TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 353, Number 5, Pages 1937–1957
S 0002-9947(00)02689-1
Article electronically published on November 29, 2000

ESSENTIAL COHOMOLOGY AND EXTRASPECIAL p-GROUPS

PHAM ANH MINH

ABSTRACT. Let p be an odd prime number and let G be an extraspecial p-group. The purpose of the paper is to show that G has no non-zero essential mod-p cohomology (and in fact that $H^*(G, \mathbb{F}_p)$ is Cohen-Macaulay) if and only if $|G| = 27$ and $exp(G) = 3$.

1. INTRODUCTION

Let p be a prime number. For every p-group K, denote by $H^*(K)$ the mod-p cohomology ring of K. A mod-p cohomology class of K is called essential if it vanishes on restriction to every proper subgroup of K. Let $Ess(K)$ be the ideal of $H^*(K)$ consisting of such classes of $H^*(K)$. As observed in [3], the study of $Ess(K)$ could provide interesting information for $H^*(K)$ (but, in contrast, it seems in general rather difficult to obtain elements of $Ess(K)$). For instance, $Ess(K) \neq \{0\}$ implies that the depth of $H^*(K)$ is just the rank of the center of K (see [3] and [5]); furthermore, with the condition that $H^*(K)$ is Cohen-Macaulay, it follows from [1] that $Ess(K) \neq \{0\}$ if and only if every element of order p of K is central (a way to obtain some element of $Ess(K)$ in this case was shown there).

We are now interested in extraspecial p-groups G. For $p = 2$, it was proved by Quillen ([17]) that $H^*(G)$ is Cohen-Macaulay and $Ess(G) = \{0\}$, except for the case $G = Q_8$, the quaternion group of order 8 (this fact also follows from Adem and Karagueuzian's result, as Q_8 is the unique group in which every element of order 2 is central). However, the situation is quite different for the case $p > 2$—*which is assumed from now on.* Consider first the case $|G| = p^3$; it follows from [8], [9], [10], [16] that $Ess(G) \neq \{0\}$ (so $H^*(G)$ is not Cohen-Macaulay) if and only if $exp(G) > 3$. In order to generalize this fact, in this note, we prove

Theorem. *If G is an extraspecial p-group, then $Ess(G) = \{0\}$ iff $exp(G) = 3$ and $|G| = 3^3$.*

It follows that the unique extraspecial p-group which has no non-zero essential cohomology is the one of order 27 and of exponent 3. In each of the remaining cases, $H^*(G)$ is not Cohen-Macaulay and the depth of $H^*(G)$ is just 1; we also point out some non-zero essential classes of G (it turns out that, if $|G| = p^5$ or $exp(G) = p^2$, there exists such a class of G belonging to $\mathrm{Im}\ \mathrm{Inf}_G^{G/Z}$ with Z the center of G).

The note is organized as follows. In Section 2, given an extraspecial p-group G of order p^{2n+1}, we shall consider G as a subgroup of the central product $\Gamma_n = C_{p^2} \bullet G$ and give a sufficient and necessary condition for the fact that $\mathrm{Res}_G^{\Gamma_n}(\xi) \neq 0$ with

Received by the editors September 10, 1999.
2000 *Mathematics Subject Classification.* Primary 20J06; Secondary 20D15, 55R40.

$\xi \in H^*(\Gamma_n)$. The proofs of the theorem for the cases $exp(G) > p$ or $|G| = p^5$, which are rather simple, will be given in Section 3. Section 4 is devoted to the case $exp(G) = p$.

2. The group Γ_n

Let us recall that an extraspecial p-group G is of order p^{2n+1} $(n \in \mathbb{N})$ and is isomorphic to one of the following central products of groups:

$$\mathbb{E}_n = \mathbb{E} \bullet \cdots \bullet \mathbb{E} \ (n \text{ times}),$$
$$\mathbb{M}_n = \mathbb{M} \bullet \mathbb{E}_{n-1},$$

where

$$\mathbb{M} = \langle a, b | a^{p^2} = b^p = 1, b^{-1}ab = a^{1+p} \rangle,$$
$$\mathbb{E} = \langle a, b | a^p = b^p = [a, b]^p = [a, [a, b]] = [b, [a, b]] = 1 \rangle$$

are extraspecial p-groups of order p^3. Note that

$$exp(G) = \begin{cases} p^2, & \text{for } G = \mathbb{M}_n, \\ p, & \text{for } G = \mathbb{E}_n, \end{cases}$$

and $\mathbb{M}_n = \mathbb{M}_{n-1} \bullet \mathbb{M}$.

These groups can be obtained cohomologically as follows. Let V be a vector space of dimension $2n + 1$ over the prime field $\mathbb{F}_p$ with basis $e, a_1, \ldots, a_{2n}$. Let $x, x_1, \ldots, x_{2n}$ be a basis of $H^1(V)$, dual to that of V, and let $y = \beta x, y_i = \beta x_i$ with β the Bockstein homomorphism, so

$$H = H^*(V) = E[x, x_1, \ldots, x_{2n}] \otimes \mathbb{F}_p[y, y_1, \ldots, y_{2n}]$$

with $E[u, v, \ldots]$ (resp. $\mathbb{F}_p[u, v, \ldots]$) the exterior (resp. polynomial) algebra over $\mathbb{F}_p$ with generators $u, v, \ldots$ of degree 1 (resp. 2). Consider the central extension

$$(\Gamma_n) \qquad 0 \to \mathbb{F}_p \xrightarrow{i} \Gamma_n \to V \to 0,$$

with factor set $z = z_n = y + x_1x_2 + \cdots + x_{2n-1}x_{2n}$. Via the inflation map, x and the x_i's can be considered as elements of $H^1(\Gamma_n)$. Given a subgroup K of Γ_n, with some abuse of notation, we also denote by x (resp. x_i) the element $\mathrm{Res}^{\Gamma_n}_K(x)$ (resp. $\mathrm{Res}^{\Gamma_n}_K(x_i)$).

It is easy to show

Lemma 1. (i) $\Gamma_n = C_{p^2} \bullet \mathbb{M}_n = C_{p^2} \bullet \mathbb{E}_n = \Gamma_{n-1} \bullet \mathbb{M}$.

(ii) $\mathbb{M}_n = \mathrm{Ker}\ (x + \alpha)$, $\mathbb{E}_n = \mathrm{Ker}\ x$ *and* $\Gamma_{n-1} \times C_p = \mathrm{Ker}\ \alpha$, *with* α *a non-zero linear combination of* $x_1, \ldots, x_{2n}$. □

Then $C_{p^2} = \bigcap_{i=1}^{2n} \mathrm{Ker}\ x_i$ is a subgroup of Γ_n. Let w be a generator of $H^2(C_{p^2})$, so

$$H^*(C_{p^2}) = E[x] \otimes \mathbb{F}_p[w].$$

Set $\mathcal{G}_n = C_{p^2} \times \mathbb{E}_n$. By the Künneth formula, we have

$$H^*(\mathcal{G}_n) = H^*(\mathbb{E}_n) \otimes E[x] \otimes \mathbb{F}_p[w].$$

As Γ_n is the central product of C_{p^2} and $\mathbb{E}_n$, there exists a central subgroup U_n of order p of $\mathcal{G}_n$ such that $\mathcal{G}_n/U_n = \Gamma_n$ and the factor set of the central extension

$$1 \to U_n \to \mathcal{G}_n \to \Gamma_n \to 1$$

is just y. Consider the following commutative diagram:

$$\tag{1} \begin{array}{ccccccccc} 1 & \longrightarrow & U_n & \longrightarrow & U_n \times \mathbb{E}_n & \longrightarrow & \mathbb{E}_n & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & U_n & \longrightarrow & \mathcal{G}_n & \longrightarrow & \Gamma_n & \longrightarrow & 1 \end{array}$$

whose rows are central extensions and whose vertical arrows are inclusion maps. Pick elements s, t of $H^*(U_n)$ satisfying $H^*(U_n) = E[s] \otimes P[t]$. It follows from [11] (see also [2]) that t can be chosen so that $\mathrm{Res}^{\mathcal{G}_n}_{U_n}(w \times 1) = t$.

We now use the following notation. Given a ring R and elements $r, s, \dots \in R$, $(r, s, \dots)$ will denote the ideal of R generated by $r, s, \dots$. The main result of this section is the following.

Proposition 1. *If $\xi \in H^*(\Gamma_n)$, then $\mathrm{Res}^{\Gamma_n}_{\mathbb{E}_n}(\xi) \neq 0$ iff $x\xi \notin (y)$.*

Proof. Set $X = \mathrm{Inf}^{\Gamma_n}_{\mathcal{G}_n}(\xi)$. As $\mathrm{Ker}\ \mathrm{Inf}^{\Gamma_n}_{\mathcal{G}_n} = (y)$, it follows that $xX = 0$ iff $x\xi \in (y)$. Write $X = \sum w^i \otimes s_i + \sum w^i x \otimes t_i$ with $s_i, t_i \in H^*(\mathbb{E}_n)$. It is clear that $\mathrm{Res}^{\Gamma_n}_{\mathbb{E}_n}(\xi) \neq 0$ iff $\mathrm{Inf}^{\mathbb{E}_n}_{U_n \times \mathbb{E}_n} \mathrm{Res}^{\Gamma_n}_{\mathbb{E}_n}(\xi) \neq 0$. So, by the commutative diagram (1), $\mathrm{Res}^{\Gamma_n}_{\mathbb{E}_n}(\xi) \neq 0$ iff $\mathrm{Res}^{\mathcal{G}_n}_{U_n \times \mathbb{E}_n}(X) \neq 0$, which is equivalent to the fact that the s_i's are not all equal to zero, or equivalently, $x\xi \notin (y)$. The proposition follows. □

For convenience, given a central extension of groups

$$\tag{K} 1 \to A \to K \to C \to 1,$$

denote by $\{E_r(K), d_r\}$ the Hochschild-Serre spectral sequence corresponding to the extension (K). We now recall some results given in [12], [13] (see also [2] for $n = 1$) concerning $\{E_r(\Gamma_n), d_r\}$. As usual, denote by $\mathcal{P}^i$ the Steenrod operations. Set $Z = i(\mathbb{F}_p) \subset C_{p^2} \subset \Gamma_n$. So $v = \mathrm{Res}^{C_{p^2}}_{Z}(w)$ is a generator of $H^2(Z)$. Let

$$\begin{aligned} X_n &= x_1 x_2 \dots x_{2n-1} x_{2n}, \\ \eta_i &= \mathcal{P}^{p^{i-2}} \dots \mathcal{P}^1 \beta z \\ &= \sum_{j=1}^{n} (x_{2j-1} y_{2j}^{p^{i-1}} - x_{2j} y_{2j-1}^{p^{i-1}}), \\ \xi_m &= \beta \mathcal{P}^{p^{m-1}} \dots \mathcal{P}^1 \beta z \\ &= \sum_{j=1}^{n} (y_{2j-1} y_{2j}^{p^m} - y_{2j} y_{2j-1}^{p^m}), \end{aligned}$$

$1 \le i \le n+1, 1 \le m \le n$, be elements of $H^*(V)$. We have

Theorem 1 ([2], as corrected in [13, Rk. 2.11(ii)], [12])**.** (i) *We have*

$$\begin{aligned} E_\infty(\Gamma_1) = &H^*(V)/(z, \eta_1, \eta_2, \xi_1) \otimes \mathbb{F}_p[v^p] \\ &\oplus (\mathbb{F}_p X_1 \oplus \mathbb{F}_p x X_1) \otimes \sum_{i=1}^{p-2} \mathbb{F}_p[v^p] v^i. \end{aligned}$$

(ii) *For $n \geq 2$,*

$$E_{2p+1}(\Gamma_n) = H^*(V)/(z, \eta_1, \xi_1, A_n\eta_2) \otimes \mathbb{F}_p[v^p] \oplus (\mathbb{F}_p X_n \oplus \mathbb{F}_p x X_n) \otimes \sum_{i=1}^{p-2} \mathbb{F}_p[v^p]v^i$$

with $A_n = \sum_{i=1}^{n} x_1x_2 \ldots \hat{x}_{2i-1}\hat{x}_{2i} \ldots x_{2n-1}x_{2n}$. □

Let W be the vector subspace of V given by $W = \mathrm{Ker}(x - x_1)$. We then have the central extension

$$(\mathbb{M}_n) \qquad 1 \to Z \to \mathbb{M}_n \to W \to 0$$

with factor set $z' = z'_n = y_1 + x_1x_2 + \cdots + x_{2n-1}x_{2n}$. Following [10], [12], we also have

Proposition 2 ([10], [12])**.** (i) *We have*

$$E_\infty(\mathbb{M}_1) = E_{2p+1}(\mathbb{M}_1) = H^*(W)/(z', \beta z') \otimes \mathbb{F}_p[v^p] \oplus (\mathbb{F}_p[y_2]x_1 \oplus \mathbb{F}_p[y_2]x_1x_2) \otimes \mathbb{F}_p[v^p]v^{p-1} \oplus (\mathbb{F}_p x_1 \oplus \mathbb{F}_p x_1x_2) \otimes \sum_{i=1}^{p-2} \mathbb{F}_p[v^p]v^i.$$

(ii) *For $n \geq 2$,*

$$E_{2p+1}(\mathbb{M}_n) = H^*(W)/(z', \beta z', \beta\mathcal{P}^1\beta z', B_n \cdot \mathcal{P}^1\beta z') \otimes \mathbb{F}_p[v^p] \oplus (\mathbb{F}_p x_1x_3x_4 \ldots x_{2n-1}x_{2n} \oplus \mathbb{F}_p X_n) \otimes \sum_{i=1}^{p-2} \mathbb{F}_p[v^p]v^i$$

with $B_n = \sum_{i=2}^{n} x_1x_3x_4 \ldots \hat{x}_{2i-1}\hat{x}_{2i} \ldots x_{2n-1}x_{2n}$. □

We also prove

Proposition 3. *If $\xi \in H^*(\Gamma_n)$ and $|\xi| < 2n+2$, then $\xi \in \mathrm{Im}\, \mathrm{Inf}_{\Gamma_n}^V$.*

The proof of the proposition is divided into the following lemmas. Set

$$\mathcal{R} = E[x_1, \ldots, x_m] \otimes \mathbb{F}_p[t_1, \ldots, t_m],$$

and let

$$\alpha_i = \sum_{j=1}^{m} x_j t_j^{p^{i-1}}, \qquad 1 \leq i \leq m,$$

be elements of $\mathcal{R}$. Denote by $\mathcal{I}_{k,m}$ the set consisting of subsets of k elements of $\{1, \ldots, m\}$. For every element $I = \{i_1, \ldots, i_k\}$ of $\mathcal{I}_{k,m}$ with $i_1 < \cdots < i_k$, set $x_I = x_{i_1} \ldots x_{i_k}$ and $x_\emptyset = 1$.

Lemma 2. *For $X \in \mathcal{R}$ and $1 \leq k \leq m$,*

(i) *if $X \cdot \alpha_1 \ldots \alpha_k = 0$, then $X \in (\alpha_1, \ldots, \alpha_k, x_I | I \in \mathcal{I}_{m-k+1,m})$;*
(ii) *if $X \cdot \alpha_k = 0$, then $X \in (\alpha_k, x_1 \ldots x_m)$.*

Proof. (i) We argue by induction on m. The case $m = 2$ is obvious. Assume that (i) holds for $m - 1$.

If $k = m$, then

$$\alpha_1 \dots \alpha_m = \begin{vmatrix} t_1 & \dots & t_m \\ t_1^p & \dots & t_m^p \\ . & \dots & . \\ t_1^{p^{m-1}} & \dots & t_m^{p^{m-1}} \end{vmatrix} x_1 \dots x_m;$$

so $X \in (x_i | 1 \leq i \leq m)$. Suppose that $k < m$. Write

$$\alpha_i = \alpha_i' + x_m t_m^{p^{i-1}}$$

with $\alpha_i' = \sum_{j=1}^{m-1} x_j t_j^{p^{i-1}}, 1 \leq i \leq m$, and

$$X = X' + X'' x_m,$$

with X', X'' free of x_m. Since $X\alpha_1 \dots \alpha_k = 0$, we have

$$0 = X'\alpha_1' \dots \alpha_k',$$

$$0 = (-1)^k X'' \alpha_1' \dots \alpha_k' + X' \sum_{i=1}^{k} (-1)^{k-i} t_m^{p^{i-1}} \alpha_1' \dots \widehat{\alpha}_i' \dots \alpha_k'.$$

By writing

$$X' = t_m^{r_1} f_1 + \dots + t_m^{r_j} f_j,$$
$$X'' = t_m^{s_1} g_1 + \dots + t_m^{s_i} g_i$$

with f_i, g_j free of $t_m, r_1 < \dots < r_j, s_1 < \dots < s_i$, we have

$$(-1)^k t_m^{s_i} g_i \alpha_1' \dots \alpha_k' + t_m^{p^{k-1}+r_j} f_j \alpha_1' \dots \alpha_{k-1}' = 0. \tag{2}$$

Consider the following cases:

- $r_j + p^{k-1} > s_i$: from (2), $f_j \alpha_1' \dots \alpha_{k-1}' = 0$. By the inductive hypothesis, $f_j \in (\alpha_1', \dots, \alpha_{k-1}', \mathcal{I}_{m-k+1,m-1})$. Since $\alpha_i' = \alpha_i - x_m t_m^{p^{i-1}}$, we have $X = t_m^{r_1} f_1 + \dots + t_m^{r_{j-1}} f_{j-1} \bmod (x_m, \alpha_1, \dots, \alpha_k, \mathcal{I}_{m-k+1,m})$. So we may suppose that $f_j = 0$.
- $r_j + p^{k-1} < s_i$: from (2), $g_i \alpha_1' \dots \alpha_k' = 0$. By the inductive hypothesis, $g_i \in (\alpha_1', \dots, \alpha_k', \mathcal{I}_{m-k,m-1})$. So $x_m g_i \in (\alpha_1, \dots, \alpha_k, \mathcal{I}_{m-k+1,m})$.
- $r_j + p^{k-1} = s_i$: from (2), $((-1)^{k+k-1} g_i \alpha_k' + f_j)\alpha_1' \dots \alpha_{k-1}' = 0$. By the inductive hypothesis, $f_j = g_i \alpha_k' \bmod (\alpha_1', \dots, \alpha_{k-1}', \mathcal{I}_{m-k+1,m-1})$. Since $\alpha_i' = \alpha_i - x_m t_m^{p^{i-1}}$, we have $f_j = g_i \alpha_k' \bmod (x_m, \alpha_1, \dots, \alpha_{k-1}, \mathcal{I}_{m-k+1,m})$. So we may suppose that $f_j = g_i \alpha_k'$. Since

$$\begin{aligned} t_m^{r_j} f_j + t_m^{s_i} g_i x_m &= t_m^{r_j}(f_j + t_m^{p^{k-1}} g_i x_m) \\ &= t_m^{r_j} g_i (\alpha_k' + t_m^{p^{k-1}} x_m) \\ &= t_m^{r_j} g_i \alpha_k, \end{aligned}$$

we may then suppose that $f_j = 0$ and $g_i = 0$.

The above arguments show that we may reduce to the case $X' = 0$. It follows that $X'' \alpha_1' \dots \alpha_k' = 0$. By the inductive hypothesis, $X'' \in (\alpha_1', \dots, \alpha_k', \mathcal{I}_{m-k,m-1})$. Hence $x_m X'' \in (\alpha_1, \dots, \alpha_k, \mathcal{I}_{m-k+1,m})$. (i) is proved.

(ii) We again use induction on m. The case $m = 1$ is trivial. Assume that (ii) holds for $m-1$. As above, write

$$X = X' + X''x_m,$$

with X', X'' free of x_m. Arguing as above, we may reduce to the case $X' = 0$. It follows that $X''\alpha'_k = 0$. Bu the inductive hypothesis, $X'' \in (\alpha'_k, x_1 \dots x_{m-1})$. Hence $x_m X'' \in (\alpha_k, x_1 \dots x_m)$.

The lemma is proved. □

Lemma 3. *Let $1 \leq k \leq n$ and let $Y_1, \dots, Y_k$ be elements of $H^*(V)$.*

(i) *If $Y_1\xi_1 + \dots + Y_k\xi_k = 0$, then $Y_k \in (\xi_1, \dots, \xi_{k-1})$.*

(ii) *Assume that*

$$Y_k = \sum_{\substack{I \subset \{1,\dots,2n\} \\ \#(I) < 2n-k+1}} x_I f_I(y, y_1, \dots, y_{2n}).$$

We have:

(iia) *if $Y_1\eta_1 + \dots + Y_k\eta_k = 0$, then $Y_k \in (\eta_1, \dots, \eta_k)$;*

(iib) *if $Y_k \in \bigcap_{i=1}^k (\eta_i)$, then $Y_k \in (\eta_1 \dots \eta_k)$;*

(iic) *if $Y_1\xi_1 + \dots + Y_{k-1}\xi_{k-1} + Y_k\eta_\ell = 0$ with $1 \leq \ell \leq n$, then*

$$Y_k \in (\eta_\ell, \xi_1, \dots, \xi_{k-1}).$$

Proof. (i) For $1 \leq i \leq k$, write

$$Y_i = \sum_{I \subset \{1,\dots,2n\}} x_I f_I^{(i)}(y, y_1, \dots, y_{2n}).$$

Then, for every I, we have

$$\sum_{i=1}^k f_I^{(i)} \xi_i = 0.$$

According to [18], $\xi_1, \dots, \xi_k$ is a regular sequence in P. So the above equality implies $f_I^{(k)} \in (\xi_1, \dots, \xi_{k-1})$. Therefore $Y_k \in (\xi_1, \dots, \xi_{k-1})$.

(iia) It follows that $Y_k\eta_1 \dots \eta_k = 0$. By Lemma 2, $Y_k \in (\eta_1, \dots, \eta_k, \mathcal{I}_{2n-k+1,2n})$. So $Y_k \in (\eta_1, \dots, \eta_k)$.

(iib) We use induction on k. For $k = 2$, $X\eta_1 + Y\eta_2 = 0$ implies $Y\eta_1\eta_2 = 0$. By Lemma 2, $Y \in (\eta_1, \eta_2, \mathcal{I}_{2n-1,2n})$. So $Y \in (\eta_1, \eta_2)$. Write $Y = a\eta_1 + b\eta_2$. Then $Y_2 = Y\eta_2 = a\eta_1\eta_2$.

Assume that (iib) holds for $k - 1 \geq 2$. As $Y_k \in \bigcap_{i<k} (\eta_i)$, it follows from the inductive hypothesis that $Y_k = Y\eta_1 \dots \eta_{k-1}$. Write $Y_k = X\eta_k$. Then $Y\eta_1 \dots \eta_k = 0$. By Lemma 2, $Y = c_1\eta_1 + \dots + c_k\eta_k$. So $Y_k = (-1)^{k-1} c_k \eta_1 \dots \eta_k$.

(iic) Again, we use induction on k. $Y_1\xi_1 + Y_2\eta_\ell = 0$ implies $Y_1\eta_\ell = 0$. By Lemma 2, $Y_1 \in (\eta_\ell)$. Write $Y_1 = c\eta_\ell$. Then $(c\xi_1 + Y_2)\eta_\ell = 0$. By Lemma 2, $c\xi_1 + Y_2 \in (\eta_\ell)$; hence $Y_2 \in (\eta_\ell, \xi_1)$.

Assume that (iic) holds for $k - 1 \geq 2$. As $Y_1\eta_\ell\xi_1 + \dots + Y_{k-1}\eta_\ell\xi_{k-1} = 0$, it follows from (i) that $Y_{k-1}\eta_\ell \in (\xi_1, \dots, \xi_{k-2})$. By the inductive hypotheses, we may write $Y_{k-1} = c_1\xi_1 + \dots + c_{k-2}\xi_{k-2} + c_{k-1}\eta_\ell$. Hence

$$(Y_1 + c_1\xi_{k-1})\xi_1 + \dots + (Y_{k-2} + c_{k-2}\xi_{k-1})\xi_{k-2} + (Y_k + c_{k-1}\xi_{k-1})\eta_\ell = 0.$$

By the inductive hypothesis, $Y_k + c_{k-1}\xi_{k-1} \in (\xi_1, \dots, \xi_{k-2}, \eta_\ell)$, and hence $Y_k \in (\xi_1, \dots, \xi_{k-1}, \eta_\ell)$. □

For $1 \leq i \leq n+1, 0 \leq k \leq n$, denote by $\Delta_{i,k}$ the ideal of $H^*(V)$ given by

$$\Delta_{i,k} = \begin{cases} (z, \eta_j, \xi_m | 1 \leq j \leq i, 1 \leq m \leq k) & \text{if } k \geq 1, \\ (z, \eta_j | 1 \leq j \leq i) & \text{if } k = 0. \end{cases}$$

Lemma 4. *If $X = \sum_{\#(I)<2n-2k+1} x_I X_I(y_1, \ldots, y_{2n})$ and $X\xi_j \in \Delta_{k,j-1}$ with $1 \leq j \leq k \leq n$, then $X \in \Delta_{k,j-1}$.*

Proof. Write

$$X\xi_j = a_0 z + \sum_{i=1}^{k} a_i \eta_i + \sum_{i=1}^{j-1} b_i \xi_i$$

with $a_i, b_\ell \in H^*(V)$. Since $y = z - x_1x_2 - \cdots - x_{2n-1}x_{2n}$, we may suppose that $a_i, b_\ell, 1 \leq i \leq k, 1 \leq \ell \leq j-1$, are free of y. It follows that $a_0 = 0$ and

$$X\xi_j \eta_1 \ldots \eta_k = \sum_{i=1}^{j-1} b_i \xi_i \eta_1 \ldots \eta_k. \tag{3}$$

We now argue by induction on j. For $j = 1$, it follows that $X\xi_1\eta_1 \ldots \eta_k = 0$. Hence $X\eta_1 \ldots \eta_k = 0$. By Lemma 2, $X \in (\eta_1, \ldots, \eta_k)$.

Assume that the lemma holds for $j - 1 \geq 1$. By Lemma 3 (i) and by (3), there exists $c_i \in H^*(V)$ such that

$$X\eta_1 \ldots \eta_k = c_1\xi_1 + \cdots + c_{j-1}\xi_{j-1}.$$

Therefore, by Lemma 3 (i), $c_{j-1}\eta_i \in (\xi_1, \ldots, \xi_{j-2})$, for every $1 \leq i \leq k$; by Lemma 3 (iic), $c_{j-1} \in \bigcap_{i \leq k}(\xi_1, \ldots, \xi_{j-2}, \eta_i)$. By writing

$$\begin{aligned} c_{j-1} &= d_1\xi_1 + \cdots + d_{j-2}\xi_{j-2} + d\eta_1 \ldots \eta_{i-1} \\ &= e_1\xi_1 + \cdots + e_{j-2}\xi_{j-2} + e\eta_i, \end{aligned}$$

we get

$$[(e_1 - d_1)\xi_1 + \cdots + (e_{j-2} - d_{j-2})\xi_{j-2}]\eta_1 \ldots \eta_i = 0.$$

By Lemma 2, $(e_1 - d_1)\xi_1 + \cdots + (e_{j-2} - d_{j-2})\xi_{j-2}$ contains $\eta_1 \ldots \eta_i$ as a factor. Hence $e\eta_i \in \bigcap_{\ell \leq i}(\eta_\ell)$. By Lemma 3 (iib), $c_{j-1} \in (\xi_1, \ldots, \xi_{j-2}, \eta_1 \ldots \eta_i)$. So we may suppose that $c_{j-1} \in (\eta_1 \ldots \eta_k)$. By writing $c_{j-1} = c\eta_1 \ldots \eta_k$, we have

$$(X - c\xi_{j-1})\eta_1 \ldots \eta_k = c_1\xi_1 + \cdots + c_{j-2}\xi_{j-2}.$$

By the inductive hypothesis, this implies $X - c\xi_{j-1} \in \Delta_{k,j-2}$. So $X \in \Delta_{k,j-1}$. The lemma follows. □

Lemma 5. *If $X = \sum_{\#(I)<2n-2k} x_I X_I(y_1, \ldots, y_{2n})$ and $X\eta_k \in \Delta_{k-1,k-1}$ with $1 \leq k \leq n+1$, then $X \in \Delta_{k,k-1}$.*

Proof. Write

$$X\eta_k = a_0 z + \sum_{i=1}^{k-1}(a_i\eta_i + b_i\xi_i).$$

Arguing as in the proof of Lemma 4, we may suppose that a_i, b_i, with $1 \leq i \leq k-1$, are free of y. It follows that $a_0 = 0$ and

$$X\eta_k = \sum_{i=1}^{k-1}(a_i\eta_i + b_i\xi_i).$$

Furthermore, we may suppose that every b_i is of form

$$b_i = \sum_{\#(I)<2n-2k+1} x_I b_I^{(i)}.$$

Therefore, applying Lemma 4 yields $b_{k-1} \in \Delta_{k,k-2}$. Hence, by induction, we need only consider the case

$$X\eta_k = b_1\xi_1 + \sum_{i=1}^{k-1} a_i\eta_i.$$

This implies $b_1\xi_1\eta_1 \ldots \eta_k = 0$. So $b_1\eta_1 \ldots \eta_k = 0$. By Lemma 2, $b_1 \in (\eta_1, \ldots, \eta_k)$. The lemma follows. □

Let us now consider the Hochschild-Serre spectral sequence $\{E_r(\Gamma_n), d_r\}$. It follows that, for $k < 2n+2$,

$$\sum_{i+j=k} E_{2p+1}^{i,j} \subset E_{2p+1}^{k,0} \oplus \bigoplus_{r\geq 1} E_{2p+1}^{*,2pr}.$$

By Kudo's transgression theorem, for $m \leq n$, $1 \otimes v^{p^m}$ (resp. $\eta_m \otimes v^{p^{m-1}(p-1)}$) survives to E_{2p^m+1} (resp. $E_{2p^{m-1}(p-1)+1}$) and

$$d_{2p^m+1}(1 \otimes v^{p^m}) = \eta_{m+1},$$
$$d_{2p^{m-1}(p-1)+1}(\eta_m \otimes v^{p^{m-1}(p-1)}) = -\xi_m.$$

Lemma 6. *For $k < 2n+2$ and $1 \leq m \leq n$, we have*

$$\sum_{i+j=k} E_{2p^m+1}^{i,j} \subset E_{2p^m+1}^{k,0} \oplus \bigoplus_{r\geq 1} E_{2p^m+1}^{*,2p^m r}.$$

Proof. By the structure of $E_{2p+1}(\Gamma_n)$, the lemma holds for $m = 1$. Suppose that the lemma holds for $m = s \geq 1$. Let $\psi = X \otimes v^{\ell p^s}$ be an element of $E_2(\Gamma_n)$ surviving to $E_{2p^{s+1}+1}$, with $1 \leq \ell \leq p-1$ and $|X| + 2\ell p^s = k < 2n+2$. So $d_{2p^s+1}(\psi) = \ell X\eta_{s+1} \otimes v^{(\ell-1)p^s}$ must be hit by images under the differentials of elements of degrees less than $2n+2$. By the inductive hypothesis and by Kudo's theorem, these images belong to the ideal $\Delta_{s,s}$; hence so does $X\eta_{s+1}$. Since in $E_3(\Gamma_n)$ we have $y = -(x_1x_2 + \cdots + x_{2n-1}x_{2n})$, we may suppose that X is free of y. As $|X| < 2n+2-2p^s < 2n-2s-2$, by Lemma 5, this means that $X \in \Delta_{s+1,s}$. So $\psi = 0$ in E_{2p^s+2} if $\ell < p-1$. If $\ell = p-1$, write $\psi = Y\eta_{s+1} \otimes v^{p^s(p-1)}$. Then $d_{2p^s(p-1)+1}(\psi) = -Y\xi_{s+1} \in \Delta_{s+1,s}$. Arguing as above, we may suppose that Y is free of y. By Lemma 4, as $|\psi| < 2n+2$, we have $Y \in \Delta_{s+1,s}$. So $\psi = 0$ in $E_{2p^s(p-1)+2}$. The lemma follows. □

Proof of Proposition 3. It follows from Lemma 6 that ξ either belongs to Im $\mathrm{Inf}_{\Gamma_n}^V$ or represents an element of $E_\infty^{*,2p^n r}$. As $2p^n > 2n+2$, the fact that $|\xi| < 2n+2$ implies $\xi \in$ Im $\mathrm{Inf}_{\Gamma_n}^V$. The proposition follows. □

3. The case $exp(G) > p$ or $|G| = p^5$

We first consider the case $G = \mathbb{M}_n$. Consider G as a subgroup of Γ_n by setting $G = \mathrm{Ker}\ (x - x_1)$. If $n = 1$, it follows from [10] (see also [4]) that $H^*(\mathbb{M})$ contains a non-zero essential element, namely X_1. Assume inductively that $0 \neq X_{n-1} \in Ess(\mathbb{M}_{n-1})$. As $\mathbb{M}_n = \mathbb{M}_{n-1} \bullet \mathbb{M}$, we have the following central extension:

$$0 \to \mathbb{F}_p \to \mathbb{M}_{n-1} \times \mathbb{M} \to \mathbb{M}_n \to 1.$$

The fact that $H^*(\mathbb{M}_n)$ contains non-zero essential elements follows from

Proposition 4. $0 \neq X_n \in Ess(\mathbb{M}_n)$.

Proof. Let K be a maximal subgroup of $\mathbb{M}_n$. As $\dim_{\mathbb{F}_p} H^1(K) = 2n-1$, it follows that the product of any $2n$ elements of $H^1(K)$ vanishes. Hence $\mathrm{Res}^{\mathbb{M}_n}_K(X_n) = 0$, which implies that $X_n \in Ess(\mathbb{M}_n)$. Furthermore, as $\mathrm{Inf}^{\mathbb{M}_n}_{\mathbb{M}_{n-1}\times\mathbb{M}}(X_n) = X_{n-1} \times x_{2n-1}x_{2n} \neq 0$ in $H^*(\mathbb{M}_{n-1}\times\mathbb{M})$ by the inductive hypothesis, it follows that $X_n \neq 0$. The proposition is proved □

By Theorem 1 (i), X_1 and xX_1 are non-zero elements of $H^*(\Gamma_1)$. By considering the central extension $0 \to \mathbb{F}_p \to \Gamma_{n-1} \times \mathbb{M} \to \Gamma_n \to 1$, and by using the same argument given in the proof of Proposition 4, we also have

Proposition 5. *The elements xX_n and X_n are non-zero elements of $H^*(\Gamma_n)$.* □

Our next task is to prove that the theorem holds for the extraspecial p-group $G = \mathbb{E}_2$. Consider $\mathbb{E}_2$ as a subgroup of Γ_2 as in Lemma 1. Let Q be the element of $H^*(V)$ defined by $Q = Q^{1,2}_{2,1} - Q^{3,4}_{2,1}$ with

$$Q^{i,j}_{2,1} = Q_{2,1}(y_i, y_j) = \frac{y_i^{p^2} y_j - y_j^{p^2} y_i}{y_i^p y_j - y_j^p y_i}$$

(so $Q^{i,j}_{2,1}$ is nothing but the Dickson invariant of order $2(p^2-p)$ with variables y_i, y_j), and set $\eta = x_1x_2Q$. It follows from [19, Th. 8.25] that $0 \neq \eta \in H^*(\mathbb{E}_2)$. The case $G = \mathbb{E}_2$ is then proved by the following:

Proposition 6. $\eta \in Ess(\mathbb{E}_2)$.

Proof. Let K be a maximal subgroup of $\mathbb{E}_2$, so $K \cong \mathbb{E}\times C_p$. If $\mathrm{Res}^{\mathbb{E}_2}_K(x_3x_4) = 0$, it is clear that $\mathrm{Res}^{\mathbb{E}_2}_K(x_1x_2Q^{1,2}_{2,1}) = 0$; we can then assume that $\mathrm{Res}^{\mathbb{E}_2}_K(x_3x_4) \neq 0$. Choose a basis u_1, u_2, u_3, u_4 of $H^1(\mathbb{E}_2/Z)$ such that $K = \mathrm{Ker}\ u_4$, $x_1x_2 + x_3x_4 = u_1u_2 + u_3u_4$, $\mathrm{Res}^{\mathbb{E}_2}_K(x_1x_2) = u_1u_2 + u_1u_3$ and $\mathrm{Res}^{\mathbb{E}_2}_K(x_3x_4) = -u_1u_3$. This implies that $u_1u_2 = 0$ in $H^*(K)$. By setting $v_i = \beta u_i$, we have

$$\begin{aligned}\mathrm{Res}^{\mathbb{E}_2}_K(\eta) &= (u_1u_2 + u_1u_3)(Q_{2,1}(v_1, v_2+v_3) - Q_{2,1}(v_3, v_1)) \\ &= u_1u_3(Q_{2,1}(v_1, v_2+v_3) - Q_{2,1}(v_3, v_1)) \qquad \text{as } u_1u_2 = 0 \text{ in } H^*(K).\end{aligned}$$

Set $Y = Q_{2,1}(v_1, v_2+v_3) - Q_{2,1}(v_3, v_1)$. Following [13, Proof of Lemma 1.10], Y contains $v_2^p - v_2v_1^{p-1}$ as a factor. As $u_1(v_2^p - v_2v_1^{p-1}) = -\mathcal{P}^1\beta(u_1u_2) + v_1^{p-1}\beta(u_1u_2)$, we have $\mathrm{Res}^{\mathbb{E}_2}_K(\eta) = 0$. So $\eta \in Ess(\mathbb{E}_2)$. The proposition is proved. □

For $exp(G) > p$ or $|G| = p^5$, Propositions 4 and 6 tell us that there exist non-zero essential cohomology classes of G which belong to $\mathrm{Im}\ \mathrm{Inf}^V_G$. Furthermore, if $G = \mathbb{M}_2$, then [12, Proposition 1.9] and [13, Theorem 3.10] tell us that

$$x_3x_4N \quad \text{and} \quad (y_3x_4 - y_4x_3)N$$

are also non-zero elements of $Ess(\mathbb{M}_2)$ with $N = (y_2^{p-1} - y_3^{p-1})(y_2^{p-1} - y_4^{p-1})$. We can then end the section by the following

Question. For $G \ncong \mathbb{E}$, is it true that $Ess(G) \cap \mathrm{Im}\ \mathrm{Inf}^V_G \neq \{0\}$?

4. The case $exp(G) = p$

We first point out some mod-p cohomology classes of Γ_n, by using the following argument given by D.J. Green [6]. Let K be a p-group containing C as a central subgroup. We have the central extension

$$(K) \qquad 1 \to C \to K \xrightarrow{\text{pr}} K/C \to 1,$$

On the other hand, by considering the extension

$$(K\times C) \qquad 1 \to C \xrightarrow{\ell} K\times C \xrightarrow{j} K \to 1$$

with $\ell(c) = (1, c), j(k, c) = k, c \in C, k \in K$, we have the commutative diagram

$$(4) \qquad \begin{array}{ccccccccc} 1 & \longrightarrow & C & \xrightarrow{\ell} & K\times C & \xrightarrow{j} & K & \longrightarrow & 1 \\ & & \| & & \downarrow{\mu} & & \downarrow{\text{pr}} & & \\ 1 & \longrightarrow & C & \longrightarrow & K & \xrightarrow{\text{pr}} & K/C & \longrightarrow & 1 \end{array}$$

with $\mu(k, c) = kc, k \in K, c \in C$. The Hochschild-Serre spectral sequences corresponding to these extensions are of the forms

$$E_2(K) = H^*(K/C) \otimes H^*(C) \Rightarrow H^*(K),$$
$$E_2(K \times C) = E_\infty(K \times C) = H^*(K) \otimes H^*(C).$$

Furthermore, vertical arrows in (4) also induce a map $\{\mu_r : E_r(K) \to E_r(K \times C)\}$ between spectral sequences with $\mu_2 = (\text{Inf}_K^{K/C}, 1_{H^*(C)})$.

The following is due to D.J. Green.

Proposition 7. *For $r \geq 2$,*

$$\text{Im}(d_r : E_r(K) \to E_r(K)) \subset \text{Ker Inf}_K^{K/C} \otimes H^*(C).$$

Proof. Let $\xi \in E_r(K)$ and write $d_r(\xi) = \sum \phi_j \otimes \psi_j, \phi_j \in H^*(K/C), \psi_j \in H^*(C)$. We can suppose that the ψ_j's are linearly independent in $H^*(C)$. From the commutative diagram (4) and from the fact that $d_r : E_r(K\times C) \to E_r(K\times C)$ vanishes, we have

$$\sum \text{Inf}_K^{K/C}(\phi_j) \otimes \psi_j = \mu_r(d_r(\xi)) = d_r(\mu_r(\xi)) = 0.$$

So $\phi_j \in \text{Ker Inf}_K^{K/C}$. The proposition follows. □

Since $d_{2p+1}(v^p) = \eta_2$ in $E_{2p+1}(\Gamma_n)$ (resp. $\mathcal{P}^1\beta z'$ in $E_{2p+1}(\mathbb{M}_n)$), it follows from Theorem 1 and Proposition 2 that $A_n \otimes v^p \in E_{2p+2}(\Gamma_n)$ and $B_n \otimes v^p \in E_{2p+2}(\mathbb{M}_n)$. We then get

Proposition 8. *For $1 \leq i \leq p-2$,*

- (i) *if $n \geq 2$ then $x_1x_3x_4 \dots x_{2n-1}x_{2n} \otimes v^i, X_n \otimes v^i, x_3x_4 \dots x_{2n-1}x_{2n} \otimes v^p$ and $B_n \otimes v^p$ represent non-zero elements of $E_\infty(\mathbb{M}_n)$;*
- (ii) *$X_n \otimes v^i, xX_n \otimes v^i$ and $A_n \otimes v^p$ represent non-zero elements of $E_\infty(\Gamma_n)$.*

Proof. Note that, in $H^*(W)$, we have

$$\begin{aligned} x_3x_4 \dots x_{2n-1}x_{2n} \cdot \mathcal{P}^1\beta z' &= (x_3x_4 \dots x_{2n-1}x_{2n})(y_1^p x_2 - y_2^p x_1) \\ &= (x_3x_4 \dots x_{2n-1}x_{2n})(z'^p x_2 + y_2^{p-1}\beta z' - y_2^{p-1} y_1 x_2) \\ &= (x_3x_4 \dots x_{2n-1}x_{2n})(z'^p x_2 + y_2^{p-1}\beta z' - y_2^{p-1} z' x_2) \\ &\in (z', \beta z'). \end{aligned}$$

So $d_{2p+1}(x_3x_4\dots x_{2n-1}x_{2n}\otimes v^p)=0$. Therefore $x_3x_4\dots x_{2n-1}x_{2n}\otimes v^p$ survives to $E_\infty(\mathbb{M}_n)$.

By Proposition 4, $X_n\neq 0$ in $H^*(\mathbb{M}_n)$ implies that $X_n, x_1x_3x_4\dots x_{2n-1}x_{2n}$ and B_n are not elements of Ker $\mathrm{Inf}^W_{\mathbb{M}_n}$. Similarly, Proposition 5 shows that X_n, xX_n and A_n are not elements of Ker $\mathrm{Inf}^V_{\Gamma_n}$. The proposition follows from Proposition 7. □

For $n\geq 1$ and for $1\leq i\leq p-2$, let us pick elements $X_{n,i}\in H^{2(n+i)-1}(\mathbb{M}_n)$ and $Y_{n,i}\in H^{2(n+i)}(\Gamma_n)$ which represent respectively $x_1x_3x_4\dots x_{2n-1}x_{2n}\otimes v^i\in E_\infty(\mathbb{M}_n)$ and $X_n\otimes v^i\in E_\infty(\Gamma_n)$; for $n\geq 2$, pick elements $X_{n,p-1}\in H^{2(n+p)-3}(\mathbb{M}_n)$, $Z_{n,p-1}\in H^{2(n+p)-2}(\mathbb{M}_n)$ and $Y_{n,p-1}\in H^{2(n+p)-2}(\Gamma_n)$ which represent respectively $B_n\otimes v^p\in E_\infty(\mathbb{M}_n), x_3x_4\dots x_{2n-1}x_{2n}\otimes v^p\in E_\infty(\mathbb{M}_n)$ and $A_n\otimes v^p\in E_\infty(\Gamma_n)$ (the existence of such elements follows from Propositions 2 and 8). In particular, define $Y_{1,p-1}$ by

$$Y_{1,p-1}=\mathcal{N}_{\mathrm{Ker}\ x_2\to\Gamma_1}(w)$$

with $\mathcal{N}$ the Evens norm map (note that $\mathrm{Ker}\,x_2\cong C_{p^2}\times C_p\subset\Gamma_1$, so, by the Künneth formula, w can be considered as an element of $H^2(\mathrm{Ker}\,x_2)$).

We now define the following subgroups of Γ_n:

$$\begin{aligned}
\mathbb{M}_n&=\mathrm{Ker}(x-x_1),\\
\Gamma'_{n-1}&=\mathrm{Ker}\,x_{2n}\cong\Gamma_{n-1}\times C_p,\\
\mathbb{M}'_{n-1}&=\mathrm{Ker}\,x_{2n}\cap\mathrm{Ker}(x-x_1)\quad\text{(so }\mathbb{M}'_{n-1}\cong\mathbb{M}_{n-1}\times C_p\text{ for }n>1),\\
{}^{‘}\Gamma_{n-1}&=\mathrm{Ker}\,x_2\cap\mathrm{Ker}(x-x_1)\cong\Gamma_{n-1},\\
{}'\Gamma_{n-2}&=\mathrm{Ker}\,x_2\cap\mathrm{Ker}\,x_{2n}\cap\mathrm{Ker}(x-x_1)\cong\Gamma_{n-2}\times C_p\ \text{(for }n\geq 2),
\end{aligned}$$

with the convention that $\Gamma_0=C_{p^2}$. Therefore ${}'\Gamma_{n-2}=\mathbb{M}'_{n-1}\cap{}^{‘}\Gamma_{n-1}$ and ${}'\Gamma_0=C_{p^2}\times C_p$. If K is one of the above subgroups, then K contains Z as a central subgroup and we have the central extension

$$(K)\qquad 1\to Z\to K\to K/Z\to 1.$$

For convenience, we also define the elements $Y_{0,i}\in H^{2i}(\Gamma_0), 1\leq i\leq p-1$, by $Y_{0,i}=w^i$. With some abuse of notation, by the Künneth formula, the $Y_{n-1,i}$ (resp. $X_{n-1,i}, Z_{n-1,i}$)'s are considered as elements of $H^*(\Gamma'_{n-1}), H^*({}'\Gamma_{n-1})$ and $H^*({}^{‘}\Gamma_{n-1})$ (resp. $H^*(\mathbb{M}'_{n-1})$). We have

Lemma 7. *For $n\geq 1$ and $1\leq i\leq p-1, 1\leq j\leq p-2$,*

(i) $\mathrm{Res}^{\Gamma_n}_{\mathbb{M}_n}(Y_{n,j})+x_2X_{n,j}\in\mathrm{Im}\ \mathrm{Inf}^W_{\mathbb{M}_n}$*; if $n>1$ then* $\mathrm{Res}^{\Gamma_n}_{\mathbb{M}_n}(Y_{n,p-1})+x_2X_{n,p-1}-Z_{n,p-1}\in\mathrm{Im}\ \mathrm{Inf}^W_{\mathbb{M}_n}$*;*

(ii) *if $n>1$ then* $\mathrm{Res}^{\mathbb{M}_n}_{{}^{‘}\Gamma_{n-1}}(X_{n,i})-xY_{n-1,i}\in\mathrm{Im}\ \mathrm{Inf}^{{}^{‘}\Gamma_{n-1}/Z}_{{}^{‘}\Gamma_{n-1}}$*;*

(iii) $\mathrm{Res}^{\Gamma_n}_{\Gamma'_{n-1}}(Y_{n,j})$ *belongs to* $\mathrm{Im}\ \mathrm{Inf}^{\Gamma'_{n-1}/Z}_{\Gamma'_{n-1}}$*; and*

(iv) $xY_{n,i}\notin(y)$*; furthermore, there exists no element $\xi\in H^*(\Gamma_n)$ satisfying* $xY_{n,i}=y\xi$ mod $\mathrm{Im}\ \mathrm{Inf}^V_{\Gamma_n}$.

Proof. (i) and (ii) follow by considering the restriction in spectral sequences and by the structures of $E_{2p+1}(\mathbb{M}_n)$ and $E_{2p+1}(\Gamma_n)$ given in Theorem 1 (ii) and Proposition 2.

(iii) Set $T_{n,i} = \mathrm{Res}^{\Gamma_n}_{\Gamma'_{n-1}}(Y_{n,i})$. Since $y_{2n-1}Y_{n,i} \in \mathrm{Im\ Inf}^V_{\Gamma_n}$, it follows that

$$y_{2n-1}T_{n,i} = \mathrm{Res}^{\Gamma_n}_{\Gamma'_{n-1}}(y_{2n-1}Y_{n,i})$$

belongs to $\mathrm{Im\ Inf}^{\Gamma'_{n-1}/Z}_{\Gamma'_{n-1}}$. As $H^*(\Gamma'_{n-1}) = H^*(\Gamma_{n-1}) \otimes E[x_{2n-1}] \otimes \mathbb{F}_p[y_{2n-1}], T_{n,i}$ also belongs to $\mathrm{Im\ Inf}^{\Gamma'_{n-1}/Z}_{\Gamma'_{n-1}}$.

(iv) Assume that there exists $\xi \in H^{2n+1}(\Gamma_n)$ such that

$$y\xi = xY_{n,1} \bmod \mathrm{Im\ Inf}^V_{\Gamma_n}.$$

By Proposition 3, $\xi \in \mathrm{Im\ Inf}^V_{\Gamma_n}$. So $y\xi \in \mathrm{Im\ Inf}^V_{\Gamma_n}$. Hence $xY_{n,1} \in \mathrm{Im\ Inf}^V_{\Gamma_n}$, a contradiction.

Assume inductively that (iv) holds for $i-1$. For $i \geq 2$, we will prove in Lemmas 10, 11, 16 and 17 that $\mathrm{tr}^{\Gamma'_n}_{\Gamma_{n+1}}(Y_{n,i}) = \lambda_i Y_{n+1,i-1} \bmod \mathrm{Im\ Inf}^V_{\Gamma_{n+1}}$ with $0 \neq \lambda_i \in \mathbb{F}_p$. Let ϕ be the element of $\mathrm{Im\ Inf}^V_{\Gamma_{n+1}}$ satisfying $\mathrm{tr}^{\Gamma'_n}_{\Gamma_{n+1}}(Y_{n,i}) = \lambda_i Y_{n+1,i-1} + \phi$. Suppose that $xY_{n,i} + \eta = y\xi$, with $\xi \in H^*(\Gamma_n)$ and $\eta \in \mathrm{Im\ Inf}^V_{\Gamma_n}$. So

$$\begin{aligned} \lambda_i x Y_{n+1,i-1} + x\phi &= x\mathrm{tr}^{\Gamma'_n}_{\Gamma_{n+1}}(Y_{n,i}) \\ &= \mathrm{tr}^{\Gamma'_n}_{\Gamma_{n+1}}(xY_{n,i} + \eta) \quad \text{since } \mathrm{tr}^{\Gamma'_n}_{\Gamma_{n+1}}(\eta) = 0 \\ &= \mathrm{tr}^{\Gamma'_n}_{\Gamma_{n+1}}(y\xi) \\ &= y\mathrm{tr}^{\Gamma'_n}_{\Gamma_{n+1}}(\xi). \end{aligned}$$

Hence $\lambda_i x Y_{n+1,i-1} = y\mathrm{tr}^{\Gamma'_n}_{\Gamma_{n+1}}(\xi) - x\phi$, which contradicts the inductive hypothesis. (iv) is then proved.

The lemma follows. □

Further properties of $X_{n,i}$ and $Y_{n,i}$ are given by the following lemmas. The first one follows from Theorem 1, Proposition 2 and [14, Theorem 1.1].

Lemma 8. $\mathrm{tr}^{\mathbb{M}'_{n-1}}_{\mathbb{M}_n}(X_{n-1,i})$ *(resp.* $\mathrm{tr}^{\Gamma'_{n-1}}_{\Gamma_n}(Y_{n-1,i})$*) represents an element of* $E^{*,2j}_\infty(\mathbb{M}_n)$ *(resp.* $E^{*,2j}_\infty(\Gamma_n)$*), with* $j < i$. □

Lemma 9. *For* $2 \leq i \leq p-1$ *we have* $\mathrm{tr}^{\Gamma'_0}_{\Gamma_1}(Y_{0,i}) = \lambda_i Y_{1,i-1} \bmod \mathrm{Im\ Inf}^V_{\Gamma_1}$*, with* $0 \neq \lambda_i \in \mathbb{F}_p$*.*

Proof. Consider the commutative diagram

$$\begin{array}{ccc} H^*(C_{p^2} \times C_p) & \xrightarrow{\mathrm{tr}} & H^*(\Gamma_1) \\ {\scriptstyle \mathrm{Res}}\downarrow & & \downarrow{\scriptstyle \mathrm{Res}} \\ H^*(Z \times C_p) & \xrightarrow[\mathrm{tr}]{} & H^*(\mathbb{E}) \end{array},$$

We have $\mathrm{Res}^{\Gamma_1}_{\mathbb{E}}\mathrm{tr}^{\Gamma'_0}_{\Gamma_1}(Y_{0,i}) = \mathrm{tr}^{Z\times C_p}_{\mathbb{E}}(v^i)$. Following [8] (see also [16]), $\mathrm{tr}^{Z\times C_p}_{\mathbb{E}}(v^i)$ is a non-zero element of $H^*(\mathbb{E})\backslash \mathrm{Im\ Inf}^{\mathbb{E}/Z}_{\mathbb{E}}$. So, by Theorem 1, $\mathrm{tr}^{\Gamma'_0}_{\Gamma_1}(Y_{0,i})$ represents an element of the form $\lambda_i x_1 x_2 \otimes v^{i-1} \in E^{2,2(i-1)}_\infty(\Gamma_1)$, with $0 \neq \lambda_i \in \mathbb{F}_p$. The lemma follows. □

In the following two lemmas, p is assumed to be greater than 3.

Lemma 10. *For $2 \le i \le p-2$,*

$$\mathrm{tr}_{\mathbb{M}_2}^{\mathbb{M}_1'}(X_{1,i}) = \lambda_i X_{2,i-1} \bmod \mathrm{Im}\ \mathrm{Inf}_{\mathbb{M}_2}^{W}$$

and

$$\mathrm{tr}_{\Gamma_2}^{\Gamma_1'}(Y_{1,i}) = \lambda_i Y_{2,i-1} \bmod \mathrm{Im}\ \mathrm{Inf}_{\Gamma_2}^{V},$$

with λ_i given in Lemma 9.

Proof. Set $Z_i = \mathrm{Res}_{{}'\Gamma_1}^{\mathbb{M}_2} \mathrm{tr}_{\mathbb{M}_2}^{\mathbb{M}_1'}(X_{1,i})$. By the double coset formula and by Lemma 7 (ii), we have

$$Z_i = \mathrm{tr}_{{}'\Gamma_1}^{{}'\Gamma_0} \mathrm{Res}_{{}'\Gamma_0}^{\mathbb{M}_1'}(X_{1,i}) = \mathrm{tr}_{{}'\Gamma_1}^{{}'\Gamma_0}(x_1 Y_{0,i}) = x_1 \mathrm{tr}_{{}'\Gamma_1}^{{}'\Gamma_0}(Y_{0,i}).$$

By Lemma 9, Z_i represents

$$\lambda_i x_1 x_3 x_4 \otimes v^{i-1} \in E_\infty^{3,2(i-1)}({}'\Gamma_1).$$

By Lemma 8 and Proposition 2, this means that $\mathrm{tr}_{\mathbb{M}_2}^{\mathbb{M}_1'}(X_{1,i})$ represents

$$\lambda_i x_1 x_3 x_4 \otimes v^{i-1} \in E_\infty^{3,2(i-1)}(\mathbb{M}_2).$$

The first part of the lemma follows from the definition of $X_{2,i-1}$.

On the other hand, by setting $Y_i = \mathrm{Res}_{\mathbb{M}_2}^{\Gamma_2} \mathrm{tr}_{\Gamma_2}^{\Gamma_1'}(Y_{1,i})$, by the double coset formula, we have

$$\begin{aligned} Y_i &= \mathrm{tr}_{\mathbb{M}_2}^{\mathbb{M}_1'} \mathrm{Res}_{\mathbb{M}_1'}^{\Gamma_1'}(Y_{1,i}) \\ &= \mathrm{tr}_{\mathbb{M}_2}^{\mathbb{M}_1'}(-x_2 X_{1,i}) \qquad \text{by Lemma 7 (i)} \\ &= -x_2 \mathrm{tr}_{\mathbb{M}_2}^{\mathbb{M}_1'}(X_{1,i}). \end{aligned}$$

As shown above, $\mathrm{tr}_{\mathbb{M}_2}^{\mathbb{M}_1'}(X_{1,i})$ represents $\lambda_i x_1 x_3 x_4 \otimes v^{i-1} \in E_\infty^{3,2(i-1)}(\mathbb{M}_2)$, so Y_i represents $\lambda_i x_1 x_2 x_3 x_4 \otimes v^{i-1} \in E_\infty^{4,2(i-1)}(\mathbb{M}_2)$. By Lemma 8 and Theorem 1 (ii), this means that $\mathrm{tr}_{\Gamma_2}^{\Gamma_1'}(Y_{1,i})$ represents $\lambda_i x_1 x_2 x_3 x_4 \otimes v^{i-1} \in E_\infty^{4,2(i-1)}(\Gamma_2)$. The last part follows from the definition of $Y_{2,i-1}$. The lemma is proved. $\square$

In general, we have

Lemma 11. *For $2 \le i \le p-2$ and $n \ge 2$,*

$$\mathrm{tr}_{\mathbb{M}_n}^{\mathbb{M}_{n-1}'}(X_{n-1,i}) = \lambda_i X_{n,i-1} \bmod \mathrm{Im}\ \mathrm{Inf}_{\mathbb{M}_n}^{W}$$

and

$$\mathrm{tr}_{\Gamma_n}^{\Gamma_{n-1}'}(Y_{n-1,i}) = \lambda_i Y_{n,i-1} \bmod \mathrm{Im}\ \mathrm{Inf}_{\Gamma_n}^{V},$$

with λ_i given in Lemma 9.

Proof. We argue by induction on n. The case $n = 2$ follows from the above lemma. Assume that the lemma holds for $n-1$. Set $Z_i = \mathrm{Res}_{{}'\Gamma_{n-1}}^{\mathbb{M}_n} \mathrm{tr}_{\mathbb{M}_n}^{\mathbb{M}_{n-1}'}(X_{n-1,i})$. By the double coset formula, we have

$$\begin{aligned} Z_i &= \mathrm{tr}_{{}'\Gamma_{n-1}}^{{}'\Gamma_{n-2}} \mathrm{Res}_{{}'\Gamma_{n-2}}^{\mathbb{M}_{n-1}'}(X_{n-1,i}) \\ &= \mathrm{tr}_{{}'\Gamma_{n-1}}^{{}'\Gamma_{n-2}}(x Y_{n-2,i}) \quad \text{by Lemma 7 (ii)} \\ &= x \mathrm{tr}_{{}'\Gamma_{n-1}}^{{}'\Gamma_{n-2}}(Y_{n-2,i}). \end{aligned}$$

By the inductive hypothesis, $\mathrm{tr}^{{}^{\backprime}\Gamma_{n-2}}_{{}^{\backprime}\Gamma_{n-1}}(Y_{n-2,i}) = \lambda_i Y_{n-1,i-1} \bmod \mathrm{Im}\,\mathrm{Inf}^{{}^{\backprime}\Gamma_{n-1}/Z}_{{}^{\backprime}\Gamma_{n-1}}$. So Z_i and $\lambda_i x Y_{n-1,i-1}$ represent the same element of $E_\infty^{2n+1,2(i-1)}({}^{\backprime}\Gamma_{n-1})$. The first part follows from Lemma 8 and Proposition 2.

Finally, by setting $Y_i = \mathrm{Res}^{\Gamma_n}_{\mathbb{M}_n} \mathrm{tr}^{\Gamma'_{n-1}}_{\Gamma_n}(Y_{n-1,i})$, we have

$$\begin{aligned} Y_i &= \mathrm{tr}^{\mathbb{M}'_{n-1}}_{\mathbb{M}_n} \mathrm{Res}^{\Gamma'_{n-1}}_{\mathbb{M}'_{n-1}}(Y_{n-1,i}) \\ &= \mathrm{tr}^{\mathbb{M}'_{n-1}}_{\mathbb{M}_n}(-x_2 X_{n-1,i}) \quad \text{by Lemma 7 (i)} \\ &= -x_2 \mathrm{tr}^{\mathbb{M}'_{n-1}}_{\mathbb{M}_n}(X_{n-1,i}). \end{aligned}$$

As shown above, $\mathrm{tr}^{\mathbb{M}'_{n-1}}_{\mathbb{M}_n}(X_{n-1,i}) = \lambda_i X_{n,i-1} \bmod \mathrm{Im}\,\mathrm{Inf}^{W}_{\mathbb{M}_n}$. So Y_i and $-\lambda_i x_2 X_{n,i-1}$ represent the same element of $E_\infty^{2n,2(i-1)}(\mathbb{M}_n)$.The last part follows from Lemma 8 and Theorem 1 (ii). The lemma is proved. ∎

We now calculate $\mathrm{tr}^{\Gamma'_{n-1}}_{\Gamma_n}(Y_{n-1,1})$. In so doing, let us recall the determination of the transfer map on bar cochain levels. Let L, K be subgroups of Γ_n with $Z \subset L \subset K$ and let $D = \{d\}$ be the set of cosets of L in K. For each d, specify a representative $\overline{d}$ of d such that $\overline{L} = 1$ and $\overline{d}\,\overline{d'}\,\overline{dd'}^{\,-1} \in Z$. The transfer map $\tilde{\mathrm{tr}}^L_K : C^*(L) \to C^*(K)$ is determined in [20] as follows:

$$\tilde{\mathrm{tr}}^L_K f(\cdot_L) = \sum_{d\in D} f(\cdot_K),$$
$$\tilde{\mathrm{tr}}^L_K f(\ell_1, \dots, \ell_n) = \sum_{d\in D} f(\overline{d}\ell_1 \overline{d\ell_1}^{\,-1}, \dots, \overline{d\ell_1 \dots \ell_{n-1}}\ell_n \overline{d\ell_1 \dots \ell_{n-1}\ell_n}^{\,-1})$$

for $f \in C^*(L), \ell_i \in K$.

Some properties of $\tilde{\mathrm{tr}}^L_K$ were given in [14]. Note that, if L is a direct factor of K, then $\tilde{\mathrm{tr}}^L_K$ is the zero map. Furthermore, if M is also a subgroup of Γ_n containing Z, we can choose representatives of the cosets of M in KM, and those of $K \cap M$ in K, so that the double coset formula

$$\mathrm{Res}^{KM}_K \tilde{\mathrm{tr}}^M_{KM} = \tilde{\mathrm{tr}}^{K\cap M}_K \mathrm{Res}^M_{K\cap M} \tag{5}$$

holds at the cochain level.

Since $v \in E_2(\Gamma'_{n-1})$ is transgressive, there exists a 2-cochain $\tilde{v}$ of Γ'_{n-1} satisfying $\tilde{v}|_Z = v, \delta\tilde{v} = \beta z_{n-1}$ (see *e.g.* [15] for a determination of such a cochain). It follows from [14, Lemma 1.4] that $\tilde{\mathrm{tr}}^{\Gamma'_{n-1}}_{\Gamma_n}(\beta z_{n-1}) = 0$, hence $\delta\tilde{\mathrm{tr}}^{\Gamma'_{n-1}}_{\Gamma_n}(\tilde{v}) = \tilde{\mathrm{tr}}^{\Gamma'_{n-1}}_{\Gamma_n}(\delta\tilde{v}) = 0$; in other words, $\tilde{\mathrm{tr}}^{\Gamma'_{n-1}}_{\Gamma_n}(\tilde{v})$ is a 2-cocycle of Γ_n. Set $\overline{v} = [\tilde{\mathrm{tr}}^{\Gamma'_{n-1}}_{\Gamma_n}(\tilde{v})] \in H^2(\Gamma_n)$ and let $\tilde{e}, \tilde{a}_1, \dots, \tilde{a}_{2n}$ be elements of Γ_n satisfying $\tilde{e}Z = e, \tilde{a}_i Z = a_i$ (recall that $e, a_1, \dots, a_{2n}$ was defined in Section 2 as a basis of V of which the dual is $x, x_1, \dots, x_{2n}$). We have

Lemma 12. $\overline{v} = -x_{2n-1}x_{2n}$.

Proof. Write

$$\overline{v} = \sum_{1\le i\le 2n} \mu_i x x_i + \sum_{1\le i<j\le 2n} \mu_{ij} x_i x_j + \sum_{1\le i\le 2n} \nu_i y_i$$

with $\mu_i, \mu_{ij}, \nu_i \in \mathbb{F}_p$ (note that, in $H^2(\Gamma_n), y = -(x_1x_2 + \dots + x_{2n-1}x_{2n})$). Consider the double coset formula (5) with $M = \Gamma'_{n-1}$ and $KM = \Gamma_n$ (this means that

$\tilde{a}_{2n} \in K$). For $K = \langle \tilde{e}, \tilde{a}_i, \tilde{a}_j, \tilde{a}_{2n}\rangle$ with $1 \leq i, j \leq 2n-2$, as $\tilde{a}_{2n}$ commutes with every element of $K \cap M$, we have $\tilde{\mathrm{tr}}_K^{K\cap M} = 0$, so $\mu_i = \mu_{2n} = \nu_i = \nu_{2n} = \mu_{ij} = \mu_{i\,2n} = 0$. For $K = \langle \tilde{e}, \tilde{a}_i, a_{2n-1}, \tilde{a}_{2n}\rangle$ with $1 \leq i \leq 2n-2$, we have $K \cong \Gamma_1 \times C_p, K \cap M = C_{p^2} \times C_p^2$ and $\mathrm{Res}^M_{K\cap M}(\tilde{v}) = w$; by a direct verification, we can show that $\mathrm{tr}_K^{K\cap M}(w) = y$, therefore $[\tilde{\mathrm{tr}}_K^{K\cap M}(w)] = y = -x_{2n-1}x_{2n}$, so $\mu_{2n-1} = \mu_{i\,2n-1} = \nu_{2n-1} = 0$ and $\mu_{2n-1\,2n} = -1$. The lemma follows. □

Lemma 13. *For $n \geq 1$, $\mathrm{tr}_{\Gamma_n}^{\Gamma'_{n-1}}(Y_{n-1,1}) = -X_n$; hence $\mathrm{Res}^{\Gamma_n}_{\mathbb{E}_n}\mathrm{tr}_{\Gamma_n}^{\Gamma'_{n-1}}(Y_{n-1,1}) = 0$.*

Proof. A cocycle representing $Y_{n-1,1}$ can be chosen as follows. Since $x_1x_2\dots x_{2n-3}\cdot x_{2n-2}\cdot\beta z_{n-1} = 0$ in $H^*(\Gamma'_{n-1}/Z)$, there exists a cochain f of Γ'_{n-1}/Z (considered as a cochain of Γ'_{n-1} via the inflation map on cochains) satisfying $\delta f = x_1x_2\dots x_{2n-3}\cdot x_{2n-2}\cdot\beta z_{n-1}$. Furthermore, it follows from the definition of $\tilde{v}$ that

$$\delta(x_1x_2\dots x_{2n-3}x_{2n-2}\cdot\tilde{v}) = x_1x_2\dots x_{2n-3}x_{2n-2}\cdot\beta z_{n-1};$$

hence $\delta(x_1x_2\dots x_{2n-3}x_{2n-2}\cdot\tilde{v} - f) = 0$. Clearly $g = x_1x_2\dots x_{2n-3}x_{2n-2}\cdot\tilde{v} - f$ is a cocycle representing $X_n \otimes v \in E_\infty(\Gamma'_{n-1})$. Hence

$$Y_{n-1,1} - [g] \in \mathrm{Im}\ \mathrm{Inf}_{\Gamma_{n-1}}^{\Gamma'_{n-1}/Z},$$

which implies that $\mathrm{tr}_{\Gamma_n}^{\Gamma'_{n-1}}(Y_{n-1,1})$ is represented by $\tilde{\mathrm{tr}}_{\Gamma_n}^{\Gamma'_{n-1}}(g)$. By [14, Lemma 1.4], $\tilde{\mathrm{tr}}_{\Gamma_n}^{\Gamma'_{n-1}}(g) = x_1x_2\dots x_{2n-3}x_{2n-2}\cdot\tilde{\mathrm{tr}}_{\Gamma_n}^{\Gamma'_{n-1}}(\tilde{v})$. So $[\tilde{\mathrm{tr}}_{\Gamma_n}^{\Gamma'_{n-1}}(g)] = x_1x_2\dots x_{2n-3}x_{2n-2}\cdot\overline{v}$. The lemma now follows from Lemma 12. □

Arguing as in the above proof, we can also choose a cocycle representing $X_{n-1}\otimes v^p$ (which is non-zero in $E_\infty(\Gamma'_{n-1})$, by Theorem 1, Propositions 5 and 7), as follows. As $v^p \in E_2(\Gamma'_{n-1})$ is transgressive and $d_{2p+1}(v^p) = \mathcal{P}^1\beta z_{n-1}$, there exists a cochain $\widetilde{v^p}$ of Γ'_{n-1} such that $\widetilde{v^p}|_Z = v^p$, and $\delta\widetilde{v^p} = \mathcal{P}^1\beta z_{n-1}$. Let h be a cochain of Γ'_{n-1}/Z satisfying $\delta h = \mathcal{P}^1\beta z_{n-1}\cdot X_{n-1}$. We have

Lemma 14. *$k = k_n = \widetilde{v^p}\cdot x_1x_2\dots x_{2n-3}x_{2n-2} - h$ is a cocycle representing $X_{n-1}\otimes v^p$ and $\mathrm{Res}^{\Gamma_n}_{\mathbb{E}_n}\mathrm{tr}_{\Gamma_n}^{\Gamma'_{n-1}}([k]) = 0$.*

Proof. It follows from the definitions of $\widetilde{v^p}$ and h that k is a cocycle representing $X_{n-1}\otimes v^p$. Set $X = \mathrm{Res}^{\Gamma_n}_{\mathbb{E}_n}\mathrm{tr}_{\Gamma_n}^{\Gamma'_{n-1}}([k])$; then $X = [\tilde{\mathrm{tr}}_{\mathbb{E}_n}^{\mathbb{E}_{n-1}\times C_p}\mathrm{Res}^{\Gamma'_{n-1}}_{\mathbb{E}_{n-1}\times C_p}(k)]$ by the double coset formula. Denote also by $\widetilde{v^p}$ (resp. h) the restriction of the cochain $\widetilde{v^p}$ (resp. h) to $\mathbb{E}_{n-1}\times C_p$. By [14, Lemma 1.4], $\tilde{\mathrm{tr}}_{\mathbb{E}_n}^{\mathbb{E}_{n-1}\times C_p}(h) = 0$; hence $X = [\tilde{\mathrm{tr}}_{\mathbb{E}_n}^{\mathbb{E}_{n-1}\times C_p}(\widetilde{v^p}\cdot x_1x_2\dots x_{2n-3}x_{2n-2})]$. Note that, in $H^*(\mathbb{E}_{n-1}\times C_p)$ we have $X_{n-1} = X_{n-2}(x_1x_2 + \cdots + x_{2n-3}x_{2n-2})$ and $x_1x_2 + \cdots + x_{2n-3}x_{2n-2} = 0$, so there exist cochains c of $(\mathbb{E}_{n-1}\times C_p)/Z$ and b of $\mathbb{E}_{n-1}\times C_p$ satisfying

$$\delta b = x_1x_2 + \cdots + x_{2n-3}x_{2n-2},$$
$$x_1x_2\dots x_{2n-3}x_{2n-2} = x_1x_2\dots x_{2n-5}x_{2n-4}\cdot\delta b + \delta c.$$

Hence

$$\begin{aligned}\widetilde{v^p}\cdot x_1x_2\dots x_{2n-3}x_{2n-2} &= \widetilde{v^p}\cdot x_1x_2\dots x_{2n-5}x_{2n-4}\cdot\delta b + \widetilde{v^p}\cdot\delta c\\ &= -\delta\widetilde{v^p}\cdot x_1x_2\dots x_{2n-5}x_{2n-4}\cdot b - \delta\widetilde{v^p}\cdot c \bmod \mathrm{Im}\ \delta.\end{aligned}$$

So $X = -[\tilde{\mathrm{tr}}_{\mathbb{E}_n}^{\mathbb{E}_{n-1}\times C_p}(\delta\widetilde{v^p}\cdot x_1x_2\dots x_{2n-5}x_{2n-4}\cdot b+\delta\widetilde{v^p}\cdot c)]$. Following [14, Lemma 1.4], $\tilde{\mathrm{tr}}_{\mathbb{E}_n}^{\mathbb{E}_{n-1}\times C_p}(\delta\widetilde{v^p}\cdot c)=0$ and $\tilde{\mathrm{tr}}_{\mathbb{E}_n}^{\mathbb{E}_{n-1}\times C_p}(\delta b)=0$. This implies that $\tilde{\mathrm{tr}}_{\mathbb{E}_n}^{\mathbb{E}_{n-1}\times C_p}(b)$ is a cocycle of $\mathbb{E}_n$ and

$$X=-\mathcal{P}^1\beta z_{n-1}\cdot X_{n-2}\cdot[\tilde{\mathrm{tr}}_{\mathbb{E}_n}^{\mathbb{E}_{n-1}\times C_p}(b)].$$

Arguing as in the proof of Lemma 12, we can show that $[\tilde{\mathrm{tr}}_{\mathbb{E}_n}^{\mathbb{E}_{n-1}\times C_p}(b)]=0$. Hence $X=0$. The lemma follows. □

With some abuse of notation, we also denote by $\tilde{v}$ (resp. $\widetilde{v^p}$) the restriction of $\tilde{v}$ (resp. $\widetilde{v^p}$) to $\mathbb{M}'_{n-1}$. So $\delta(\tilde{v})=\beta z'_{n-1}$ and $\delta(\widetilde{v^p})=\mathcal{P}^1\beta z'_{n-1}$ in $C^*(\mathbb{M}'_{n-1})$. Let $\tilde{u}$ be a 1-cochain of $\mathbb{M}'_{n-1}$ satisfying $\delta(\tilde{u})=z'_{n-1}$. It follows from the proof of Proposition 8 that there exists a cochain d of $\mathbb{M}'_{n-1}/Z$ such that

$$\begin{aligned}\delta d&=x_3x_4\dots x_{2n-3}x_{2n-2}(\mathcal{P}^1\beta z'_{n-1}-x_2z'^p_{n-1}-y_2^{p-1}\beta z'_{n-1}+y_2^{p-1}x_2z'_{n-1})\\&=\delta(x_3x_4\dots x_{2n-3}x_{2n-2}(\widetilde{v^p}+x_2z'^{p-1}_{n-1}\tilde{u}-y_2^{p-1}\tilde{v}-y_2^{p-1}x_2\tilde{u})).\end{aligned}$$

So, for $n\ge 3$, $q=x_3x_4\dots x_{2n-3}x_{2n-2}(\widetilde{v^p}+x_2z'^{p-1}_{n-1}\tilde{u}-y_2^{p-1}\tilde{v}-y_2^{p-1}x_2\tilde{u})-d$ is a cocycle of $\mathbb{M}'_{n-1}$ representing $Z_{n-1,p-1}$. We have

Lemma 15. *For $n\ge 3$,*

$$\mathrm{tr}_{\mathbb{M}_n}^{\mathbb{M}'_{n-1}}(Z_{n-1,p-1})\in \mathrm{Im}\mathrm{Inf}_{\mathbb{M}_n}^W .$$

Proof. It follows that $\mathrm{tr}_{\mathbb{M}_n}^{\mathbb{M}'_{n-1}}(Z_{n-1,p-1})=[\tilde{\mathrm{tr}}_{\mathbb{M}_n}^{\mathbb{M}'_{n-1}}(q)]$. By [14, Lemma 1.4],

$$\begin{aligned}\mathrm{tr}_{\mathbb{M}_n}^{\mathbb{M}'_{n-1}}(Z_{n-1,p-1})=x_3x_4\dots x_{2n-3}x_{2n-2}([\tilde{\mathrm{tr}}_{\mathbb{M}_n}^{\mathbb{M}'_{n-1}}(\widetilde{v^p})]+x_2z'^{p-1}_{n-1}[\tilde{\mathrm{tr}}_{\mathbb{M}_n}^{\mathbb{M}'_{n-1}}(\tilde{u})]\\-y_2^{p-1}[\tilde{\mathrm{tr}}_{\mathbb{M}_n}^{\mathbb{M}'_{n-1}}(\tilde{v})]-y_2^{p-1}x_2[\tilde{\mathrm{tr}}_{\mathbb{M}_n}^{\mathbb{M}'_{n-1}}(\tilde{u})])\end{aligned}$$

(note that $\tilde{\mathrm{tr}}_{\mathbb{M}_n}^{\mathbb{M}'_{n-1}}$ maps each of $\widetilde{v^p},\tilde{u},\tilde{v}$ to a cocycle). Since each of $\tilde{\mathrm{tr}}_{\mathbb{M}_n}^{\mathbb{M}'_{n-1}}(\widetilde{v^p})$, $\tilde{\mathrm{tr}}_{\mathbb{M}_n}^{\mathbb{M}'_{n-1}}(\tilde{v})$, $\tilde{\mathrm{tr}}_{\mathbb{M}_n}^{\mathbb{M}'_{n-1}}(\tilde{u})$ is of degree $\le 2p$, it follows from the structure of $E_{2p+1}(\mathbb{M}_n)$ that $[\tilde{\mathrm{tr}}_{\mathbb{M}_n}^{\mathbb{M}'_{n-1}}(\tilde{v})],[\tilde{\mathrm{tr}}_{\mathbb{M}_n}^{\mathbb{M}'_{n-1}}(\tilde{u})]$ and the cup-product of $x_3x_4\dots x_{2n-3}x_{2n-2}$ with $[\tilde{\mathrm{tr}}_{\mathbb{M}_n}^{\mathbb{M}'_{n-1}}(\widetilde{v^p})]$ belong to Im $\mathrm{Inf}_{\mathbb{M}_n}^W$. The lemma follows. □

Lemma 16. *There exists a non-zero $\lambda\in\mathbb{F}_p$ such that $\mathrm{tr}_{\Gamma_2}^{\Gamma'_1}(Y_{1,p-1})-\lambda Y_{2,p-2}\in$ Im $\mathrm{Inf}_{\Gamma_2}^V$.*

Proof. Set $K=\mathrm{Ker}\,x_2\cap\mathrm{Ker}(x-x_4)\subset\Gamma_2$ and $X=\mathrm{tr}_{\Gamma_2}^{\Gamma'_1}(Y_{1,p-1})$. So $K\cong\mathbb{M}\times C_p, K\cap\Gamma'_1\cong C_p^3$ and $\mathrm{Res}_K^{\Gamma_2}(X)=\mathrm{tr}_K^{\Gamma'_1\cap K}\mathrm{Res}_{\Gamma'_1\cap K}^{\Gamma'_1}(Y_{1,p-1})$. As $\mathrm{Res}_{\Gamma'_1\cap K}^{\Gamma'_1}(Y_{1,p-1})=v^p-vy_1^{p-1}$, we have

$$\mathrm{Res}_K^{\Gamma_2}(X)=\mathrm{tr}_K^{\Gamma'_1\cap K}(v^p-vy_1^{p-1})=\mathrm{tr}_K^{\Gamma'_1\cap K}(v^p)-\mathrm{tr}_K^{\Gamma'_1\cap K}(vy_1^{p-1}).$$

A direct verification shows that $\mathrm{tr}_K^{\Gamma'_1\cap K}(v)=y_4$, so $\mathrm{Res}_K^{\Gamma_2}(X)=-y_4y_1^{p-1}\neq 0$. Hence $X\neq 0$.

Suppose that $X\in$ Im $\mathrm{Inf}_{\Gamma_2}^V$. Since $y_4X=0$, y_4X must belong to (z,η_1,η_2,ξ_1). Write

$$y_4X=az+b\eta_1+c\eta_2+\mu\xi_1 \tag{6}$$

with $a, b, c \in H^*(V)$ and $\mu \in \mathbb{F}_p$. Multiplying (6) by $x_1x_2x_3x_4$ yields $\mu\xi_1 \in (y, y_4)$. Hence $\mu = 0$. Multiplying (6) by η_2 yields $y_4X\eta_2 \in (z, \eta_1)$. So, by [13, Lemma 2.4], $X\eta_2 \in (z, \eta_1, X_2)$. Since $X\eta_2$ is of degree > 4, it follows that $X\eta_2 \in (z, \eta_1)$. By [13, Lemma 2.14], $X = ey \bmod (z, \eta_1)$ with $e \in H^{2p-2}(V)$. Write

$$eyy_4 = a_1z + b_1\eta_1 + c_1\eta_2. \tag{7}$$

Multiplying (7) by $\eta_1\eta_2$ yields

$$\begin{aligned} eyy_4\eta_1\eta_2 &= a_1z\eta_1\eta_2 \\ &= a_1y\eta_1\eta_2 - a_1X_2\xi_1. \end{aligned}$$

So $a_1 \in (y, x_1, \ldots, x_4)$. Therefore $b_1 \in (y, x_ix_j)$ and $c_1 = 0$. By [13, Lemma 2.4], we have $ey \in (z, \eta_1, X_2)$. Since ey is of degree > 4, it follows that $ey \in (z, \eta_1)$. So $X \in (z, \eta_1)$, and hence $X = 0$ in $H^*(\Gamma_2)$, a contradiction. The lemma follows. □

Lemma 17. *For $n \geq 3$,*

$$\mathrm{tr}_{\mathbb{M}_n}^{\mathbb{M}'_{n-1}}(X_{n-1,p-1}) = \lambda X_{n,p-2} \bmod \mathrm{Im}\ \mathrm{Inf}_{\mathbb{M}_n}^{W}$$

and

$$\mathrm{tr}_{\Gamma_n}^{\Gamma'_{n-1}}(Y_{n-1,p-1}) = \lambda Y_{n,p-2} \bmod \mathrm{Im}\ \mathrm{Inf}_{\Gamma_n}^{V},$$

with λ given in Lemma 16.

Proof. Consider the case $n = 3$. Set $X = \mathrm{Res}_{{}^{\backprime}\Gamma_2}^{\mathbb{M}_3}\mathrm{tr}_{\mathbb{M}_3}^{\mathbb{M}'_2}(X_{2,p-1})$. By the double coset formula, we have

$$\begin{aligned} X &= \mathrm{tr}_{{}^{\backprime}\Gamma_2}^{{}^{\backprime}\Gamma'_1}\mathrm{Res}_{{}^{\backprime}\Gamma'_1}^{\mathbb{M}'_2}(X_{2,p-1}) \\ &= \mathrm{tr}_{{}^{\backprime}\Gamma_2}^{{}^{\backprime}\Gamma'_1}(xY_{1,p-1}) \quad \text{by Lemma 7 (ii)} \\ &= x\mathrm{tr}_{{}^{\backprime}\Gamma_2}^{{}^{\backprime}\Gamma'_1}(Y_{1,p-1}). \end{aligned}$$

It follows from Lemma 16 that X and $\lambda xY_{2,p-2}$ represent the same element of $E_\infty^{5,2(p-2)}({}^{\backprime}\Gamma_2)$. By Lemma 8 and Proposition 2, it follows that $\mathrm{tr}_{\mathbb{M}_3}^{\mathbb{M}'_2}(X_{2,p-1}) = \lambda X_{3,p-2} \bmod \mathrm{Im}\ \mathrm{Inf}_{\mathbb{M}_3}^{W}$. Similarly, by setting $Y = \mathrm{Res}_{\mathbb{M}_3}^{\Gamma_3}\mathrm{tr}_{\Gamma_3}^{\Gamma'_2}(Y_{2,p-1})$, we have

$$\begin{aligned} Y &= \mathrm{tr}_{\mathbb{M}_3}^{\mathbb{M}'_2}\mathrm{Res}_{\mathbb{M}'_2}^{\Gamma'_2}(Y_{2,p-1}) \\ &= \mathrm{tr}_{\mathbb{M}_3}^{\mathbb{M}'_2}(-x_2X_{2,p-1} + Z_{2,p-1}) \quad \text{by Lemma 7 (i)} \\ &= -x_2\mathrm{tr}_{\mathbb{M}_3}^{\mathbb{M}'_2}(X_{2,p-1}) + \mathrm{tr}_{\mathbb{M}_3}^{\mathbb{M}'_2}(Z_{2,p-1}). \end{aligned}$$

As shown above, $\mathrm{tr}_{\mathbb{M}_3}^{\mathbb{M}'_2}(X_{2,p-1}) = \lambda X_{3,p-2} \bmod \mathrm{Im}\ \mathrm{Inf}_{\mathbb{M}_3}^{W}$. So, by Lemma 15, Y and $-\lambda x_2X_{3,p-2}$ represent the same element of $E_\infty^{6,2(p-2)}(\mathbb{M}_3)$. By Lemma 8 and Proposition 2, it follows that $\mathrm{tr}_{\Gamma_3}^{\Gamma'_2}(Y_{2,p-1}) = \lambda Y_{3,p-2} \bmod \mathrm{Im}\ \mathrm{Inf}_{\Gamma_3}^{V}$.

Assume that the lemma holds for $n-1$. Set $Z = \mathrm{Res}_{{}^{\backprime}\Gamma_{n-1}}^{\mathbb{M}_n}\mathrm{tr}_{\mathbb{M}_n}^{\mathbb{M}'_{n-1}}(X_{n-1,p-1})$. By the double coset formula, we have

$$\begin{aligned} Z &= \mathrm{tr}_{{}^{\backprime}\Gamma_{n-1}}^{{}^{\backprime}\Gamma'_{n-2}}\mathrm{Res}_{{}^{\backprime}\Gamma'_{n-2}}^{\mathbb{M}'_{n-1}}(X_{n-1,p-1}) \\ &= \mathrm{tr}_{{}^{\backprime}\Gamma_{n-1}}^{{}^{\backprime}\Gamma'_{n-2}}(xY_{n-2,p-1}) \quad \text{by Lemma 7 (ii)} \\ &= x\mathrm{tr}_{{}^{\backprime}\Gamma_{n-1}}^{{}^{\backprime}\Gamma'_{n-2}}(Y_{n-2,p-1}). \end{aligned}$$

By the inductive hypothesis, $\mathrm{tr}^{'\Gamma_{n-2}}_{'\Gamma_{n-1}}(Y_{n-2,p-1}) = \lambda Y_{n-1,p-2}$ mod Im $\mathrm{Inf}^{'\Gamma_{n-1}/Z}_{'\Gamma_{n-1}}$. So Z and $\lambda x Y_{n-1,p-2}$ represent the same element of $E_\infty^{2n-1,2(p-2)}('\Gamma_{n-1})$. The first part follows from Lemma 8 and Proposition 2.

Finally, by setting $Y = \mathrm{Res}^{\Gamma_n}_{\mathbb{M}_n}\mathrm{tr}^{\Gamma'_{n-1}}_{\Gamma_n}(Y_{n-1,p-1})$, we have

$$\begin{aligned} Y &= \mathrm{tr}^{\mathbb{M}'_{n-1}}_{\mathbb{M}_n}\mathrm{Res}^{\Gamma'_{n-1}}_{\mathbb{M}'_{n-1}}(Y_{n-1,p-1}) \\ &= \mathrm{tr}^{\mathbb{M}'_{n-1}}_{\mathbb{M}_n}(-x_2 X_{n-1,p-1}) + \mathrm{tr}^{\mathbb{M}'_{n-1}}_{\mathbb{M}_n}(Z_{n-1,p-1}) \quad \text{by Lemma 7 (i)} \\ &= -x_2\mathrm{tr}^{\mathbb{M}'_{n-1}}_{\mathbb{M}_n}(X_{n-1,p-1}) + \mathrm{tr}^{\mathbb{M}'_{n-1}}_{\mathbb{M}_n}(Z_{n-1,p-1}). \end{aligned}$$

As shown above, $\mathrm{tr}^{\mathbb{M}'_{n-1}}_{\mathbb{M}_n}(X_{n-1,p-1}) = \lambda X_{n,p-2}$mod Im $\mathrm{Inf}^{W}_{\mathbb{M}_n}$. So, by Lemma 15, Y and $-\lambda x_2 X_{n,p-2}$ represent the same element of $E_\infty^{2n,2(p-2)}(\mathbb{M}_n)$. The last part follows from Lemma 8 and Theorem 1 (ii). The lemma is proved. □

Let

$$\cdots \supset F^iC^*(\Gamma'_{n-1}) \supset F^{i+1}C^*(\Gamma'_{n-1}) \supset \ldots$$

be the filtration of $C^*(\Gamma'_{n-1})$ introduced by Hochschild and Serre ([7]) corresponding to the central extension (Γ'_{n-1}). Let us recall that

$$F^iC^*(\Gamma'_{n-1}) = \begin{cases} C^*(\Gamma'_{n-1}) & \text{for } i \le 0, \\ \sum_{m=0}^{\infty} F^iC^m(\Gamma'_{n-1}) & \text{for } i > 0, \end{cases}$$

where $F^iC^m(\Gamma'_{n-1}) = 0$ for $i > m$; and for $0 < i \le m$, $F^iC^m(\Gamma'_{n-1})$ is the group of all m-cochains f for which $f(g_1, \ldots, g_m) = 0$ whenever $m-i+1$ of the arguments g_k belong to Z. It is clear that the conjugation by $a = a_{2n}$ on $C^*(\Gamma'_{n-1})$ is compatible with the Hochschild-Serre filtration. We then have the induced conjugation on the Hochschild-Serre spectral sequence $\{E_r(\Gamma'_{n-1})\}$. As the action of a on $E_2^{*,*}(\Gamma'_{n-1})$ satisfies ${}^a x_k = x_k, 1 \le k \le 2n-1$, and ${}^a v = v + y_{2n-1}$, it follows from the structure of $E_{2p+1}(\Gamma'_{n-1})$ that $Y_{n-1,i}$ and ${}^aY_{n-1,i}$ represent the same element of $E_\infty(\Gamma'_{n-1})$. Hence

$$(8) \qquad Y_{n-1,i} -^a Y_{n-1,i} = \sum_{0<j<i} \mu_j Y_{n-1,j} y_{2n-1}^{i-j} + \sum_{0<j<i} \nu_j Y_{n-1,j} y_{2n-1}^{i-j-1} x x_{2n-1} \quad \text{mod Im } \mathrm{Inf}^{\Gamma'_{n-1}/Z}_{\Gamma'_{n-1}},$$

with $\mu_j, \nu_j \in \mathbb{F}_p$. We have

Lemma 18. *For $n \ge 2$ we have $Y_{n-1,1} -^a Y_{n-1,1} = 0$.*

Proof. Set $K = \mathrm{Ker}\, x_{2n-2} \cap \Gamma'_{n-1}$. Since the transfer commutes with the conjugation and Im $\mathrm{Inf}^{\Gamma'_{n-1}/Z}_{\Gamma_{n-1}}$ is invariant under the action of a, by Lemmas 9, 10, 11, 16 and 17, we have

$$Y_{n-1,1} -^a Y_{n-1,1} = \mathrm{tr}^K_{\Gamma'_{n-1}}(Y_{n-2,2} -^a Y_{n-2,2})$$

up to a non-zero constant multiple. By Lemma 13, $\mathrm{tr}^K_{\Gamma'_{n-1}}(Y_{n-2,1}) = -X_{n-1}$; hence $\mathrm{tr}^K_{\Gamma'_{n-1}}(Y_{n-2,1}y_{2n-3}) = 0$ and $\mathrm{tr}^K_{\Gamma'_{n-1}}(Y_{n-2,1}x_{2n-3}) = 0$ in $H^*(\Gamma'_{n-1})$. The lemma follows from (8) and from the fact that $\mathrm{tr}^K_{\Gamma'_{n-1}}\mathrm{Inf}^{K/Z}_K = 0$. □

We now have

Lemma 19. *For $n \geq 2$ and $1 \leq i \leq p-1$,*

$$Y_{n-1,i} +^{a} Y_{n-1,i} + \cdots +^{a^{p-1}} Y_{n-1,i} = 0;$$

hence

$$\mathrm{Res}^{\Gamma_n}_{\Gamma'_{n-1}} \mathrm{tr}^{\Gamma'_{n-1}}_{\Gamma_n}(Y_{n-1,i}) = 0.$$

Proof. Since $1 + a + \cdots + a^{p-1} = (1-a)^{p-1}$, we need prove that ${}^{(1-a)^{p-1}}Y_{n-1,i} = 0$. For $1 \leq k \leq p-1$, by (8) and by Lemma 18, ${}^{(1-a)^k}Y_{n-1,k} = 0$. Since $\mathrm{Res}^{\Gamma_n}_{\Gamma'_{n-1}} \mathrm{tr}^{\Gamma'_{n-1}}_{\Gamma_n}(Y_{n-1,i}) = ^{(1-a)^{p-1}} Y_{n-1,i}$, the lemma follows. □

For $n \geq 2$ and $1 \leq i \leq p-2$, set $\kappa_{n,i} = \mathrm{Res}^{\Gamma_n}_{\mathbb{E}_n} \mathrm{tr}^{\Gamma'_{n-1}}_{\Gamma_n}(Y_{n-1,i+1})$. The proof of the theorem is completed by the following fact.

Proposition 9. $0 \neq \kappa_{n,i} \in Ess(\mathbb{E}_n)$ *with* $1 \leq i < p-2$ *for* $p > 3$, *and* $i = 1$ *for* $p = 3$.

Proof. It follows from Proposition 1, Lemmas 7 (iv), 9, 11 and 17 that $\kappa_{n,i} \neq 0$ in $H^*(\mathbb{E}_n)$. Let K be a maximal subgroup of $\mathbb{E}_n$. K is then of the form $\mathbb{E}_{n-1} \times C_p$. Let L be the central product of K and $C_{p^2} = \bigcap_{j=1}^{2n} \mathrm{Ker}\, x_j$. It follows that L is a subgroup of Γ_n containing K and $L \cong \Gamma_{n-1} \times C_p$. Therefore

$$\begin{aligned}\mathrm{Res}^{\mathbb{E}_n}_K(\kappa_{n,i}) &= \mathrm{Res}^{\Gamma_n}_K \mathrm{tr}^{\Gamma'_{n-1}}_{\Gamma_n}(Y_{n-1,i+1}) \\ &= \mathrm{Res}^L_K \mathrm{Res}^{\Gamma_n}_L \mathrm{tr}^{\Gamma'_{n-1}}_{\Gamma_n}(Y_{n-1,i+1}).\end{aligned}$$

Hence, if $\Gamma_n = \Gamma'_{n-1}L$, it follows from the double coset formula that

$$\mathrm{Res}^{\mathbb{E}_n}_K(\kappa_{n,i}) = \mathrm{Res}^L_K \mathrm{tr}^{L\cap\Gamma'_{n-1}}_L \mathrm{Res}^{\Gamma'_{n-1}}_{L\cap\Gamma'_{n-1}}(Y_{n-1,i+1}).$$

By Lemma 1, there exists a non-zero linear combination α of $x_1, \ldots, x_{2n}$ such that $L = \mathrm{Ker}\,\alpha$. Consider the following cases:

- $\alpha = x_{2n-1} + \gamma$ with γ a linear combination of $x_1, \ldots, x_{2n-2}, x_{2n}$: it follows that $\Gamma_n = \Gamma'_{n-1}L$ and $L \cap \Gamma'_{n-1} \cong \Gamma_{n-1}$ is a direct factor of L. Hence $\mathrm{tr}^{L\cap\Gamma'_{n-1}}_L$ is the zero map. We have

$$\begin{aligned}\mathrm{Res}^{\mathbb{E}_n}_K(\kappa_{n,i}) &= \mathrm{Res}^L_K \mathrm{tr}^{L\cap\Gamma'_{n-1}}_L \mathrm{Res}^{\Gamma'_{n-1}}_{L\cap\Gamma'_{n-1}}(Y_{n-1,i+1}) \\ &= 0;\end{aligned}$$

- $\alpha = \mu x_{2n} + \gamma$ with γ a non-zero linear combination of $x_1, \ldots, x_{2n-2}$ and $\mu \in \mathbb{F}_p$: it follows that $L \cap \Gamma'_{n-1} = H \times \langle \tilde{a}_{2n-1} \rangle$ for a subgroup H of Γ_{n-1} with $H \cong \Gamma'_{n-2}$. If $p > 3$, it follows from the proof of Lemma 7(iii) that $\mathrm{Res}^{\Gamma'_{n-1}}_{L\cap\Gamma'_{n-1}}(Y_{n-1,i+1})$ belongs to the ideal generated by Im $\mathrm{Inf}^{(L\cap\Gamma'_{n-1})/Z}_{L\cap\Gamma'_{n-1}}$; since

$$\mathrm{Im}\mathrm{Inf}^{(L\cap\Gamma'_{n-1})/Z}_{L\cap\Gamma'_{n-1}} \subset \mathrm{Ker}\mathrm{tr}^{L\cap\Gamma'_{n-1}}_L,$$

it follows that

$$\mathrm{Res}^{\mathbb{E}_n}_K(\kappa_{n,i}) = 0.$$

If $p = 3$, by Lemma 14, there exist $\phi \in H^2(L \cap \Gamma'_{n-1}), \psi \in H^1(L \cap \Gamma'_{n-1})$ such that $\mathrm{Res}^{\Gamma'_{n-1}}_{L\cap\Gamma'_{n-1}}(Y_{n-1,2})$ is a linear combination of

$$[k_{n-1}], \quad Y_{n-2,1} \cdot \phi, \quad Y_{n-2,1} \cdot x\psi$$

and an element of Im $\mathrm{Inf}^{(L\cap\Gamma'_{n-1})/Z}_{L\cap\Gamma'_{n-1}}$; since ϕ, ψ belong to Im $\mathrm{Inf}^{(L\cap\Gamma'_{n-1})/Z}_{L\cap\Gamma'_{n-1}}$, by Lemmas 13 and 14, it follows that

$$\mathrm{Res}^L_K\, \mathrm{tr}_L^{L\cap\Gamma'_{n-1}} \mathrm{Res}^{\Gamma'_{n-1}}_{L\cap\Gamma'_{n-1}}(Y_{n-1,2}) = 0,$$

so $\mathrm{Res}^{\mathbb{E}_n}_K(\kappa_{n,1}) = 0$.

Finally, the case $\alpha = x_{2n}$ follows from Lemma 19. The proposition is proved. □

Acknowledgments

Most of the results of this paper were obtained during a stay at the University of Essen and the ETH-Zentrum in Autumn 1997. I would like to thank Eckart Viehweg, Hélène Esnault and Urs Stammbach for making the visits possible. Many thanks to David John Green for valuable comments.

References

1. A. Adem, D. Karagueuzian, *Essential cohomology of finite groups*, Comment. Math. Helv. **72** (1997), 101-109. MR **98f:**20038
2. D.J. Benson, J.F. Carlson, *The cohomology of extra-special groups*, Bull. London Math. Soc. **24** (1992), 209-235; **25** (1993), 498. MR **93b:**20087; MR **94f:**20099
3. J. Carlson, *Depth and transfer map in the cohomology of groups*, Math. Z. **218** (1995), 461-468. MR **95m:**20058
4. T. Diethelm, *The mod p cohomology rings of the nonabelian split metacyclic p-groups*, Arch. Math. (Basel) **44** (1985), 29-38. MR **86e:**20057
5. J. Duflot, *Depth and equivariant cohomology*, Comment. Math. Helv. **56** (1981), 627-637. MR **83g:**57029
6. D.J. Green, *Private communication.*
7. G. Hochschild and J. P. Serre, *Cohomology of group extensions*, Trans. Amer. Math. Soc. **74** (1953), 110-143. MR **14:**619b
8. I.J. Leary, *The mod p cohomology rings of some p-groups*, Math. Proc. Camb. Phil. Soc. **112** (1992), 63-75. MR **94a:**20087
9. R.J. Milgram, M. Tezuka, *The geometry and cohomology of M_{12}: II*, Bol. Soc. Mat. Mex. **1** (1995), 91-108. MR **97f:**20064
10. P.A. Minh, H. Mui, *The mod p cohomology algebra of the group $M(p^n)$*, Acta Math. Vietnamica **7** (1982), 17-26. MR **85i:**20051
11. P.A. Minh, *Hochschild-Serre spectral sequences, modular invariant theory and cohomology algebras of extra-special p-groups*, Thesis, Univ. of Hanoi (1990).
12. ———, *On the* mod p *cohomology groups of extra-special* p*-groups*, Japan. J. Math. **18** (1992), 139-154. MR **93m:**20074
13. ———, *The mod p cohomology group of extra-special p-group of order p^5 and of exponent p^2*, Math. Proc. Camb. Phil. Soc. **120** (1996), 423-440. MR **97b:**20076
14. ———, *Transfer map and Hochschild-Serre spectral sequences*, J. Pure Appl. Alg. **104** (1995), 89-95. MR **96h:**20099
15. ———, *On a conjecture of Kahn for Stiefel-Whitney classes of a regular representation*, J. Alg. **188** (1997), 590-609. MR **98f:**20039
16. H. Mui, *The mod p cohomology algebra of the group $E(p^3)$*, Unpublished essay.
17. D. Quillen, *The mod 2 cohomology rings of extra-special 2-groups and the spinor groups*, Math. Ann. **194** (1971), 197-212. MR **44:**7582
18. M. Tezuka, N. Yagita, *The varieties of the mod p cohomology rings of extra-special p-groups for an odd prime p*, Math. Proc. Camb. Phil. Soc. **94** (1983), 449-459. MR **85g:**20069

19. ———, *Calculations in mod p cohomology of extra special p-groups I*, Contemp. Math. **158** (1994), 281-306. MR **95b:**20074

20. E. Weiss, *Cohomology of groups*, Academic Press, New York, 1969. MR **41:**8499

Department of Mathematics, College of Sciences, University of Hue, Dai hoc Khoa hoc, Hue, Vietnam

E-mail address: paminh@dng.vnn.vn

Current address: 53 Craig Road, Stockport SK4 2AP, England

E-mail address: minhp@vol.vnn.vn