \epsilon = 0 \text{ or } 1, (zy)^j z^\epsilon \in I\} \\ &\vdots \\ Z_k &= \{(zy)^j z^\epsilon : j \geq 0, \epsilon = 0 \text{ or } 1, x^k(zy)^j z^\epsilon \in I\} \\ &\vdots \\ Z_{\beta-1} &= \{(zy)^j z^\epsilon : j \geq 0, \epsilon = 0 \text{ or } 1, x^{\beta-1}(zy)^j z^\epsilon \in I\}. \end{aligned}$$

Let v_k be the element of Z_k of minimal degree for $0 \leq k \leq \beta - 1$.

Let

$$E_y = \{m \in I : m = x^i(yz)^j y^\epsilon, i \geq \alpha, \text{ and } i - \alpha \text{ is even}\},$$

and let

$$O_y = \{m \in I : m = x^i(yz)^j y^\epsilon, i > \alpha, \text{ and } i - \alpha \text{ is odd}\}.$$

Note that for any two elements $x^{i_1}(yz)^{j_1} y^{\epsilon_1}$ and $x^{i_2}(yz)^{j_2} y^{\epsilon_2}$ of O_y , $i_1 - i_2$ is even.

Let

$$E_z = \{m \in I : m = x^i(zy)^j z^\epsilon, i \geq \beta, \text{ and } i - \beta \text{ is even}\},$$

and let

$$O_z = \{m \in I : m = x^i(zy)^j z^\epsilon, i > \beta, \text{ and } i - \beta \text{ is odd}\}.$$

Let

$$S = \{(yz)^j y^\epsilon : x^i(yz)^j y^\epsilon \in O_y\}$$

and let

$$T = \{(zy)^j z^\epsilon : x^i(zy)^j z^\epsilon \in O_z\}.$$

Let w_1 be the element in S of minimal degree and w_2 the element in T of minimal degree. Then there exist γ, δ such that $x^\gamma w_1 \in O_y$ and $x^\delta w_2 \in O_z$. Let

$$\begin{aligned} U_1 &= \{(yz)^j y^\epsilon : x^{\alpha+1}(yz)^j y^\epsilon \in O_y\} \\ &\vdots \\ U_k &= \{(yz)^j y^\epsilon : x^{\alpha+k}(yz)^j y^\epsilon \in O_y\} \\ &\vdots \\ U_{(\gamma-\alpha-2)} &= \{(yz)^j y^\epsilon : x^{\gamma-2}(yz)^j y^\epsilon \in O_y\}. \end{aligned}$$

Let the element of minimal degree in U_k be s_k for $1 \leq k \leq \gamma - \alpha - 2$.

Let

$$\begin{aligned} V_1 &= \{(zy)^j z^\epsilon : x^{\beta+1}(zy)^j z^\epsilon \in O_z\} \\ &\vdots \\ V_k &= \{(zy)^j z^\epsilon : x^{\beta+k}(zy)^j z^\epsilon \in O_z\} \\ &\vdots \\ V_{(\delta-\beta-2)} &= \{(zy)^j z^\epsilon : x^{\delta-2}(zy)^j z^\epsilon \in O_z\}. \end{aligned}$$

Let the element of minimal degree in V_k be t_k for $1 \leq k \leq \delta - \beta - 2$.

Now let

$$\begin{aligned} G' &= \{x^\alpha u, x^\beta v, x^k u_k \text{ for } 0 \leq k \leq \alpha - 1, \\ &\quad x^k v_k \text{ for } 0 \leq k \leq \beta - 1, \\ &\quad x^\gamma w_1, x^\delta w_2, \\ &\quad x^{\alpha+k} s_k \text{ for } 1 \leq k \leq \gamma - \alpha - 2, \\ &\quad x^{\beta+k} t_k \text{ for } 1 \leq k \leq \delta - \beta - 2\}. \end{aligned}$$

We show that G' generates I as a right ideal.

Let $m \in G$.

Case 1. Suppose $m = x^i (yz)^j y^\epsilon$ for some $i \geq 0, j \geq 0, \epsilon = 0$ or 1 .

Subcase (a). Suppose $i < \alpha$. Then $(yz)^j y^\epsilon \in Y_i$ and there is an $a \in K\langle x, y, z \rangle$ such that $u_i a = (yz)^j y^\epsilon$. So $x^i u_i \in G'$ and $(x^i u_i) a = x^i (yz)^j y^\epsilon$.

Subcase (b). Suppose $i \geq \alpha$ and $i - \alpha$ is even; i.e., $m \in E_y$. Then there is an $a \in K\langle x, y, z \rangle$ such that $u a = (yz)^j y^\epsilon$. So $x^\alpha u (x^{i-\alpha} a) = x^i (yz)^j y^\epsilon$.

Subcase (c). Suppose $i > \alpha$ and $i - \alpha$ is odd; i.e., $m \in O_y$. If $\alpha + 1 \leq i < \gamma$, then there is an $a \in K\langle x, y, z \rangle$ such that $s_i a = (yz)^j y^\epsilon$. So $(x^i s_i) a = x^i (yz)^j y^\epsilon$. If $i \geq \gamma$, then there is an $a \in K\langle x, y, z \rangle$ such that $w_1 a = (yz)^j y^\epsilon$. Note that $i - \gamma$ is even. So $(x^\gamma w_1)(x^{i-\gamma} a) = x^i (yz)^j y^\epsilon$.

Case 2. Suppose $m = x^i (zy)^j z^\epsilon$. This case is similar to Case 1.

We have shown that any monomial right ideal of A is finitely generated. Note that we have chosen our generators so that we will always multiply them by even powers of x . It is now easy to show that an arbitrary right ideal of A is finitely generated using the procedure given in the proof of Theorem 6.15.

In conclusion, we note that two very natural questions remain unanswered. First, Theorem 6.15 shows that if a pseudopolynomial ring is almost commutative on

cycles, then it is Noetherian; however, by Example 6.16 we know that a Noetherian pseudopolynomial ring need not be almost commutative on cycles. So what must be true of any Noetherian pseudopolynomial ring? That is, what are some necessary conditions on a pseudopolynomial ring if it is to be Noetherian? This question is answered by Gateva-Ivanova for some binomial skew polynomial rings in [GI3] (here cited as Theorem 5.8) and in [GI4].

Second, recall from Definition 5.5 that an algebra $A = K\langle x_1, \dots, x_n \rangle / I$ is strictly ordered if

$$\deg(\overline{x_i x_j}) = 2 \text{ for } 1 \leq i, j \leq n,$$

and if for $u, v, w \in N(I)$ such that $v < w$, we have $uv < uw$ and $vu < wu$. In a pseudopolynomial ring B that is almost commutative on cycles, we get similar, though somewhat weaker, conditions. That is, for B we have

- (1) if x_i, x_j are vertices in distinct cycles of the Ufnarovskii graph, then $\overline{x_i x_j} \neq 0$ (from Remark 6.6); and
- (2) for $u, v, w \in N$ such that $v < w$, we have
 - $\overline{uw} \neq 0$ implies $uv < uw$, and
 - $\overline{wu} \neq 0$ implies $vu < wu$ from Lemma 6.10.

Since B is graded by degree, condition (1) says that for x_i, x_j from distinct cycles in $\Gamma(B)$, we have $\deg(\overline{x_i x_j}) = 2$. If $u, w \in N$ correspond to paths in distinct cycles in $\Gamma(B)$, then Lemma 6.9 shows that $\overline{uw} \neq 0$ and $\overline{wu} \neq 0$. So condition (2) says that if $u, v, w \in N$ such that $v < w$ and u, w correspond to paths in distinct cycles in $\Gamma(B)$, we have $uv < uw$ and $vu < wu$. Thus, we call any pseudopolynomial ring which satisfies (1) and (2) *strictly ordered on cycles*. Clearly, any pseudopolynomial ring which is almost commutative on cycles must be strictly ordered on cycles. The question is whether a pseudopolynomial ring which is strictly ordered on cycles must be almost commutative on cycles. If not, then one might ask if a pseudopolynomial ring which is strictly ordered on cycles and left Noetherian must be almost commutative on cycles.

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