

SHARP SOBOLEV INEQUALITIES WITH LOWER ORDER REMAINDER TERMS

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ABSTRACT. Given a smooth compact Riemannian n -manifold (M, g) , this paper deals with the sharp Sobolev inequality corresponding to the embedding of $H_1^2(M)$ in $L^{2n/(n-2)}(M)$ where the L^2 remainder term is replaced by a lower order term.

Let (M, g) be a smooth compact Riemannian n -dimensional manifold, $n \geq 3$. Let $H_1^2(M)$ be the standard Sobolev space, defined as the completion of $C^\infty(M)$ with respect to

$$\|u\|_{H_1^2}^2 = \|\nabla u\|_2^2 + \|u\|_2^2$$

where $\|\cdot\|_p$, $p \geq 1$, stands for the L^p -norm. By the Sobolev embedding theorem, one has that $H_1^2(M) \subset L^{2^*}(M)$, where $2^* = 2n/(n-2)$ is critical. Given $\theta \in [1, 2]$, there exist $A, B \in \mathbb{R}$ such that, for any $u \in H_1^2(M)$,

$$(S_\theta) \quad \|u\|_{2^*}^2 \leq A\|\nabla u\|_2^2 + B\|u\|_\theta^2$$

This easily follows from the Sobolev-Poincaré inequality. Now define K_n as the best constant K for the standard H_1^2 -Sobolev inequality in \mathbb{R}^n :

$$\|u\|_{2^*} \leq K\|\nabla u\|_2$$

As is well known, see [1] and [24],

$$K_n = \sqrt{\frac{4}{n(n-2)\omega_n^{2/n}}}$$

where ω_n stands for the volume of the unit sphere S^n of \mathbb{R}^{n+1} . Mimicking what was done in [15], proposition 4.5, one easily checks that any constant A in (S_θ) has to be greater than or equal to K_n^2 . Conversely, one may prove that for any $\varepsilon > 0$, there exists B_ε such that for any $u \in H_1^2(M)$,

$$\|u\|_{2^*}^2 \leq (K_n^2 + \varepsilon)\|\nabla u\|_2^2 + B_\varepsilon\|u\|_\theta^2$$

This follows from standard arguments, by locally comparing the Riemannian metric with the Euclidean metric, and then by gluing together the local inequalities to get the global one.

These few remarks lead to the consideration of the following sharp inequality: for any $u \in H_1^2(M)$,

$$(S_{opt}^\theta) \quad \|u\|_{2^*}^2 \leq K_n^2\|\nabla u\|_2^2 + B\|u\|_\theta^2$$

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where $B \in \mathbb{R}$. In what follows, we say that (S_{opt}^θ) is valid on (M, g) if there exists $B \in \mathbb{R}$ such that (S_{opt}^θ) holds for all $u \in H_1^2(M)$. As one can easily check, if (S_{opt}^θ) is valid for some $\theta_0 \in [1, 2]$, then it is also valid for all $\theta \in [\theta_0, 2]$. For $\theta = 1$, one recovers a sharp inequality of Sobolev-Poincaré type.

Given $\theta \in [1, 2]$, and (M, g) a smooth compact Riemannian n -manifold, $n \geq 3$, this article deals with the question of how we can know whether (S_{opt}^θ) is valid on (M, g) or not. Among others, such a question was considered in Aubin and Li [3]. Since the work of Druet [10], we know that a key point when studying sharp Sobolev inequalities is to prove that such inequalities are localisable. They may then be affected by the geometry. In other words, sometimes they are valid and sometimes not, depending on the manifold. This will be referred to as the effect of geometry. In addition to the original [10], see also [11] and [12]. Another effect that may occur when studying elliptic type problems, first pointed out by Brezis and Nirenberg in their celebrated paper [6], is the so-called low dimensions effect: a situation may change when passing from low dimensions to high dimensions. A particularly interesting feature when studying the scale (S_{opt}^θ) of sharp inequalities, $\theta \in [1, 2]$, is that it mixes both types of phenomena. For $\theta = 2$, we know by Hebey and Vaugon [19] that (S_{opt}^2) is always valid, without any assumptions on the manifold. For $\theta < 2$, it appears that (S_{opt}^θ) is always localisable, with the property that it is affected by the geometry when $n \geq 4$, but not when $n = 3$.

Concerning the terminology, a Cartan-Hadamard n -dimensional manifold is a complete simply-connected n -dimensional Riemannian manifold of nonpositive sectional curvature. We then refer to the n -dimensional Cartan-Hadamard conjecture as follows: given a Cartan-Hadamard n -dimensional manifold (\tilde{M}, \tilde{g}) and a smooth bounded domain Ω in \tilde{M} , we ask for the inequality

$$\frac{Area_{\tilde{g}}(\partial\Omega)}{Vol_{\tilde{g}}(\Omega)^{(n-1)/n}} \geq n \left(\frac{\omega_{n-1}}{n} \right)^{\frac{1}{n}}$$

that is the value one gets for the above ratio in the Euclidean n -dimensional space when Ω is a ball. Such a conjecture has been proved to be true for $n = 2$ by Weil [26], for $n = 3$ by Kleiner [22], and for $n = 4$ by Croke [8]. Our first result is the following. We let S_g and K_g be the scalar and sectional curvatures of g .

Theorem 1. *Let (M, g) be a smooth compact Riemannian n -manifold, $n \geq 3$. For $\theta = 2$, (S_{opt}^2) is valid on (M, g) without any assumption. On the contrary, for $\theta \in [1, 2)$, (S_{opt}^θ) is valid on (M, g) if either K_g is a nonpositive constant, or K_g is nonpositive everywhere on M and the n -dimensional Cartan-Hadamard conjecture is true, but (S_{opt}^θ) is false if $n \geq 4$ and S_g is positive somewhere on M .*

A main step in the proof of this theorem is to show that (S_{opt}^θ) is localisable—in other words to prove that if for any $x \in M$, there exist an open neighborhood Ω_x of x , and $B_x \in \mathbb{R}$, such that for any $u \in C_c^\infty(\Omega_x)$,

$$\|u\|_{2^*}^2 \leq K_n^2 \|\nabla u\|_2^2 + B_x \|u\|_\theta^2$$

then (S_{opt}^θ) is valid on (M, g) . For $\theta = 2$, a partition of unity easily gives the localisation. For $\theta < 2$, it is proved here by PDE type arguments.

Regarding Theorem 1, one sees that for $n \geq 4$, and $\theta < 2$, (S_{opt}^θ) is valid depending on the sign of the curvature. This is the illustration in our context of the first type of phenomenon mentioned above: the effect of geometry. For $n = 3$, as we see in Theorem 2 below, (S_{opt}^θ) is valid for any θ on any manifold. One

recovers there the second type of phenomenon mentioned above: the effect of low dimensions. Theorems 1 and 2 were announced in [13].

Theorem 2. *For any smooth compact Riemannian 3-manifold (M, g) , and for any $\theta \in [1, 2]$, (S_{opt}^θ) is valid on (M, g) .*

Now define $B_0(g)$ in (S_{opt}^2) by

$$B_0(g) = \inf \{B \in \mathbb{R} \text{ for which } (S_{opt}^2) \text{ is true}\}$$

Consider then the optimal Sobolev inequality: for all $u \in H_1^2(M)$,

$$(S_{OPT}^2) \quad \|u\|_{2^*}^2 \leq K_n^2 \|\nabla u\|_2^2 + B_0(g) \|u\|_2^2$$

Such an inequality is always valid. By definition, we say that $u \in H_1^2(M)$, $u \neq 0$, is an extremal function for (S_{OPT}^2) if u realizes the equality in (S_{OPT}^2) . The existence of extremal functions for (S_{OPT}^2) was studied by Hebey [16], and more recently by Djadli and Druet [9]. It was proved in [16] that if $n \geq 4$, and g is a conformal metric to the standard metric h on the unit sphere S^n , then (S_{OPT}^2) with respect to g has extremal functions if and only if g and h are (up to a constant) isometric. In particular, there exist manifolds of positive scalar curvature for which (S_{OPT}^2) does not have extremal functions. Among other results, it was proved in [9] that when $n \geq 4$, (S_{OPT}^2) possesses extremal functions if either S_g is nonpositive, or S_g is constant. As a concluding remark, following an idea that was developed by Humbert [21], and quoting a theorem that was stated in the very recent announcement [18] by Hebey, Humbert and Vaugon, we wish to point out the influence that (S_{opt}^1) has on the existence of extremal functions for the optimal Sobolev inequality (S_{OPT}^2) . This is the subject of Theorem 3 below.

Theorem 3. *Let (M, g) be a smooth compact Riemannian n -manifold, $n \geq 5$. Suppose that (S_{opt}^1) is true. Then there exist extremal functions for (S_{OPT}^2) . In particular, (S_{OPT}^2) possesses extremal functions on flat tori and compact hyperbolic spaces, $n \geq 5$, or again on compact Riemannian n -manifolds of nonpositive sectional curvature, $n \geq 5$, if the n -dimensional Cartan-Hadamard conjecture is true.*

The rest of the article is devoted to the proof of these three theorems.

1. PROOF OF THEOREM 1

As already mentioned, a main step when proving Theorem 1 is to show that (S_{opt}^θ) is localisable. This is done through PDE arguments. A preliminary technical lemma is needed.

Lemma 1. *Let (M, g) be a smooth compact Riemannian n -manifold, $n \geq 3$, and let $u \in H_1^2(M)$, $u \geq 0$, be such that for any nonnegative $\varphi \in H_1^2(M)$,*

$$(1.1) \quad \int_M \langle \nabla u, \nabla \varphi \rangle dv_g \leq \int_M u^{2^*-1} \varphi dv_g$$

Then $u \in L^\infty(M)$. Moreover, for any x in M , any $\Lambda > 0$, any $p > 0$, and any $q > 2^$, there exists $\delta > 0$ such that if u is some nonnegative function of $H_1^2(M)$ satisfying (1.1) and*

$$\int_{B_x(2\delta)} u^q dv_g \leq \Lambda$$

then

$$\sup_{y \in B_x(\delta)} u(y) \leq \tilde{C} \left(\int_{B_x(2\delta)} u^p dv_g \right)^{1/p}$$

where $\tilde{C} > 0$ does not depend on u .

We omit the proof of this lemma since it goes in a very standard way. Mimicking what was done in Trudinger [25] when studying the regularity for the Yamabe problem, one first gets that $u \in L^k(M)$ for some $k > 2^*$. The lemma then follows from the classical De Giorgi-Nash-Moser theory when studying inequations of the type

$$\Delta_g u + cu \leq f$$

where $\Delta_g u = -\operatorname{div}_g(\nabla u)$ stands for the Laplacian of u with respect to g . We let $c = -u^{2^*-2}$ and $f = 0$ here.

Going on with Theorem 1, we prove now that (S_{opt}^θ) is localisable.

Lemma 2. *Let (M, g) be a smooth compact Riemannian n -manifold, $n \geq 3$, and let $\theta \in [1, 2]$. Suppose that for any x in M , there exist an open neighborhood Ω_x of x , and $B_x \in \mathbb{R}$, such that for any $u \in C_c^\infty(\Omega_x)$,*

$$\|u\|_{2^*}^2 \leq K_n^2 \|\nabla u\|_2^2 + B_x \|u\|_\theta^2$$

Then (S_{opt}^θ) is valid on (M, g) .

Proof. For any $\alpha > 0$, let I_α be the functional defined on $H_1^2(M) \setminus \{0\}$ by

$$I_\alpha(u) = \frac{\|\nabla u\|_2^2 + \alpha \|u\|_\theta^2}{\|u\|_{2^*}^2}$$

Under the assumptions of the lemma, namely that (S_{opt}^θ) is locally valid, the lemma reduces to the existence of some α_0 such that

$$\inf_{H_1^2(M) \setminus \{0\}} I_{\alpha_0}(u) \geq \frac{1}{K_n^2}$$

We proceed by contradiction, and assume that for any $\alpha > 0$,

$$(1.2) \quad \inf_{H_1^2(M) \setminus \{0\}} I_\alpha(u) < \frac{1}{K_n^2}$$

The proof then goes in several steps.

Step 1. We claim that (1.2) implies the existence of a nonnegative minimizer for

$$\mu_\alpha = \inf_{H_1^2(M) \setminus \{0\}} I_\alpha(u)$$

If $\theta > 1$, the proof of such a claim is quite standard. If $\theta = 1$, one may proceed as follows. For $q < 2^*$, let $\theta_q > 1$ be given with the property that θ_q goes to 1 as q goes to 2^* . We fix $\alpha > 0$ and for $q < 2^*$ we let

$$\lambda_q = \inf_{H_1^2(M) \setminus \{0\}} \frac{\|\nabla u\|_2^2 + \alpha \|u\|_{\theta_q}^2}{\|u\|_q^2}$$

Since the embedding of $H_1^2(M)$ in $L^q(M)$ is compact, and since the above functional is homogeneous, there exists a nonnegative minimizer u_q for λ_q such that $\|u_q\|_q = 1$.

Clearly, u_q is a weak solution of

$$(1.3) \quad \Delta_g u_q + \alpha \left(\int_M u_q^{\theta_q} dv_g \right)^{\frac{2}{\theta_q}-1} u_q^{\theta_q-1} = \lambda_q u_q^{q-1}$$

where Δ_g stands for the Laplacian with respect to g . As one can easily check, up to a subsequence we may assume that for some $\lambda_\alpha \leq \mu_\alpha$, the sequence (λ_q) goes to λ_α as q goes to 2^* . Noting that (u_q) is bounded in $H_1^2(M)$, we see that there exists $u_\alpha \in H_1^2(M)$ such that, up to a subsequence, (u_q) converges weakly to u_α in $H_1^2(M)$, strongly to u_α in $L^2(M)$, and almost everywhere. Moreover, one can assume that

$$u_q^{q-1} \rightharpoonup u_\alpha^{2^*-1} \quad \text{in } L^{2^\sharp}(M)$$

where $2^\sharp = 2n/(n+2)$ is the conjugate exponent of 2^* . By (1.2), and since for any $\varepsilon > 0$ there exists B_ε such that for any $u \in H_1^2(M)$,

$$\|u\|_{2^*}^2 \leq (K_n^2 + \varepsilon) \|\nabla u\|_2^2 + B_\varepsilon \|u\|_1^2$$

one has that $u_\alpha \neq 0$. This is by now standard. Let $\varepsilon_q = \theta_q - 1$. Clearly, $(u_q^{\varepsilon_q})$ is bounded in $L^p(M)$ for any $p > 1$. Concerning such an assertion, just note that for $q \gg 1$,

$$(1.4) \quad \left(\int_M u_q^{p\varepsilon_q} dv_g \right)^{1/p} \leq \left(\int_M u_q^2 dv_g \right)^{\varepsilon_q/2} V_g^{\frac{1}{p} - \frac{\varepsilon_q}{2}}$$

where V_g stands for the volume of M with respect to g . Since L^p -spaces are reflexive for $p > 1$, there exists $\Sigma_\alpha \in \bigcap_{p>1} L^p(M)$ such that for any $p > 1$, and up to a subsequence,

$$u_q^{\varepsilon_q} \rightharpoonup \Sigma_\alpha \quad \text{in } L^p(M)$$

Passing to the limit as q goes to 2^* in (1.4), one gets that for any $p > 1$,

$$\|\Sigma_\alpha\|_p \leq V_g^{1/p}$$

As an easy consequence, $\Sigma_\alpha \in L^\infty(M)$ and $0 \leq \Sigma_\alpha \leq 1$. Another easy claim is that $\Sigma_\alpha \varphi = \varphi$ for any $\varphi \in H_1^2(M)$ having the property that $|\varphi| \leq C u_\alpha$ on M for some constant $C > 0$. By passing to the limit in (1.3), one gets that u_α is a weak solution of

$$(1.5) \quad \Delta_g u_\alpha + \alpha \left(\int_M u_\alpha dv_g \right) \Sigma_\alpha = \lambda_\alpha u_\alpha^{2^*-1}$$

Clearly, $\|u_\alpha\|_{2^*} \leq 1$. Multiplying (1.5) by u_α and integrating over M gives

$$\frac{\|\nabla u_\alpha\|_2^2 + \alpha \|u_\alpha\|_2^2}{\|u_\alpha\|_{2^*}^2} = \lambda_\alpha \|u_\alpha\|_{2^*}^{2^*-2}$$

As one can easily check, this implies that $\|u_\alpha\|_{2^*} = 1$ and $\lambda_\alpha = \mu_\alpha$. In particular, u_α is a minimizer for μ_α . The above claim is proved.

From now on, let $u_\alpha \in H_1^2(M)$, $u_\alpha \geq 0$, $u_\alpha \neq 0$, be the minimizer for μ_α given by step 1. As shown in the proof of step 1, we can choose u_α such that

$$\int_M u_\alpha^{2^*} dv_g = 1$$

and for some $\Sigma_\alpha \in L^\infty(M)$, with the property that $0 \leq \Sigma_\alpha \leq 1$, u_α is a solution of

$$(E_\alpha) \quad \Delta_g u_\alpha + \alpha \left(\int_M u_\alpha^\theta dv_g \right)^{\frac{2}{\theta}-1} \Sigma_\alpha = \mu_\alpha u_\alpha^{2^*-1}$$

Moreover, $\Sigma_\alpha = u_\alpha^{\theta-1}$ if $\theta > 1$, and if $\theta = 1$, then $\Sigma_\alpha \varphi = \varphi$ for any $\varphi \in H_1^2(M)$ having the property that $|\varphi| \leq C u_\alpha$ on M for some constant $C > 0$. By lemma 1, $u_\alpha \in L^\infty(M)$. Hence, $u_\alpha \in H_2^p(M)$ for any $p > 1$, and u_α is actually in $C^{1,\lambda}$ for any $\lambda \in (0, 1)$. As another remark, the sequence (u_α) is bounded in $H_1^2(M)$.

By definition (this is by now classical) we say that $x \in M$ is a concentration point for the sequence (u_α) if for any $\delta > 0$,

$$\limsup_{\alpha \rightarrow +\infty} \int_{B_x(\delta)} u_\alpha^{2^*} dv_g > 0$$

Since M is compact, the existence of at least one such point is easy to get. We prove the uniqueness of the concentration point in step 2.

Step 2. We claim that, up to a subsequence, (u_α) has one and only one concentration point. Given $x \in M$ and $\delta > 0$, δ small, let $\eta \in C_c^\infty(B_x(\delta))$ be such that $0 \leq \eta \leq 1$ and $\eta = 1$ in $B_x(\delta/2)$. Multiplying (E_α) by $\eta^2 u_\alpha^k$, $k \geq 1$ real, and integrating over M lead to

$$\begin{aligned} (1.6) \quad & \int_M \eta^2 u_\alpha^k \Delta_g u_\alpha dv_g + \alpha \left(\int_M u_\alpha^\theta dv_g \right)^{\frac{2}{\theta}-1} \int_M \eta^2 u_\alpha^{k+\theta-1} dv_g \\ & = \mu_\alpha \int_M \eta^2 u_\alpha^{2^*+k-1} dv_g \end{aligned}$$

As one can easily check,

$$\begin{aligned} \int_M \eta^2 u_\alpha^k \Delta_g u_\alpha dv_g & = \frac{4k}{(k+1)^2} \int_M |\nabla(\eta u_\alpha^{(k+1)/2})|^2 dv_g \\ & - \frac{2(k-1)}{(k+1)^2} \int_M \eta(\Delta_g \eta) u_\alpha^{k+1} dv_g - \frac{2}{k+1} \int_M |\nabla \eta|^2 u_\alpha^{k+1} dv_g \end{aligned}$$

while, by Hölder's inequality,

$$\int_M \eta^2 u_\alpha^{2^*+k-1} dv_g \leq \left(\int_M (\eta u_\alpha^{(k+1)/2})^{2^*} dv_g \right)^{2/2^*} \left(\int_{B_x(\delta)} u_\alpha^{2^*} dv_g \right)^{(2^*-2)/2^*}$$

According to Hebey and Vaugon [19], there exists $B > 0$ such that for any $u \in H_1^2(M)$,

$$\left(\int_M |u|^{2^*} dv_g \right)^{2/2^*} \leq K_n^2 \int_M |\nabla u|^2 dv_g + B \int_M u^2 dv_g$$

Coming back to (1.6), and since the second term in the left hand side of (1.6) is nonnegative, one gets that

$$\begin{aligned} (1.7) \quad & A_\alpha(k, \delta) \left(\int_M (\eta u_\alpha^{(k+1)/2})^{2^*} dv_g \right)^{2/2^*} \leq \frac{k-1}{2k} K_n^2 \int_M \eta(\Delta_g \eta) u_\alpha^{k+1} dv_g \\ & + \frac{k+1}{2k} K_n^2 \int_M |\nabla \eta|^2 u_\alpha^{k+1} dv_g + B \int_M \eta^2 u_\alpha^{k+1} dv_g \end{aligned}$$

where

$$A_\alpha(k, \delta) = 1 - \frac{(k+1)^2}{4k} \mu_\alpha K_n^2 \left(\int_{B_x(\delta)} u_\alpha^{2^*} dv_g \right)^{(2^*-2)/2^*}$$

Suppose now that x is a concentration point for (u_α) . Given $\delta > 0$, let

$$\limsup_{\alpha \rightarrow +\infty} \int_{B_x(\delta)} u_\alpha^{2^*} dv_g = \lambda_\delta$$

Then $\lambda_\delta > 0$ and $\lambda_\delta \leq 1$. Assume that $\lambda_\delta < 1$ for some $\delta > 0$. Together with (1.2), we may then choose $k > 1$ sufficiently close to 1 such that

$$1 - \frac{(k+1)^2}{4k} \mu_\alpha K_n^2 \lambda_\delta^{(2^*-2)/2^*} > 0$$

The right hand side of (1.7) being bounded for $k > 1$ close to 1, we get with (1.7) the existence of $K > 0$ such that for $\alpha \gg 1$,

$$\int_M (\eta u_\alpha^{(k+1)/2})^{2^*} dv_g \leq K$$

By Hölder's inequality, writing $2^* = (2^* - k - 1) + (k + 1)$,

$$\begin{aligned} \int_{B_x(\delta/2)} u_\alpha^{2^*} dv_g &\leq \left(\int_M u_\alpha^{2^* - \frac{2^*(k-1)}{2^*-2}} dv_g \right)^{(2^*-2)/2^*} \left(\int_M (\eta u_\alpha^{(k+1)/2})^{2^*} dv_g \right)^{2/2^*} \\ &\leq K^{2/2^*} \left(\int_M u_\alpha^{2^* - \frac{2^*(k-1)}{2^*-2}} dv_g \right)^{(2^*-2)/2^*} \end{aligned}$$

Noting that for $k > 1$ close to 1,

$$1 < 2^* - \frac{2^*(k-1)}{2^*-2} < 2^*$$

one gets

$$(1.8) \quad \lim_{\alpha \rightarrow +\infty} \int_{B_x(\delta/2)} u_\alpha^{2^*} dv_g = 0$$

Concerning such an assertion, just note that multiplying (E_α) by u_α and integrating over M gives that the L^1 -norm of u_α goes to 0 as α goes to $+\infty$, so that the L^q -norm of u_α also goes to 0 for any $1 < q < 2^*$ (by Hölder's inequality and since the L^{2^*} -norm of u_α is 1). Since (1.8) contradicts the definition of a concentration point, one actually has that $\lambda_\delta = 1$ for any $\delta > 0$. As one will easily check, up to the extraction of a subsequence, this implies that a concentration point must be unique. The above claim is proved.

We let x_0 be the concentration point for (u_α) . Given $x \neq x_0$, one gets with (1.7) that for $\delta > 0$ small, the $L^{(2^*)^2/2}$ -norm of u_α in $B_x(\delta)$ is bounded. Together with lemma 1, noting that $(2^*)^2/2 > 2^*$, this implies that

$$(1.9) \quad u_\alpha \rightarrow 0 \quad \text{in } C_{loc}^0(M \setminus \{x_0\})$$

as α goes to $+\infty$. Mimicking what was done in Druet [10], the final argument in the proof of lemma 2 goes in the following way.

Step 3. We claim that (1.2) is impossible for α sufficiently large. By assumption there exist $B \in \mathbb{R}$ and $\delta > 0$ such that for any $u \in H_{0,1}^2(B_\delta)$,

$$(1.10) \quad \left(\int_M |u|^{2^*} dv_g \right)^{2/2^*} \leq K_n^2 \int_M |\nabla u|^2 dv_g + B \left(\int_M |u|^\theta dv_g \right)^{2/\theta}$$

where $B_\delta = B_{x_0}(\delta)$ and $H_{0,1}^2(B_\delta)$ stands for the completion of $C_c^\infty(B_\delta)$ with respect to $\|\cdot\|_{H_1^2}$. We let $\eta \in C_c^\infty(B_\delta)$ be such that $0 \leq \eta \leq 1$ and $\eta = 1$ in $B_{\delta'}$ for some $\delta' \in (0, \delta)$. If we set $\eta' = 1 - \eta$, (1.10) leads in particular to

$$\left(\int_{B_{\delta'}} u_\alpha^{2^*} dv_g \right)^{2/2^*} \leq K_n^2 \int_M |\nabla((1 - \eta')u_\alpha)|^2 dv_g + B \left(\int_M u_\alpha^\theta dv_g \right)^{2/\theta}$$

Clearly, there exists $C > 0$, independent of α , such that

$$\begin{aligned} \int_M |\nabla((1-\eta')u_\alpha)|^2 dv_g &\leq \int_M |\nabla u_\alpha|^2 dv_g + C \int_{M \setminus B_{\delta'}} |\nabla u_\alpha|^2 dv_g \\ &\quad + C \int_{M \setminus B_{\delta'}} u_\alpha |\nabla u_\alpha| dv_g + C \int_{M \setminus B_{\delta'}} u_\alpha^2 dv_g \end{aligned}$$

Multiplying (E_α) by u_α , and integrating over M , gives

$$\int_M |\nabla u_\alpha|^2 dv_g + \alpha \left(\int_M u_\alpha^\theta dv_g \right)^{2/\theta} = \mu_\alpha$$

Hence,

$$\begin{aligned} \left(\int_{B_{\delta'}} u_\alpha^{2^*} dv_g \right)^{2/2^*} &\leq K_n^2 \mu_\alpha - \alpha K_n^2 \left(\int_M u_\alpha^\theta dv_g \right)^{2/\theta} \\ &\quad + C \int_{M \setminus B_{\delta'}} |\nabla u_\alpha|^2 dv_g + C \int_{M \setminus B_{\delta'}} u_\alpha |\nabla u_\alpha| dv_g \\ &\quad + C \int_{M \setminus B_{\delta'}} u_\alpha^2 dv_g + B \left(\int_M u_\alpha^\theta dv_g \right)^{2/\theta} \end{aligned}$$

for some other constant $C > 0$ independent of α . Clearly,

$$1 - \left(\int_{B_{\delta'}} u_\alpha^{2^*} dv_g \right)^{2/2^*} \leq \int_{M \setminus B_{\delta'}} u_\alpha^{2^*} dv_g$$

while

$$\int_{M \setminus B_{\delta'}} u_\alpha |\nabla u_\alpha| dv_g \leq \left(\int_{M \setminus B_{\delta'}} u_\alpha^2 dv_g \right)^{1/2} \left(\int_{M \setminus B_{\delta'}} |\nabla u_\alpha|^2 dv_g \right)^{1/2}$$

Since $\mu_\alpha K_n^2 < 1$, one gets

$$\begin{aligned} \alpha K_n^2 - B &\leq \frac{\int_{M \setminus B_{\delta'}} u_\alpha^{2^*} dv_g}{\left(\int_M u_\alpha^\theta dv_g \right)^{2/\theta}} + C \frac{\int_{M \setminus B_{\delta'}} |\nabla u_\alpha|^2 dv_g}{\left(\int_M u_\alpha^\theta dv_g \right)^{2/\theta}} + C \frac{\int_{M \setminus B_{\delta'}} u_\alpha^2 dv_g}{\left(\int_M u_\alpha^\theta dv_g \right)^{2/\theta}} \\ (1.11) \quad &+ C \left(\frac{\int_{M \setminus B_{\delta'}} u_\alpha^2 dv_g}{\left(\int_M u_\alpha^\theta dv_g \right)^{2/\theta}} \right)^{1/2} \left(\frac{\int_{M \setminus B_{\delta'}} |\nabla u_\alpha|^2 dv_g}{\left(\int_M u_\alpha^\theta dv_g \right)^{2/\theta}} \right)^{1/2} \end{aligned}$$

By lemma 1,

$$\begin{aligned} \int_{M \setminus B_{\delta'}} u_\alpha^2 dv_g &\leq V_g \left(\sup_{M \setminus B_{\delta'}} u_\alpha \right)^2 \\ &\leq C \left(\int_M u_\alpha^\theta dv_g \right)^{2/\theta} \end{aligned}$$

where V_g stands for the volume of M with respect to g , and $C > 0$ is independent of α . As a consequence,

$$(1.12) \quad \frac{\int_{M \setminus B_{\delta'}} u_\alpha^2 dv_g}{\left(\int_M u_\alpha^\theta dv_g \right)^{2/\theta}} \leq C$$

Together with (1.9),

$$\frac{\int_{M \setminus B_{\delta'}} u_{\alpha}^{2^*} dv_g}{\left(\int_M u_{\alpha}^{\theta} dv_g\right)^{2/\theta}} \leq C \left(\sup_{M \setminus B_{\delta'}} u_{\alpha}\right)^{2^*-2}$$

so that

$$(1.13) \quad \lim_{\alpha \rightarrow +\infty} \frac{\int_{M \setminus B_{\delta'}} u_{\alpha}^{2^*} dv_g}{\left(\int_M u_{\alpha}^{\theta} dv_g\right)^{2/\theta}} = 0$$

For $\delta'' \in (0, \delta')$, let $0 \leq \eta'' \leq 1$ be a smooth function on M such that $\eta'' = 0$ on $B_{\delta''}$ and $\eta'' = 1$ on $M \setminus B_{\delta''}$. Multiplying (E_{α}) by $(\eta'')^2 u_{\alpha}$, and integrating over M , gives

$$\int_M (\eta'')^2 |\nabla u_{\alpha}|^2 dv_g + 2 \int_M \eta'' u_{\alpha} \langle \nabla \eta'', \nabla u_{\alpha} \rangle dv_g \leq K_n^{-2} \int_M (\eta'')^2 u_{\alpha}^{2^*} dv_g$$

In particular,

$$\begin{aligned} & \int_M (\eta'')^2 |\nabla u_{\alpha}|^2 dv_g \\ & \leq C \int_M (\eta'')^2 u_{\alpha}^{2^*} dv_g + C \left(\int_M |\nabla \eta''|^2 u_{\alpha}^2 dv_g \right)^{1/2} \left(\int_M (\eta'')^2 |\nabla u_{\alpha}|^2 dv_g \right)^{1/2} \end{aligned}$$

for some constant $C > 0$ independent of α . Hence,

$$\begin{aligned} & \frac{\int_M (\eta'')^2 |\nabla u_{\alpha}|^2 dv_g}{\left(\int_M u_{\alpha}^{\theta} dv_g\right)^{2/\theta}} \\ & \leq C \frac{\int_M (\eta'')^2 u_{\alpha}^{2^*} dv_g}{\left(\int_M u_{\alpha}^{\theta} dv_g\right)^{2/\theta}} + C \left(\frac{\int_M |\nabla \eta''|^2 u_{\alpha}^2 dv_g}{\left(\int_M u_{\alpha}^{\theta} dv_g\right)^{2/\theta}} \right)^{1/2} \left(\frac{\int_M (\eta'')^2 |\nabla u_{\alpha}|^2 dv_g}{\left(\int_M u_{\alpha}^{\theta} dv_g\right)^{2/\theta}} \right)^{1/2} \end{aligned}$$

By (1.13),

$$\lim_{\alpha \rightarrow +\infty} \frac{\int_M (\eta'')^2 u_{\alpha}^{2^*} dv_g}{\left(\int_M u_{\alpha}^{\theta} dv_g\right)^{2/\theta}} = 0$$

while by (1.12),

$$\frac{\int_M |\nabla \eta''|^2 u_{\alpha}^2 dv_g}{\left(\int_M u_{\alpha}^{\theta} dv_g\right)^{2/\theta}} \leq C$$

for some $C > 0$ independent of α . Noting that

$$\frac{\int_{M \setminus B_{\delta'}} |\nabla u_{\alpha}|^2 dv_g}{\left(\int_M u_{\alpha}^{\theta} dv_g\right)^{2/\theta}} \leq \frac{\int_M (\eta'')^2 |\nabla u_{\alpha}|^2 dv_g}{\left(\int_M u_{\alpha}^{\theta} dv_g\right)^{2/\theta}}$$

one gets the existence of $C > 0$ independent of α such that

$$(1.14) \quad \frac{\int_{M \setminus B_{\delta'}} |\nabla u_\alpha|^2 dv_g}{\left(\int_M u_\alpha^\theta dv_g\right)^{2/\theta}} \leq C$$

Combining (1.11) with (1.12)–(1.14), leads to a contradiction. This ends the proof of the lemma. \square

We prove now Theorem 1.

Proof of Theorem 1. Given a smooth compact Riemannian n -manifold (M, g) , suppose first that its sectional curvature K_g is nonpositive, and that the n -dimensional Cartan-Hadamard conjecture is true. Let (\tilde{M}, \tilde{g}) be the universal Riemannian covering of (M, g) . Then for any smooth bounded domain Ω in \tilde{M} ,

$$(1.15) \quad \frac{Area_{\tilde{g}}(\partial\Omega)}{Vol_{\tilde{g}}(\Omega)^{(n-1)/n}} \geq n \left(\frac{\omega_{n-1}}{n}\right)^{\frac{1}{n}}$$

By standard arguments (see for instance Aubin-Druet-Hebey [2] or Hebey [17]), (1.15) implies that for any $u \in C_c^\infty(\tilde{M})$,

$$(1.16) \quad \left(\int_{\tilde{M}} |u|^{2^*} dv_{\tilde{g}}\right)^{2/2^*} \leq K_n^2 \int_{\tilde{M}} |\nabla u|^2 dv_{\tilde{g}}$$

Since (M, g) is locally isometric to (\tilde{M}, \tilde{g}) , for any $\theta \in [1, 2]$, (1.16) implies that (S_{opt}^θ) is locally valid on (M, g) . By lemma 2, this implies that (S_{opt}^θ) is valid on (M, g) . The same argument leads to the same conclusion if K_g is a nonpositive constant, since (1.15) holds for hyperbolic space (and Euclidean space).

Conversely, let us prove that (S_{opt}^θ) is false if $\theta \in [1, 2)$, $n \geq 4$ and $S_g > 0$ somewhere on M . We can do this very simply. The result was also announced in Aubin and Li [3]. Given $x \in M$ such that $S_g(x)$ is positive, we let $r > 0$ be such that $r < i_g(x)$, the injectivity radius at x . In geodesic normal coordinates,

$$\frac{1}{\omega_{n-1}} \int_{S(r)} \sqrt{\det(g_{ij})} ds = 1 - \frac{1}{6n} S_g(x) r^2 + O(r^4)$$

where $S(r)$ stands for the sphere of radius r and center x in M . For $\varepsilon > 0$, we define

$$u_\varepsilon = (\varepsilon + r^2)^{1-n/2} - (\varepsilon + \delta^2)^{1-n/2} \text{ if } r \leq \delta$$

$$u_\varepsilon = 0 \text{ otherwise}$$

where $\delta \in (0, i_g(x))$ is given and $r = d_g(x, \cdot)$. Easy computations lead to

$$\begin{aligned} \int_M |\nabla u_\varepsilon|^2 dv_g &= \frac{(n-2)^2 \omega_{n-1}}{2} I_n^{n/2} \varepsilon^{1-n/2} \\ &\quad \times \left(1 - \frac{(n+2)}{6n(n-4)} S_g(x) \varepsilon + o(\varepsilon)\right) \text{ if } n > 4 \\ &= \frac{(n-2)^2 \omega_{n-1}}{2} \varepsilon^{1-n/2} \\ &\quad \times \left(I_n^{n/2} + \frac{1}{6n} S_g(x) \varepsilon \text{Log } \varepsilon + o(\varepsilon \text{Log } \varepsilon)\right) \text{ if } n = 4 \end{aligned}$$

and

$$\begin{aligned} \int_M u_\varepsilon^{2^*} dv_g &\geq \frac{(n-2)\omega_{n-1}}{2n} I_n^{n/2} \varepsilon^{-n/2} \\ &\quad \times \left(1 - \frac{1}{6(n-2)} S_g(x) \varepsilon + o(\varepsilon)\right) \quad \text{if } n > 4 \\ &\geq \frac{(n-2)\omega_{n-1}}{2n} I_n^{n/2} \varepsilon^{-n/2} \\ &\quad \times \left(1 + o(\varepsilon \text{Log } \varepsilon)\right) \quad \text{if } n = 4 \end{aligned}$$

where $I_p^q = \int_0^{+\infty} (1+t)^{-p} t^q dt$. As one can easily check,

$$\frac{\omega_n}{2^{n-1}\omega_{n-1}} = I_n^{n/2-1} = \frac{(n-2)}{n} I_n^{n/2}$$

Hence,

$$\frac{(n-2)^2\omega_{n-1}}{2} I_n^{n/2} = \frac{1}{K(n,2)^2} \left(\frac{(n-2)\omega_{n-1}}{2n} I_n^{n/2}\right)^{(n-2)/n}$$

Independently, for $\theta \in [1, 2)$,

$$\begin{aligned} \int_M u_\varepsilon^\theta dv_g &= o(\varepsilon^{2-n/2}) \quad \text{if } n > 4 \\ &= o(\text{Log } \varepsilon) \quad \text{if } n = 4 \end{aligned}$$

Given $B \in \mathbb{R}$, this leads to

$$\begin{aligned} \frac{\|\nabla u_\varepsilon\|_2^2 + B\|u_\varepsilon\|_\theta^2}{\|u_\varepsilon\|_{2^*}^2} &\leq K_n^{-2} \left(1 - \frac{S_g(x)}{n(n-4)} \varepsilon + o(\varepsilon)\right) \quad \text{if } n > 4 \\ &\leq K_4^{-2} \left(1 + \frac{1}{8} S_g(x) \varepsilon \text{Log } \varepsilon + o(\varepsilon \text{Log } \varepsilon)\right) \quad \text{if } n = 4 \end{aligned}$$

As a consequence, for $n \geq 4$ and any $B \in \mathbb{R}$,

$$\frac{\|\nabla u_\varepsilon\|_2^2 + B\|u_\varepsilon\|_\theta^2}{\|u_\varepsilon\|_{2^*}^2} < \frac{1}{K_n^2}$$

provided that $\varepsilon > 0$ is small. Clearly, this ends the proof of Theorem 1. □

2. PROOF OF THEOREM 2

An interesting situation where Theorem 2 is easy to get is when the manifold considered is conformally flat. This includes the case of the standard 3-sphere. The result is then an easy consequence of the following inequality obtained by Brezis and Nirenberg [6]: for Ω a smooth bounded domain in \mathbb{R}^3 , and for any $u \in C_c^\infty(\Omega)$,

$$(2.1) \quad \|u\|_{2^*}^2 \leq K_3^2 \|\nabla u\|_2^2 - \lambda |\Omega|^{-2/3} \|u\|_2^2$$

where $|\Omega|$ stands for the volume of Ω , and $\lambda > 0$, known explicitly, does not depend on Ω . If ξ stands for the Euclidean metric, and (M, g) is conformally flat, then for any x in M , there exist $r_x > 0$ and a smooth positive function φ_x on M such that in some chart at x whose domain contains $\Omega_x = B_x(r_x)$ we have $\xi = \varphi_x^{-4} g$ on Ω_x . As one can easily check, for $u \in C_c^\infty(\Omega_x)$,

$$\int_M |\nabla(u\varphi_x)|^2 dx = \int_M |\nabla u|^2 dv_g + \frac{1}{8} \int_M S_g u^2 dv_g$$

where S_g stands for the scalar curvature of g . Coming back to (2.1), for any $u \in C_c^\infty(\Omega_x)$,

$$\begin{aligned} & \left(\int_M u^6 dv_g \right)^{1/3} + \lambda |\Omega_x|^{-2/3} \int_M \frac{u^2}{\varphi_x^4} dv_g \\ & \leq K_3^2 \int_M |\nabla u|^2 dv_g + \frac{1}{8} K_3^2 \int_M S_g u^2 dv_g \end{aligned}$$

Choosing $r_x > 0$ small enough so that

$$\lambda |\Omega_x|^{-2/3} (\max \varphi_x^{-4}) \geq \frac{1}{8} K_3^2 (\max S_g)$$

one then gets that for any $x \in M$, there exists an open neighborhood Ω_x of x such that for any $u \in C_c^\infty(\Omega_x)$,

$$\left(\int_M u^6 dv_g \right)^{1/3} \leq K_3^2 \int_M |\nabla u|^2 dv_g$$

By lemma 2, this proves the above claim.

For an arbitrary smooth compact 3-manifold, we need to be more subtle. The argument there follows estimates that were obtained by Druet [11] in his resolution of the conjecture on sharp Sobolev constants. Clearly, it suffices to prove the result for $\theta = 1$. As when proving that sharp Sobolev inequalities with lower order remainder terms are localisable (lemma 2), we proceed by contradiction. We assume therefore that for any $\alpha > 0$,

$$(2.2) \quad \inf_{H_1^2(M) \setminus \{0\}} I_\alpha(u) < \frac{1}{K_3^2}$$

where

$$I_\alpha(u) = \frac{\|\nabla u\|_2^2 + \alpha \|u\|_1^2}{\|u\|_6^2}$$

As in the proof of lemma 2, (2.2) leads to the existence of a minimizer $u_\alpha \in H_1^2(M)$ with $u_\alpha \geq 0$ and of norm 1 in $L^6(M)$. If μ_α stands for the above infimum, one has in addition that

$$\Delta_g u_\alpha + \alpha \left(\int_M u_\alpha dv_g \right) \Sigma_\alpha = \mu_\alpha u_\alpha^5$$

where $\Sigma_\alpha \in L^\infty(M)$ is such that $0 \leq \Sigma_\alpha \leq 1$ and $\Sigma_\alpha \varphi = \varphi$ for any $\varphi \in H_1^2(M)$ having the property that $|\varphi| \leq C u_\alpha$ on M for some constant $C > 0$. Moreover, u_α is actually $C^{1,\lambda}$ for any $\lambda \in (0, 1)$, and the sequence (u_α) is bounded in $H_1^2(M)$.

Multiplying (E_α) by u_α , and integrating over M , we get that

$$\|\nabla u_\alpha\|_2^2 + \alpha \|u_\alpha\|_1^2 = \mu_\alpha$$

As a consequence, $\|u_\alpha\|_1 \rightarrow 0$ as $\alpha \rightarrow +\infty$. By Hölder's inequality, and since u_α is of norm 1 in L^6 , this implies that $\|u_\alpha\|_2 \rightarrow 0$ as $\alpha \rightarrow +\infty$. By Hebey and Vaugon [19], there exists $B \in \mathbb{R}$ such that for any $u \in C^\infty(M)$,

$$\|u\|_6^2 \leq K_3^2 \|\nabla u\|_2^2 + B \|u\|_2^2$$

Taking $u = u_\alpha$ in this inequality, we get

$$1 \leq K_3^2 \|\nabla u_\alpha\|_2^2 + B \|u_\alpha\|_2^2 \leq K_3^2 \mu_\alpha + B \|u_\alpha\|_2^2$$

Together with (2.2), this implies that

$$(2.3) \quad \lim_{\alpha \rightarrow +\infty} \mu_\alpha = \frac{1}{K_3^2}$$

Similarly,

$$1 - B\|u_\alpha\|_2^2 \leq K_3^2 \|\nabla u_\alpha\|_2^2 = K_3^2 \mu_\alpha - \alpha \|u_\alpha\|_1^2$$

so that

$$(2.4) \quad \lim_{\alpha \rightarrow +\infty} \alpha \|u_\alpha\|_1^2 = 0$$

Now define a concentration point x for (u_α) by the property that for any $\delta > 0$,

$$\limsup_{\alpha \rightarrow +\infty} \int_{B_x(\delta)} u_\alpha^6 dv_g > 0$$

As in the proof of lemma 2, we get that, up to a subsequence, (u_α) has one and only one concentration point x_0 . One may then assume that for any $\delta > 0$,

$$(2.5) \quad \lim_{\alpha \rightarrow +\infty} \int_{B_{x_0}(\delta)} u_\alpha^6 dv_g = 1$$

Moreover,

$$(2.6) \quad u_\alpha \rightarrow 0 \quad \text{in } C_{loc}^0(M \setminus \{x_0\})$$

as α goes to $+\infty$.

We let $x_\alpha \in M$ and $\lambda_\alpha \in \mathbb{R}$ be such that

$$u_\alpha(x_\alpha) = \|u_\alpha\|_\infty = \lambda_\alpha^{-1/2}$$

According to what we just said, $x_\alpha \rightarrow x_0$ and $\lambda_\alpha \rightarrow 0$ as $\alpha \rightarrow +\infty$. By (2.4), noting that

$$1 = \|u_\alpha\|_6^6 \leq \|u_\alpha\|_\infty^5 \|u_\alpha\|_1$$

one gets that

$$(2.7) \quad \lim_{\alpha \rightarrow +\infty} \alpha \lambda_\alpha^{5/2} \|u_\alpha\|_1 = 0$$

Here again, the proof now splits into several steps.

Step 1. We claim that for any $R > 0$,

$$(2.8) \quad \lim_{\alpha \rightarrow +\infty} \int_{B_{x_\alpha}(R\lambda_\alpha)} u_\alpha^6 dv_g = 1 - \varepsilon_R$$

where $\varepsilon_R > 0$ is such that $\varepsilon_R \rightarrow 0$ as $R \rightarrow +\infty$. We let \exp_{x_α} be the exponential map at x_α . There clearly exists $\delta > 0$, independent of α , such that for any α , \exp_{x_α} is a diffeomorphism from $B_0(\delta) \subset \mathbb{R}^3$ onto $B_{x_\alpha}(\delta)$. For $x \in B_0(\lambda_\alpha^{-1}\delta)$, set

$$\begin{aligned} \tilde{g}_\alpha(x) &= (\exp_{x_\alpha}^* g)(\lambda_\alpha x) \\ \tilde{u}_\alpha(x) &= \lambda_\alpha^{1/2} u_\alpha(\exp_{x_\alpha}(\lambda_\alpha x)) \\ \tilde{\Sigma}_\alpha &= \Sigma_\alpha(\exp_{x_\alpha}(\lambda_\alpha x)) \end{aligned}$$

As we can easily check,

$$(\tilde{E}_\alpha) \quad \Delta_{\tilde{g}_\alpha} \tilde{u}_\alpha + \alpha \|u_\alpha\|_1 \lambda_\alpha^{5/2} \tilde{\Sigma}_\alpha = \mu_\alpha \tilde{u}_\alpha^5$$

Moreover,

$$(2.9) \quad \tilde{u}_\alpha(0) = \|\tilde{u}_\alpha\|_\infty = 1$$

and, if ξ stands for the Euclidean metric of \mathbb{R}^3 ,

$$(2.10) \quad \lim_{\alpha \rightarrow +\infty} \tilde{g}_\alpha = \xi \quad \text{in } C^2(K)$$

for any compact subset K of \mathbb{R}^3 . By (2.7), (2.9), and theorem 8.24 of Gilbarg and Trudinger [14], (\tilde{u}_α) is equicontinuous on any compact subset of \mathbb{R}^3 . By Ascoli's theorem, one gets the existence of some $\tilde{u} \in C^0(\mathbb{R}^3)$ such that for any compact subset K of \mathbb{R}^3 ,

$$(2.11) \quad \lim_{\alpha \rightarrow +\infty} \tilde{u}_\alpha = \tilde{u} \quad \text{in } L^\infty(K)$$

Clearly, $\tilde{u}(0) = 1$ and $\tilde{u} \not\equiv 0$. It is easy to check that $\tilde{u} \in H_{0,1}^2(\mathbb{R}^3)$, where $H_{0,1}^2(\mathbb{R}^3)$ stands for the completion of $C_c^\infty(\mathbb{R}^3)$ with respect to

$$\|u\|_{H_{0,1}^2} = \sqrt{\int_{\mathbb{R}^3} |\nabla u|^2 dx}$$

Indeed, let $\eta \in C_c^\infty(\mathbb{R}^3)$, $0 \leq \eta \leq 1$, be such that $\eta = 1$ in $B_0(\delta/4)$ and $\eta = 0$ in $\mathbb{R}^3 \setminus B_0(\delta/2)$. We set $\eta_\alpha(x) = \eta(\mu_\alpha x)$, and

$$\varphi_\alpha(x) = \eta_\alpha(x) \tilde{u}_\alpha(x)$$

Then, $\varphi_\alpha \in C_c^1(\mathbb{R}^3)$, and $\varphi_\alpha \rightarrow \tilde{u}$ in $L^\infty(K)$ for any compact subset K of \mathbb{R}^3 . Clearly, there exists $C > 0$ such that for any α ,

$$\begin{aligned} \|\varphi_\alpha\|_{H_{0,1}^2} &\leq C \int_{\mathbb{R}^3} |\nabla \varphi_\alpha|^2 dv_{\tilde{g}_\alpha} \\ &\leq C \int_{\mathbb{R}^3} \eta_\alpha^2 |\nabla \tilde{u}_\alpha|^2 dv_{\tilde{g}_\alpha} + C \lambda_\alpha^2 \int_{\mathbb{R}^3} |\nabla \eta(\mu_\alpha x)|^2 \tilde{u}_\alpha^2 dv_{\tilde{g}_\alpha} \end{aligned}$$

On the one hand,

$$\int_{B_0(\delta \lambda_\alpha^{-1})} \tilde{u}_\alpha^2 dv_{\tilde{g}_\alpha} = \lambda_\alpha^{-2} \int_{B_{x_\alpha}(\delta)} u_\alpha^2 dv_g$$

On the other hand,

$$\int_{\mathbb{R}^3} \eta_\alpha^2 |\tilde{u}_\alpha|^2 dv_{\tilde{g}_\alpha} \leq \int_{B_0(\delta \lambda_\alpha^{-1})} |\tilde{u}_\alpha|^2 dv_{\tilde{g}_\alpha} = \int_{B_{x_\alpha}(\delta)} |\nabla u_\alpha|^2 dv_g$$

Hence, (φ_α) is bounded in $H_{0,1}^2(\mathbb{R}^3)$, and since $H_{0,1}^2(\mathbb{R}^3)$ is reflexive, $\tilde{u} \in H_{0,1}^2(\mathbb{R}^3)$. This proves the above assertion. By passing to the limit as α goes to $+\infty$ in (\tilde{E}_α) , according to (2.3), (2.7), (2.10), and (2.11), one now gets that \tilde{u} is a solution of

$$(2.12) \quad \Delta_\xi \tilde{u} = \frac{1}{K_3^2} \tilde{u}^5$$

By Caffarelli-Gidas-Spruck [7], or also Obata [23],

$$\tilde{u}(x) = \left(\frac{3K_3^2}{3K_3^2 + |x|^2} \right)^{1/2}$$

since $\tilde{u}(0) = 1$. Noting that \tilde{u} is of norm 1 in $L^6(\mathbb{R}^3)$, and that for any $R > 0$,

$$\int_{B_{x_\alpha}(R \lambda_\alpha)} u_\alpha^6 dv_g = \int_{B_0(R)} \tilde{u}_\alpha^6 dv_{\tilde{g}_\alpha}$$

one gets that

$$\lim_{\alpha \rightarrow +\infty} \int_{B_{x_\alpha}(R \lambda_\alpha)} u_\alpha^6 dv_g = 1 - \int_{\mathbb{R}^3 \setminus B_0(R)} \tilde{u}^6 dx$$

Clearly, this proves (2.8) and the claim we made in step 1.

Step 2. We claim that there exists $C > 0$, independent of α , such that for any α , and any x ,

$$(2.13) \quad d_g(x_\alpha, x)^{1/2} u_\alpha(x) \leq C$$

where d_g stands for the distance with respect to g . In order to prove this, set

$$v_\alpha(x) = d_g(x_\alpha, x)^{1/2} u_\alpha(x)$$

and assume by contradiction that, for some subsequence,

$$(2.14) \quad \lim_{\alpha \rightarrow +\infty} \|v_\alpha\|_\infty = +\infty$$

Let y_α be some point in M where v_α is maximum. By (2.6), $y_\alpha \rightarrow x_0$ as $\alpha \rightarrow +\infty$, while by (2.14),

$$(2.15) \quad \lim_{\alpha \rightarrow +\infty} \frac{d_g(x_\alpha, y_\alpha)}{\lambda_\alpha} = +\infty$$

Fix $\delta > 0$ small, and set

$$\Omega_\alpha = u_\alpha(y_\alpha)^2 \exp_{y_\alpha}^{-1}(B_{x_\alpha}(\delta))$$

For $x \in \Omega_\alpha$, define

$$\tilde{v}_\alpha(x) = u_\alpha(y_\alpha)^{-1} u_\alpha(\exp_{y_\alpha}(u_\alpha(y_\alpha)^{-2}x))$$

and

$$h_\alpha(x) = (\exp_{y_\alpha}^* g)(u_\alpha(y_\alpha)^{-2}x)$$

Clearly,

$$(2.16) \quad \lim_{\alpha \rightarrow +\infty} h_\alpha = \xi \quad \text{in } C^2(B_0(2))$$

Moreover, as one can easily check,

$$(2.17) \quad \Delta_{h_\alpha} \tilde{v}_\alpha \leq \mu_\alpha \tilde{v}_\alpha^5$$

Since $v_\alpha(y_\alpha)$ goes to $+\infty$ as α goes to $+\infty$, using (2.14), one gets that for α large, and all $x \in B_0(2)$,

$$(2.18) \quad d_g(x_\alpha, \exp_{y_\alpha}(u_\alpha(y_\alpha)^{-2}x)) \geq \frac{1}{2} d_g(x_\alpha, y_\alpha)$$

This implies that

$$\begin{aligned} \tilde{v}_\alpha(x) &\leq \sqrt{2} d_g(x_\alpha, y_\alpha)^{-1/2} u_\alpha(y_\alpha)^{-1} v_\alpha(\exp_{y_\alpha}(u_\alpha(y_\alpha)^{-2}x)) \\ &\leq \sqrt{2} d_g(x_\alpha, y_\alpha)^{-1/2} u_\alpha(y_\alpha)^{-1} v_\alpha(y_\alpha) \end{aligned}$$

so that for α large,

$$(2.19) \quad \sup_{x \in B_0(2)} \tilde{v}_\alpha(x) \leq \sqrt{2}$$

By (2.15) and (2.18), given $R > 0$, and for α large,

$$B_{y_\alpha}(2u_\alpha(y_\alpha)^{-2}) \cap B_{x_\alpha}(R\lambda_\alpha) = \emptyset$$

Noting that

$$\int_{B_0(2)} \tilde{v}_\alpha^6 dv_{h_\alpha} = \int_{B_{y_\alpha}(2u_\alpha(y_\alpha)^{-2})} u_\alpha^6 dv_g$$

and using (2.8), one gets that

$$(2.20) \quad \lim_{\alpha \rightarrow +\infty} \int_{B_0(2)} \tilde{v}_\alpha^6 dv_{h_\alpha} = 0$$

By (2.16), (2.17), (2.19), (2.20), and the De Giorgi-Nash-Moser theory, as used in lemma 1, we get

$$\lim_{\alpha \rightarrow +\infty} \sup_{x \in B_0(1)} \tilde{v}_\alpha(x) = 0$$

But $\tilde{v}_\alpha(0) = 1$, so that (2.14) must be false. This proves (2.13) and the claim we made in step 2.

Step 3. We prove the theorem, showing that (2.2) leads to a contradiction. We take $\delta > 0$ small (to be fixed later on), and for any α , we let $\eta_\alpha \in C_c^\infty(B_{x_\alpha}(4\delta))$ be such that $0 \leq \eta_\alpha \leq 1$, $\eta_\alpha = 1$ in $B_{x_\alpha}(2\delta)$, and $|\nabla \eta_\alpha| \leq C/\delta$. Here, and in what follows, C denotes a constant independent of α and δ . By Brezis and Nirenberg [6], inequality (2.1), and passing through geodesic normal coordinates,

$$(2.21) \quad \left(\int_{B_{x_\alpha}(2\delta)} u_\alpha^6 dx \right)^{1/3} \leq K_3^2 \int_{B_{x_\alpha}(4\delta)} |\nabla(\eta_\alpha u_\alpha)|_\xi^2 dx - \frac{\lambda}{\delta^2} \int_{B_{x_\alpha}(4\delta)} (\eta_\alpha u_\alpha)^2 dx$$

where $\lambda > 0$ does not depend on α and δ . When confusion is possible, we write $|\cdot|_\xi$ and $|\cdot|_g$ to specify the metric with respect to which norms are taken. Starting from the Cartan expansion of g in such coordinates, we get

$$|\nabla(\eta_\alpha u_\alpha)|_\xi^2 \leq (1 + Cd_g(x_\alpha, x)^2) |\nabla(\eta_\alpha u_\alpha)|_g^2$$

and

$$(1 - Cd_g(x_\alpha, x)^2) dv_g \leq dx \leq (1 + Cd_g(x_\alpha, x)^2) dv_g$$

Hence,

$$(2.22) \quad \int_{B_{x_\alpha}(4\delta)} |\nabla(\eta_\alpha u_\alpha)|_\xi^2 dx \leq \int_{B_{x_\alpha}(4\delta)} (1 + Cd_g(x_\alpha, x)^2) |\nabla(\eta_\alpha u_\alpha)|_g^2 dv_g$$

On the one hand,

$$\begin{aligned} \int_{B_{x_\alpha}(4\delta)} |\nabla(\eta_\alpha u_\alpha)|_g^2 dv_g &\leq \int_M |\nabla u_\alpha|_g^2 dv_g + \frac{C}{\delta^2} \int_{M \setminus B_{x_\alpha}(2\delta)} u_\alpha^2 dv_g \\ &\quad + \frac{C}{\delta} \int_{M \setminus B_{x_\alpha}(2\delta)} u_\alpha |\nabla u_\alpha| dv_g \end{aligned}$$

Multiplying (E_α) by u_α , and integrating over M , gives

$$(2.23) \quad \begin{aligned} \int_{B_{x_\alpha}(4\delta)} |\nabla(\eta_\alpha u_\alpha)|_g^2 dv_g &\leq \mu_\alpha - \alpha \left(\int_M u_\alpha dv_g \right)^2 \\ &\quad + \frac{C}{\delta^2} \int_{M \setminus B_{x_\alpha}(2\delta)} u_\alpha^2 dv_g + \frac{C}{\delta} \int_{M \setminus B_{x_\alpha}(2\delta)} u_\alpha |\nabla u_\alpha|_g dv_g \end{aligned}$$

On the other hand,

$$(2.24) \quad \begin{aligned} \int_{B_{x_\alpha}(4\delta)} d_g(x_\alpha, x)^2 |\nabla(\eta_\alpha u_\alpha)|_g^2 dv_g &\leq C \int_{M \setminus B_{x_\alpha}(2\delta)} u_\alpha^2 dv_g \\ &+ 2 \int_{B_{x_\alpha}(4\delta)} \eta_\alpha^2 |\nabla u_\alpha|_g^2 d_g(x_\alpha, x)^2 dv_g \end{aligned}$$

Integrating by parts, and using (E_α) , we get

$$(2.25) \quad \begin{aligned} \int_{B_{x_\alpha}(4\delta)} \eta_\alpha^2 |\nabla u_\alpha|_g^2 d_g(x_\alpha, x)^2 dv_g \\ \leq C \int_{B_{x_\alpha}(4\delta)} d_g(x_\alpha, x)^2 \eta_\alpha^2 u_\alpha^6 dv_g + C \int_{M \setminus B_{x_\alpha}(2\delta)} u_\alpha |\nabla u_\alpha|_g dv_g \\ + C \int_{M \setminus B_{x_\alpha}(2\delta)} u_\alpha^2 dv_g + C \int_{B_{x_\alpha}(4\delta)} \eta_\alpha^2 u_\alpha^2 dv_g \end{aligned}$$

By (2.13),

$$(2.26) \quad \int_{B_{x_\alpha}(4\delta)} d_g(x_\alpha, x)^2 \eta_\alpha^2 u_\alpha^6 dv_g \leq C \int_{B_{x_\alpha}(4\delta)} \eta_\alpha^2 u_\alpha^2 dv_g$$

Combining (2.24), (2.25), and (2.26), we get

$$(2.27) \quad \begin{aligned} \int_{B_{x_\alpha}(4\delta)} d_g(x_\alpha, x)^2 |\nabla(\eta_\alpha u_\alpha)|_g^2 dv_g &\leq C \int_{B_{x_\alpha}(4\delta)} \eta_\alpha^2 u_\alpha^2 dv_g \\ + \frac{C}{\delta} \int_{M \setminus B_{x_\alpha}(2\delta)} u_\alpha |\nabla u_\alpha|_g dv_g &+ \frac{C}{\delta^2} \int_{M \setminus B_{x_\alpha}(2\delta)} u_\alpha^2 dv_g \end{aligned}$$

Independently,

$$\int_{B_{x_\alpha}(2\delta)} u_\alpha^6 dx \geq \int_{B_{x_\alpha}(2\delta)} u_\alpha^6 dv_g - C \int_{B_{x_\alpha}(2\delta)} d_g(x_\alpha, x)^2 u_\alpha^6 dv_g$$

so that, again by (2.13),

$$\int_{B_{x_\alpha}(2\delta)} u_\alpha^6 dx \geq \int_{B_{x_\alpha}(2\delta)} u_\alpha^6 dv_g - C \int_{B_{x_\alpha}(2\delta)} u_\alpha^2 dv_g$$

For α large, noting that $B_{x_0}(\delta) \subset B_{x_\alpha}(2\delta)$, one gets from (2.5) and the fact that $\|u_\alpha\|_2 \rightarrow 0$ as $\alpha \rightarrow +\infty$, that the right hand side in this inequality is positive. Since it is also less than 1,

$$\left(\int_{B_{x_\alpha}(2\delta)} u_\alpha^6 dx \right)^{1/3} \geq \int_{B_{x_\alpha}(2\delta)} u_\alpha^6 dv_g - C \int_{B_{x_\alpha}(2\delta)} u_\alpha^2 dv_g$$

and

$$(2.28) \quad \left(\int_{B_{x_\alpha}(2\delta)} u_\alpha^6 dx \right)^{1/3} \geq 1 - \int_{M \setminus B_{x_\alpha}(2\delta)} u_\alpha^6 dv_g - C \int_{B_{x_\alpha}(4\delta)} \eta_\alpha^2 u_\alpha^2 dv_g$$

By (2.2), (2.21), (2.22), (2.23), (2.27), and (2.28), one gets

$$\begin{aligned}
 & \alpha K_3^2 \left(\int_M u_\alpha dv_g \right)^2 \\
 (2.29) \quad & \leq \int_{M \setminus B_{x_\alpha}(2\delta)} u_\alpha^6 dv_g + \frac{C}{\delta^2} \int_{M \setminus B_{x_\alpha}(2\delta)} u_\alpha^2 dv_g \\
 & \quad + \frac{C}{\delta} \int_{M \setminus B_{x_\alpha}(2\delta)} u_\alpha |\nabla u_\alpha| dv_g + \left(C - \frac{\lambda}{\delta^2} \right) \int_{B_{x_\alpha}(4\delta)} \eta_\alpha^2 u_\alpha^2 dv_g
 \end{aligned}$$

We now fix $\delta > 0$ sufficiently small so that

$$C - \frac{\lambda}{\delta^2} < 0$$

Noting that $B_{x_0}(\delta) \subset B_{x_\alpha}(2\delta)$, and writing by Hölder’s inequality that

$$\int_{M \setminus B_{x_\alpha}(2\delta)} u_\alpha |\nabla u_\alpha| dv_g \leq \sqrt{\int_{M \setminus B_{x_\alpha}(2\delta)} u_\alpha^2 dv_g} \sqrt{\int_{M \setminus B_{x_\alpha}(2\delta)} |\nabla u_\alpha|^2 dv_g}$$

one gets with (2.29) the existence of some constant $C > 0$, independent of α , such that

$$\begin{aligned}
 (2.30) \quad \alpha K_3^2 & \leq \frac{\int_{M \setminus B_{x_0}(\delta)} u_\alpha^6 dv_g}{\left(\int_M u_\alpha dv_g \right)^2} + C \frac{\int_{M \setminus B_{x_0}(\delta)} u_\alpha^2 dv_g}{\left(\int_M u_\alpha dv_g \right)^2} \\
 & \quad + C \left(\frac{\int_{M \setminus B_{x_0}(\delta)} u_\alpha^2 dv_g}{\left(\int_M u_\alpha dv_g \right)^2} \right)^{1/2} \left(\frac{\int_{M \setminus B_{x_0}(\delta)} |\nabla u_\alpha|^2 dv_g}{\left(\int_M u_\alpha dv_g \right)^2} \right)^{1/2}
 \end{aligned}$$

As in the proof of lemma 2, see (1.12)–(1.14), the right hand side in (2.30) is bounded by some positive constant independent of α . Since the left hand side of (2.30) goes to $+\infty$ as α goes to $+\infty$, we get a contradiction. This ends the proof of Theorem 2.

3. PROOF OF THEOREM 3

Given $\alpha > 0$, let I_α be the functional defined on $H_1^2(M)$ by

$$I_\alpha(u) = \|\nabla u\|_2^2 + \alpha \|u\|_2^2$$

and let \mathcal{H} be the set of functions $u \in H_1^2(M)$ such that $\|u\|_{2^*} = 1$. According to the definition of $B_0(g)$, if $\alpha < B_0(g)K_n^{-2}$, then

$$(3.1) \quad \inf_{u \in \mathcal{H}} I_\alpha(u) < \frac{1}{K_n^2}$$

By standard arguments, this leads to the existence of a smooth and positive solution $u_\alpha \in \mathcal{H}$ of the equation

$$(3.2) \quad \Delta_g u_\alpha + \alpha u_\alpha = \lambda_\alpha u_\alpha^{2^*-1}$$

where λ_α is the infimum in the left hand side of (3.1). Suppose now that (S_{opt}^1) is true, and let B be some constant in (S_{opt}^1) . Multiplying (3.2) by u_α , and integrating over M , gives us

$$\begin{aligned}
 1 & = \|u_\alpha\|_{2^*}^2 \\
 & \leq K_n^2 \|\nabla u_\alpha\|_2^2 + B \|u_\alpha\|_1^2 \\
 & = \lambda_\alpha K_n^2 - \alpha K_n^2 \|u_\alpha\|_2^2 + B \|u_\alpha\|_1^2
 \end{aligned}$$

Since $\lambda_\alpha K_n^2 < 1$, and $\alpha K_n^2 < B_0(g)$, it follows that

$$(3.3) \quad \|u_\alpha\|_2^2 \leq \frac{B}{\alpha K_n^2} \|u_\alpha\|_1^2 \leq \frac{2B}{B_0(g)} \|u_\alpha\|_1^2$$

for α close to $B_0(g)K_n^{-2}$. Independently, integrating (3.2) gives that

$$\alpha \|u_\alpha\|_1 = \lambda_\alpha \|u_\alpha\|_{2^*-1}^{2^*-1}$$

When $n \geq 6$ we have $2^* - 1 \leq 2$, and by Hölder's inequality,

$$(3.4) \quad \|u_\alpha\|_{2^*-1}^{2^*-1} \leq \|u_\alpha\|_2^{\frac{n+2}{n-2}} V_g^{\frac{n-6}{2(n-2)}}$$

Combining (3.3) and (3.4), we get

$$\|u_\alpha\|_2^2 \leq \frac{2B\lambda_\alpha^2}{\alpha^2 B_0(g)} V_g^{\frac{n-6}{(n-2)}} \|u_\alpha\|_2^{\frac{2(n+2)}{n-2}}$$

Clearly, this leads to the existence of $C > 0$, independent of α , such that

$$(3.5) \quad \|u_\alpha\|_2 \geq C$$

If $n = 5$, we write, by Hölder's inequality,

$$\|u_\alpha\|_{2^*-1}^{2^*-1} \leq \|u_\alpha\|_2^{\frac{3}{2}} \|u_\alpha\|_{2^*}^{\frac{5}{6}}$$

Since $u_\alpha \in \mathcal{H}$, it follows that $\|u_\alpha\|_{2^*} = 1$, and one gets

$$(3.6) \quad \|u_\alpha\|_{2^*-1}^{2^*-1} \leq \|u_\alpha\|_2^{\frac{3}{2}}$$

Combining (3.3) and (3.6), we find that

$$\|u_\alpha\|_2^2 \leq \frac{2B\lambda_\alpha^2}{\alpha^2 B_0(g)} \|u_\alpha\|_2^3$$

Hence, again, there exists $C > 0$, independent of α , such that

$$(3.7) \quad \|u_\alpha\|_2 \geq C .$$

We now let α go to $B_0(g)K_n^{-2}$. Clearly, (u_α) is a bounded sequence in $H_1^2(M)$. Up to a subsequence, it then converges weakly to some u in $H_1^2(M)$, and strongly to u in $L^p(M)$, $p < 2^*$. Moreover, one may also assume that $\lambda_\alpha \rightarrow \lambda$. By (3.5) and (3.7), u is nonzero. Standard arguments then give that u is a smooth positive solution of the equation

$$(3.8) \quad K_n^2 \Delta_g u + B_0(g)u = \lambda K_n^2 u^{2^*-1}$$

Clearly, $\|u\|_{2^*} \leq 1$. Set

$$\alpha_0 = \frac{B_0(g)}{K_n^2} \quad \text{and} \quad v = \frac{u}{\|u\|_{2^*}}$$

Then, $I_{\alpha_0}(v) \geq K_n^{-2}$, and hence, by (3.8),

$$1 \leq \lambda K_n^2 \|u\|_{2^*}^{2^*-2}$$

Since $\lambda K_n^2 \leq 1$ and $\|u\|_{2^*} \leq 1$, this implies that $\lambda K_n^2 = 1$ and $\|u\|_{2^*} = 1$. In particular, u is a nonzero extremal function of (S_{OT}^2) . The theorem is proved.

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