

## CONFORMALLY INVARIANT MONGE-AMPÈRE EQUATIONS: GLOBAL SOLUTIONS

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ABSTRACT. In this paper we will examine a class of fully nonlinear partial differential equations which are invariant under the conformal group  $SO(n+1, 1)$ . These equations are elliptic and variational. Using this structure and the conformal invariance, we will prove a global uniqueness theorem for solutions in  $\mathbf{R}^n$  with a quadratic growth condition at infinity.

### 1. INTRODUCTION

Let  $u$  be a positive function in  $C^2(\mathbf{R}^n)$ ,  $n \geq 3$ , and let  $k$  be an integer  $1 \leq k \leq n$ . We will let  $\sigma_k(A)$  denote the  $k$ th elementary symmetric function of the eigenvalues of the matrix  $A$ , and  $\delta_{ij}$  will denote the Kronecker delta symbol. In this paper we will study the following nonlinear partial differential equations:

$$(1) \quad \sigma_k \left( u \cdot \frac{\partial^2 u}{\partial x^i \partial x^j} - \frac{|\nabla u|^2}{2} \delta_{ij} \right) = C,$$

where  $C > 0$  is a constant. These equations are not arbitrary, but arise naturally in the study of conformal geometry, and they are conformally invariant: if  $T : \mathbf{R}^n \rightarrow \mathbf{R}^n$  is a conformal transformation, and  $u(x)$  is a solution of (1), then

$$(2) \quad v(x) = |J(x)|^{-1/n} u(Tx)$$

is also a solution, where  $J$  is the Jacobian of  $T$ . This property will be demonstrated in Section 7 below.

For  $1 \leq k \leq n-1$ , a global uniqueness theorem for solutions of (1) was proved in [5]. We will extend this uniqueness result to the case  $k = n$ :

**Theorem 1.1.** *Let  $u(x) \in C^2(\mathbf{R}^n)$  be a positive solution to (1) for some  $k$  with  $1 \leq k \leq n$ . Suppose that  $\tilde{u}(y) = |y|^2 \cdot u(\frac{y^1}{|y|^2}, \dots, \frac{y^n}{|y|^2})$  is  $C^2$  and*

$$\lim_{y \rightarrow 0} \tilde{u}(y) > 0.$$

*Then*

$$u(x) = a|x|^2 + b_i x^i + c$$

*where  $a$ ,  $b_i$ , and  $c$  are constants.*

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Received by the editors November 19, 1998.

2000 *Mathematics Subject Classification.* Primary 35J60, 53A30.

*Key words and phrases.* Monge-Ampère equations, conformally invariant, global solutions.

This will be proved by showing that we can use the Alexandroff reflection principle, as employed by Gidas-Ni-Nirenberg in [4], to get a priori rotational symmetry of solutions. The equation then reduces to an ODE, and we will analyze the solutions to arrive at the theorem.

**Acknowledgements.** This material is based on work supported under a Sloan Dissertation Fellowship, and represents part of the author's doctoral dissertation at Princeton University.

## 2. ASYMPTOTIC CONDITION

We begin with a short explanation of the geometric origin of the equations (1), to explain why the condition in Theorem 1.1 is a natural geometric condition. We let  $(S^n, g_0)$  be the  $n$ -sphere with  $g_0$  the standard metric. If  $g$  is another metric in the conformal class of  $g_0$ , then we consider the equations

$$(3) \quad \sigma_k \left( Ric - \frac{R}{2(n-1)} \cdot g \right) = \text{constant},$$

where  $Ric$  and  $R$  are the Ricci tensor and scalar curvature of  $g$ , respectively, and  $\sigma_k$  is taken with respect to  $g$ .

We will let  $x = (x^1, \dots, x^n)$  be the coordinates on  $S^n$  corresponding to stereographic projection from  $(0, \dots, 0, 1)$  and  $y = (y^1, \dots, y^n)$  be those corresponding to projection from  $(0, \dots, 0, -1)$ . In the  $x$  coordinates, we write the metric as  $g = u(x)^{-2} g_{\text{flat}}$ , and the equations become (see Section 3)

$$(4) \quad \sigma_k \left( u \cdot \frac{\partial^2 u}{\partial x^i \partial x^j} - \frac{|\nabla u|^2}{2} \delta_{ij} \right) = \text{constant}.$$

The round metric is represented by the function  $1 + |x|^2$ .

What is the condition on the function  $u$ , so that  $g$  will give a  $C^2$  metric on  $S^n$ ? We have

$$g = u(x)^{-2} g_{\text{flat}} = u(x)^{-2} (1 + |x|^2)^2 \frac{1}{(1 + |x|^2)^2} g_{\text{flat}} = u(x)^{-2} (1 + |x|^2)^2 g_0,$$

so we require that  $w(x) = u(x)(1 + |x|^2)^{-1}$  should extend to be a positive  $C^2$  function on  $S^n$ . We write  $g = \bar{w}^{-2} g_0$  where  $\bar{w}$  is a function on  $S^n$ , and  $w(x)$  is just the function  $\bar{w}$  in the  $x$  coordinates. We will let  $\tilde{w}(y)$  be the function  $\bar{w}$  expressed in the  $y$  coordinates. We then have

$$\tilde{w}(y) = w\left(\frac{y}{|y|^2}\right) = u\left(\frac{y}{|y|^2}\right) \left(1 + \frac{1}{|y|^2}\right)^{-1} = u\left(\frac{y}{|y|^2}\right) |y|^2 (1 + |y|^2)^{-1}.$$

So then in the  $y$  coordinates, the metric pulls back to

$$g = \tilde{w}^{-2}(y) (1 + |y|^2)^{-2} g_{\text{flat}} = \left(u\left(\frac{y}{|y|^2}\right) |y|^2\right)^{-2} g_{\text{flat}}.$$

So the function  $\tilde{u}(y) = u\left(\frac{y}{|y|^2}\right) |y|^2$  is the conformal factor to the flat metric in the  $y$  coordinates. Therefore we have a  $C^2$  metric on  $S^n$  if and only if  $\tilde{u}(y)$  is  $C^2$  and positive.

3. ELLIPTICITY

In this subsection we will show that our equations are elliptic at any global solution, and moreover, in order to apply the version of the maximum principle required in Theorem 4.1 below, we show that if  $u(x)$  and  $v(x)$  are global solutions, then we have ellipticity for all functions  $(1 - t)v(x) + tu(x)$  with  $t \in [0, 1]$ .

We let

$$F_k(u, u_i, u_{ij}) = \sigma_k \left( u \cdot u_{ij} - \frac{1}{2} \left( \sum_l u_l^2 \right) \delta_{ij} \right) - C.$$

To simplify the notation, we let

$$(5) \quad \bar{u}_{ij} = u \cdot u_{ij} - \frac{1}{2} \left( \sum_l u_l^2 \right) \delta_{ij}.$$

**Definition 3.1.** Let  $a_{ij}$  be the components of an  $n \times n$  matrix. Then for  $0 \leq q \leq n$ , the  $q$ th Newton transformation associated with  $a_{ij}$  is defined to be

$$T_q^{ij}(a_{**}) = \frac{1}{q!} \delta_{j_1 \dots j_q}^{i_1 \dots i_q} a_{i_1 j_1} \cdots a_{i_q j_q},$$

where  $\delta_{j_1 \dots j_q}^{i_1 \dots i_q}$  is the generalized Kronecker delta symbol, and we sum on all repeated indices.

We then have

$$(6) \quad \frac{\partial}{\partial u_{ij}} F_k = u \cdot T_{k-1}^{ij}(\bar{u}_{**}).$$

We now claim that, at  $u$ , this is positive definite, i.e., the equations are elliptic at  $u$ .

**Definition 3.2.** Let  $(\lambda_1, \dots, \lambda_n) \in \mathbf{R}^n$ . We view the elementary symmetric functions as functions on  $\mathbf{R}^n$

$$\sigma_k(\lambda_1, \dots, \lambda_n) = \sum_{i_1 < \dots < i_k} \lambda_{i_1} \cdots \lambda_{i_k},$$

and we let

$$\Gamma_k^+ = \text{component of } \{\sigma_k > 0\} \text{ containing the positive cone.}$$

For a symmetric  $n \times n$  matrix  $A$ , the notation  $A \in \Gamma_k^+$  will mean that the eigenvalues lie in the set.

We have the following proposition, whose proof may be found in [1], [2], and [6].

**Proposition 3.3.** *Each set  $\Gamma_k^+$  is an open convex cone with vertex at the origin, and we have the following sequences of inclusions:*

$$\Gamma_n^+ \subset \Gamma_{n-1}^+ \subset \cdots \subset \Gamma_1^+.$$

*For symmetric matrices  $A \in \Gamma_k^+$ ,  $B \in \Gamma_k^+$ , and  $t \in [0, 1]$ , we have the following inequality:*

$$\{\sigma_k((1 - t)A + tB)\}^{1/k} \geq (1 - t)\{\sigma_k(A)\}^{1/k} + t\{\sigma_k(B)\}^{1/k}.$$

*Furthermore, if  $a_{ij} \in \Gamma_k^+$ , then  $T_{k-1}^{ij}(a_{**})$  is positive definite.*

Since our solution  $u$  is positive, and  $u \rightarrow \infty$  as  $|x| \rightarrow \infty$ , there is a minimum. At this minimum,  $\bar{u}_{ij}$  is positive semi-definite. By continuity we must have  $\bar{u}_{ij} \in \Gamma_k^+$  everywhere. From the above proposition, and (6), we have that equations (1) are elliptic at  $u$ .

We let  $w_t(x) = (1 - t)v(x) + tu(x)$ . We will now show that if  $\bar{u}_{ij} \in \Gamma_k^+$  and  $\bar{v}_{ij} \in \Gamma_k^+$ , then  $(\bar{w}_t)_{ij} \in \Gamma_k^+$ , i.e.,  $F_k$  is elliptic at  $w_t$  for  $t \in [0, 1]$ . We have

$$\begin{aligned} (\bar{w}_t)_{ij} &= w_t(w_t)_{ij} - \frac{|\nabla w_t|^2}{2} \delta_{ij} \\ &= ((1 - t)v + tu)((1 - t)v_{ij} + tu_{ij}) - \frac{|\nabla((1 - t)v + tu)|^2}{2} \delta_{ij} \\ &= (1 - t)^2 v v_{ij} + t^2 u u_{ij} + t(1 - t)(v u_{ij} + u v_{ij}) \\ &\quad - ((1 - t)^2 \frac{|\nabla v|^2}{2} + t(1 - t) \nabla u \cdot \nabla v + t^2 \frac{|\nabla u|^2}{2}) \delta_{ij} \\ &= (1 - t)^2 \bar{v}_{ij} + t^2 \bar{u}_{ij} + t(1 - t) \left( \frac{v}{u} (u u_{ij} - \frac{|\nabla u|^2}{2} \delta_{ij}) + \frac{|\nabla u|^2}{2} \delta_{ij} \right) \\ &\quad + \frac{u}{v} (v v_{ij} - \frac{|\nabla v|^2}{2} \delta_{ij} + \frac{|\nabla v|^2}{2} \delta_{ij}) - \nabla u \cdot \nabla v \delta_{ij} \\ &= (1 - t) \left( (1 - t) \bar{v}_{ij} + t \frac{u}{v} \bar{v}_{ij} \right) + t \left( t \bar{u}_{ij} + (1 - t) \frac{v}{u} \bar{u}_{ij} \right) \\ &\quad + \frac{t(1 - t)}{2uv} (v^2 |\nabla u|^2 + u^2 |\nabla v|^2 - 2v \nabla u \cdot u \nabla v) \delta_{ij} \\ &= (1 - t) \left( (1 - t) \bar{v}_{ij} + t \frac{u}{v} \bar{v}_{ij} \right) + t \left( t \bar{u}_{ij} + (1 - t) \frac{v}{u} \bar{u}_{ij} \right) \\ &\quad + \frac{t(1 - t)}{2uv} (|v \nabla u - u \nabla v|^2 \delta_{ij}). \end{aligned}$$

From Proposition 3.3, the first two terms together are in  $\Gamma_k^+$ . It is easy to see that if  $a_{ij} \in \Gamma_k^+$  and  $\lambda \geq 0$ , then  $a_{ij} + \lambda \delta_{ij} \in \Gamma_k^+$ , so we are done.

#### 4. ROTATIONAL SYMMETRY

We will now show how to apply the ideas of [4] to prove

**Theorem 4.1.** *Suppose that  $u(x)$  satisfies the conditions in Theorem 1.1. Then  $u$  is rotationally symmetric at some point and  $u_r > 0$  for  $r > 0$  where  $r$  is the radial coordinate at that point.*

*Proof.* Let  $v(x) = u^{-1}(x)$ . From the assumption we have that  $\tilde{v}(y) = v(\frac{y}{|y|^2})|y|^{-2}$  is  $C^2$  and positive. Since the equations are invariant under translation, and  $\tilde{v}(y) \rightarrow 0$  and  $y \rightarrow \infty$ , we may assume that  $v(y)$  has a global maximum at 0. From Taylor's Theorem near the origin we have

$$\begin{aligned} \tilde{v}(y) &= a_0 + \frac{1}{2} \tilde{v}_{ij}(0) y_i y_j + o(|y|^2), \text{ and} \\ \tilde{v}_i(y) &= \tilde{v}_i(0) y_j + o(|y|). \end{aligned}$$

From this it follows that  $v(x)$  satisfies the following asymptotic expansions for large  $|x|$ :

$$(7) \quad \begin{aligned} v &= \frac{1}{|x|^2} \left( a_0 + \frac{a_{jk}x_jx_k}{|x|^4} + o\left(\frac{1}{|x|^2}\right) \right), \\ v_{x_i} &= -\frac{2a_0}{|x|^4}x_i - O\left(\frac{1}{|x|^5}\right). \end{aligned}$$

Following [4], we will prove that  $u(x)$  is rotationally symmetric at the origin and that  $u_r > 0$  for  $r > 0$ . Since the equation is invariant under rotations, we just need to prove symmetry under reflection in the hyperplane  $x_1 = 0$  and that  $u_{x_1} > 0$  if  $x_1 > 0$ . As in [4], we let  $T_\lambda$  denote the hyperplane  $x_1 = \lambda$ . For  $\lambda > 0$  and for any  $x = (x_1, x')$  we denote by  $x^\lambda$  the reflection of  $x$  in the plane  $x_1 = \lambda$ .

**Lemma 4.2.** *For any  $\lambda > 0, \exists R = R(\lambda)$  depending only on  $\min(1, \lambda)$  (as well as on  $u$ ) such that for  $x = (x_1, x'), y = (y_1, y')$  satisfying*

$$x_1 < y_1, \quad x_1 + y_1 \geq 2\lambda, \quad |x| \geq R,$$

*we have*

$$u(x) < u(y).$$

*Proof.* Since  $v(x)$  satisfies the expansions (7), Lemma 4.1 of [4] applies to  $v$ , to conclude that  $v(x) > v(y)$ . Therefore  $u(x) < u(y)$ . □

We then have the following lemma, analogous to Lemma 4.2 of [4]

**Lemma 4.3.** *There exists  $\lambda_0 \geq 1$  such that  $\forall \lambda \geq \lambda_0$ ,*

$$(8) \quad u(x) < u(x^\lambda) \quad \text{if } x_1 < \lambda.$$

The following lemma is where ellipticity and the invariance of the equation under reflection in  $T_\lambda$  enter the argument.

**Lemma 4.4.** *Assume that for some  $\lambda > 0$*

$$u(x) \leq u(x^\lambda), \quad u(x) \not\equiv u(x^\lambda), \quad \text{for } x_1 < \lambda.$$

*Then  $u(x) < u(x^\lambda)$  if  $x_1 < \lambda$ , and*

$$(9) \quad u_1(x) > 0 \quad \text{on } T_\lambda.$$

*Proof.* From invariance under reflections, the function  $w(x) = u(x^\lambda)$  is also a solution in  $x_1 < \lambda$  and  $w \geq u$  there. We have that

$$z(x) = u(x) - w(x) \leq 0, \quad z(x) \not\equiv 0.$$

From the discussion of ellipticity above, we then have that  $z(x)$  satisfies a linear elliptic equation of the form

$$Lz = 0 \quad \text{in } x_1 \leq \lambda.$$

On compact subsets,  $L$  is uniformly elliptic, therefore the lemma follows from the maximum principle and the Hopf boundary point lemma. □

Using the above lemmas, as in [4], we can conclude that the reflection property (8) holds for all  $\lambda \in (0, \infty)$ . We then have that  $u(x) \leq u(x^0)$ , and  $u_1 < 0$ , for all  $x$  with  $x_1 < 0$ . Since the direction was arbitrary, the theorem follows. □

5. RADIAL SOLUTIONS

In this section we will analyze the radial solutions of (1). We assume  $u(x)$  is a solution and  $u = u(r)$ . We let  $r = e^t$ , and

$$\phi(t) = e^{-t}u(e^t).$$

The equations (1) become

$$(10) \quad \sigma_k \begin{pmatrix} \phi\phi'' - \frac{\phi^2}{2} - \frac{(\phi')^2}{2} & 0 \\ 0 & (\frac{\phi^2}{2} - \frac{(\phi')^2}{2})I_{n-1} \end{pmatrix} = C_k,$$

where  $C_k$  is a constant. This can be verified by direct computation, but it is easier to use cylindrical coordinates; see [5]. Note that  $\phi(t) = \cosh(t)$  is a solution for all  $k$ . This corresponds to  $u(x) = 1 + |x|^2$ . We will now fix  $C_k = 2^{-k} \binom{n}{k}$  corresponding to this solution in order to get rid of scaling.

From [5], we know that equations (10) are variational, with Lagrangian given by ( $k \neq n/2$ )

$$L = \frac{1}{\phi^n} \left( \frac{1}{(n-2k)} \sigma_k \begin{pmatrix} \phi\phi'' - \frac{\phi^2}{2} - \frac{(\phi')^2}{2} & 0 \\ 0 & (\frac{\phi^2}{2} - \frac{(\phi')^2}{2})I_{n-1} \end{pmatrix} - \frac{C_k}{n} \right).$$

For any one-dimensional functional of the form

$$\int L(\phi, \phi', \phi'') dt,$$

then we have the first integral (see [3])

$$L - \phi'(L_{\phi'} - \frac{d}{dt}L_{\phi''}) - \phi''L_{\phi''} = \text{constant}.$$

To compute the conservation law, since we are at a solution we use (10) for solving  $\phi''$  and substitute this in to get a first order Hamiltonian involving only  $\phi$  and  $\phi'$ . The conservation law takes the form

$$(11) \quad 1 - (\phi^2 - (\phi')^2)^k = D_{k,n}\phi^n,$$

where  $D_{k,n}$  is a constant parametrizing the solutions. Instead of computing it this way, we will just show directly that this is indeed a conservation law by substitution. Note that the above Lagrangian is valid for  $k \neq n/2$ , but for  $k = n/2$  the conservation law still works.

In the following we consider only positive solutions. Assume we have a function  $\phi$  that satisfies the conservation law (11). Since (11) is invariant under  $\phi' \rightarrow -\phi'$ , we may also assume that  $\phi' \geq 0$ , i.e., we just need to do the computation in the upper half phase space. Solving (11) for  $\phi'$  (see below why taking the  $k$ th root is justified) we get

$$(12) \quad \phi' = \sqrt{\phi^2 - (1 - D_{k,n}\phi^n)^{1/k}}.$$

Differentiating this, we find

$$(13) \quad \begin{aligned} \phi'' &= \frac{1}{2} \cdot \frac{1}{\sqrt{\phi^2 - (1 - D_{k,n}\phi^n)^{1/k}}} \cdot \frac{d}{dt}(\phi^2 - (1 - D_{k,n}\phi^n)^{1/k}) \\ &= \frac{1}{2\phi'}(2\phi + \frac{1}{k}(1 - D_{k,n}\phi^n)^{\frac{1}{k}-1} \cdot nD_{k,n}\phi^{n-1})\phi' \\ &= \phi + \frac{n}{2k}(1 - D_{k,n}\phi^n)^{\frac{1-k}{k}} D_{k,n}\phi^{n-1}. \end{aligned}$$

We will now show that  $\phi$  necessarily solves the original equations (10). Expand out (10) to get

$$\binom{n-1}{k-1} \left(\frac{\phi^2}{2} - \frac{\phi'^2}{2}\right)^{k-1} \left(\phi\phi'' - \frac{\phi^2}{2} - \frac{\phi'^2}{2}\right) + \binom{n-1}{k} \left(\frac{\phi^2}{2} - \frac{\phi'^2}{2}\right)^k = C_k.$$

Notice that for  $k > 1$ ,  $\phi^2 - \phi'^2$  factors out of this equation. Since  $C_k > 0$ , we must have  $\phi^2 > \phi'^2$ . This is why we were able to take the  $k$ th root above. Therefore solving this for  $\phi''$  we have

$$\phi'' = \frac{1}{\phi} \left( \frac{-(n-2k)}{2k} \phi^2 + \frac{n}{2k} \phi'^2 + \frac{C_k}{\binom{n-1}{k-1}} \cdot \frac{1}{\left(\frac{\phi^2}{2} - \frac{\phi'^2}{2}\right)^{k-1}} \right).$$

We substitute (12) into this to get

$$\begin{aligned} \phi'' &= \frac{1}{\phi} \left( \frac{2k-n}{2k} \phi^2 + \frac{n}{2k} (\phi^2 - (1 - D_{k,n} \phi^n)^{1/k}) + \frac{2^{k-1} C_k}{\binom{n-1}{k-1}} \cdot (1 - D_{k,n} \phi^n)^{\frac{k-1}{k}} \right) \\ &= \frac{1}{\phi} \left( \phi^2 - \frac{n}{2k} (1 - D_{k,n} \phi^n)^{1/k} + 2^{k-1} \frac{C_k}{\binom{n-1}{k-1}} \cdot (1 - D_{k,n} \phi^n)^{\frac{1}{k}-1} \right) \\ &= \frac{1}{\phi} \left( \phi^2 - (1 - D_{k,n} \phi^n)^{\frac{1}{k}-1} \left( \frac{n}{2k} (1 - D_{k,n} \phi^n) - \frac{2^{k-1} C_k}{\binom{n-1}{k-1}} \right) \right) \\ &= \frac{1}{\phi} \left( \phi^2 - (1 - D_{k,n} \phi^n)^{\frac{1}{k}-1} \left( \frac{n}{2k} (-D_{k,n} \phi^n) \right) \right) \\ &= \phi + \frac{n}{2k} (1 - D_{k,n} \phi^n)^{\frac{1-k}{k}} D_{k,n} \phi^{n-1}, \end{aligned}$$

which equals (13) above.

### 6. COMPLETION OF PROOF OF THEOREM 1.1

We have that  $u(r) = r\phi(\ln r)$ . In order for  $u(r)$  to give a valid solution, we need it to be positive at zero, and that  $u_r(0) = 0$ . For the first condition, we need

$$\infty > \lim_{r \rightarrow 0} u(r) = \lim_{r \rightarrow 0} r\phi(\ln r) = \lim_{t \rightarrow -\infty} e^t \phi(t) > 0.$$

So we must have for some constant  $c_1 > 0$

$$(14) \quad \phi(t) = (c_1 + o(1)) \frac{e^{-t}}{2} \text{ as } t \rightarrow -\infty.$$

In particular, we must have  $\phi(t) \rightarrow \infty$  as  $t \rightarrow -\infty$ .

For the second condition we have

$$0 = \lim_{r \rightarrow 0} u_r(r) = \lim_{r \rightarrow 0} (r\phi(\ln r))_r = \lim_{r \rightarrow 0} (\phi(\ln r) + \phi'(\ln r)) = \lim_{t \rightarrow -\infty} (\phi(t) + \phi'(t)).$$

Therefore we must have

$$(15) \quad \phi'(t) = -\phi(t) + \epsilon(t),$$

where

$$(16) \quad \epsilon(t) \rightarrow 0 \text{ as } t \rightarrow -\infty.$$

Plugging this into the conservation law (11) we have

$$\begin{aligned} (\phi^2 - \phi'^2)^k &= \left(\phi^2 - (-\phi + \epsilon)^2\right)^k \\ &= \left(\phi^2 - (\phi^2 - 2\epsilon\phi + \epsilon^2)\right)^k \\ &= \left(2\epsilon\phi - \epsilon^2\right)^k = 1 - D_{k,n}\phi^n. \end{aligned}$$

Since  $\phi > 0$ , dividing by  $\phi^n$  we have

$$\frac{1}{\phi^{n-k}} \left(2\epsilon - \frac{\epsilon^2}{\phi}\right)^k = \frac{1}{\phi^n} - D_{k,n}.$$

Since  $\phi(t) \rightarrow \infty$  and  $\epsilon(t) \rightarrow 0$  as  $t \rightarrow -\infty$ , the left hand side goes to zero. The right hand side approaches  $D_{k,n}$ , therefore we must have  $D_{k,n} = 0$ . It follows easily that  $\phi(t) = \cosh(t - t_0)$ , or  $u(x) = r_0 + \frac{|x|^2}{r_0}$ . The images of this under the conformal group are all of the form

$$u(x) = a|x|^2 + b_i x^i + c$$

where  $a$ ,  $b_i$ , and  $c$  are constants. This proves Theorem 1.1.

### 7. CONFORMAL GEOMETRY

In this section we will explain the geometric origin of the equations and prove the invariance property (2). If  $(N, g_0)$  is a Riemannian manifold, we consider the equations

$$(17) \quad \sigma_k \left( Ric - \frac{R}{2(n-1)}g \right) = \text{constant}$$

for metrics  $g$  in the conformal class of  $g_0$ , where  $Ric$  and  $R$  are the Ricci tensor and scalar curvature of the metric  $g$ , respectively (note that we are using the metric to view the tensor as a  $(1, 1)$  tensor). If we let  $g = u^{-2}g_0$ , then we have the following transformation formula (see [5]):

$$\frac{1}{n-2} \left( Ric - \frac{R}{2(n-1)}g \right) = w\nabla^2 w + \frac{w^2}{n-2} \left( Ric_0 - \frac{R_0}{2(n-1)}g_0 \right) - \frac{|\nabla w|^2}{2}g_0.$$

Letting  $N = \mathbf{R}^n$ , and  $g_0$  be the flat metric, we get the equations (1).

If  $T : N \rightarrow N$  is a conformal transformation, we have that  $T^*g_0 = \lambda(x)g_0$  for some positive function  $\lambda$ . The Jacobian of  $T$  with respect to the metric  $g_0$  is defined by  $T^*\text{vol}_0 = J \cdot \text{vol}_0$ . It is easy to verify that  $\lambda = J^{2/n}$ . If  $g = u^{-2}g_0$  is a metric solving (17), we let  $\bar{g} = T^*g$ . The map  $T$  is then an isometry from  $(N, \bar{g})$  to  $(N, g)$ , therefore  $\bar{g}$  also solves the equations (17). We then have

$$\bar{g} = T^*g = T^*(u^{-2}g_0) = (u \circ T)^{-2}J^{2/n}g_0 = (u \circ T \cdot |J|^{-1/n})^{-2}g_0,$$

which is the invariance property (2).



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