ON POSITIVITY OF LINE BUNDLES
ON ENRIQUES SURFACES

TOMASZ SZEMBERG

ABSTRACT. We study linear systems on Enriques surfaces. We prove rationality of Seshadri constants of ample line bundles on Enriques surfaces and provide lower bounds on these numbers. We show the nonexistence of k-very ample line bundles on Enriques surfaces of degree $4k + 4$ for $k \geq 1$, thus answering an old question of Ballico and Sommese.

INTRODUCTION

In this note we answer a question posed by Ballico and Sommese [1] (see also Terakawa [14]) concerning the existence of k–very ample line bundles of degree $4k + 4$ on Enriques surfaces. We show that such line bundles exist if and only if $k = 0$ and that in fact the degree of a k–very ample line bundle on an Enriques surface is subject to a much more restrictive bound (Theorem 2.4). This has strong consequences for the local positivity of line bundles on Enriques surfaces. We address this questions in section 3. In particular we show that Seshadri constants of ample line bundles on Enriques surfaces are always rational.

1. AUXILIARY MATERIAL AND NOTATION

The basic reference for this paper is the monograph [6] by Cossec and Dolgachev. Here we briefly recall properties of Enriques surfaces needed for our considerations. First of all, an Enriques surface is a surface Y of Kodaira dimension zero with no differential forms. In particular, $h^0(K_Y) = 0$ and the canonical bundle K_Y is a 2–torsion element in $\mathrm{Pic}^0(Y)$, i.e. $2K_Y \cong \mathcal{O}_Y$. To alleviate notation it is convenient to use the following

Convention. In the sequel we make frequent use of vanishing theorems and statements on adjoint line bundles. Since the canonical divisor K_Y is numerically trivial, the adjoint line bundle $K_Y + L$ has the same numerical properties as L. In particular, if L is nef and big (ample) then so is $K_Y + L$ (by the Nakai-Moishezon criterion). So in order to get vanishing statements for L we apply vanishing theorems to $K_Y + L$ and use the fact that $2K_Y$ is trivial. In the sequel we do so without comment, and hope to cause no confusion this way.

The group of numerical equivalence classes on an Enriques surface together with the bilinear symmetric form induced by the intersection of divisors is a unimodular lattice $\mathbb{E}$ of rank 10 isomorphic to $\mathbb{H} \oplus E_8$, where $\mathbb{H}$ resp. E_8 are the even unimodular lattices of signature $(1,1)$, resp. $(0,8)$. We denote by $\mathbb{E}_0$ the set of all isotropic vectors in the lattice and by $\mathbb{E}^+$ the set of all $x \in \mathbb{E}$ with $x^2 > 0$. The function

$$\phi : \mathbb{E} \ni x \longrightarrow \min_{f \in \mathbb{E}_0 \setminus \{0\}} |x.f| \in \mathbb{Z}_{\geq 0}.$$

studied in [6] turned out to be important in understanding line bundles of small degree on Enriques surfaces. We extend its applicability to k–very ample line bundles. The important properties of the function ϕ are summarized in the following

Proposition 1.1 (Looijenga, Cossec-Dolgachev). *For any vector $x \in \mathbb{E}^+$ we have*

$$0 < (\phi(x))^2 \leq x^2.$$

Moreover, there exists $x \in \mathbb{E}^+$ such that $(\phi(x))^2 = x^2$.

For the proof we refer to [6, Proposition 2.7.1 and Corollary 2.7.1].

An isotropic vector $f \in \mathbb{E}_0$ is represented by an effective class F. If F is reduced and irreducible, then either F or $2F$ gives rise to an elliptic pencil on Y. The existence of elliptic fibrations on Enriques surfaces is one of their most important features. If $2F$ is an elliptic pencil, then it contains exactly two double fibers $2F$ and $2F'$. The underlying reduced curves F, F' are called halfpencils. It is clear from the definition that $\phi(x)$ is computed by halfpencils. The general fiber of an elliptic pencil $2F$ is a smooth elliptic curve. If Y contains no effective (-2)–curves (is *unnodal*) then *every* fiber of $2F$ is irreducible. A generic Enriques surface is unnodal.

We work throughout the paper over the field of complex numbers $\mathbb{C}$.

2. Higher order embeddings

Definition 2.1. Let (X, L) be a smooth polarized variety. We say that the line bundle L is k–very ample if for all zero-dimensional subschemes $(Z, \mathcal{O}_Z) \subset (X, \mathcal{O}_X)$ of length $\leq k+1$ the restriction mapping

$$H^0(L) \longrightarrow H^0(L \otimes \mathcal{O}_Z)$$

is surjective.

Note that a line bundle is 0–very ample iff it is globally generated, and 1–very ample iff it is very ample. Geometrically, for $k \geq 1$, k–very ampleness means that under the embedding defined by L there are no $(p+1)$–secant $(p-1)$–planes to X for $p \leq k$. Equivalently, any 0–dimensional subscheme Z of length p with $p \leq k+1$ imposes independent conditions on global sections of L, i.e.

$$h^0(L \otimes \mathcal{I}_Z) = h^0(L) - \text{length}(Z).$$

Definition 2.1 can be extended verbatim to singular varieties. In particular, note that it follows at once that, given a subvariety $Y \subset X$ and a k–very ample line bundle L on X, the restriction $L|_Y$ is also k–very ample. In the sequel we shall need a bound for the degree of a k–very ample line bundle on a singular curve. For the clarity of the exposition we give an ad hoc argument dealing with the case we are really interested in, rather than developing a systematic theory.

Lemma 2.2. *Let C be an irreducible singular curve of arithmetic genus 1 and let L be a k–very ample line bundle on C. Then $\deg(L) \geq k + 2$.*

Proof. C is either a nodal or a cuspidal curve. We give the argument for the nodal curve, the cuspidal case being similar. Let $\sigma : \mathbb{P}^1 \cong \widetilde{C} \longrightarrow C$ be the normalization with $\sigma(P_1) = \sigma(P_2) = P$, where P is the node on C. Let $M = \sigma^* L$; then $M = \mathcal{O}_{\mathbb{P}^1}(d)$ with $d = \deg(L)$. Note that the subspace

$$V_d = \{s \in H^0(\mathcal{O}_{\mathbb{P}^1}(d)) \text{ such that } s(P_1) = s(P_2)\}$$

is bijective via σ to the space of global sections of L. It remains to show that the linear system V_d fails to be k–very ample if $d \leq k + 1$. To this end it is enough to consider the case $d = k + 1$. Taking on $\widetilde{C}$ coordinates $(x : y)$ such that $P_1 = (0 : 1)$ and $P_2 = (1 : 0)$, we see that all sections in V_{k+1} are of the form

$$s(x : y) = \alpha_0(x^{k+1} + y^{k+1}) + \alpha_1 x^k y + \cdots + \alpha_k x y^k$$

with $(\alpha_0 : \cdots : \alpha_k) \in \mathbb{P}^k$. Then any set of points $(x_i : 1)$, $i = 0, \ldots, k$, satisfying $x_0 \cdots x_k = (-1)^{k+1}$ cannot be separated by sections in V_{k+1}. This shows that V_d and hence L is not k–very ample for $d \leq k + 1$. $\square$

On surfaces the following result due to Beltrametti and Sommese [4] generalizes Reider's Theorem [13] to k–very ample line bundles and provides a useful criterion for k–very ampleness of adjoint line bundles.

Proposition 2.3. *Let X be a smooth surface and let L be an ample line bundle on X such that $L^2 \geq 4k + 5$. Then either $K_X + L$ is k–very ample, or there exists an effective divisor D satisfying the following conditions:*

(1) *$L - 2D$ is $\mathbb{Q}$-effective, i.e. there exists a positive integer m such that*

$$|m(L - 2D)| \neq \emptyset.$$

(2) *D contains a subscheme Z of the length $k + 1$, such that the mapping*

$$H^0(K_X \otimes L) \to H^0(K_X \otimes L \otimes \mathcal{O}_Z)$$

is not surjective.

(3) *$LD - k - 1 \leq D^2 < \frac{LD}{2} < k + 1$.*

In the view of the above proposition the problem of k–very ampleness of adjoint line bundles breaks up naturally into two cases: when $L^2 \geq 4k + 5$ and when $L^2 \leq 4k + 4$. The latter case was studied by Ballico and Sommese [1]. In particular it remained an open question if there exist polarized Enriques surfaces (Y, L) such that $L^2 = 4k + 4$ and $K_Y + L$ is k–very ample [1, Theorem 6.1 and Remark 6.2]. This problem appeared (unsolved) recently also in a paper by Terakawa [14]. Here we show that the answer is negative if $k \geq 1$. Existence of globally generated line bundles of degree 4 on Enriques surfaces was established by Verra [15]. On the other hand it was known, e.g. thanks to the classification of Okonek [12], that there are no embeddings of Enriques surfaces of degree 8. Thus the negative answer was expected. Yet it is somewhat surprising that the degree of k–very ample line bundles on Enriques surfaces is subject to a much stronger restriction.

Theorem 2.4. *Let Y be a smooth Enriques surface and L a k–very ample line bundle on Y. Then $\phi(L) \geq k + 2$. In particular, $L^2 \geq (k + 2)^2$.*
If Y is unnodal, then the converse is also true.

Proof. Let f be an isotropic vector in $\mathbb{E}_0$ such that $\phi(L) = L.f$. Without loss of generality we can assume that f represents a connected and reduced curve F of arithmetic genus 1. All we need to show is that $L.F \geq k + 2$.

Now, the proof amounts only to remarking that the restriction of L to any curve $C \subset Y$ is k–very ample. If F is a smooth elliptic curve then we are done immediately. On the other hand, singular elliptic fibers are classified, see e.g. [2, V. table 3]. If F is a chain of rational curves, then L restricted to any of the components has degree $\geq k$, and the claim follows. In fact we need here $k \geq 2$, but for $k = 1$ the assertion follows from Okonek's classification [12]. In the remaining cases F is a rational curve with a node or a cusp, and the assertion follows from Lemma 2.2. Proposition 1.1 implies immediately that in this case $L^2 \geq (k+2)^2$.

For the converse, assume that Y is unnodal and L is an ample line bundle on Y with $L^2 \geq (k+2)^2$. Since $K_Y + L$ satisfies the same numerical conditions as L (in particular is ample by the Nakai-Moishezon criterion), we can apply Proposition 2.3 to prove the k–very ampleness of L itself. So assume that L fails to be k–very ample; then there exists an effective divisor $D \subset Y$ such that

$$(2.1) \qquad LD - k - 1 \leq D^2 < \frac{LD}{2} < k + 1.$$

If $D^2 \geq 4$, then the Hodge Index Theorem and the last of the above inequalities implies that

$$(2k+4)^2 = 4(2+k)^2 \leq L^2 D^2 \leq (L.D)^2 \leq (2k+1)^2,$$

a contradiction.

If $D^2 = 2$, then by the Hodge Index Theorem again we have $L.D \geq (2+k)\sqrt{2}$, and the first inequality in (2.1) gives

$$(2+k)\sqrt{2} - k - 1 \leq 2,$$

which is possible only if $k = 0$. But then $L.D = 1$, and this contradicts the middle inequality.

Hence $D^2 = 0$, since Y is unnodal. Then we get $L.D \leq k + 1$, which contradicts our assumption on $\phi(L)$. $\qquad\square$

It is well-known that if L is an ample line bundle on an Enriques surface then $2L$ is globally generated and $3L$ is very ample. This generalizes easily to the following

Proposition 2.5. *Let L be an ample line bundle on an Enriques surface. Then for $n \geq k + 2$ the line bundle nL is k–very ample.*

Proof. Since nL intersects every curve with multiplicity $\geq k + 2$, we have $\phi(L) \geq k+2$. The proof of Theorem 2.4 goes through, as the first inequality in (2.1) asserts that the k–very ampleness of L cannot fail on a (-2)–curve. $\qquad\square$

Remark 2.6. Proposition 2.5 gives a naive criterion for generation of jets by a multiple of an ample line bundle L on an Enriques surface Y. Indeed, from a result of Langer [8, Theorem 2.2] it follows that nL generates s–jets at $x \in Y$ provided $n \geq \left\lfloor \frac{s(s+4)}{4} \right\rfloor + 2$. Using different techniques, we significantly improve this bound in Theorem 4.2.

3. Seshadri constants

In recent years there has been considerable interest in understanding the local positivity of ample line bundles on algebraic varieties. Seshadri constants, introduced by Demailly [7], emerged as a natural measure of the local positivity of a line bundle.

Let X be a smooth projective variety of dimension n, and L an ample line bundle on X. Then the real number

$$\varepsilon(L, x) =_{\text{def}} \inf_{C \ni x} \frac{L \cdot C}{\text{mult}_x C}$$

is the *Seshadri constant of L at $x \in X$*. (The infimum is taken over all irreducible curves passing through x.) The Seshadri constant of L is defined in turn as

$$\varepsilon(L) = \inf_{x \in X} \varepsilon(L, x).$$

These invariants are very hard to control and their exact value is known only in a very few cases. Therefore it is reasonable to ask for bounds on Seshadri constants. For an exposition and main results on Seshadri constants on surfaces we refer to [3]. Here we begin with the lemma giving a criterion for rationality of $\varepsilon(L)$ (we conjecture that $\varepsilon(L)$ is always rational).

Lemma 3.1. *Let (X, L) be a polarized surface. If there exists a point $x \in X$ such that the Seshadri constant of L at x is submaximal, $\varepsilon(L, x) < \sqrt{L^2}$, then $\varepsilon(L)$ is a rational number.*

Proof. The following argument was suggested by Thomas Bauer. The claim follows also from recent results of Oguiso [11, Corollary 2].

Suppose that $\alpha = \varepsilon(L)$ and there exists a sequence (C_n, x_n) of irreducible curves and points on X such that

$$\alpha_n = \frac{L.C}{\text{mult}_{x_n} C_n} \longrightarrow \alpha.$$

Let β be a rational number in the interval $\alpha < \beta < \sqrt{L^2}$. Without loss of generality we can assume that $\alpha_n < \beta$ for all n. It follows from the Riemann-Roch Theorem that there exist a positive integer q (not depending on n) and a sequence of divisors $D_n \in |qL|$ such that

$$\frac{L.D_n}{\text{mult}_{x_n} D_n} < \sqrt{L^2} + \delta,$$

where δ satisfies $0 < \delta < \frac{L^2 - \sqrt{L^2}\beta}{\beta}$. Assuming that C_n is not a component of D_n, we have

$$qL.C_n = D_n.C_n \geq \text{mult}_{x_n} D_n \cdot \text{mult}_{x_n} C_n > \frac{qL^2}{\sqrt{L^2} + \delta} \text{mult}_{x_n} C_n,$$

which gives $\frac{L.C_n}{\text{mult}_{x_n} C_n} > \beta$, a contradiction. Hence every curve C_n is a component of D_n. This shows that the degree of C_n is uniformly bounded. But then there are only finitely many possible multiplicities of curves C_n, so that the sequence α_n, being convergent, must in fact stabilize. This shows that α is a rational number. $\square$

Corollary 3.2. *Let (X, L) be a polarized surface. If L^2 is a square then $\varepsilon(L)$ is a rational number.*

Proof. Either $\varepsilon(L)$ is maximal, and then $\varepsilon(L) = \sqrt{L^2}$; or it is submaximal and Lemma 3.1 applies. $\qquad\square$

Theorem 3.3. *Let (Y, L) be a polarized Enriques surface. Then $\varepsilon(L)$ is rational.*

Proof. In the view of the above lemma and its corollary it is enough to consider L with $\sqrt{L^2}$ irrational and to find in this case a point $x \in Y$ with $\varepsilon(L, x)$ submaximal. Since $\phi(L) \leq \sqrt{L^2}$ by Proposition 1.1 and $\phi(L) < \sqrt{L^2}$ by irrationality, there exists an elliptic pencil $|E|$ with $\frac{1}{2}L.E < \sqrt{L^2}$.

Let x be an arbitrary point on a halfpencil E_0 of E. Then $\varepsilon(L, x) \leq L.E_0 < \sqrt{L^2}$. $\qquad\square$

Now, we introduce the following notation. For a non–negative integer g let $\varepsilon_g(L, x)$ be a genus g Seshadri constant of L at x, i.e.

$$\varepsilon_g(L, x) := \inf \frac{L.C}{\mathrm{mult}_x C},$$

where the infimum is taken over all irreducible curves of arithmetic genus g passing through x. Note that $\varepsilon_g(L, x) \geq \frac{1}{m}$, where m is the maximal integer such that $m(m - 1) \leq 2g$. The following theorem gives a lower bound for the value of Seshadri constants on Enriques surfaces.

Theorem 3.4. *Let (Y, L) be a polarized Enriques surface and $x \in Y$ an arbitrary point. Then*

$$\varepsilon(L, x) \geq \min \left\{ \varepsilon_0(L, x), \varepsilon_1(L, x), \frac{1}{4}\sqrt{L^2} \right\}.$$

Proof. By Proposition 1.1 there exists a genus one pencil E such that $\frac{1}{2}E.L \leq \sqrt{L^2}$. Let $E_x \in E$ be the fiber passing through x. Given $\delta > 0$ and a rational number $\frac{p}{q}$ such that

$$\frac{\sqrt{L^2}}{4} - \delta \leq \frac{p}{q} \leq \frac{\sqrt{L^2}}{4},$$

we consider the divisor

$$N = qL - pE.$$

Our first claim is that the linear system $|N|$ is non–empty. Indeed by Riemann-Roch and [2, Prop. VIII.16.1.ii] it is enough to check that $N^2 \geq 0$ and $N.L > 0$. This is easily verified.

Let $f : \widetilde{Y} \longrightarrow Y$ be the blowing up of x with exceptional divisor Z. We claim that either $M := qf^*L - pZ$ is nef or there exists a rational or an elliptic curve spoiling the nefness of M.

Clearly $M.Z = p > 0$, so we need to test the nefness on irreducible curves D coming from Y, i.e. on the curves of the form $D = f^*C - \mathrm{mult}_x C \cdot Z$. For D we have

$$M.D = qL.C - p \cdot \mathrm{mult}_x C.$$

Taking $R \in |N|$, we have $R + pE_x \in |qL|$ and

$$qL.C = R.C + pE_x.C.$$

If C is not a (-2)–curve then $R.C \geq 0$, and if $C \neq E_x$ then of course $C.E_x \geq \text{mult}_x C$, and the claim follows. This shows that

$$\varepsilon(L,x) \geq \min\left\{\varepsilon_0(L,x), \varepsilon_1(L,x), \frac{p}{q}\right\}.$$

Letting δ converge to zero, we get the assertion. □

For an ample line bundle L on an Enriques surface, $2L$ is globally generated by a result of Cossec [5, Corollary 8.3.3]. Since the Seshadri constant of an ample and globally generated line bundle is ≥ 1 (cf. [9, Exercise 5.5]), we have

$$\text{(3.1)} \qquad\qquad\qquad \varepsilon(L,x) \geq \frac{1}{2}$$

for any $x \in Y$ and L an ample line bundle on Y. In view of this inequality the bound given in Theorem 3.4 might appear weak if the degree of L is small. This is fixed by the following proposition, which shows that $\varepsilon(L)$ strictly smaller than 1 can be only caused by the existence of a singular elliptic curve on Y of degree 1 with respect to L.

Proposition 3.5. *Let (Y,L) be a polarized Enriques surface. Then $\varepsilon(L) < 1$ if and only if there exist $x \in Y$ and an irreducible curve E with $L.E = 1$, $p_a(E) = 1$ and $\text{mult}_x E = 2$. Moreover, $\varepsilon(L) = \frac{1}{2}$ in this case.*

Proof. To begin with we assume that $\phi(L) = 1$. Let $x \in Y$ be a point with $\varepsilon(L,x) < 1$. Let F be an elliptic curve with $L.F = 1$ and let $E_x \in |2F|$ be the fiber passing through x. Then

$$N = 2L - \frac{L^2}{2}E_x$$

is effective and the arguments of the proof of Theorem 3.4 go through to the effect that $\varepsilon(L,x)$ must be computed by a component of E_x. In fact, in order to satisfy

$$\frac{L.E_x}{\text{mult}_x E_x} = \frac{2}{\text{mult}_x E_x} < 1$$

E_x must be a double elliptic curve with a double point at x, as asserted.

If $\phi(L) \geq 2$ then [6, Theorem 4.4.1] implies that L is globally generated; hence $\varepsilon(L) \geq 1$ in this case. □

4. Generation of jets

Seshadri constants studied in the previous section are closely related to the generation of jets by an ample line bundle L on a surface X. More precisely, if $\varepsilon(L,x) \geq s+2$ and $L^2 > (s+2)^2$, then $K_X + L$ generates s–jets at x [9, Proposition 5.7]. This fact combined with (3.1) leads to the following criterion which already improves our previous estimate given in Remark 2.6.

Corollary 4.1. *Let (Y,L) be a polarized Enriques surface. Then nL generates s–jets at an arbitrary point of Y for $n \geq 2(s+2) = 2s + 4$.*

Though close to optimal, this bound can still be improved.

Theorem 4.2. *Let (Y,L) be a polarized Enriques surface and let $x \in Y$ be a point with $\varepsilon(L,x) = \frac{1}{2}$. If $L^2 \geq 4$ then nL generates s–jets at x provided $n \geq 2s + 1$. If $L^2 = 2$ then $n \geq 2s + 2$ is necessary. Moreover, both bounds are sharp.*

Proof. Let $d = L^2$ and assume $d \geq 4$. As in the proof of Proposition 3.5, let E_x be the elliptic curve singular at x with $L.E_x = 1$. Then the linear system $N = L - \frac{d}{2}E_x$ is effective. Let $R \in |N|$ be an effective divisor. Note that since $R.E_x = L.E_x = 1$, the curve R doesn't pass through x.

Let $\tilde{Y} \longrightarrow Y$ be the blowing up of Y at x with exceptional divisor Z. It is enough to show the vanishing

$$(4.1) \quad \begin{aligned} &H^1(Y, nL \otimes \mathrm{m}_x^{s+1}) \\ &= H^1(\tilde{Y}, nf^*L - (s+1)Z) = H^1(\tilde{Y}, K_{\tilde{Y}} + nf^*L - (s+2)Z) = 0, \end{aligned}$$

where the first equality follows from the Leray spectral sequence. Consider

$$A = nf^*L - (s+1)Z$$

and

$$D = \frac{3}{2d-1}\left(f^*\left(\frac{d}{2}E_x + R\right) - \frac{d+1}{3}Z\right).$$

The $\mathbb{Q}$–divisor

$$A - D = \left(n - \frac{3}{2d-1}\right)f^*L - \left(s+1 - \frac{d+1}{2d-1}\right)Z$$

is nef since $n - \frac{3}{2d-1} \geq 2(s+1 - \frac{d+1}{2d-1})$, and big since $(A - D)^2 > 0$. Hence by the Nadel Vanishing Theorem (see [10, Theorem 4.5])

$$(4.2) \quad H^1(\tilde{Y}, \mathcal{O}_{\tilde{Y}}(K_{\tilde{Y}} \otimes A) \otimes \mathcal{J}(D)) = 0,$$

where $\mathcal{J}(D)$ denotes the multiplier ideal sheaf of D.

Now, we have

$$D = \frac{3d}{4d-2}E_x' + \frac{3}{2d-1}R' + Z,$$

where E_x' and R' denote proper transforms. It follows that $\mathrm{mult}_Z D = 1$ and $\mathrm{mult}_y D < 1$ for $y \notin Z$. Indeed R' has at most double points (again since $R.E_x = 1$) and $\frac{3}{2d-1} < \frac{1}{2}$. By [10, Proposition 5.11 and Example 3.4] $\mathcal{J}(D) = \mathcal{I}_Z$, and (4.2) implies the vanishing (4.1) as desired.

Now we show that nL doesn't generate s–jets at x if $n \leq 2s$. Since L is generated at x, it is sufficient to prove the assertion for $n = 2s$. The exact sequence

$$0 \longrightarrow 2sL \otimes \mathrm{m}_x^{s+1} \longrightarrow 2sL \longrightarrow 2sL \otimes \mathcal{O}_Y/\mathrm{m}_x^{s+1} \longrightarrow 0$$

together with the vanishing $H^1(2sL) = 0$ (cf. [6, Theorem 1.5.1]) implies that our claim is equivalent to showing that

$$H^1(Y, 2sL \otimes \mathrm{m}_x^{s+1})$$

doesn't vanish. This in turn is equivalent to non–vanishing of

$$H^1(\tilde{Y}, 2sf^*L - (s+1)Z).$$

Note that E_x' is nothing but the normalization of E_x; hence $E_x' \cong \mathbb{P}^1$. Consider the exact sequence

$$(4.3) \quad 0 \longrightarrow \mathcal{O}_{\tilde{Y}}(2sf^* - E'_x - (s+1)Z)$$
$$\longrightarrow \mathcal{O}_{\tilde{Y}}(2sf^*L - (s+1)Z) \longrightarrow \mathcal{O}_{E'_x}(2sf^*L - (s+1)Z) \longrightarrow 0.$$

Since $f^*L.E'_x = 1$ and $E'_x.Z = 2$, we have

$$\mathcal{O}_{E'_x}(2sf^*L - (s+1)Z) \cong \mathcal{O}_{\mathbb{P}^1}(-2).$$

Moreover, $H^2(\mathcal{O}_{\tilde{Y}}(2sf^* - E'_x - (s+1)Z)) = 0$ by Serre duality and negative intersection of the resulting divisor with the nef line bundle f^*L. Hence the long cohomology sequence associated to (4.3) splits and our assertion follows from

$$h^1(2sf^*L - (s+1)Z) = h^1(2sf^* - E'_x - (s+1)Z) + 1 > 0.$$

The case $d = 2$ requires small modifications of the above argument and is left to an interested reader. $\square$

Remark 4.3. Finally we observe that if $x \in Y$ is a point with $\varepsilon(L, x) \geq 1$, then nL generates s–jets at x for $n \geq s + 2$ and that without further assumptions on L this bound cannot be improved. This follows again from [9, Proposition 5.7].

Acknowledgment

References

[1] Ballico, E., Sommese, A.: Projective surfaces with k-very ample line bundles of degree $\leq 4k + 4$, Nagoya Math. J. 136, 57-79 (1994) MR **96d:**14005

[2] Barth, W., Peters, C., van de Ven, A.: Compact complex surfaces. Ergeb. Math. Grenzgeb. (3) 4, Springer-Verlag, 1984. MR **86c:**32026

[3] Bauer, Th.: Seshadri constants on algebraic surfaces. Habilitationsschrift, Univ. Erlangen-Nürnberg, 1998. Math. Ann. 313, 547-583 (1999) MR **2000d:**14006

[4] Beltrametti, M.C., Sommese, A.J.: Zero cycles and k-th order embeddings. Projective surfaces and their classification, Symp. Math., INDAM **32**, 33-48, Academic Press 1991 MR **95d:**14005

[5] Cossec, F.: Projective models of Enriques surfaces, Math. Ann. 265, 283-334 (1983) MR **86d:**14035

[6] Cossec, F., Dolgachev I.: Enriques surfaces I, Progress in Mathematics 76, Birkhäuser 1989 MR **90h:**14052

[7] Demailly, J.-P.: Singular Hermitian metrics on positive line bundles. Complex algebraic varieties (Bayreuth, 1990), Lect. Notes Math. 1507, Springer-Verlag, 1992, pp. 87-104. MR **93g:**32044

[8] Langer, A.: A note on k-jet ampleness on surfaces, in *Algebraic geometry: Hirzebruch 70 (Warsaw, 1998)*, 273–282, Contemp. Math., 241, Amer. Math. Soc., Providence, RI, 1999 MR **2000j:**14013

[9] Lazarsfeld, R.: Lectures on linear series. Park City/IAS Mathematics series vol. 3, 161–219 (1997) MR **98h:**14008

[10] ――――, Multiplier Ideals for Algebraic Geometers, `http://www.math.lsa.umich.edu/~rlaz/MIAG.ps`

[11] Oguiso, K.: Seshadri constants for a family of surfaces, preprint, Essen 2000, `http://www.expmath.uniessen.de/~forscher/alggeo.html`

[12] Okonek, Ch.: Flächen vom Grad 8 in $\mathbb{P}^4$, Math. Z. 191, 207-223 (1986) MR **87m:**14058

[13] Reider, I.: Vector bundles of rank 2 and linear systems on algebraic surfaces. Ann. Math. **127**, 309-316 (1988) MR **89e:**14038

[14] Terakawa, H.: Higher order embeddings of algebraic surfaces of Kodaira dimension zero, Math. Z. 229, 417-433 (1998) MR **99m:**14013

[15] Verra, A.: On Enriques surfaces as a fourfold cover of $\mathbb{P}^2$, Math. Ann. 266, 241-250 (1983) MR **85b:**14051

INSTITUTE OF MATHEMATICS, REYMONTA 4, PL-30-059 KRAKÓW, POLAND

Current address: Department of Mathematics, University of Essen, FB6 Mathematik, D-45117 Essen, Germany

E-mail address: szemberg@im.uj.edu.pl

URL: http://www.im.uj.edu.pl/~szemberg