WANDERING VECTOR MULTIPLIERS FOR UNITARY GROUPS

DEGUANG HAN AND D. LARSON

ABSTRACT. A wandering vector multiplier is a unitary operator which maps
the set of wandering vectors for a unitary system into itself. A special case
of unitary system is a discrete unitary group. We prove that for many (and
perhaps all) discrete unitary groups, the set of wandering vector multipliers is
itself a group. We completely characterize the wandering vector multipliers for
abelian and ICC unitary groups. Some characterizations of special wandering
vector multipliers are obtained for other cases. In particular, there are simple
characterizations for diagonal and permutation wandering vector multipliers.
Similar results remain valid for irrational rotation unitary systems. We also
obtain some results concerning the wandering vector multipliers for those uni-
tary systems which are the ordered products of two unitary groups. There are
applications to wavelet systems.

0. INTRODUCTION

The close relationship between standard wavelet theory and wandering vector
theory for unitary systems has been systematically studied in [DL]. For both theo-
retical and practical reasons, the most interesting unitary systems are those which
are ordered products of two (or more) unitary groups. The study of this kind of
unitary system is necessarily based on the study of the unitary groups. Also, the
case of a "group unitary system" has independent interest, and some open problems
remain. In this paper we will focus on the situation where the underlying Hilbert
space is separable and the unitary system is countable. We will investigate the
structure of the set of wandering vector multipliers for the group case, and for the
ordered product of two groups. The latter includes wavelet systems and irrational
rotation unitary systems. This article represents the final product of a project of
the authors that was begun in the Spring of 1996.

Let H be a separable Hilbert space. A *unitary system* $\mathcal{U}$ is a set of unitary
operators acting on H which contains the identity operator I in $B(H)$. If $\mathcal{G}_1$ and
$\mathcal{G}_2$ are unitary groups, by the *ordered product* of $\mathcal{G}_1$ and $\mathcal{G}_2$, we will mean the unitary
system $\mathcal{U} = \{g_1 g_2 : g_i \in \mathcal{G}_i, i = 1, 2\}$. A *wandering subspace* for $\mathcal{U}$ is a closed linear
subspace M of H with the property that $UM \perp VM$ for any different U and V in
$\mathcal{U}$. If in addition $H = \overline{span}\,\mathcal{U}M$, we call M a *complete wandering subspace* for $\mathcal{U}$.
In the case that $M = \mathbb{C}x$ for some unit vector x, we call x a *(complete) wandering
vector* for $\mathcal{U}$ if M is a (complete) wandering subspace for $\mathcal{U}$. The set of all complete

wandering vectors for $\mathcal{U}$ is denoted by $\mathcal{W}(\mathcal{U})$. If $\mathcal{U} = \{D^n T^m : n, m \in \mathbb{Z}\}$ with $(Tf)(t) = f(t-1)$ and $(Df)(t) = \sqrt{2}f(2t)$ for all $f \in L^2(\mathbb{R})$, then the complete wandering vectors are precisely the standard orthogonal (or orthonormal) wavelets, and more generally the complete wandering subspaces (rather, orthonormal bases for them) correspond to the orthogonal multi-wavelets.

Suppose that $x \in \mathcal{W}(\mathcal{U})$. The *local commutant* $\mathcal{C}_x(\mathcal{U})$ of $\mathcal{U}$ at x, which is defined [DL] to be the set $\{A \in B(H) : (AU - UA)x = 0, U \in \mathcal{U}\}$, provides a good abstract machinery for creating new complete wandering vectors for $\mathcal{U}$. Specifically, for every complete wandering vector y of $\mathcal{U}$, there is a unique unitary operator $V \in \mathcal{C}_x(\mathcal{U})$ such that $y = Vx$ and vice versa (see Proposition 1.3 in [DL]). Another way to attempt to understand the wandering vector theory for a given unitary system $\mathcal{U}$ is to try to characterize all the unitary operators A in $B(H)$ such that $A\mathcal{W}(\mathcal{U}) \subseteq \mathcal{W}(\mathcal{U})$. A unitary operator with this property was called a *wandering vector multiplier* for $\mathcal{U}$ in [DL]. We use $M_{\mathcal{U}}$ to denote the set of all such operators. It is obvious that $M_{\mathcal{U}}$ is always a semi-group. A natural question is : Is $M_{\mathcal{U}}$ a group? This seems hard to answer in general. A "good" characterization of the wandering vector multipliers for a specific unitary system $\mathcal{U}$ should be one which is strong enough to answer this basic test question. We conjecture that the answer is yes if $\mathcal{U}$ is a group. We will show that it is true for many groups. We will also show that it is true for irrational rotation unitary systems. It remains open for the orthogonal dyadic wavelet system, and this problem is one of the main motivations for this project, although there is much that has been accomplished here.

Unitary groups with complete wandering vectors are exactly the left regular representation groups (under unitary equivalence, see the preliminaries section). A simple example is: Let $H = L^2(\mathbb{T})$, where $\mathbb{T}$ is the unit circle with normalized Lebesgue measure. Let $U = M_z$ be the unitary operator corresponding to multiplication by z. Then the constant function 1 is a complete wandering vector for $\mathcal{U} = \{M_z^n : n \in \mathbb{Z}\}$. An elementary argument shows that a function $f \in L^2(\mathbb{Z})$ is a complete wandering vector for $\mathcal{U}$ if and only if f is unimodular; i.e., $|f(z)| = 1$, a.e. $z \in \mathbb{T}$. Therefore the set of all the complete wandering vectors for $\mathcal{U}$ can be viewed as the set of all the unitary elements in the abelian von Neumann algebra $L^\infty(\mathbb{T})$ viewed as vectors in $L^2(\mathbb{T})$. (This idea can be generalized to the arbitrary discrete unitary group case.) Hence in this case the wandering vector multipliers are precisely those unitary operators on $L^2(\mathbb{T})$ which map unimodular functions to unimodular functions. Included are multiplication operators with unimodular symbols, and the composition operators with measure-preserving symbols. Even in this simple case the fact that $M_{\mathcal{U}}$ is a group is perhaps not so obvious (see Example 2.10). It is true because an element of $M_{\mathcal{U}}$ factors as a product of these two types (see Proposition 2.11 for the general abelian case), and the inverse of a composition operator with measure-preserving symbol is also a composition operator with measure preserving symbol. The general (non-abelian) factorization result (see Remark 2.7) replaces the measure-preserving point mapping of the underlying measure space in the abelian case with an appropriate injective Jordan homomorphism of $w^*(\mathcal{G})$ into itself which is isometric in the (noncommutative) Hilbert-Schmidt metric.

For the unitary group case, the local commutant for a unitary group at a fixed complete wandering vector for the group is just the commutant of the group ([DL], Proposition 1.1). This suggests that there is a close relationship between wandering subspace theory and the von Neumann algebras generated by the left and the right regular representations of the group. The main result of this paper is in section 2 in

which we obtain characterizations of wandering vector multipliers for some of the interesting groups including abelian and free groups. The methods also apply to the irrational rotation unitary systems that were studied in [Ha]. The results we get and the techniques involved show that wandering vector multipliers are interesting objects to study from the operator algebra point of view, independent of possible applications.

Given a complete wandering vector for a unitary group, we show that the diagonal associated (resp. permutation) unitary operator (see definitions in section 3) are wandering vector multipliers if and only if they are induced by characters (resp. isomorphisms or anti-isomorphisms) of the group. The same results hold for the irrational rotation unitary systems. In particular, the last question in [Ha] which asks for a characterization of all the wandering vector multipliers for irrational rotation unitary systems is answered. The wandering vector multipliers for the wavelet system $\mathcal{U} = \{D^n T^m : \ n, m \in \mathbb{Z}\}$ are called *wavelet multipliers*, and were first studied in [DL]. There is a special class of wavelet multipliers, namely functional wavelet multipliers, which play an essential role in understanding the phases of wavelets. These have been completely characterized (see [DGLL], [Wut]). This motivated us to try to characterize all the (unitary) wavelet multipliers, which is a problem posed in [DL]. While this general problem remains open, in section 5 we present several partial results on this.

1. Preliminaries

Two unitary systems $\mathcal{U}_1$ and $\mathcal{U}_2$ acting on Hilbert spaces H_1 and H_2, respectively, are said to be *unitarily equivalent* if there is a unitary operator $T : H_1 \to H_2$ such that $T\mathcal{U}_1 T^* = \mathcal{U}_2$. In this case, $T\mathcal{W}(\mathcal{U}_1) = \mathcal{W}(\mathcal{U}_2)$ and $TM_{\mathcal{U}_1} T^* = M_{\mathcal{U}_2}$. Given a group $\mathcal{G}$ and two unitary representations π_1 and π_2, as usual we call π_1 and π_2 *unitarily equivalent* if there is a corresponding unitary operator T satisfying $T\pi_1(g)T^* = \pi_2(g)$ for all $g \in \mathcal{G}$. We say a vector ϕ is a complete wandering vector for a unitary representation π of a group $\mathcal{G}$ if it is a complete wandering vector for the unitary system $\{\pi(g) : \ g \in \mathcal{G}\}$.

Let $\mathcal{G}$ be a discrete group with identity element e. For each $g \in \mathcal{G}$, let t_g denote the characteristic function at $\{g\}$. Then $\{t_g : g \in \mathcal{G}\}$ is an orthonormal basis for $l_2(\mathcal{G})$. Define a linear map l_g (resp. r_g), for each g, by $l_g t_h = t_{gh}$ (resp. $r_g t_h = t_{hg^{-1}}$), then both l_g and r_g are unitary operators on $l_2(\mathcal{G})$. The unitary representation $g \to l_g$ (resp. $g \to r_g$) is called the left (resp. right) regular representation of $\mathcal{G}$. Let H be a Hilbert space. For any subset $\mathcal{S}$ of $B(H)$, we will use $w^*(\mathcal{S})$ to denote the von Neumann algebra generated by $\mathcal{S}$ and use $\mathcal{S}'$ to denote the commutant of $\mathcal{S}$; i.e., $\mathcal{S}' = \{T \in B(H) : TS = ST, S \in \mathcal{S}\}$. The notations $L_\mathcal{G}$ and $R_\mathcal{G}$ are reserved to denote the von Neumann algebras $w^*(l_g : g \in \mathcal{G})$ and $w^*(r_g : g \in \mathcal{G})$, respectively. A group $\mathcal{G}$ is said to be an ICC group if the conjugacy class of each element other than the identity is infinite. It is a well-known fact that $L_\mathcal{G}' = R_\mathcal{G}$ for all groups $\mathcal{G}$, and also that both $L_\mathcal{G}$ and $R_\mathcal{G}$ are II_1 factors when $\mathcal{G}$ is an ICC group (see [KR]).

The following results will be used in section 2.

Proposition 1.1. *Let $\mathcal{G}$ be a unitary group on a Hilbert space H which has a complete wandering subspace M. Then $w^*(\mathcal{G})$ is a finite von Neumann algebra. Moreover, $\mathcal{G}'$ is finite if and only if $\dim M < \infty$.*

Proof. Let $\{x_i\}$ be an orthonormal basis for M. By defining a unitary operator $W : H \rightarrow l_2(\mathcal{G}) \otimes M$ such that

$$W g x_i = l_g \chi_e \otimes x_i, \qquad \forall i, \forall g \in \mathcal{G},$$

we can assume that $\mathcal{G}$ has the form of $\{l_g \otimes I : g \in \mathcal{G}\}$, where I is the identity operator on M. Since the commutant of $\{l_g \otimes I : g \in \mathcal{G}\}$ is $\{l_g : g \in \mathcal{G}\}' \otimes B(M)$, and $\{l_g : g \in \mathcal{G}\}'$ and $w^*(l_g : g \in \mathcal{G})$ are finite von Neumann algebras by [KR], the lemma follows. □

Corollary 1.2. *Let $\mathcal{G}$ be a unitary group which has a complete wandering subspace M. If M has finite dimension n, then every n-dimensional wandering subspace N is complete. In general, if even $dim M = \infty$, every complete wandering subspace for $\mathcal{G}$ has the same dimension as M.*

Proof. Let $\{x_i : i = 1, \ldots, n\}$ and $\{y_i : i = 1, \ldots, n\}$ be orthonormal bases for M and N, respectively. Define $W \in B(H)$ by

$$W g x_i = g y_i, \qquad i = 1, \ldots, n, g \in \mathcal{G}.$$

Then W is an isometry since $\{g x_i : g \in \mathcal{G}, i = 1, \ldots, n\}$ is an orthonormal basis for H and $\{g y_i : g \in \mathcal{G}, i = 1, \ldots, n\}$ is an orthonormal set. For any $g, h \in \mathcal{G}$, we have $W g h x_i = g h y_i = g W h x_i$ for $i = 1, 2, \ldots, n$. Hence $W \in \mathcal{G}'$. By Proposition 1.1, $\mathcal{G}'$ is a finite von Neumann algebra, which implies that W is a unitary operator. So $N = WM$ must be a complete wandering subspace.

For the second part, let $n = dim M \leq \infty$ and let N be a complete wandering subspace of dimension m for $\mathcal{G}$. Suppose that $n \neq m$. We may assume that $m < n$. Then $m < \infty$. Take an m-dimensional subspace K of M. Then K is an m-dimensional wandering subspace, hence it is complete by the first paragraph, which contradicts the completeness of M. Thus $n = m$. □

Let $\hat{\mathcal{G}}$ be the dual group of an abelian group $\mathcal{G}$, i.e., the compact group of all characters of $\mathcal{G}$. Let $\theta : \mathcal{G} \rightarrow \hat{\hat{\mathcal{G}}}$ be the duality correspondence between $\mathcal{G}$ and its second dual $\hat{\hat{\mathcal{G}}}$, i.e., $\theta(g) = \chi_g$ for any $g \in \mathcal{G}$, where $\chi_g(\omega) = \omega(g)$, $\omega \in \hat{\mathcal{G}}$.

The following fact is well-known in spectral theory (cf. Prop. 2.2 in [BKM]) and will be used in section 2.

Lemma 1.3. *Let $\mathcal{G}$ be an abelian unitary group which has a cyclic unit vector f in H. Then H is unitarily equivalent to the complex Hilbert space $L^2(\hat{\mathcal{G}}, \sigma)$, where σ is a Borel probability measure on $\hat{\mathcal{G}}$. More specifically, there is a unitary operator $W : H \rightarrow L^2(\hat{\mathcal{G}}, \sigma)$ such that*
  (i) $W f = 1$, where 1 denotes the function on $\hat{\mathcal{G}}$ identically equal to 1;
  (ii) for ang $g \in \mathcal{G}$, the operator $\pi(g) = W g W^*$ is the multiplication operator (on $L^2(\hat{\mathcal{G}}, \sigma)$) by the character χ_g:

$$\pi(g)\psi(\omega) = \chi_g(\omega)\psi(\omega)$$

for all $\psi \in L^2(\hat{\mathcal{G}}, \sigma)$.

Recall that a Jordan homomorphism from one C*-algebra into another C*-algebra is a linear mapping ϕ which satisfies $\phi(a^*) = (\phi(a))^*$ and $\phi(a^2) = \phi(a)^2$ for all a. The second condition is equivalent to the condition $\phi(ab + ba) = \phi(a)\phi(b) + \phi(b)\phi(a)$ for all a, b.

Lemma 1.4 ([BR], Proposition 3.2.2 or see [KR], 10.5.24)**.** *Let H be a Hilbert space, and suppose that ϕ is a Jordan homomorphism from a C^*-algebra $\mathcal{A}$ into $B(H)$. Let $\mathcal{B}$ be the C^*-algebra generated by $\phi(\mathcal{A})$. Then there is a projection P in $\mathcal{B}' \cap \mathcal{B}''$ such that*

$$a \rightarrow \phi(a)P$$

is a *-homomorphism, and

$$a \rightarrow \phi(a)(I - P)$$

is an *-antihomomorphism.

Lemma 1.5 (Lemma 6 in [Ka])**.** *Suppose that η is a Jordan homomorphism from a C*-algebra $\mathcal{A}$ into a C*-algebra $\mathcal{B}$. Then for any element $a, b \in \mathcal{A}$ and any positive integer n, we have*
   (i) $\eta((ab)^n + (ba)^n) = (\eta(a)\eta(b))^n + (\eta(b)\eta(a))^n$,
   (ii) $\eta(aba) = \eta(a)\eta(b)\eta(a)$

We also need the following lemma with the proof due to Kadison (see the proof of Theorem 5 in [Ka]).

Lemma 1.6. *Let η be a Jordan homomorphism from a C*-algebra $\mathcal{A}$ into a C*-algebra $\mathcal{B}$. Suppose that $||\eta|| \leq 1$ and η preserves the norm for self-adjoint elements. Then η is an isometry.*

Proof. Let $a \in \mathcal{A}$ be an arbitrary element of norm 1. Since both $(aa^*)^n$ and $(a^*a)^n$ are self-adjoint, we have

$$1 = ||a||^{4n} = ||aa^*||^{2n} = ||(aa^*)^{2n}||$$

$$= ||\frac{1}{2}((aa^*)^n + i(a^*a)^n + ((aa^*)^n + i(a^*a)^n)^*)|| \cdot$$

$$||\frac{1}{2}((aa^*)^n - i(a^*a)^n + ((aa^*)^n - i(a^*a)^n)^*)||$$

$$\leq ||(aa^*)^n + i(a^*a)^n|| \cdot ||(aa^*)^n - i(a^*a)^n||$$

$$= ||(aa^*)^{2n} + (a^*a)^{2n} + i((a^*a)^n(aa^*)^n - (aa^*)^n(a^*a)^n)||$$

$$= ||\eta((aa^*)^{2n} + (a^*a)^{2n}) + i\eta((a^*a)^n(aa^*)^n - (aa^*)^n(a^*a)^n)||$$

$$= ||(\eta(a)\eta(a)^*)^{2n} + (\eta(a^*)\eta(a))^{2n} + i(\eta(a)^*\eta(a))^n(\eta(a)\eta(a)^*)^n$$
$$- (\eta(a)\eta(a)^*)^n(\eta(a)^*\eta(a))^n)||$$

$$\leq 4||\eta(a)||^{4n}.$$

for arbitrary positive integer n, where we use Lemma 1.5. Thus $||\eta(a)|| \geq 1$. Therefore $||\eta(a)|| = 1$ since η is a contraction. $\qquad\square$

2. Wandering Vector Multipliers

In this section, we will consider wandering vector multipliers for unitary groups. Since we will assume in this section that the unitary group $\mathcal{G}$ has a complete wandering vector, as we have seen in the proof of Proposition 1.1, it must be unitarily equivalent to $\{l_g : g \in \mathcal{G}\}$ on $L^2(\mathcal{G})$. So we can assume $\mathcal{G}$ has the form $\{l_g : g \in \mathcal{G}\}$. Sometimes we do not distinguish l_g from g when we consider g as a unitary operator. By Corollary 1.2, every wandering vector for $\mathcal{G}$ must be complete. Therefore a unit

vector $\eta = \sum_{g \in \mathcal{G}} \lambda_g t_g$ is a complete wandering vector if and only if $\{l_h \eta : h \in \mathcal{G}\}$ is an orthonormal set, which, in turn, is equivalent to the condition

$$\sum_{g \in \mathcal{G}} \lambda_{h^{-1}g} \overline{\lambda_g} = \begin{cases} 1, & h = e, \\ 0, & h \neq e, \end{cases}$$

where t_g is the characteristic function of $\{g\}$.

Lemma 2.1. *The two unitary groups $\{l_g : g \in \mathcal{G}\}$ and $\{r_g : g \in \mathcal{G}\}$ have the same complete wandering vectors.*

Proof. First it is clear that t_e is a complete wandering vector for both unitary groups. By Proposition 1.3 in [DL], η is a complete wandering vector for $\{r_g : g \in \mathcal{G}\}$ if and only if there is a unique unitary U in its commutant $L_{\mathcal{G}}$ such that $\eta = Ut_e$. It is well known (see [KR]) that there is a conjugate linear unitary operator J on $l_2(\mathcal{G})$ such that $J^2 = I, JAt_e = A^*t_e$ for all $A \in L_{\mathcal{G}}$ and $\pi : A \to JAJ$ is a *-anti-isomorphism from $L_{\mathcal{G}}$ onto $R_{\mathcal{G}}$. Thus $\eta = Ut_e = JU^*t_e = JU^*Jt_e$ with JU^*J being a unitary in $R_{\mathcal{U}}$. Therefore, again by Proposition 1.3 in [DL], η is a complete wandering vector for $\{l_g : g \in \mathcal{G}\}$. Similarly, every complete wandering vector for $\{l_g : g \in \mathcal{G}\}$ is also a complete wandering vector for $\{r_g : g \in \mathcal{G}\}$. □

To characterize the wandering vector multipliers, it is convenient to first consider the abelian group case. As we mentioned in the introduction (also see Example 2.10), the set of all the complete wandering vectors for $\{M_z^n : n \in \mathbb{Z}\}$ is precisely the set of all unimodular functions. Similarly if $\mathcal{G}$ is abelian, then, by Lemma 1.3, we can assume $\mathcal{G} = \{M_{\chi_g} : g \in \mathcal{G}\}$, acting on $L^2(\hat{\mathcal{G}}, \sigma)$, with the complete wandering vector 1. By Corollary 1.2, every (unit) wandering vector is complete. It is easily seen that if $|\psi(\omega)| = 1$ a.e., then ψ is a wandering vector. Conversely if $\psi \in L^2(\hat{\mathcal{G}}, \sigma)$ is a wandering (unit) vector, then

$$\langle \pi(g)\psi, \psi \rangle = \int_{\hat{\mathcal{G}}} \chi_g(\omega)|\psi(\omega)|^2 d\sigma = 0$$

when g is not the identity of $\mathcal{G}$. Since 1 is a complete wandering vector for $\pi(\mathcal{G})$, the set of vectors $\{\chi_g\}_{g \in \mathcal{G}}$ is an orthonormal basis for $L^2(\hat{\mathcal{G}}, \sigma)$. It follows that $|\psi|^2$ is a constant function, hence $|\psi|$ is a constant. Since ψ is a unit vector, this constant is 1. Hence $|\psi(\omega)| = 1$ a.e. So a unitary operator is a wandering vector multiplier if and only if $|(Uf)(\omega)| = 1$ a.e. whenever $|f(\omega)| = 1$ a.e. Thus the question of how to characterize all the wandering vector multipliers becomes the problem of characterizing all such operators U in a simple way. For the general (non-abelian) case, there is an analogue using some von Neumann algebra techniques.

Let $\mathcal{G}$ be a unitary group acting on a Hilbert space H which has a complete wandering vector ψ and let $\mathcal{M}$ be the von Neumann algebra generated by $\mathcal{G}$. Then $\mathcal{M}$ and its commutant $\mathcal{M}'$ are finite von Neumann algebras which have a joint separating and cyclic trace vector ψ. Let τ be the faithful trace defined by $\tau(T) = \langle T\psi, \psi \rangle$. Then $\mathcal{M}$ becomes a pre-Hilbert space with the inner product $\langle S, T \rangle = \tau(T^*S)$. Let $L^2(\mathcal{M}, \tau)$ be the completion of $\mathcal{M}$ with respect to the norm $\|T\|_2 = (\tau(T^*T))^{1/2}$. Then $L^2(\mathcal{M}, \tau)$ is a Hilbert space. There is a natural way to represent $\mathcal{G}$ on $L^2(\mathcal{M}, \tau)$. Specifically, let $\pi(g) \in B(L^2(\mathcal{M}, \tau))$ be defined by $\pi(g)T = gT$ for all $T \in \mathcal{M}$, and then extend it to $L^2(\mathcal{M}, \tau)$ by using the inequality $\|gT\|_2 \leq \|T\|_2$. It follows that $\pi(g)$ is a unitary and I is a complete wandering vector for $\pi(\mathcal{G})$. Note that ψ is a separating vector for $\mathcal{M}$. We can define $W : \mathcal{M}\psi \to \mathcal{M} \subset L^2(\mathcal{M}, \tau)$ by

$WA\psi = A$. Then W is isometric and can be extended to a unitary from H onto $L^2(\mathcal{M}, \tau)$. Also $\pi(\mathcal{G})$ is unitarily equivalent to $\mathcal{G}$ via this unitary operator W. For this reason we will consider $\pi(\mathcal{G})$ instead of $\mathcal{G}$ when necessary.

Lemma 2.2. *In the above notations, if $x \in L^2(\mathcal{M}, \tau)$ is a complete wandering vector for $\pi(\mathcal{G})$, then x is a unitary element in $\mathcal{M}$, viewed as a vector of $L^2(\mathcal{M}, \tau)$.*

Proof. If x is a complete wandering vector for $\pi(\mathcal{G})$, then W^*x is a complete wandering vector for $\mathcal{G}$. Thus, by Lemma 2.1, there exists a unitary operator $U \in w^*(\mathcal{G})$ satisfying $W^*x = U\psi$. So $x = WU\psi$. However, by the definition of W, we have $WU\psi = U$. Hence $x = U$, as required.   □

Proposition 2.3. *With the notations above, an element $x \in L^2(\mathcal{M}, \tau)$ is a complete wandering vector for $\pi(\mathcal{G})$ if and only if x is a unitary element in $\mathcal{M}$.*

Proof. Let x be a unitary element in $\mathcal{M}$. Then

$$\langle \pi(g)x, x \rangle = \tau(x^*gx) = \tau(gxx^*) = \tau(g) = \begin{cases} 0, & g \neq I, \\ 1, & g = I, \end{cases}$$

where I is the identity operator in $\mathcal{G}$. Thus x is a wandering vector for $\pi(\mathcal{G})$ and hence complete by Corollary 1.2.

The converse follows from Lemma 2.2.   □

By Proposition 2.3, a wandering vector multiplier corresponds precisely to a unitary operator on $L^2(\mathcal{M}, \tau)$ that sends all the unitary elements in $\mathcal{M}$ to unitary elements in $\mathcal{M}$. In the paper [Ka] concerning linear mappings from C*-algebras to C*-algebras, Kadison proved the following famous result: If ϕ is a linear isometry from a C*-algebra **onto** a C*-algebra, then ϕ is a Jordan isomorphism followed by left multiplication by a fixed unitary element, viz., $\phi(I)$. And a Jordan isomorphism is, in turn by Lemma 1.4, the (direct) sum of a *-homomorphism and a *-antihomomorphism. We are dealing with unitary-preserving linear mappings here and we do not assume surjectivity. We will first prove that this kind of mapping is also a Jordan homomorphism followed by left multiplication by a fixed unitary element. The techniques used in [Ka] can be applied to this situation, yielding Proposition 2.5. It will, of course, follow from this that it is a linear isometry (Corollary 2.6) , but we do not see a way of directly proving it is an isometry and then obtain this decomposition using Kadison's theorem.

The elegant proof of the next lemma is taken from [Ka] and is included for completeness.

Lemma 2.4. *Let ϕ be a linear mapping from a C*-algebra $\mathcal{A}$ into a C*-algebra $\mathcal{B}$ such that $\|\phi\| \leq 1$ and $\phi(I_{\mathcal{A}}) = I_{\mathcal{B}}$. Then $\phi(a^*) = \phi(a)^*$ for all $a \in \mathcal{A}$.*

Proof. First suppose that a is a self-adjoint element of norm 1 and let $\phi(a) = b + ic$ with b and c self-adjoint. If $c \neq 0$, then there is a non-zero number β in the spectrum of c. We can assume that $\beta > 0$. Then for n large enough, we have $(1 + n^2)^{1/2} < \beta + n$. If we note that

$$\|c + nI\| \leq \|b + i(c + nI)\| = \|\phi(a + inI)\| \leq \|a + inI\|,$$

then we have

$$\|a + inI\| = (1 + n^2)^{1/2} < \beta + n \leq \|c + nI\| \leq \|a + inI\|.$$

This is a contradiction. Hence $c = 0$. So $\phi(a)$ is self-adjoint. Now let $a = a_1 + ia_2$ be arbitrary with a_1 and a_2 self-adjoint. Then $\phi(a)^* = \phi(a_1) - i\phi(a_2) = \phi(a^*)$.   □

The proof of the next proposition is an adaptation of the elegant argument in [Ka] to the unitary-preserving situation.

Proposition 2.5. *If ϕ is a unitary-preserving linear mapping from a C*-algebra $\mathcal{A}$ into a C*-algebra $\mathcal{B}$ such that $\phi(I_{\mathcal{A}}) = I_{\mathcal{B}}$, then ϕ is a Jordan homomorphism.*

Proof. First we prove that ϕ has norm one. In fact for any $a \in \mathcal{A}$ with $\|a\| \leq 1$, there are four unitary elements u_1, u_2, u_3, u_4 satisfying $x = \frac{1}{4}(u_1 + u_2 + u_3 + u_4)$ (cf. Theorem 4.1.7 in [KR]). Thus $\|\phi(a)\| \leq 1$ since $\phi(u_1)$, $\phi(u_2)$, $\phi(u_3)$, $\phi(u_4)$ are unitaries. Hence $\|\phi\| = 1$ since $\phi(I_{\mathcal{A}}) = I_{\mathcal{B}}$. Therefore, by Lemma 2.4, ϕ preserves adjoints. Moreover ϕ preserves order. In fact if a is a positive element with norm one, then $\phi(a)$ is self-adjoint and

$$\|\phi(a) - I\| = \|\phi(a - I)\| \leq \|a - I\| \leq 1.$$

Thus $\phi(a)$ has non-negative spectrum. So it is positive.

Now we verify that ϕ preserves the power structure. First let a be a self-adjoint element with norm less than or equal to one. Then $u = a^2 + i(I - a^2)^{1/2}$ is a unitary element. Hence so is $\phi(u)$ by hypothesis. Thus we have

$$I = \phi(u)\phi(u)^*$$
$$= (\phi(a) + i\phi((I - a^2)^{1/2}))(\phi(a) - i\phi((I - a^2)^{1/2}))$$
$$= \phi(a)^2 + \phi((I - a^2)^{1/2})^2 + i(\phi((I - a^2)^{1/2})\phi(a) - \phi(a)\phi((I - a^2)^{1/2})).$$

Therefore $I = \phi(a)^2 + \phi((I-a^2)^{1/2})^2$, which implies that $\phi((I-a^2)^{1/2})^2 = I - \phi(a)^2$. Since $\phi((I - a^2)^{1/2})$ is a positive element, we obtain $\phi((I - a^2)^{1/2}) = (I - \phi(a)^2)^{1/2}$.

For an arbitrary self-adjoint element $a \in \mathcal{A}$. Let α be non-zero real number such that αa has norm less than one. By using the binomial expressions:

$$(I - (\alpha a)^2)^{1/2} = I - \frac{1}{2}\alpha^2 a^2 - \frac{1}{8}\alpha^4 a^4 + \dots,$$
$$(I - \phi(\alpha a)^2)^{1/2} = I - \frac{1}{2}\alpha^2 \phi(a)^2 - \frac{1}{8}\alpha^4 \phi(a)^4 - \dots,$$

and the relation

$$\phi((I - (\alpha a)^2)^{1/2}) = (I - \phi(\alpha a)^2)^{1/2},$$

we have

$$\frac{1}{2}\phi(a^2) + \frac{1}{8}\alpha^2 \phi(a^4) + \dots = \frac{1}{2}\phi(a)^2 + \frac{1}{8}\alpha^2 \phi(a)^4 + \dots.$$

Thus it follows that $\phi(a^2) = \phi(a)^2$ by letting α tend to 0.

For an arbitrary element a, write $a = b + ic$ with b, c self-adjoint. Then

$$\phi(b + c)^2 = \phi((b + c)^2), \quad \phi(b)^2 = \phi(b^2) \quad \text{and} \quad \phi(c)^2 = \phi(c^2).$$

Hence $\phi(bc + cb) = \phi(b)\phi(c) + \phi(c)\phi(b)$. Therefore

$$\phi(a^2) = \phi(b^2 - c^2 + i(bc + cb))$$
$$= \phi(a)^2 - \phi(c)^2 + i(\phi(b)\phi(c) + \phi(c)\phi(b))$$
$$= \phi(a)^2.$$

The proof is complete.   □

Corollary 2.6. *If ϕ is a unitary-preserving linear mapping from one C*-algebra $\mathcal{A}$ into another C*-algebra $\mathcal{B}$ such that ϕ is one-to-one, then ϕ is an isometry.*

Proof. Let $u = \phi(I)$. Then u is a unitary element. By considering $\eta = u^{-1}\phi$, then η is a unitary-preserving mapping such that $\eta(I_{\mathcal{A}}) = I_{\mathcal{B}}$. Thus η is a one-to-one Jordan homomorphism by Proposition 2.5. Let $a \in \mathcal{A}$ be a self-adjoint element and let $\mathcal{C}$ be the C*-algebra generated by a and I. Since η preserves adjoints, and the power structure, and η is a contraction as proved in Proposition 2.5, η induces a one-to-one Jordan homomorphism from $\mathcal{C}$ into $\mathcal{C}$. We claim that η preserves norm for self-adjoint elements. In fact, let $a \in \mathcal{A}$ be a self-adjoint operator and let $b = ||\eta(a)||I + a$. Then $\eta(b) \geq 0$. Note that b is self-adjoint. Let $|b|$ be the positive square root of b. Then $\eta(|b|)^2 = \eta(|b|^2) = \eta(b^2) = \eta(b)^2$. Since $\eta(|b|)$ is positive (see Theorem 2.5), we have that $\eta(|b|) = \eta(b)$ because of the uniqueness of positive square roots. Thus $b = |b|$, which implies that $a \geq -||\eta(a)||I$. Similarly, we have $a \leq ||\eta(a)||I$. Therefore $||a|| \leq ||\eta(a)||$. We already know that η is a contraction. Hence η preserves norm for self-adjoint operators. Thus, by Lemma 1.6, we have that η is an isometry, and hence ϕ is an isometry. $\qquad\square$

Remark 2.7. From Proposition 2.5 we have that any wandering vector multiplier for $\pi(\mathcal{G})$ can be factorized in the form $M_U C_J$, where U is a unitary operator in $w^*(\mathcal{G})$, and J is a Jordan homomorphism from $w^*(\mathcal{G})$ into $w^*(\mathcal{G})$ such that $J(I) = I$ and with the additional property that J can be extended to a unitary operator C_J on $L^2(w^*(\mathcal{G}), \tau)$. In fact, let $\Phi \in B(L^2(w^*(\mathcal{G}), \tau))$ be a wandering vector multiplier for $\pi(\mathcal{G})$. Then $U := \Phi(I)$ is a unitary by Proposition 2.3. Let $C_J = M_{U^*}\Phi$ and let J be the restriction of C to $w^*(\mathcal{G})$. Then C is also a wandering vector multiplier and $C_J(I) = I$. Thus J is a unitary-preserving linear map from $w^*(\mathcal{G})$ into itself, and $J(I) = I$. Therefore, by Proposition 2.5, J is a Jordan homomorphism which satisfies the required requirements. The converse is also true. For the converse, by 10.5.22 in [KR], we have that J is unitary-preserving if J is a Jordan homomorphism and $J(I) = I$. Thus $M_U C_J$ is a unitary operator on $L^2(w^*(\mathcal{G}), \tau)$ which maps unitary elements in $w^*(\mathcal{G})$ to unitary elements. Hence it is a wandering vector multiplier by Proposition 2.3.

Lemma 2.8. *Let ϕ be a Jordan homomorphism from a von Neumann algebra $\mathcal{A}$ into itself. If either $\mathcal{A}$ is abelian, or $w^*(\phi(\mathcal{A}))$ is a factor, then $\phi(\mathcal{A})$ is a *-subalgebra.*

Proof. Let $\mathcal{B} = w^*(\phi(\mathcal{A}))$. If $\mathcal{A}$ is abelian, then ϕ is a *-homomorphism and hence $\phi(\mathcal{A})$ is clearly a *-subalgebra. In the general case, by Proposition 1.4, there is projection E in $\mathcal{B} \cap \mathcal{B}'$ such that $a \to \phi(a)E$ is a *-homomorphism and $a \to \phi(a)E^{\perp}$ is a *-anti-homomorphism. Hence if $\mathcal{B}$ is a factor, then ϕ is either a homomorphism or an anti-homomorphism. In either case, we have that $\phi(\mathcal{A})$ is a *-subalgebra. $\qquad\square$

Let $\mathcal{M}$ be a von Neumann algebra on a Hilbert space H. A closed operator A on H is said to be *affiliated with* $\mathcal{M}$ if $\mathcal{M}'D(A) \subseteq D(A)$ and $AB \subseteq BA$ for all $B \in \mathcal{M}'$ (see [BR] or [KR]), where $D(A)$ is the domain of A. If ψ is a complete wandering vector for $\mathcal{G}$, then ψ is a trace, cyclic and separating vector for both $\mathcal{M} = w^*(\mathcal{G})$ and $\mathcal{M}'$. Thus we can define two conjugate linear operators S_0 and F_0 on $\mathcal{M}\psi$ by

$$S_0 A\psi = A^*\psi$$

for all $A \in \mathcal{M}$ and

$$F_0 B\psi = B^*\psi$$

for all $B \in \mathcal{M}'$. Using the properties of the trace vector, it follows that the closures $\overline{S}_0$ and $\overline{F}_0$ are conjugate linear unitary operators.

Theorem 2.9. *If $\mathcal{G}$ is either finite or abelian or an ICC group acting on a Hilbert space H, and if $\mathcal{G}$ has a complete wandering vector, then $M_\mathcal{G}$ is a group.*

Proof. Let $\mathcal{M}$, H, τ, ψ and π be as in Lemma 2.2. Let Φ be a unitary operator on $L^2(\mathcal{M}, \tau)$ such that $\Phi \in M_{\pi(\mathcal{G})}$. In order to prove that $M_{\pi(\mathcal{G})}$ is a group, it suffices to show that Φ^{-1} is also a wandering vector multiplier. We will complete the proof by several steps. Since Φ is unitary-preserving, $\Phi(I)$ is a unitary in $\mathcal{M}$. Replacing Φ by $\Phi(I)^{-1}\Phi$, we can assume that $\Phi(I) = I$. By Proposition 2.3, Proposition 2.5 and Corollary 2.6, we know that Φ is an isometry as well as a Jordan homomorphism from $\mathcal{M}$ into $\mathcal{M}$.

(1) Let $\mathcal{N}$ be the von Neumann algebra generated by $\Phi(\mathcal{M})$. We claim that $\mathcal{N} = \mathcal{M}$. In fact, by Proposition 2.3, Φ sends unitary operators in $\mathcal{M}$ to unitary operators in $\mathcal{M}$. Thus $\mathcal{N} \subseteq \mathcal{M}$ since every element of $\mathcal{M}$ is a linear combination of four unitaries in $\mathcal{M}$. Since Φ is a unitary operator on $L^2(\mathcal{M}, \tau)$ and $\mathcal{M}$ is τ-dense in $L^2(\mathcal{M}, \tau)$, we have that $\Phi(\mathcal{M})$ is τ-dense in $L^2(\mathcal{M}, \tau)$. This implies that $\Phi(\mathcal{M})\psi$ is dense in H. Thus ψ is a cyclic and separating trace vector for $\mathcal{N}$. (It is separating for $\mathcal{N}$ because $\mathcal{N} \subset \mathcal{M}$ and it is separating for $\mathcal{M}$.) Hence we have S_0 and F_0 related to $\mathcal{N}$ and ψ. Let $A \in \mathcal{M}$. By Proposition 2.5.9 in [BR], for $A\psi \in D(\overline{S}_0) = H$, there exists a closed operator Q such that $Q\psi = A\psi$ and Q is affiliated with $\mathcal{N}$. Thus for any $B \in \mathcal{M}' \subseteq \mathcal{N}'$, we have that

$$AB\psi = BA\psi = BQ\psi = QB\psi.$$

Since ψ is cyclic for $\mathcal{M}'$ and Q is closed, we obtain that Q is a bounded operator and $A = Q$. Since Q is a bounded operator affiliated with $\mathcal{N}$, we have that $Q \in \mathcal{N} \subseteq \mathcal{M}$. Thus $Q = A$ since ψ is a separating vector for $\mathcal{M}$. Hence $A \in \mathcal{N}$, and so we have $\mathcal{N} = \mathcal{M}$, as required.

(2) We secondly claim that $\Phi(\mathcal{M}) = \mathcal{M}$. Let $A \in \mathcal{M}$ with norm one. By Lemma 2.8 and (1), $\Phi(\mathcal{M})$ is a *-subalgebra of $\mathcal{M}$ for the abelian and ICC group case and equal to $\mathcal{M}$ for the finite group case. Thus, by (1) in either case, we have that $\mathcal{M}$ is the strong closure of the *-subalgebra $\Phi(\mathcal{M})$. Using Kaplansky's density theorem, there is a net $\{A_\lambda\}$ in $\mathcal{M}$ such that $\Phi(A_\lambda)$ is in the unit ball of $\Phi(\mathcal{M})$ and $\Phi(A_\lambda)$ converges to A in the strong operator topology. Thus $\{A_\lambda\}$ is contained in the unit ball of $\mathcal{M}$ because Φ is an isometry for the norm topology of $\mathcal{M}$. Since $\Phi(A_\lambda)\psi \to A\psi$, we have that $\{\Phi(A_\lambda)\}$ is a Cauchy net in $L^2(\mathcal{M}, \tau)$. Note that Φ is viewed as a unitary operator on $L^2(\mathcal{M}, \tau)$. Thus $\{A_\lambda\}$ is also a Cauchy net in $L^2(\mathcal{M}, \tau)$, which implies that $\{A_\lambda\psi\}$ converges.

We know that ψ is cyclic for $\mathcal{M}'$. Thus $\{A_\lambda\}$ is a Cauchy net in the topology of pointwise convergence on the dense subset $\mathcal{M}'\psi$ since $\{A_\lambda\}$ is a bounded net in norm. Therefore $\{A_\lambda\}$ is a Cauchy net in the strong operator topology and thus converges to some operator $B \in \mathcal{M}$. So $\|\Phi(A_\lambda)\psi - \Phi(B)\psi\| = \|\Phi(A_\lambda - B)\psi\| = \|(A_\lambda - B)\psi\| \to 0$. Hence $A\psi = \Phi(B)\psi$. Since ψ separates $\mathcal{M}$ and $A, \Phi(B) \in \mathcal{M}$, we have $A = \Phi(B)$. Therefore $\mathcal{M} = \Phi(\mathcal{M})$, as required.

(3) By (2) and Corollary 2.6, Φ is a surjective isometry on $\mathcal{M}$. So the inverse of Φ is an isometry. A surjective isometry between unital C*-algebras preserves unitary elements [Ka]. Therefore, by Proposition 2.3, Φ^{-1} is a wandering vector multiplier. The proof is complete. $\square$

Example 2.10. Now we consider the example mentioned in the introduction. Let $H = L^2(\mathbb{T})$, $\mathcal{U} = \{M_z^n : n \in \mathbb{Z}\}$ and let M_z be the unitary operator corresponding to multiplication by z. Then a function is a complete wandering vector for $\mathcal{U}$ if and only if it is unimodular. Any multiplication unitary operator M_f with f unimodular belongs to $M_{\mathcal{U}}$. There are others. Let σ be a measure-preserving bijective mapping from $\mathbb{T}$ to itself. Define a unitary operator A_σ on $L^2(\mathbb{T})$ by $(A_\sigma f)(z) = f(\sigma^{-1}(z))$ for all $f \in L^2(\mathbb{T})$. Then $A_\sigma \in M_{\mathcal{U}}$. Thus every element in the group generated by all the M_f and all the A_σ is a wandering vector multiplier. We claim that every wandering vector multiplier for $\mathcal{U}$ can be factored as the product of A_σ and M_f for some measure-preserving bijection mapping σ and some unimodular function f.

In fact, suppose that $T \in M_{\mathcal{U}}$. Identify $w^*(\mathcal{U})$ with $L^\infty(\mathbb{T})$. Then, by the proof of Theorem 2.9, $T = V\Phi$, for some unitary operator V in $w^*(\mathcal{U})$ (hence some unimodular function in $L^\infty(\mathbb{T})$) and some $\Phi : L^\infty(\mathbb{T}) \to L^\infty(\mathbb{T})$ such that Φ is a *-isomorphism of $L^\infty(\mathbb{T})$ and $\|\Phi(g)\|_2 = \|g\|_2$ for all $g \in L^\infty(\mathbb{T})$. Then Φ induces a homomorphism of the Borel algebra $(\mathcal{B}, \mathbb{T})$, where $\mathcal{B}$ is the set of all Borel sets of $\mathbb{T}$. Therefore there is a measurable one-to-one (modulo a null set) (see Theorem 4.7 on page 17 in [Pe]) mapping σ from $\mathbb{T}$ onto $\mathbb{T}$ such that $\Phi(f)(t) = f(\sigma(t))$, a.e. on $\mathbb{T}$. However the condition $\|\Phi(g)\|_2 = \|g\|_2$ for all $g \in L^\infty(\mathbb{T})$ implies that σ is measure-preserving. Hence we get the equality. This example can be extended to the arbitrary abelian group case in the following way.

Proposition 2.11. *Let $\mathcal{G}$ be an abelian unitary group which has a complete wandering vector and let $\hat{\mathcal{G}}$, σ and π be as in Lemma 1.3. Then an operator T on $L^2(\hat{\mathcal{G}}, \sigma)$ is a wandering vector multiplier for $\pi(\mathcal{G})$ if and only if $T(f)(\omega) = h(\omega)f(\Phi(\omega))$ for all $f \in L^2(\hat{\mathcal{G}}, \sigma)$ for some unimodular function h and some measure-preserving mapping Φ from $\hat{\mathcal{G}}$ onto $\hat{\mathcal{G}}$.*

Proof. Since $L^\infty(\hat{\mathcal{G}}, \sigma)$, the von Neumann algebra generated by $\pi(\mathcal{G})$, is maximal abelian, it follows that it is unitarily equivalent to $L^\infty(X, \mu)$ on a Hilbert space $L^2(X, \mu)$ for some compact set X in the complex plane and some finite Borel measure μ (cf. Corollary 7.14 in [RR]). Thus, by Theorem 4.6 in [Pe, p.17], $(\mathcal{B}, \hat{\mathcal{G}}, \sigma)$ is Lebesgue in the sense of Definition 4.5 in [Pe, p.16], where $\mathcal{B}$ denotes the measure algebra.

Now let $T \in M_{\pi(\mathcal{G})}$. Then, by the proof of Theorem 2.9, there is a unitary element h in $w^*(\pi(\mathcal{G}))$ $(=L^\infty(\hat{\mathcal{G}}, \sigma))$ and an *-isomorphism α on $w^*(\pi(\mathcal{G}))$ satisfying $T(f) = h\alpha(f)$ for all $f \in L^\infty(\hat{\mathcal{G}}, \sigma)$ and

$$\int_{\hat{\mathcal{G}}} |\alpha(f)(\omega)|^2 d\sigma = \int_{\hat{\mathcal{G}}} |f(\omega)|^2 d\sigma$$

for all $f \in L^2(\hat{\mathcal{G}}, \sigma)$.

The *-isomorphism α induces an isomorphism from the measure algebra $(\mathcal{B}, \hat{\mathcal{G}}, \sigma)$ onto itself. Since $(\mathcal{B}, \hat{\mathcal{G}}, \sigma)$ is a Lebesgue space, by Theorem 4.7 in [Pe, p.17], there is a measurable one-to-one point mapping (modulo a measure zero set) Φ such that $\alpha(f)(\omega) = f(\Phi(\omega))$. By taking f to be a characteristic function on a set $E \in \mathcal{B}$,

we have

$$\mu(\Phi^{-1}(E)) = \int_{\hat{\mathcal{G}}} |f(\Phi(\omega))|^2 d\omega$$

$$= \int_{\hat{\mathcal{G}}} |\alpha(f)(\omega)|^2 d\omega$$

$$= \int_{\hat{\mathcal{G}}} |f(\omega)|^2 d\omega$$

$$= \mu(E).$$

Thus Φ is a measure-preserving mapping and the proof is complete. $\qquad\square$

Let $\mathcal{I}$ be the group of all surjective isometries on $w^*(\mathcal{G})$. We have

Theorem 2.12. *Let $\mathcal{G}$ be an ICC unitary group acting on a Hilbert space H which has a complete wandering vector. Then $M_{\mathcal{G}}$ is group isomorphic to $\mathcal{I}$*

Proof. Let $\mathcal{M} = w^*(\mathcal{G})$. By the proof of Theorem 2.9, we only need to show that every Jordan isomorphism of $\mathcal{M}$ induces a wandering vector multiplier.

Suppose that ϕ is a Jordan isomorphism on $\mathcal{M}$. Then it must be either a *-automorphism or a *-anti-isomorphism since $\mathcal{M}$ is a factor. Assume that it is an automorphism. Then, by 9.6.27 in [KR], there is a unitary operator U acting on H such that $\phi(A) = UAU^*$ for each $A \in \mathcal{M}$. We claim that $\|\phi(T)\|_2 = \|T\|_2$ for all $T \in \mathcal{M}$, where $\|T\|_2^2 = \langle T\psi, T\psi \rangle$ for the given complete wandering vector ψ. Define $\tau_U(T) = \langle UTU^*\psi, \psi \rangle$ for each $T \in \mathcal{M}$. Then τ_U is a normalized faithful normal trace of $\mathcal{M}$. By the corollary of Theorem 3 in [Di, p.103], we have that $\tau_U = \tau$. Thus $\|\phi(T)\|_2 = \|T\|_2$ for all $T \in \mathcal{M}$. This implies that ϕ induces a unitary operator on H. Hence it induces a wandering vector multiplier since it preserves the unitary elements of $\mathcal{M}$.

If ϕ is a *-anti-isomorphism, the proof is similar. $\qquad\square$

Corollary 2.13. *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be ICC groups. If $R_{\mathcal{G}_1}$ is *-isomorphic to $R_{\mathcal{G}_2}$, then $M_{\mathcal{G}_1}$ is group isomorphic to $M_{\mathcal{G}_2}$.*

Let $\mathcal{M}$ be a von Neumann algebra acting on H. A *normalizer* of $\mathcal{M}$ is a *unitary* operator A on H such that $A^*\mathcal{M}A = \mathcal{M}$. We use $N_{\mathcal{G}}$ to denote the set of all normalizers for $w^*(\mathcal{G})$. For convenience, we use $NC_{\mathcal{G}}$ to denote the set of all unitary operators U with the property that $U^*w^*(\mathcal{G})U \subseteq \mathcal{G}'$.

Theorem 2.14. *Let $\mathcal{G}$ be an ICC unitary group with a complete wandering vector. Then $M_{\mathcal{G}}$ is equal to the group generated by $N_{\mathcal{G}} \cup NC_{\mathcal{G}}$.*

Proof. Let $\psi \in \mathcal{W}(\mathcal{G})$ be arbitrary. We have known that $\tau_{\psi}(T) = \langle T\psi, \psi \rangle$ defines a faithful normalized trace on both $w^*(\mathcal{G})$ and $\mathcal{G}'$.

Suppose that $A \in N_{\mathcal{G}}$. Define $\tau_{A\psi}(T) = \langle A^*TA\psi, \psi \rangle$. By using the fact that $A^*w^*(\mathcal{G})A = w^*(\mathcal{G})$, it follows that $\tau_{A\psi}$ is also a faithful normalized trace for $w^*(\mathcal{G})$. Since $w^*(\mathcal{G})$ is a factor, we have that $\tau_{\psi} = \tau_{A\psi}$ (cf. [Di]), which implies that $A\psi$ is a wandering vector for $\mathcal{G}$ and hence complete by Corollary 1.2. Thus $A \in M_{\mathcal{G}}$.

If $A \in NC_{\mathcal{G}}$. Define $\tau_{A\psi}$ as above. Using the fact that ψ is also a faithful trace vector for $\mathcal{G}'$ and that $A^*TA \in \mathcal{G}'$ for all $T \in w^*(\mathcal{G})$, it follows that $\tau_{A\psi}$ is a faithful normalized trace for $w^*(\mathcal{G})$. Thus $\tau_{A\psi} = \tau$, which implies that $A\psi \in \mathcal{W}(\mathcal{G})$. Hence $A \in M_{\mathcal{G}}$.

From Theorem 2.9, $M_{\mathcal{G}}$ is a group. Thus the group generated by $N_{\mathcal{G}}$ and $NC_{\mathcal{G}}$ is contained in $M_{\mathcal{G}}$.

Conversely, let $A \in M_{\mathcal{G}}$. Fix a complete wandering vector ψ. Then $A\psi \in \mathcal{W}(\mathcal{G})$. Thus there is a unitary operator $U \in w^*(\mathcal{G})$ such that $A\psi = U\psi$. Note that U is a normalizer for $w^*(\mathcal{G})$. Replacing A by $U^{-1}A$, we can assume that $A\psi = \psi$.

By the proof of Theorem 2.9, there is a Jordan isomorphism Φ on $w^*(\mathcal{G})$ such that $AT\psi = \Phi(T)\psi$ for all $T \in w^*(\mathcal{G})$. Since $w^*(\mathcal{G})$ is a factor, it follows that Φ is either a *-isomorphism or a *-anti-isomorphism.

If Φ is a *-isomorphism, then for all $S, T \in w^*(\mathcal{G})$ we have

$$ATS\psi = \Phi(TS)\psi = \Phi(T)\Phi(S)\psi = \Phi(T)AS\psi.$$

Thus $AT = \Phi(T)A$ for all $T \in w^*(\mathcal{G})$ since ψ is cyclic for $w^*(\mathcal{G})$. Therefore $ATA^* = \Phi(T) \in w^*(\mathcal{G})$. Since $A^*\psi = \psi$ and $A^* \in M_{\mathcal{G}}$, by the same argument we also have $A^*TA \in w^*(\mathcal{G})$. Therefore A is a normalizer for $w^*(\mathcal{G})$.

If Φ is an *-anti-isomorphism, we need to show that $A^*TA \in \mathcal{G}'$ for all $T \in w^*(\mathcal{G})$. Let $S, T, U, V \in w^*(\mathcal{G})$ be arbitrary. Then

$$\begin{aligned}
\langle U\psi,\ A^*TASV\psi \rangle &= \langle AU\psi,\ TASV\psi \rangle \\
&= \langle \Phi(U)\psi, T\Phi(V)\Phi(S)\psi \rangle \\
&= tr(\Phi(S)^*\Phi(V)^*T^*\Phi(U))
\end{aligned}$$

and

$$\begin{aligned}
\langle U\psi,\ SA^*TAV\psi \rangle &= \langle AS^*U\psi,\ TAV\psi \rangle \\
&= \langle \Phi(U)\Phi(S^*)\psi,\ T\Phi(V)\psi \rangle \\
&= tr(\Phi(V)^*T^*\Phi(U)\Phi(S)^*).
\end{aligned}$$

Since ψ is a trace vector for $w^*(\mathcal{G})$, it follows that

$$\langle U\psi,\ A^*TASV\psi \rangle = \langle U\psi,\ SA^*TAV\psi \rangle.$$

Hence $(A^*TA)S = S(A^*TA)$ for all $S \in w^*(\mathcal{G})$ since ψ is cyclic for $w^*(\mathcal{G})$. Therefore $A^*TA \in \mathcal{G}'$ for all $T \in w^*(\mathcal{G})$, as required. $\qquad\square$

The proof of Theorem 2.14 also shows that if $\mathcal{G}$ is abelian, then $M_{\mathcal{G}} \subseteq N_{\mathcal{G}}$. The following example tells us that the equality is not necessarily true.

Example 2.15. Let $H = L^2([0,1])$ and let

$$\mathcal{G} = \{M_{e^{2n\pi i s}}:\ n \in \mathbb{Z}\}.$$

Then, as we mentioned before, a function $f \in H$ is a complete wandering vector for $\mathcal{G}$ if and only if $|f(s)| = 1$, a.e. $s \in [0,1]$. Let $\sigma: [0,1] \to [0,1]$ be a measurable bijection defined by

$$\sigma(s) = \begin{cases} \dfrac{1}{2}s, & 0 \le s \le \dfrac{1}{2}, \\[2mm] \dfrac{3}{2}s - \dfrac{1}{2}, & \dfrac{1}{2} < s \le 1, \end{cases}$$

and define an invertible operator B by

$$(Bf)(s) = f(\sigma(s)),\quad f \in L^2([0,1]).$$

Let $h \in L^\infty([0,1])$ such that

$$h(s) = \begin{cases} \dfrac{1}{\sqrt{2}}, & 0 \le s \le \dfrac{1}{2}, \\[2mm] \sqrt{\dfrac{3}{2}}, & \dfrac{1}{2} < s \le 1. \end{cases}$$

Then $A = M_h B$ is a unitary which is a normalizer of $w^*(\mathcal{G})$ ($= \{M_f : f \in L^\infty([0,1]))$. In fact, that the operator A is unitary follows from

$$\begin{aligned} ||Af||^2 &= \int_0^1 |h(s)|^2 |f(\sigma(s))|^2 ds \\ &= \frac{1}{2} \int_0^{\frac{1}{2}} |f(\frac{1}{2}s)|^2 ds + \frac{3}{2} \int_{\frac{1}{2}}^1 |f(\frac{3}{2}s - \frac{1}{2})|^2 ds \\ &= \int_0^{\frac{1}{4}} |f(t)|^2 dt + \int_{\frac{1}{4}}^1 |f(u)|^2 du \\ &= ||f||^2, \quad f \in L^2([0,1]). \end{aligned}$$

For any $f \in L^\infty([0,1])$, we have $A^* M_f A = M_{f(\sigma^{-1}(s))}$ and $A M_f A^* = M_{f(\sigma(s))}$. Thus A is a normalizer for $w^*(\mathcal{G})$. But A is not a wandering vector multiplier for $\mathcal{G}$ since Af is clearly not unimodular if f is unimodular.

However we have the following simple observation:

Proposition 2.16. *Let $\mathcal{G}$ be a unitary group and let A be a normalizer for $w^*(\mathcal{G})$ such that $A\psi \in \mathcal{W}(\mathcal{G})$ for some complete wandering vector $\psi \in \mathcal{W}(\mathcal{G})$. Then A is a wandering vector multiplier for $\mathcal{G}$.*

Proof. Let η be an arbitrary complete wandering vector for $\mathcal{G}$. Then there is a unitary operator $U \in w^*(\mathcal{G})$ (cf. proof of Lemma 2.1) so that $\eta = U\psi$. Thus $A\eta = AU\psi = (AUA^*)A\psi$ is a complete wandering vector since AUA^* is a unitary in $w^*(\mathcal{G})$ and $A\psi \in \mathcal{W}(\mathcal{G})$. $\square$

Remark 2.17. (i) If $\mathcal{G}$ is a unitary group such that $\mathcal{W}(\mathcal{G})$ is nonempty, then all the unitary operators in $w^*(\mathcal{G})$ and $\mathcal{G}'$ are contained in $M_\mathcal{G}$. Thus $M_\mathcal{G}$ generates $B(H)$ as a von Neumann algebra if $w^*(\mathcal{G})$ is a factor. When $\mathcal{G}$ is abelian, it is easy to prove that this is also true by using Proposition 2.11. We conjecture that this is true in general.

(ii) A non-unitary operator can also map complete wandering vectors to complete wandering vectors. For instance if σ is a measurable bijection on $\mathbb{T}$ which is not measure-preserving, then the corresponding composition operator is not a unitary, but it maps unimodular functions to unimodular functions, hence complete wandering vectors to complete wandering vectors. We call an invertible operator *an invertible wandering vector multiplier* if it maps complete wandering vectors to complete wandering vectors. Similar to the unitary wandering vector multiplier case, we also have the following characterization for invertible wandering vector multipliers. From Proposition 2.5, every invertible wandering vector multiplier for $\pi(\mathcal{G})$ can be factorized in the form $M_U C_J$, where U is a unitary operator in $w^*(\mathcal{G})$ and J is Jordan homomorphism from $w^*(\mathcal{G})$ into $w^*(\mathcal{G})$ such that $J(I) = I$ and J can be extended to a invertible operator C_J on $L^2(w^*(\mathcal{G}), \tau)$. The converse is also true. See also the *another direction* commentary at the end of section 5.

3. Diagonal and Permutation Wandering Vector Multipliers

In this section we will study a special class of wandering vector multipliers which are either diagonal or permutation unitary operators induced by any fixed wandering vector for a unitary group $\mathcal{G}$. It turns out that this class of multipliers can be completely characterized by either the dual group of $\mathcal{G}$ or the automorphism (resp. anti-automorphism) group of $\mathcal{G}$.

For a function $f : \mathcal{G} \to \mathbb{T}$ (resp. a bijection σ of $\mathcal{G}$), we can define a unitary operator B_f (resp. A_σ) by $B_f t_g = f(g)t_g$ (resp. $A_\sigma t_g = t_{\sigma(g)}$) for all $g \in \mathcal{G}$. We will call f (resp. σ) a *wandering vector multiplier function* (resp. *wandering vector multiplier mapping*) on $\mathcal{G}$ if B_f (resp. A_σ) is a wandering vector multiplier for $\mathcal{G}$. Using Proposition 2.5, we have the following characterization of wandering vector multiplier functions and mappings.

Theorem 3.1. *(i) f is a wandering vector multiplier function if and only if there exists a modulus one complex number λ and a character ω of $\mathcal{G}$ such that $f(g) = \lambda\omega(g)$ for all $g \in \mathcal{G}$.*

(ii) σ is a wandering vector multiplier mapping of $\mathcal{G}$ if and only if there exists an element $h \in \mathcal{G}$ and an automorphism or anti-automorphism τ such that $\sigma(g) = h\tau(g)$ for all $g \in \mathcal{G}$.

Proof. (i) For necessity, by considering $f(e)^{-1}f$, we can assume that $f(e) = 1$. Suppose that $B_f \in M_{\mathcal{G}}$. By Proposition 2.5, B_f is a Jordan homomorphism on $w^*(\mathcal{G})$. Thus, for any $g, h \in \mathcal{G}$, we have that $A_f(gh + hg) = A_f(g)A_f(h) + A_f(h)A_f(g)$. That is $f(gh)gh + f(hg)hg = f(g)f(h)gh + f(h)f(g)hg$. If $gh = hg$, we obtain $f(gh) = f(g)f(h)$. Suppose that $gh \neq hg$, then by

$$\langle (f(gh)gh + f(hg)hg)t_e, t_{gh} \rangle = \langle (f(g)f(h)gh + f(h)f(g)hg)t_e, t_{gh} \rangle,$$

we have

$$\langle f(gh)t_{gh} + f(hg)t_{hg}, \ t_{gh} \rangle = \langle f(g)f(h)t_{gh}, \ t_{gh} \rangle + \langle f(h)f(g)t_{hg}, \ t_{gh} \rangle.$$

Note that $\langle t_{gh}, \ t_{hg} \rangle = 0$ and $\langle t_{gh}, \ t_{gh} \rangle = 1$. We get $f(gh) = f(g)f(h)$. Thus $f(gh) = f(g)f(h)$ for all $g, h \in \mathcal{G}$. Therefore f is a character of $\mathcal{G}$.

For sufficiency, let $\eta = \sum_{g \in \mathcal{G}} \lambda_g t_g$ be a complete wandering vector for $\{l_g : g \in \mathcal{G}\}$. Then, for any $h \in \mathcal{G}$, we have

$$
\begin{aligned}
\langle l_h B_\omega \eta, B_\omega \eta \rangle &= \langle \sum_{g \in \mathcal{G}} \lambda_g \omega(g) t_{hg}, \sum_{g \in \mathcal{G}} \lambda_g \omega(g) t_g \rangle \\
&= \langle \sum_{g \in \mathcal{G}} \lambda_{h^{-1}g} \omega(h^{-1}g) t_g, \sum_{g \in \mathcal{G}} \lambda_g \omega(g) t_g \rangle \\
&= \sum_{g \in [G} \lambda_{h^{-1}g} \omega(h^{-1}g) \overline{\lambda_g \omega(g)} \\
&= \sum_{g \in \mathcal{G}} \omega(h^{-1}) \lambda_{h^{-1}g} \overline{\lambda_g} \\
&= \omega(h^{-1}) \langle l_h \eta, \eta \rangle \\
&= \begin{cases} 1, & h = e, \\ 0, & h \neq e. \end{cases}
\end{aligned}
$$

Thus $B_\omega \eta$ is a wandering vector which also complete by Lemma 1.2. Hence B_ω is a wandering vector multiplier.

(ii) For necessity we can also assume that $\sigma(e) = e$. If A_σ is a wandering vector multiplier, then A_σ induces a Jordan homomorphism on $L_\mathcal{G}$. Thus $\sigma(gh) + \sigma(hg) = \sigma(g)\sigma(h) + \sigma(h)\sigma(g)$. So

$$\langle t_{\sigma(gh)} + t_{\sigma(hg)}, t_{\sigma(gh)} \rangle = \langle t_{\sigma(g)\sigma(h)} + t_{\sigma h \sigma(g)}, t_{\sigma(gh)} \rangle.$$

This equality, together with $\langle t_a, t_b \rangle = \delta_{a,b}$ for all $a, b \in \mathcal{G}$, implies that either $\sigma(gh) = \sigma(g)\sigma(h)$ or $\sigma(gh) = \sigma(h)\sigma(g)$. Thus, by a result due to L. K. Hua (see Lemma 1 in [JR]), we know that σ is either an automorphism or an anti-automorphism.

Conversely, we assume that σ is an automorphism. Let $\eta = \sum_{g \in \mathcal{G}} \lambda_g t_g$ be a complete wandering vector. Then $A_\sigma \eta = \sum_{g \in \mathcal{G}} \lambda_g t_{\sigma(g)}$. Thus, for each $h \in \mathcal{G}$, we have

$$\begin{aligned}
\langle l_h A_\sigma \eta, A_\sigma \eta \rangle &= \langle \sum_{g \in \mathcal{G}} \lambda_g t_{h\sigma(g)}, \sum_{g \in \mathcal{G}} \lambda_g t_{\sigma(g)} \rangle \\
&= \langle \sum_{g \in \mathcal{G}} \lambda_{\sigma^{-1}(h^{-1}g)} t_g, \sum_{g \in \mathcal{G}} \lambda_{\sigma^{-1}(g)} t_g \rangle \\
&= \sum_{g \in \mathcal{G}} \lambda_{\sigma^{-1}(h^{-1})\sigma^{-1}(g)} \overline{\lambda_{\sigma^{-1}(g)}} \\
&= \langle l_{\sigma(h^{-1})} \eta, \eta \rangle.
\end{aligned}$$

Thus $\{l_h A_\sigma \eta\}$ is an orthonormal set, which implies that $A_\sigma \eta$ is a complete wandering vector by Lemma 1.2. If σ is an anti-automorphsim, we use Lemma 2.1, and the proof is similar. $\square$

Let $\mathcal{U}$ be a unitary system which has a complete wandering vector. If $\mathcal{U}$ has the the property that $\mathcal{U} = \mathcal{U}\mathcal{U}_0$ with $\mathcal{U}_0$ being a subset of $\mathcal{U}$ which is a group, it is known that the product of a unitary in $\mathcal{U}'$ and a unitary in $w^*(\mathcal{U}_0)$ is a wandering vector multiplier (see [DL]), where $w^*(\mathcal{U}_0)$ is the von Neumann algebra generated by $\mathcal{U}_0$. Dai and Larson asked in [DL] whether the converse is true. This was called the factorization problem. There are some counterexamples for the non-group case (see [LMT], [Ha]). The following proposition provides some counterexamples for the group case (i.e. the case where $\mathcal{U}$ itself is a group).

The following proposition tells us that many A_σ, B_f's do not have the factorization property.

Proposition 3.2. *Let $\mathcal{U} = \{l_g : g \in \mathcal{G}\}$ and let $f : \mathcal{G} \to \mathbb{C}$ (resp. $\sigma : \mathcal{G} \to \mathcal{G}$) be a unimodular function (resp. bijection).*

(i) If $\mathcal{G}$ is an abelian group, then A_σ (resp. B_f) belongs to $\mathcal{U}'w^*(\mathcal{U})$ if and only if $\sigma = id$ (resp. $f(g) = 1$ for all $g \in \mathcal{G}$).

(ii) If σ is either (a) an antimorphism and $\{\sigma(h^{-1})h : h \in \mathcal{G}\}$ is an infinite set, or (b) a non-identity automorphism and $\{\sigma(h^{-1})gh : h \in \mathcal{G}\}$ is an infinite set for all but finite number of $g \in \mathcal{G}$, then A_σ does not belong to $\mathcal{U}'w^*(\mathcal{U})$,

Proof. (i) Suppose that A_σ has the factorization property. Then it must belong to $L_\mathcal{G}$ $(= L'_\mathcal{G})$ since $L_\mathcal{G}$ is an abelian von Neumann algebra with a cyclic vector which implies that it is a maximal von Neumann algebra. Hence

$$A_\sigma l_g = l_g A_\sigma$$

for all $g \in \mathcal{G}$. Apply t_e to the above equality, we get $t_{\sigma(g)} = A_\sigma t_g = A_\sigma l_g t_e = l_g A_\sigma t_e = l_g t_{\sigma(e)} = t_g$ for each $g \in \mathcal{G}$. Hence $\sigma(g) = g$ for all $g \in \mathcal{G}$, which implies that $\sigma = id$.

Similarly, suppose B_f has the factorization property. Then

$$B_f l_g = l_g B_f$$

for all $g \in \mathcal{G}$, which implies that (by applying t_e to the equality) $f(g)t_g = f(e)t_g$. Hence $f(g) = f(e) = 1$.

(ii) Assume (a) holds. If A_σ has the factorization property, then there is a unitary operator U in $L_\mathcal{G}$ such that $A_\sigma U \in R_\mathcal{G}$. Thus $A_\sigma U l_g = l_g A_\sigma U$ for all $g \in \mathcal{G}$. Let $U t_e = \sum_{h \in \mathcal{G}} \lambda_h t_h$. Then $U t_g = U r_{g^{-1}} t_e = r_{g^{-1}} U t_e = \sum_{h \in \mathcal{G}} \lambda_h t_{hg} = \sum_{h \in \mathcal{G}} \lambda_{hg^{-1}} t_h$. So

$$A_\sigma U l_g t_e = A_\sigma U t_g = \sum_{h \in \mathcal{G}} \lambda_{hg^{-1}} t_{\sigma h} = \sum_{h \in \mathcal{G}} \lambda_{\sigma^{-1}(h)g^{-1}} t_h,$$

and

$$l_g A_\sigma U t_e = \sum_{h \in \mathcal{G}} \lambda_h t g \sigma h = \sum_{h \in \mathcal{G}} \lambda \sigma^{-1}(g^{-1}h) t_h.$$

Thus we get

$$\lambda_{\sigma^{-1}(h)g^{-1}} = \lambda_{\sigma^{-1}(g^{-1}h)}$$

for all $g, h \in \mathcal{G}$. Write $\sigma^{-1}(g^{-1}h) = k$. we get $h = g\sigma(k)$. So $\sigma^{-1}(h)g^{-1} = k\sigma^{-1}(g)g^{-1}$. This implies that $\lambda_k = \lambda_{k\sigma^{-1}(g)g^{-1}}$ holds for all $k, g \in \mathcal{G}$. Since $\{\sigma^{-1}(g)g^{-1}; g \in \mathcal{G}\}$ is an infinite set, we have $\lambda_k = 0$ because $\sum_{h \in \mathcal{G}} |l_h|^2 < \infty$, which implies that $U t_e = 0$, a contradiction. Hence A_σ does not have the factorization property.

The proof is similar for the case (b). $\qquad\square$

4. Irrational Rotation Unitary Systems

In [Ha], the first named author of this paper studied wandering vectors of irrational rotation unitary systems $\mathcal{U}_\theta$. These are not unitary groups. Some results concerning wandering subspaces and wandering vector multipliers for irrational rotation unitary systems were obtained in [Ha]. In this section, we point out that most of the results in section 2 remain true for irrational rotation unitary systems. In particular we answer the last question raised in [Ha].

Let θ be an irrational number in $(0, 1)$. Recall that ([Ha]) an irrational rotation unitary system $\mathcal{U}$ is a system of the form $\{U^n V^m : n, m \in \mathbb{Z}\}$ with $UV = e^{2\pi i\theta}VU$. If ψ is a complete wandering vector for $\mathcal{U}$, then, by Theorem 1 and Theorem 12 in [Ha], we know that $\mathcal{C}_\psi(\mathcal{U}) = \mathcal{U}'$, ψ is a faithful trace vector for $w^*(\mathcal{G})$ and $\mathcal{U}'$, and $w^*(\mathcal{U})$ is a finite factor. These properties imply that Lemma 2.1, 2.2, 2.3, Theorem 2.9, Theorem 2.12 and Theorem 2.14 remain true for irrational rotation unitary systems $\mathcal{U}$. We leave the routine checking of this to the reader. Thus the following answers the last problem posted in [Ha], which asks for a complete characterization of wandering vector multipliers for irrational rotation unitary systems.

Theorem 4.1. *Suppose that $\mathcal{U}$ is an irrational rotation unitary system such that $\mathcal{W}(\mathcal{U})$ is non-empty. Then $M_\mathcal{U}$ is a group. Moreover, it is equal to the group generated by all the normalizers of $w^*(\mathcal{U})$ and all the unitaries A with the property $A^* w^*(\mathcal{U})A \subseteq \mathcal{U}'$.*

Similar to the group case, we also have a complete characterization for the diagonal wandering vector multipliers with respect to a fixed complete wandering vector. Let $f : \mathbb{Z} \otimes \mathbb{Z} \to \mathbb{T}$ be a function and let B_f be a unitary defined by $B_f U^n V^m \psi = f(n,m) U^n V^m \psi$ on the orthonormal basis $\{U^n V^m \psi : n, m \in \mathbb{Z}\}$. Then we have

Proposition 4.2. *The unitary operator B_f is a wandering vector multiplier if and only if there exist two characters ω and σ of $\mathbb{Z}$ such that $f(n,m)/f(0,0) = \omega(n)\sigma(m)$ for all $n, m \in \mathbb{Z}$.*

Proof. Sufficiency is contained in Theorem 10 in [Ha]. So we only need to prove the necessity. Suppose that $B_f \in M_{\mathcal{G}}$. Clearly we can assume that $f(0,0) = 1$. By Proposition 2.3 and Proposition 2.5 for the irrational rotation unitary system case, B_f induces a Jordan homomorphism on $w^*(\mathcal{U})$. Thus it must be an automorphism or anti-automorphism since $w^*(\mathcal{U})$ is a factor. This implies that $\omega(n) := f(n,0)$ and $\sigma(n) := f(0,n)$ define two characters of $\mathbb{Z}$. From

$$f(n,m)U^n V^m = B_f(U^n V^m) = B_f U^n B_f V^m = f(n,0)f(0,m)U^n V^m,$$

we get that $F(n,m) = \omega(n)\sigma(m)$ for all $n, m \in \mathbb{Z}$. Thus the proof is complete. $\square$

Let α and τ be two bijections on $\mathbb{Z}$. Fix a complete wandering vector $\psi \in \mathcal{W}(\mathcal{U})$. Define a unitary operator $A_{\alpha,\tau}$ by

$$A_{\alpha,\tau} U^n V^m \psi = U^{\alpha(n)} V^{\tau(m)} \psi$$

for all $n, m \in \mathbb{Z}$. We have

Proposition 4.3. *Suppose that $\alpha(0) = \tau(0) = 0$. Then $A_{\alpha,\tau}$ is a wandering vector multiplier if and only if both α and τ are group isomorphisms of $\mathbb{Z}$.*

Proof. For convenience let $A = A_{\alpha,\tau}$. First assume that A is a wandering vector multiplier. By Proposition 2.3 and Proposition 2.5 (for the irrational rotation unitary system case), A induces a Jordan homomorphism Φ on $w^*(\mathcal{U})$ such that

$$A T \psi = \Phi(T) \psi$$

for all $T \in w^*(\mathcal{U})$. Since $w^*(\mathcal{U})$ is a factor, it follows from Lemma 2.8 that Φ is either a *-homomorphism or a *-antihomomorphism.

If Φ is a *-homomorphism, then

$$U^{\alpha(n+m)} = A U^{n+m} \psi = \Phi(U^{n+m})\psi = \Phi(U^n)\Phi(U^m)\psi$$

for all $n, m \in \mathbb{Z}$. Thus $U^{\alpha(n+m)} = \Phi(U^n)\Phi(U^m)$ since ψ separates $w^*(\mathcal{G})$. In particular $U^{\alpha(n)} = \Phi(U^n)$ for all $n \in \mathbb{Z}$. Thus $U^{\alpha(n+m)} = U^{\alpha(n)}U^{\alpha(m)}$, which implies that $\alpha(n+m) = \alpha(n) + \alpha(m)$ for all $n, m \in \mathbb{Z}$. Hence α is an automorphism of $\mathbb{Z}$. If Φ is an *-antihomomorphism, the same argument works. Similarly, τ is an automorphism of $\mathbb{Z}$.

The converse is trivial since there are only two automorphisms of $\mathbb{Z}$, the identity and the inverse automorphism: $n \to -n$. $\square$

5. Wavelet Multipliers

For a general unitary system it is not clear how to characterize the set of all the wandering vector multipliers. In this section we investigate some special wandering vector multipliers for the unitary systems which are the ordered product of two unitary groups. This is an interesting class which contains the wavelet systems in

$L^2(\mathbb{R}^n)$. When we consider wavelet systems, for simplicity in this paper we will only consider the one-dimensional dyadic wavelet system.

The Fourier transform, $\hat{f}$, of a function $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is defined by

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(s)e^{-is\xi}ds,$$

where ds means integration with respect to Lebesgue measure. This transformation can be uniquely extended to an unitary operator $\mathcal{F}$ on $L^2(\mathbb{R})$. We write $\hat{D} =: \mathcal{F}D\mathcal{F}^{-1}$ and $\hat{T} =: \mathcal{F}T\mathcal{F}^{-1}$. It is not hard to check that $\hat{D} = D^{-1}$ and $\hat{T} = M_{e^{-is}}$, where $M_{e^{-is}}$ is the multiplication unitary operator with symbol e^{-is}. The unitary system $\{\hat{D}^m\hat{T}^n : n, m \in \mathbb{Z}\}$ is denoted by $\mathcal{U}_{\hat{D},\hat{T}}$. As in [DL], the wandering vector multipliers for $\mathcal{U}_{D,T}$ will be called *wavelet multipliers*.

For a unitary system $\mathcal{U} = \mathcal{U}_0\mathcal{U}_1$ which is the ordered product of two unitary groups $\mathcal{U}_0$ and $\mathcal{U}_1$, we always assume in this section that $\mathcal{U}_0 \cap \mathcal{U}_1 = \{I\}$ and that $\mathcal{W}(\mathcal{U})$ is non-empty.

Lemma 5.1. *Let $\mathcal{U} = \mathcal{U}_0\mathcal{U}_1$ be a unitary system on a Hilbert space H of the above form. If A normalizes $w^*(\mathcal{U}_1)$ and $U_0AU_0^*A^* \in w^*(\mathcal{U}_1)$ for all $U_0 \in \mathcal{U}_0$, then $A\psi$ is a cyclic vector for $\mathcal{U}$ whenever ψ is.*

Proof. Note that A^* also satisfies the above condition. Thus for any $U_0 \in \mathcal{U}_0$, we have

$$\begin{aligned}
U_0w^*(\mathcal{U}_1)A\psi &= U_0A(A^*w^*(\mathcal{U}_1)A)\psi \\
&= U_0Aw^*(\mathcal{U}_1)\psi \\
&= AU_0(U_0^*A^*U_0A)w^*(\mathcal{U}_1)\psi \\
&= AU_0w^*(\mathcal{U}_1)\psi.
\end{aligned}$$

Hence $A\psi$ is cyclic for $\mathcal{U}$ if ψ is. $\square$

Proposition 5.2. *Let $\mathcal{U}$ be as in Lemma 5.1. Suppose that $A \in B(H)$ is a unitary operator satisfying*

(i) $A \in \mathcal{U}_1'$,
(ii) $U_0AU_0^*A^* \in w^*(\mathcal{U})$ for all $U_0 \in \mathcal{U}_0$.
Then A is a wandering vector multiplier for $\mathcal{U}$.

Proof. By Lemma 5.1, it suffices to check the orthonormality of $\{U_0U_1A\psi : U_0 \in \mathcal{U}_0, U_1 \in \mathcal{U}_1\}$.

If $U_0 = I$, then $\{U_1A\psi : U_1 \in \mathcal{U}_1\}$ is clearly an orthonormal set since $U_1A = AU_1$ and $\{U_1\psi : U_1 \in \mathcal{U}_1\}$ is an orthonormal set. Now let $U_0 \in \mathcal{U}_0$ such that $U_0 \neq I$. Write $U_0AU_0^* = BA = AB$ for some unitary operator $B \in w^*(\mathcal{U}_1)$. Then for all $U, V \in \mathcal{U}_1$ we have

$$\begin{aligned}
\langle U_0UA\psi, V\psi \rangle &= \langle U_0AU\psi, AV\psi \rangle \\
&= \langle U_0AU_0^*U_0U\psi, AV\psi \rangle \\
&= \langle ABU_0U\psi, AV\psi \rangle \\
&= \langle BU_0U\psi, V\psi \rangle \\
&= \langle U_0U\psi, B^*V\psi \rangle.
\end{aligned}$$

Note that $B^* \in w^*(\mathcal{U}_1)$. Thus $\langle U_0UA\psi, V\psi \rangle = \langle U_0U\psi, B^*V\psi \rangle = 0$, as required. $\square$

If f is a unimodular function, then it was proved in [DGLL] and [Wut], by using the fundamental equation characterizations of wavelets, that M_f is a wavelet multiplier if and only if $f(2s)/f(s)$ is 2π-periodic (a.e. $s \in \mathbb{R}$). Proposition 5.2 does not imply this since the 2π-periodic property of $f(2s)/f(s)$ can only guarantee that $\hat{D}^{-n} M_f \hat{D}^n M_f^{-1} \in w^*(\hat{T})$ for all $n \geq 0$.

For some special unitary group $\mathcal{U}_1$, Proposition 5.2 can be extended as follows:

Theorem 5.3. *Let $\mathcal{U}$ be as in Lemma 5.1 such that $\mathcal{U}_1$ is an ICC group. Suppose that $A \in B(H)$ is a unitary operator such that*
  (i) A normalizes $w^*(\mathcal{U}_1)$, and
  (ii) $U_0 A U_0^* A^* \in w^*(\mathcal{U}_1)$ for all $U_0 \in \mathcal{U}_0$.
  Then A is a wandering vector multiplier for $\mathcal{U}$.

Proof. Let $\psi \in \mathcal{W}(\mathcal{U})$ be arbitrary. By Lemma 5.1 and the proof of Proposition 5.2, it suffices to prove the orthonormality of the set $\{UA\psi : U \in \mathcal{U}_1\}$. Let $H_\psi = [w^*(\mathcal{U}_1)\psi]$, the closed subspace generated by $\mathcal{U}_1\psi$. Then H_ψ is an invariant subspace of $\mathcal{U}_1$. Let $\mathcal{G}$ be the restriction unitary group of $\mathcal{U}_1$ to H_ψ. From the orthonormality of $\{U_1\psi : U_1 \in \mathcal{U}_1\}$, we have that $\mathcal{G}$ is also an ICC group which is group isomorphic to $\mathcal{U}_1$. Also observe that ψ is a complete wandering vector for $\mathcal{G}$. Thus $\mathcal{M}$, the von Neumann algebra generated by $\mathcal{G}$, is a factor.

Define a linear mapping $\Psi : \mathcal{M} \to \mathcal{M}$ by

$$\Phi(m) = A^* m A|_{H_\psi}.$$

Note that $A^* m A \in w^*(\mathcal{U}_1)$. Then Φ is a *-isomorphism. Since ψ is both cyclic and separating for $\mathcal{M}$, it follows from Theorem 7.2.9 in [KR] that there is a unitary operator $B \in B(H_\psi)$ such that $\Phi(m) = B^* m B$. Thus B is a normalizer of $\mathcal{M}$. By Theorem 2.14, B is a wandering vector multiplier for $\mathcal{G}$. So $B\psi \in \mathcal{W}(\mathcal{G})$. Therefore for any $U \in \mathcal{U}_1$ such that $U \neq I$, we obtain

$$\langle UA\psi,\ A\psi \rangle\ =\ \langle A^* UA\psi,\ \psi \rangle\ =\ \langle B^* UB\psi,\ \psi \rangle$$
$$=\ \langle UB\psi,\ B\psi \rangle = 0,$$

as required.   □

If we consider the special case that $\mathcal{U}_0$ is the trivial group (i.e., $\mathcal{U}_0 = \{I\}$), then Example 2.15 tells us that Theorem 5.3 is not valid in general even when $\mathcal{G}_1$ is abelian. However in the rest of this section we will show that Theorem 5.3 is true for wavelet systems $\mathcal{U}_{\hat{D}, \hat{T}}$.

Lemma 5.4. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be a measurable bijection such that*
  (i) $\phi(2s) = 2\phi(s)$, a.e. $s \in \mathbb{R}$, and
  (ii) both $e^{in\phi(s)}$ and $e^{in\phi^{-1}(s)}$ are 2π-periodic almost everywhere for all $n \in \mathbb{Z}$.
  Then ϕ is measure preserving.

Proof. Let μ denote Lebesgue measure on $\mathbb{R}$. We first claim that $\mu(\phi(0, 2\pi]) = 2\pi$.
  For every $n \in \mathbb{Z}$, let

$$F_n = \{s \in (0, 2\pi] : \phi(s) - 2n\pi \in (0, 2\pi]\}.$$

Then $\{F_n : n \in \mathbb{Z}\}$ forms a partition of $(0, 2\pi]$. In fact, if $s \in F_n \cap F_m$ and $n \neq m$, then $\phi(s) \in ((0, 2\pi] + 2n\pi) \cap ((0,\ 2\pi] + 2m\pi)$, which is impossible. Thus $\{F_n : n \in \mathbb{Z}\}$ is a disjoint family. Also for any $s \in (0, 2\pi]$, since $\bigcup_{k \in \mathbb{Z}}((0, 2\pi] + 2k\pi) = \mathbb{R}$, there exists n such that $\phi(s) - 2n\pi \in (0, 2\pi]$. Hence $s \in F_n$.

We check that $\{\phi(F_n) - 2n\pi : n \in \mathbb{Z}\}$ also forms a partition of $(0, 2\pi]$. Suppose that $\phi(s) - 2n\pi = \phi(t) - 2m\pi$ for some $n \neq m$ and for some $s \in F_n$ and $t \in F_m$. Then

$$\phi^{-1}(\psi(s) - 2n\pi) = \phi^{-1}(\phi(t) - 2m\pi).$$

So, by the assumption (ii), there is $l \in \mathbb{Z}$ such that $s - t = 2l\pi$. This implies $l = 0$ since $s, t \in (0, 2\pi]$, which is a contradiction. Thus $\{\phi(F_n) + 2n\pi : n \in \mathbb{Z}\}$ is a disjoint family. To show that the union is $(0, 2\pi]$, let $t \in (0, 2\pi]$. Then there is r such that $\phi(r) = t$. Write $r = s + 2m\pi$ with $s \in (0, 2\pi]$. Then by (ii) we have

$$t = \phi(r) = \phi(s + 2m\pi) = \phi(s) - 2l\pi$$

for some $l \in \mathbb{Z}$. Hence $s \in F_l$ and so $t \in \phi(F_l) - 2l\pi$, as required.

Now we have

$$\mu(\phi(0, 2\pi]) = \mu(\phi(\bigcup_{n \in \mathbb{Z}} F_n)) = \mu(\bigcup_{n \in \mathbb{Z}} \phi(F_n))$$

$$= \sum_{n \in \mathbb{Z}} \mu(\phi(F_n)) = \sum_{n \in \mathbb{Z}} \mu(\phi(F_n) - 2n\pi)$$

$$= \mu(\bigcup_{n \in \mathbb{Z}} (\phi(F_n) - 2n\pi)) = \mu((0, 2\pi]) = 2\pi.$$

Next we claim that $\mu(\phi(E + k\pi)) = \mu(\phi(E))$ for any measurable set E and any $k \in \mathbb{Z}$.

For any $s \in E$, conditions (i) and (ii) imply that there is $l \in \mathbb{Z}$ such that

$$2\phi(s + k\pi) = \phi(2s + 2k\pi) = \phi(2s) + 2l\pi = 2\phi(s) + 2l\pi.$$

Hence $\phi(s + k\pi) = \phi(s) + l\pi$.

Let $E_l = \{s \in E : \phi(s + k\pi) = \phi(s) + l\pi\}$. Then $\{E_l\}$ is a partition of E. Thus

$$\mu(\phi(E + k\pi)) = \mu(\bigcup_{l \in \mathbb{Z}} \phi(E_l + k\pi))$$

$$= \bigcup_{l \in \mathbb{Z}} \mu(\phi(E_l + k\pi))$$

$$= \bigcup_{l \in \mathbb{Z}} \mu(\phi(E_l))$$

$$= \mu(\phi(\bigcup_{l \in \mathbb{Z}} E_l)) = \mu(\phi(E)).$$

Finally, we claim that ϕ is measure preserving.

For any $k \in \mathbb{Z}$, by the above two claims, we have

$$\mu(\phi((k\pi, (k+1)\pi])) = \mu(\phi((0, \pi] + k\pi)) = \mu(\phi((0, \pi]))$$

$$= \mu(\frac{1}{2}\phi((0, 2\pi])) = \frac{1}{2}\mu(\phi((0, 2\pi])) = \pi.$$

Thus

$$\mu(\phi((\frac{k}{2^n}\pi, \frac{k+1}{2^n}\pi])) = \mu(\frac{1}{2^n}\phi((k\pi, (k+1)\pi])) = \frac{\pi}{2^n}$$

for all $n, k \in \mathbb{Z}$. Note that

$$\{(\frac{k}{2^n}\pi, \frac{k+1}{2^n}\pi] : k, n \in \mathbb{Z}\}$$

generates the Borel structure of $\mathbb{R}$. Thus ϕ is measure preserving. $\square$

We say that two measurable sets E and F of $\mathbb{R}$ are *translation congruent modulo* 2π if there exists a measurable bijection $\sigma : E \to F$ such that $\sigma(s) - s$ is an integral multiple of 2π for each $s \in E$.

Lemma 5.5. *Suppose that ϕ is as in Lemma 5.4. Then $\phi((0, \ 2\pi])$ is 2π-translation congruent to $(0, \ 2\pi]$.*

Proof. Let $t \in (0, \ 2\pi]$ and let $\phi(r) = t$. Write $r = s + 2m\pi$ for some (unique) $s \in (0, \ 2\pi]$ and some (unique) $m \in \mathbb{Z}$. Since $e^{i\phi(s)}$ is 2π-periodic, there is $k \in \mathbb{Z}$ such that $t = \phi(s + 2m\pi) = \phi(s) + 2k\pi$. So $t - \phi(s) \in 2\mathbb{Z}\pi$. Since ϕ is measure-preserving, we get that $\phi((0, \ 2\pi])$ is 2π-translation congruent to $(0, \ 2\pi]$. $\qquad\square$

Lemma 5.6. *Let A be a unitary operator such that*
    (i) A normalizes $w^*(\hat{T})$,
    (ii) $\hat{D}^n A \hat{D}^{-n} A^{-1} \in w^*(\hat{T})$.
    Then there exists a measurable bijection $\phi : \mathbb{R} \to \mathbb{R}$ and a function $f \in L^\infty(\mathbb{R})$ such that $A = B_\phi M_f$, where $(B_\phi h)(s) = h(\phi^{-1}(s))$ for all $h \in L^2(\mathbb{R})$.

Proof. Since $(D^n T^m D^{-n})h(s) = h(s + \frac{m}{2^n})$ and the set of all the dyadic numbers $\{\frac{m}{2^n} : n, m \in \mathbb{Z}\}$ is dense in $\mathbb{R}$, it follows that $\{D^n T^m D^{-n} : \ n, m \in \mathbb{Z}\}$ generates a maximal abelian von Neumann algebra. Thus the von Neumann algebra $\mathcal{M}$ generated by $\{D^{-n} M_{e^{ims}} D^n : \ m, n \in \mathbb{Z}\}$ is $\{M_g : \ g \in L^\infty(\mathbb{R})\}$.

For any $n \in \mathbb{Z}$, write $A^{-1} D^{-n} = D^{-n} A^{-1} M_g$ for some 2π-periodic unimodular function g. Then

$$A^{-1}(D^{-n} w^*(\hat{T}) D^n) A = D^{-n} A^{-1} M_g w^*(\hat{T}) M_g^* A D^n$$
$$= D^{-n} A^{-1} w^*(\hat{T}) A D^n = D^{-n} w^*(\hat{T}) D^n.$$

Thus A normalizes $\mathcal{M}$.

Define an isomorphism Φ on $\mathcal{M}$ by

$$\Phi(M_g) = A^* M_g A, \quad g \in L^\infty(\mathbb{R}).$$

Then, by Theorem 4.7 in [Pe, p. 16], there is a measurable bijective mapping $\phi : \mathbb{R} \to \mathbb{R}$ such that $\Phi(M_g) = M_{g(\phi(s))}$ for all $g \in L^\infty(\mathbb{R})$. That is, $A^{-1} M_g A = B_\phi^{-1} M_g B_\phi$. Therefore $B_\phi^{-1} A$ is in the commutant $\mathcal{M}'$ of $\mathcal{M}$. Since $\mathcal{M} = \mathcal{M}'$, it follows that there is a function $f \in L^\infty(\mathbb{R})$ such that $A = B_\phi M_f$, as required. $\qquad\square$

Theorem 5.7. *Let A be as in Lemma 5.5. Then $\mathcal{F}^{-1} A \mathcal{F}$ is a wavelet multiplier.*

Proof. Let $A = B_\phi M_f$ be as in Lemma 5.5. From $DAD^{-1} A^{-1} \in w^*(\hat{T})$, there is a 2π-periodic unimodular function $k(s)$ such that

$$(DAD^{-1} A^{-1} g)(s) = k(s) g(s)$$

for all $g \in L^2(\mathbb{R})$. Equivalently,

$$\frac{f(\phi^{-1}(2s))}{f(\frac{\phi^{-1}(2s)}{2})} g(\phi(\frac{\phi^{-1}(2s)}{2})) = k(s) g(s)$$

for all $g \in L^2(\mathbb{R})$. Let E be an arbitrary set of finite measure and let

$$F = E \cup \phi(\frac{\phi^{-1}(2E)}{2}).$$

Applying $g = \chi_F$ to the above equality, we get

$$k(s) = \frac{f(\phi^{-1}(2s))}{f(\frac{\phi^{-1}(2s)}{2})}, \quad s \in E.$$

Since E is arbitrary we have the equality for all $S \in \mathbb{R}$. Thus

$$h(\phi(\frac{\phi^{-1}(2s)}{2})) = h(s)$$

for $h \in L^2(\mathbb{R})$. Therefore $\phi(2s) = 2\phi(s)$, and thus $B_\phi \in \{D\}'$.

Since A normalizes $w^*(\hat{T})$ and $M_f \in w^*(\hat{T})'$, we have that $B_\psi^{-1} w^*(\hat{T}) B_\phi = w^*(\hat{T})$. This implies that both $e^{il\phi(s)}$ and $e^{il\phi^{-1}(s)}$ are 2π-periodic. Therefore, by Lemma 5.5, ϕ is measure preserving, and hence both B_ϕ and M_f are unitary. We need to show that both $\mathcal{F}^{-1} B_\phi \mathcal{F}$ and $\mathcal{F}^{-1} M_f \mathcal{F}$ are wavelet multipliers.

Since

$$D^n B_\phi M_f D^n M_f^{-1} B_\phi^{-1} D^{-n} = B_\phi (D^n M_f D^{-n} M_f^{-1}) B_\phi^{-1} \in w^*(\hat{T})$$

and $B_\phi^{-1} w^*(\hat{T}) B_\phi = w^*(\hat{T})$, it follows that $D^n M_f D^{-n} M_f^{-1} \in w^*(\hat{T})$. Thus, by Corollary 5.3 (or Proposition 5.2), $\mathcal{F}^{-1} M_f \mathcal{F}$ is a wavelet multiplier.

Observe that we already know that B_ϕ also satisfies the conditions (i) and (ii). To show that $\mathcal{F}^{-1} B_\phi \mathcal{F}$ is a wavelet multiplier, by the proof of Theorem 5.3, it suffices to show that $\{\hat{T}^l B_\phi \hat{h} : l \in \mathbb{Z}\}$ is an orthonormal set for any wavelet h. Note that $\sum_{k \in \mathbb{Z}} |h(s + 2k\pi)|^2 = 1$, a.e. $s \in \mathbb{R}$. Using Lemma 5.4, Lemma 5.5 and the fact that $e^{il\phi(s)}$ is 2π-periodic, we have

$$
\begin{aligned}
\langle \hat{T}^l B_\phi \hat{h}, B_\phi \hat{h} \rangle &= \int_{\mathbb{R}} e^{ils} |h(\phi^{-1}(s))|^2 ds \\
&= \int_{\mathbb{R}} e^{il\phi(s)} |h(s)|^2 ds \\
&= \sum_{k \in \mathbb{Z}} \int_{2k\pi}^{2(k+1)\pi} e^{il\phi(s)} |h(s)|^2 ds \\
&= \int_0^{2\pi} e^{il\phi(s)} \sum_{k \in \mathbb{Z}} |h(s + 2k\pi)|^2 ds \\
&= \int_0^{2\pi} e^{il\phi(s)} = \int_0^{2\pi} e^{ils} ds.
\end{aligned}
$$

Thus $\{\hat{T}^l B_\phi h : l \in \mathbb{Z}\}$ is an orthonormal set. $\square$

Questions. Does every wavelet multiplier A have the form $\hat{A} = B_\sigma M_f$ for some function $f \in L^\infty(\mathbb{R})$ and some measurable bijection $\sigma : \mathbb{R} \to \mathbb{R}$? Are the only measurable bijections σ for which $\mathcal{F}^{-1} B_\sigma \mathcal{F}$ is a wavelet multiplier the identity bijection $s \to s$ and the inverse bijection $s \to -s$?

Another Direction. The wandering vector multipliers for a unitary system $\mathcal{U}$ that were investigated in [DL], [Ha] and in the present article are by definition *unitary operators* on the underlying Hilbert space that map complete wandering vectors for $\mathcal{U}$ to complete wandering vectors for $\mathcal{U}$. However, as pointed out in Remark 2.17 (ii), a bounded invertible operator which maps complete wandering vectors to complete wandering vectors need not be unitary. It might be worthwhile to try to

extend known results for unitary wandering vector multipliers to wandering vector multipliers which are not necessarily unitary whenever possible. For instance, in Example 2.10, let σ be a measurable bijection of $\mathbb{T}$ which is not measure-preserving but is such that the composition operator A_σ is bounded and invertible. Then A_σ maps unimodular functions to unimodular functions on $\mathbb{T}$, so is a wandering vector multiplier which is essentially different from the type we have considered in this article. In the general group unitary system case the same type of non-unitary wandering vector multiplier can arise from a Jordan isomorphism of $w^*(\mathcal{G})$ which is not isometric in the Hilbert-Schmidt metric on $w^*(\mathcal{G})$ but is bounded above and below so extends to a (non-unitary) bounded invertible operator on $L^2(w^*(\mathcal{G}), \tau)$. In fact, as pointed out in Remark 2.17 (ii), there is a general factorization result for this case. In [Wut] it was *not* taken as a matter of definition that wavelet multiplier functions were unimodular, but rather, it was *proven* that a wavelet multiplier function *must* be unimodular. This is in the same spirit. (Guido Weiss was the member of the WUTAM CONSORTIUM who suggested that the unimodularity should be proven and not simply assumed.) The following interesting *open question* remains:

Question. Is an (operator) wavelet multiplier which is a bounded linear operator necessarily a unitary operator?

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Department of Mathematics, University of Central Florida, Orlando, Florida 32816-1364

E-mail address: `dhan@pegasus.cc.ucf.edu`

Department of Mathematics, Texas A&M University, College Station, Texas 77843

E-mail address: `David.Larson@math.tamu.edu`