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TOPOLOGICAL MIXING IN $CAT\,(-1)$-SPACES

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Abstract. If X is a proper $CAT\,(-1)$-space and Γ a non-elementary discrete group of isometries acting properly discontinuously on X, it is shown that the geodesic flow on the quotient space $Y = X/\Gamma$ is topologically mixing, provided that the generalized Busemann function has zeros on the boundary ∂X and the non-wandering set of the flow equals the whole quotient space of geodesics $GY := GX/\,\Gamma$ (the latter being redundant when Y is compact). Applications include the proof of topological mixing for (A) compact negatively curved polyhedra, (B) compact quotients of proper geodesically complete $CAT\,(-1)$-spaces by a one-ended group of isometries and (C) finite n-dimensional ideal polyhedra.

1. Introduction and preliminary results

The extensive study of the geodesic flow, primarily on Riemannian manifolds, has been concerned, among other properties, with the establishment of topological transitivity and topological mixing. For compact manifolds of negative curvature, topological transitivity of the geodesic flow was proved by Anosov in [1]. Topological mixing, a stronger property, has been shown for the class of compact manifolds with non-positive curvature by P. Eberlein in [16]. In this paper we establish topological mixing of the geodesic flow in certain classes of spaces which are quotients of proper $CAT\,(-1)$-spaces by a non-elementary discrete group of isometries. A $CAT\,(-1)$-space is a geodesic metric space in which every hyperbolic triangle is thinner than its associated comparison triangle in the hyperbolic plane (for definitions and basic properties see [2], [3], [4] and [17]). The $CAT\,(-1)$ property is (locally) one among many possible generalizations to singular spaces of the notion of negative curvature. Important examples of $CAT\,(-1)$-spaces include Riemannian manifolds of sectional curvature ≤ -1, metric trees and simply connected cell complexes of negative curvature.

Throughout this paper we will use the letter X to denote a proper $CAT\,(-1)$-space. Let Γ be a *non-elementary discrete* group of isometries of X, i.e., the cardinality of the limit set of the action of Γ on X is > 2 (see section 1.2 below) and consider the quotient metric space $Y = X/\,\Gamma$. Recall that the space of geodesics GX consists of all isometries $g : \mathbb{R} \to X$ and its topology is that of uniform convergence on compact sets. The action of Γ on X induces an isometric action of Γ on the space GX which is also properly discontinuous. Hence, the space of geodesics GY is defined as the quotient metric space $GX/\,\Gamma$ (see section 1.2 below for details).

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We will use the letter p to denote both projections $X \to Y$ and $GX \to GY$. As the action of Γ on X is not necessarily free, observe that an element $g \in GY$ is not a geodesic in the usual sense; it is just a continuous map $g : \mathbb{R} \to Y$ for which there exists an isometry $\overline{g} : \mathbb{R} \to X$ such that $g = p \circ \overline{g}$. The geodesic flow on X is defined by the map

$$\mathbb{R} \times GX \to GX$$

where the action of $\mathbb{R}$ is given by a right translation, i.e., for all $t \in \mathbb{R}$ and $g \in GX$, $(t, g) \to t \cdot g$ where $t \cdot g : \mathbb{R} \to X$ is the geodesic defined by $(t \cdot g)(s) = g(s + t), s \in \mathbb{R}$. If $t \in \mathbb{R}$ and $g \in GY$ define the geodesic flow on GY by setting

$$t \cdot g = p\big(t \cdot \overline{g}\big)$$

where $\overline{g}$ is any lift of g in GX.

Definition 1.1. The geodesic flow $\mathbb{R} \times GY \to GY$ is topologically mixing if given any open sets $\mathcal{O}$ and $\mathcal{U}$ in GY there exists a real number $t_0 > 0$ such that for all $|t| \geq t_0$, $t \cdot \mathcal{O} \cap \mathcal{U} \neq \emptyset$.

A point g in GY belongs to the *non-wandering set* Ω of the geodesic flow $\mathbb{R} \times GY \to GY$ if there exist sequences $\{g_n\} \subset GY$ and $\{t_n\} \subset \mathbb{R}$, such that $t_n \to \infty$, $g_n \to g$ and $t_n \cdot g_n \to g$.

The *generalized Busemann function* is a continuous function $\alpha : (\partial X \cup X) \times X \times X \to \mathbb{R}$ whose restriction on $X \times X \times X$ is given by

$$\alpha(y, x, x') := d(x', y) - d(x, y)$$

for $(\xi, x, x') \in X \times X \times X$ (see section 2 for a precise definition). A main result of this paper, from which we deduce topological mixing for the classes of spaces (A), (B) and (C) explained below, is the following:

Theorem 1.2. *Let X be a proper $CAT(-1)$-space and Γ a non-elementary discrete group of isometries of X. Assume*

(1) $\forall x, x' \in X$ there exist $\xi \in \partial X$ such that $\alpha(\xi, x, x') = 0$, and

(2) the non-wandering set Ω equals GY.

Then the geodesic flow on the quotient space $Y = X/\Gamma$ is topologically mixing.

Remark 1.3. If the quotient space $Y = X/\Gamma$ is compact, the limit set $\Lambda(\Gamma)$ of the action of Γ on X is the whole boundary ∂X (a proof of this is included in the proof of corollary 5.1 below). The property $\Lambda(\Gamma) = \partial X$ is equivalent to $\Omega = GY$ (see proposition 3.2 below). Hence, if Y is compact, the second assumption in the above theorem, which is required for the proof of topological mixing even in the case of manifolds with non-negative curvature, is redundant.

The above theorem implies that the geodesic flow is topologically mixing for the following classes of spaces:

(A) compact negatively curved polyhedra, i.e., a finite union of hyperbolic simplices glued together isometrically along faces of the same dimension so that, with the induced metric, it has curvature ≤ -1.

(B) compact quotients X/Γ where X is a proper geodesically complete $CAT(-1)$-space and Γ a discrete one-ended group of isometries of X.

(C) Y is an n-dimensional ideal polyhedron, i.e., a finite union of ideal hyperbolic n-polytopes glued together isometrically along their $(n-1)$-faces with at least two germs of polytopes along each $(n-1)$-face so that with the induced metric Y is a complete length space of curvature ≤ -1.

The structure of this paper is as follows: in the present section we include basic definitions and prove certain properties of $CAT(-1)$-spaces needed in the sequel. Moreover, the notion of a non-elementary group Γ of isometries of X is explained and the action of such Γ on GX is analyzed. Finally, the first section is concluded by (counter)examples, which are 1-dimensional simplicial complexes, in which the geodesic flow is not topologically mixing. These examples justify the fact that 1-dimensional simplices are excluded when proving topological mixing for negatively curved polyhedra. In section 2 Busemann functions are discussed and we use them to study strong stable sets in the space of geodesics. Although topological transitivity follows from topological mixing, we show in section 3 that the geodesic flow on Y is topologically transitive because we need this property in the proof of theorem 1.2 given in section 4. Finally, in section 5 the above-mentioned classes (A, B and C) of spaces are defined and it is shown that all assumptions posited in theorem 1.2 above are satisfied.

1.1. **Preliminaries on $CAT(-1)$-spaces.** The letter X will always denote a proper $CAT(-1)$-space. Recall that a metric space is proper if the closed balls are compact. For definitions and basic properties of $CAT(-1)$-spaces we refer the reader to [2], [3] and [17]. We recall here basic properties of the spaces GX and ∂X. GX consists of all isometric maps $g:\mathbb{R}\to X$ and its topology is the topology of uniform convergence on compact sets. In addition, we note here that GX is metrizable and the metric is given by the formula

$$d_{GX}(g_1,g_2):=\int_{-\infty}^{+\infty} e^{-|t|} d\big(g_1(t),g_2(t)\big)\,dt. \tag{1.1}$$

If $g\in GX$ we will denote by $-g$ the geodesic defined by $(-g)(s)=g(-s)$ and, similarly, if $\mathcal{A}\subset GX$, then $-\mathcal{A}:=\{-g \mid g\in\mathcal{A}\}$.

The (visual) boundary ∂X of a $CAT(-1)$-space can be defined, since X is assumed to be proper, as the space of equivalence classes of asymptotic geodesic rays starting at a fixed point in X. If g is a geodesic, we will denote by $g(+\infty)$ the boundary point determined by the geodesic ray $g|_{[0,+\infty)}$ and similarly for $g(-\infty)$. We need the following two conditions called (U) and (C), which are standard for studying flows (cf. [19]). Recall that two geodesic rays g_1,g_2 (or geodesics) are called *asymptotic* if $d\big(g_1(t),g_2(t)\big)$ is bounded for all $t\in\mathbb{R}^+$.

Condition (U) For any two points $x_1,x_2\in X\cup\partial X$ there exists a unique geodesic joining them.

Condition (C) For any two asymptotic geodesic rays (or geodesics) g_1,g_2 there exists a real number d such that

$$\lim_{t\to\infty} d\big(g_1(t),g_2(t+c)\big)=0.$$

It is well known (see for example [11]) that a $CAT(-1)$-space satisfies condition (U). This implies, in particular, that a complete $CAT(-1)$-space is contractible. We next show that a proper, complete $CAT(-1)$-space satisfies condition (C).

Proposition 1.4. *A proper, $CAT(-1)$-space X satisfies condition (C).*

Proof. Let $g_1,g_2:[0,\infty)\to X$ be two asymptotic geodesic rays. Denote by ξ the common boundary point $g_1(+\infty)=g_2(+\infty)$. Let $\{t_n\}_{n\in\mathbb{N}}\subset\mathbb{R}$ be a sequence converging to $+\infty$. For each $n\in\mathbb{N}$, set $y_n=g_2(t_n)$. The sequence $\{s_n\}_{n\in\mathbb{N}}\subset\mathbb{R}$ given by $s_n=d\big(g_1(0),y_n\big)$ converges to $+\infty$. Denote by x_n the unique point on

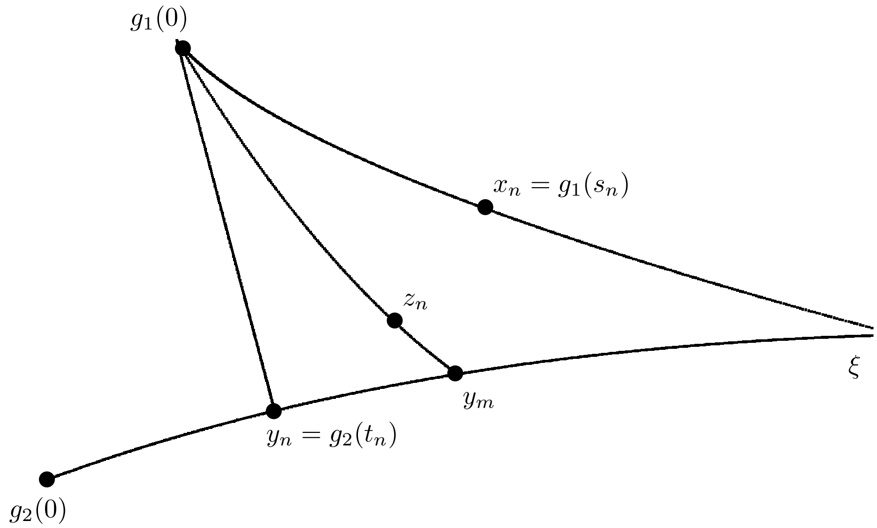

FIGURE 1.

$\mathrm{Im}g_1$ such that $d\big(g_1(0), x_n\big) = d\big(g_1(0), y_n\big)$, i.e., $x_n = g_2(s_n)$. For the reader's convenience we have gathered all the above notation in figure 1.

Let $c_n = t_n - s_n$, $n \in \mathbb{N}$. This sequence is increasing and bounded above by $d\big(g_1(0), g_2(0)\big)$. If c is the real number such that $c_n \to c$, we will show that

$$\lim_{t\to\infty} d\big(g_1(t), g_2(t+c)\big) = 0. \tag{1.2}$$

We will need the notion of the angle in $CAT(-1)$-spaces. We refer the reader to [2, Ch.I Sec.3] for definitions and basic properties. If (x, y, z) is a geodesic triangle in X, we denote the angle subtended at x by $\measuredangle_x(y, z)$. Recall that if $(\overline{x}, \overline{y}, \overline{z})$ is the comparison triangle in the hyperbolic space $\mathbb{H}^2$ of the geodesic triangle (x, y, z), then

$$\measuredangle_x(y, z) \leq \measuredangle_{\overline{x}}(\overline{y}, \overline{z}).$$

We first show that

$$\measuredangle_{y_n}\big(g_1(0), g_2(0)\big) \to 0 \;\; as \;\; n \to \infty. \tag{1.3}$$

For each $n \in \mathbb{N}$, let $\left(\overline{g_1(0)}, \overline{g_2(0)}, \overline{y_n}\right)$ be the comparison triangle in the hyperbolic space $\mathbb{H}^2$ of the geodesic triangle $(g_1(0), g_2(0), y_n)$. Since $s_n \to \infty$ and $t_n \to \infty$ as $n \to \infty$, it is not possible to have both angles $\measuredangle_{\overline{g_1(0)}}\left(\overline{g_2(0)}, \overline{y_n}\right)$ and $\measuredangle_{\overline{g_2(0)}}\left(\overline{g_1(0)}, \overline{y_n}\right)$ converging to 0 as $n \to \infty$. Without loss of generality, assume that $\measuredangle_{\overline{g_2(0)}}\left(\overline{g_1(0)}, \overline{y_n}\right)$ is bounded away from zero for all n. Then using the law of cosines

$$\frac{\sinh d\left(\overline{g_1(0)}, \overline{g_2(0)}\right)}{\sin\left(\measuredangle_{\overline{y_n}}\left(\overline{g_1(0)}, \overline{g_2(0)}\right)\right)} = \frac{\sinh s_n}{\sin\left(\measuredangle_{\overline{g_2(0)}}\left(\overline{g_1(0)}, \overline{y_n}\right)\right)}$$

it follows that

$$\measuredangle_{\overline{y_n}}\left(\overline{g_1(0)}, \overline{g_2(0)}\right) \to 0 \ or, \ \pi \ as \ n \to \infty.$$

If $\measuredangle_{\overline{y_n}}\left(\overline{g_1(0)}, \overline{g_2(0)}\right) \to \pi$, then $s_n + t_n \to d\big(\overline{g_1(0)}, \overline{g_2(0)}\big)$ which is impossible (because $\{s_n\}, \{t_n\} \to +\infty$). Thus, $\measuredangle_{\overline{y_n}}\left(\overline{g_1(0)}, \overline{g_2(0)}\right) \to 0$. Since $\measuredangle_{y_n}(g_1(0), g_2(0)) \le \measuredangle_{\overline{y_n}}\left(\overline{g_1(0)}, \overline{g_2(0)}\right)$, equation (1.3) is proved.

Our next step is to show that

$$d(x_n, y_n) \to 0 \ as \ n \to \infty. \tag{1.4}$$

The sequence of geodesic segments $[g_1(0), y_n]$ converges to the geodesic ray g_1 uniformly on compact sets. Thus for each $n \in \mathbb{N}$, we may find $m > n$ such that the neighborhood (in the compact open topology) around g_1 determined by the compact set $[0, s_n]$ and the positive number $1/n$ contains the segment $[g_1(0), y_m]$. In particular, if z_n is the unique point on $[g_1(0), y_m]$ with $d(g_1(0), z_n) = s_n$ we have

$$d(z_n, x_n) < 1/n. \tag{1.5}$$

In order to prove equation (1.4) above it suffices to show that

$$d(z_n, y_n) \to 0 \ as \ n \to \infty. \tag{1.6}$$

For each $n \in \mathbb{N}$, let $\left(\overline{g_1(0)}, \overline{y_n}, \overline{y_m}\right)$ be the comparison triangle of the geodesic triangle $(g_1(0), y_n, y_m)$. Let $\overline{z_n}$ be the point corresponding to z_n. Denote by ϕ_n the angles $\measuredangle_{\overline{y_n}}\left(\overline{g_1(0)}, \overline{z_n}\right) = \measuredangle_{\overline{z_n}}\left(\overline{g_1(0)}, \overline{y_n}\right)$. Apparently, $\phi_n < \pi/2$ for all n. If $\{\phi_n\}$, or a subsequence, converges to ϕ, for some $\phi < \pi/2$, then using the facts

$$\measuredangle_{\overline{y_n}}\left(\overline{g_1(0)}, \overline{y_m}\right) \to \pi \ \ \big(by \ \ (1.3)\big),$$
$$\measuredangle_{\overline{z_n}}\left(\overline{g_1(0)}, \overline{y_m}\right) = \pi,$$

it follows that

$$\measuredangle_{\overline{y_n}}(\overline{z_n}, \overline{y_m}) + \measuredangle_{\overline{z_n}}(\overline{y_n}, \overline{y_m}) > \pi/2 + \pi/2,$$

a contradiction. Therefore, $\phi_n \to \pi/2$. Using this and the second law of cosines we obtain that

$$\cosh d(\overline{z_n}, \overline{y_m}) \to 1 \ as \ n \to \infty;$$

hence, $d(\overline{z_n}, \overline{y_n}) \to 0$. By comparison, $d(z_n, y_m) \le d(\overline{z_n}, \overline{y_m})$ which proves equation (1.6) and, in consequence, proves equation (1.4).

We proceed now to show equation (1.2). Since the function

$$t \to d\big(g_1(t), g_2(t+c)\big)$$

is convex with respect to t (see [3, Ch. III]), it suffices to show that for each $\varepsilon > 0$ there exists a positive real number $T = T(\varepsilon)$ such that

$$d\big(g_1(T), g_2(T+c)\big) < \varepsilon.$$

Let $\varepsilon > 0$ be arbitrary. Choose $N \in \mathbb{N}$ such

$$d(x_N, y_N) < \varepsilon/2, \quad |c_N - c| < \varepsilon/2.$$

For the number $T = s_N$ we have

$$\begin{aligned} d\big(g_1(T), g_2(T+c)\big) &\leq d\big(g_1(s_N), g_2(s_N+c_N)\big) + d\big(g_2(s_N+c_N), g_2(s_N+c)\big) \\ &= d\big(g_1(s_N), g_2(t_N)\big) + |c_N - c| \\ &= d(x_N, y_N) + |c_N - c| < \varepsilon \end{aligned}$$

which completes the proof of the proposition. □

We will also need the following well-known lemma which asserts that the projection of a point onto a geodesic always exists. For a proof see, for example, [13].

Lemma 1.5. *Let g be a geodesic in $G\widetilde{X}$ (or a geodesic segment) and x_0 a point in $\widetilde{X}$. There exists a unique real number s such that $g(s)$ realizes the distance of x_0 from $\operatorname{Im} g$, i.e., $dist(x_0, \operatorname{Im} g) = d(x_0, g(s))$.*

As usual, set $\partial^2 X = \{(\xi, \eta) \in \partial X \times \partial X : \xi \neq \eta\}$. Condition (U) asserts that the fiber bundle

$$\rho : GX \to \partial^2 X$$

given by $\rho(g) = \big(g(-\infty), g(+\infty)\big)$ has a single copy of $\mathbb{R}$ as fiber. Moreover, this bundle is trivial (see for example [8, Th. 4.8]). To define a trivialization, let x_0 be a base point and let

$$H : G\widetilde{X} \xrightarrow{\approx} \partial^2 \widetilde{X} \times \mathbb{R} \tag{1.7}$$

be the trivialization of ρ with respect to x_0 defined by

$$H(g) = (g(-\infty), g(+\infty), s)$$

where $-s$ is the real number provided by lemma 1.5.

It is shown in [8, Prop. 4.8] that the conjugation of the geodesic flow with H is simply the map

$$(\xi_1, \xi_2, s) \to (\xi_1, \xi_2, s+t), \ for\ all\ \ (\xi_1, \xi_2) \in \partial^2 \widetilde{X}\ \ and\ \ s \in \mathbb{R}. \tag{1.8}$$

1.2. **The quotient space of geodesics.** In this section we define the space of geodesics for the quotient space $Y = X/\Gamma$ and prove certain properties of it. We first recall the notion of a non-elementary group of isometries. If X is a $CAT(-1)$-space and Γ a discrete group of isometries acting on X, the *limit set* $\Lambda(\Gamma)$ of the action of Γ is defined to be $\Lambda(\Gamma) = \overline{\Gamma x} \cap \partial X$, where x is arbitrary in X. The limit set has been studied extensively (see [14, Ch. II], [15, Ch. 2.1] for a detailed exposition) using the classification of the isometries of X into three types, namely, elliptic, parabolic and hyperbolic. If ϕ is hyperbolic, then $\phi^n(x)$ converges to a point $\phi(+\infty) \in \partial X$ (resp. $\phi(-\infty) \in \partial X$) as $n \to +\infty$ (resp. $n \to -\infty$) with $\phi(+\infty) \neq \phi(-\infty)$. Moreover,

$$\begin{aligned} &\forall\, \xi \in \partial X \setminus \{\phi(+\infty)\} \big(resp.\ \partial X \setminus \{\phi(-\infty)\}\big) \Longrightarrow \\ &\phi^n(\xi) \to \phi(+\infty)\ \ \big(resp.\ \phi(-\infty)\big)\ as\ n \to \infty\ \ (resp.\ -\infty). \end{aligned} \tag{1.9}$$

The cardinality of the limit set is $0, 1, 2$ or infinite. A group Γ acting on a $CAT(-1)$-space X is said to be *non-elementary* if the cardinality of $\Lambda(\Gamma)$ is infinite. In this case, the following result is shown in [14]:

$$\{(\phi(+\infty), \phi(-\infty)) : \phi \in \Gamma\ \ is\ \ hyperbolic\}\ \ is\ dense\ in\ \Lambda(\Gamma) \times \Lambda(\Gamma). \tag{1.10}$$

Note here that, as X is assumed to be proper, discreteness of the group Γ is equivalent to requiring that Γ acts properly discontinuously on X, i.e., for any compact $K \subset X$ the set $\{\gamma \in \Gamma \mid \gamma K \cap K \neq \emptyset\}$ is finite (see [22, Th. 5.3.5]). The following proposition is a well-known fact. We include its proof here since we cannot find a reference for it.

Proposition 1.6. *Let Γ be a group of isometries of X acting properly discontinuously on X. Then Γ acts by isometries and properly discontinuously on the space of geodesics GX.*

Proof. We have assumed that Γ acts by isometries on X. Therefore, if $f, g \in GX$ and $\gamma \in \Gamma$, we have

$$\int_{-\infty}^{+\infty} e^{-|t|} d(\gamma f(t), \gamma g(t))\, dt = \int_{-\infty}^{+\infty} e^{-|t|} d(f(t), g(t))\, dt$$

which implies that $d_{GX}(\gamma f, \gamma g) = d_{GX}(f, g)$. This shows that Γ acts on GX by isometries. Moreover, we have assumed that Γ acts properly discontinuously on X, i.e.,

$$\forall \text{ compact } K \subset X, \{\gamma \in \Gamma \mid \gamma K \cap K \neq \emptyset\} \text{ is finite.} \tag{1.11}$$

We proceed to show that Γ acts properly discontinuously on GX. Let $\mathcal{K}$ be an arbitrary compact set in GX. Set $d = diam(\mathcal{K})$ and choose $g \in \mathcal{K}$ arbitrary. Using the triangle inequality in X one can show that

$$d\big(\gamma g(0), g(0)\big) - 2|t| \leq d\big(\gamma g(t), g(t)\big) \leq d\big(\gamma g(0), g(0)\big) + 2|t|$$

from which it follows, after integration, that

$$d\big(\gamma g(0), g(0)\big) - 4 \leq d_{GX}(\gamma g, g) \leq d\big(\gamma g(0), g(0)\big) + 4. \tag{1.12}$$

As the space X is assumed to be proper, the closure of the ball $B = B\big(g(0), 2d+4\big)$ centered at $g(0)$ and radius $2d+4$ is compact and, by (1.11), the set

$$A = \{\gamma \in \Gamma \mid \gamma \overline{B} \cap \overline{B} \neq \emptyset\} \text{ is finite.}$$

This together with equation (1.12) implies that for all but a finite number of elements $\gamma \in \Gamma$,

$$d_{GX}(\gamma g, g) \geq d\big(\gamma g(0), g(0)\big) - 4 > 2d + 4 - 4 = 2d.$$

Now let f be an arbitrary element of $\mathcal{K}$. Then, since $d_{GX}(f, g) < d = diam(\mathcal{K})$, we have that for all but a finite number of elements $\gamma \in \Gamma$,

$$\begin{aligned} d_{GX}(\gamma f, g) &\geq |d_{GX}(\gamma f, \gamma g) - d_{GX}(\gamma g, g)| \\ &= d_{GX}(\gamma g, g) - d_{GX}(\gamma f, \gamma g) \\ &> 2d - d = d \end{aligned}$$

which implies that

$$\forall f \in \mathcal{K} \Rightarrow \gamma f \notin \mathcal{K}$$

for all but a finite number of elements $\gamma \in \Gamma$. In other words, the set

$$\{\gamma \in \Gamma \mid \gamma \mathcal{K} \cap \mathcal{K} \neq \emptyset\}$$

is finite. □

Define now GY to be the orbit space $\{\Gamma g \mid g \in GX\}$ of the action of Γ on GX. The space GY can be viewed as the set of all continuous functions $g : \mathbb{R} \to Y$ for which there exists an isometry $\overline{g} : \mathbb{R} \to X$ satisfying $p \circ \overline{g} = g$. By abuse of language, we will be calling the elements of GY geodesics in Y.

Remark 1.7. If, in addition, the action of Γ on X is free, so that X would be homeomorphic to the universal cover of Y and the metric space Y would have curvature ≤ -1, then GX/Γ is the space of all local geodesics $\mathbb{R} \longrightarrow Y$ (i.e., maps which are locally isometric) which is, in fact, the natural definition for GY.

Since the action of Γ on GX is properly discontinuous, each Γ-orbit is a closed subset of GX (cf. [22, Th. 5.3.4]). Using this, the distance function

$$d_{GY} : GY \times GY \to \mathbb{R}$$

defined by the formula

$$d_{GY}\left(\Gamma g, \Gamma f\right) := \inf \left\{ d\left(x, y\right) \bigm| x \in \Gamma g, y \in \Gamma f \right\}$$

becomes a metric on GY (cf. [22, Th. 6.5.1]). The topology induced by the metric on GY coincides with the quotient topology on GY (see [22, Th. 6.5.2]). Moreover, it can be shown easily that the compact open topology on GY coincides with the quotient topology. Define the geodesic flow on GY by the map

$$\mathbb{R} \times GY \to GY : (t, g) \to t \cdot g$$

where $t \cdot g = p\big(t \cdot \overline{g}\big)$ and $\overline{g}$ is any lift of g in GX. It is easy to check that this definition does not depend on the choice of the lift $\overline{g}$.

Since ∂X is compact and $\partial^2 X$ is an open subset of $\partial X \times \partial X$, $\partial^2 X$ is separable. Moreover, GX being, by (1.7), homeomorphic to $\partial^2 X \times \mathbb{R}$, is also a separable metric space. Thus, its continuous image GY is separable, hence, the metric space

$$GY \textit{ is 2nd countable.} \tag{1.13}$$

1.3. **Non-mixing example.** We conclude this section by describing examples in which the geodesic flow is not topologically mixing. In the example which is discussed in detail below and also appears in [2, Ch. II, Remark 3.6], the space X is a simplicial tree and the quotient space is a finite graph. In fact, as it was pointed out to us by the referee, in the case of a graph it is possible to characterize exactly when the geodesic flow is mixing by looking at the lengths of the closed geodesics (see Remark 1.9 below).

Example 1.8. Let Y be a plane graph homeomorphic to the figure-eight $S^1 \vee S^1$ consisting of seven vertices denoted A, B, C, D, E, F, G and eight edges AB, BC,CD, DA, CE, EF, FG, GC all with length 1 (see figure 2), let X be its universal cover and Γ the free group on two generators acting on X so that $Y = X/\Gamma$. Then the geodesic flow on GY is not topologically mixing.

Proof. Observe first that as X is a tree, it is a proper geodesically complete $CAT\,(-1)$-space and its boundary is totally connected. As Y is compact, the second assumption of theorem 1.2 is also satisfied. Let $g_1 : \mathbb{R} \to Y$ be the closed geodesic with period $\omega = 4$ satisfying $g_1(0) = A$, $g_1(1) = B$, $g_1(2) = C$ and $g_1(3) = D$. Similarly, let $g_2 : \mathbb{R} \to Y$ be the closed geodesic with period $\omega = 4$ satisfying $g_2(0) = C$ and $g_2(1) = E$. Observe that g_1, g_2 are well defined by the above requirements. Moreover, observe that for any $k \in \mathbb{Z}$

$$g_1(4k) = A, g_2(4k) = C \;\; \textit{and} \;\; g_1(4k+1) = B.$$

Consider neighborhoods, in the compact open topology, $\mathcal{O}_1$ and $\mathcal{O}_2$ of g_1 and g_2 respectively, determined by some compact set in $\mathbb{R}$, say $[-1/4, 1/4]$, and the number

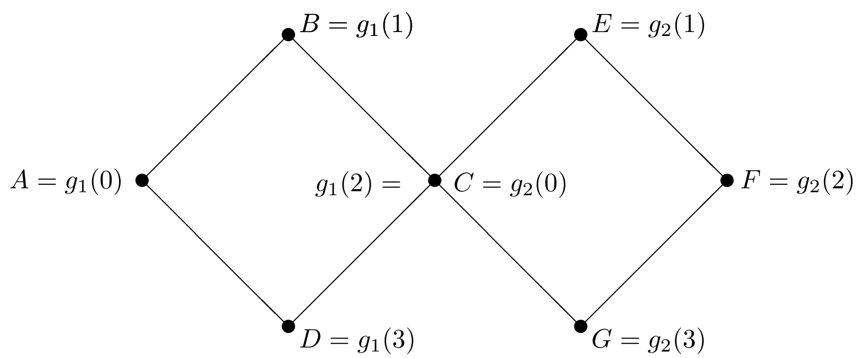

FIGURE 2.

1/8, i.e.,

$$h \in \mathcal{O}_i \Leftrightarrow d\big(h(t), g_i(t)\big) < 1/8 \text{ for all } t \in [-1/4, 1/4], \tag{1.14}$$

$i = 1, 2$. We proceed to show that there exists a sequence $\{t_n\} \subset \mathbb{R}$ converging to $+\infty$ such that $t_n \cdot \mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset$ (cf. definition 1.1 above). Let $\{t_n\}$ be the sequence $t_n = 4n + 1$, $n \in \mathbb{N}$.

From (1.14) it is apparent that

$$h \in \mathcal{O}_2 \Longrightarrow d\big(h(0), C\big) < 1/8. \tag{1.15}$$

Moreover, it is straightforward to check that if $h' \in \mathcal{O}_1$, then $h'(t_n)$ lies within a distance 1/8 from either B, D or, E or, G. If $h \in t_n \cdot \mathcal{O}_1$, then $h = t_n \cdot h'$ for some $h' \in \mathcal{O}_1$, thus, $h(0) = h'(t_n)$. Hence,

$$h \in t_n \cdot \mathcal{O}_1 \Longrightarrow dist\big(h(0), \{B, D, E, G\}\big) < 1/8. \tag{1.16}$$

If $h \in t_n \cdot \mathcal{O}_1 \cap \mathcal{O}_2$, combining equations (1.15) and (1.16) above, we obtain that

$$\begin{aligned} 1 &= d\big(C, \{B, D, E, G\}\big) \\ &\leq d\big(h(0), C\big) + dist\big(h(0), \{B, D, E, G\}\big) < 1/8 + 1/8 = 1/4. \end{aligned}$$

This contradiction shows that $t_n \cdot \mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset$ for all $n \in \mathbb{N}$ completing the proof that the geodesic flow on the figure-eight is not topologically mixing. □

Remark 1.9. A modification of the above argument can be used to show that the geodesic flow on any finite graph Y (not homeomorphic to the circle S^1) is not topologically mixing provided that the following condition holds:

(1.17)

$$\text{for any two closed geodesics in } Y \text{ with periods } \ell_1 \text{ and } \ell_2, \text{ the ratio } \frac{\ell_1}{\ell_2} \in \mathbb{Q}.$$

Moreover, as it was pointed out to us by the referee, the following converse statement can be shown: if Y is any (finite or infinite) graph not homeomorphic to the circle S^1, then the geodesic flow on Y is topologically mixing if

(1.18)

$$\text{there exist two closed geodesics in } Y \text{ with periods } \ell_1 \text{ and } \ell_2, \text{ such that } \frac{\ell_1}{\ell_2} \notin \mathbb{Q}.$$

The proof of the latter statement utilizes the fact that $\frac{\ell_1}{\ell_2} \notin \mathbb{Q}$ implies that the set $\mathbb{Z}\ell_1 + \mathbb{Z}\ell_2$ is dense in $\mathbb{R}$, which, in turn, implies that there exists a positive integer $M \in \mathbb{N}$ such that the distance of the set $A = \{n\ell_1 + m\ell_2 \mid -M \leq n, m \leq M\}$ can be made arbitrarily small for all $s \in [0, \ell_1 + \ell_2]$ uniformly. Thus, the question of topological mixing on graphs can be settled by looking at the subgroup of $\mathbb{R}$ generated by the lengths of the closed loops.

2. Stable and strong stable sets

In this section we will define and study stable and strong stable sets in GX and GY. For this we will use the *generalized Busemann function* (for more details see [2, p. 27], [19, Sec. 2]) whose definition we recall briefly. As usual, X will denote a proper $CAT(-1)$-space and Γ a non-elementary group of isometries of X.

Define a function $\alpha : X \times X \times X \to \mathbb{R}$ by letting

$$\alpha(\xi, x, x') := d(x', \xi) - d(x, \xi) \tag{2.1}$$

for $(\xi, x, x') \in X \times X \times X$. It is shown in [2, Ch. II, Sec. 2] that this function extends to a continuous function

$$(\partial X \cup X) \times X \times X \to \mathbb{R}$$

denoted again by α, called the *generalized Busemann function*.

This function, in fact, generalizes the classical Busemann function whose definition makes sense in our context. To see this, let $\gamma : [0, +\infty) \to X$ be a geodesic ray; the Busemann function associated to γ is a function b_γ on X defined by

$$b_\gamma(x) = \lim_{t\to\infty}[d(x, \gamma(t)) - t].$$

It is easy to see that for any $x \in X$,

$$\begin{aligned} \alpha\big(\gamma(+\infty), \gamma(0), x\big) &= \lim_{t\to\infty}\alpha\big(\gamma(t), \gamma(0), x\big) \\ &= \lim_{t\to\infty}[d(x, \gamma(t)) - t] = b_\gamma(x) \end{aligned}$$

and, therefore, the Busemann function b_γ coincides with $\alpha\big(\gamma(+\infty), \gamma(0), \cdot\big)$, i.e., the restriction of α on $\{\gamma(+\infty)\} \times \{\gamma(0)\} \times X$. Conversely, for arbitrary $\xi \in \partial X$ and $y \in X$, the restriction $\alpha(\xi, y, \cdot) \equiv \alpha|_{\{\xi\}\times\{y\}\times X}$ is simply the Busemann function $b_{\gamma_{y\xi}}$ associated to the unique geodesic ray $\gamma_{y\xi}$ with $\gamma_{y\xi}(0) = y$ and $\gamma_{y\xi}(+\infty) = \xi$.

The generalized Busemann function α is Lipschitz with respect to the second and third variable with Lipschitz constant 1. The latter means, in particular, that any Busemann function is Lipschitz with constant 1. To check the Lipschitz property, let $\xi \in \partial X$ and choose a sequence $\{z_n\} \subset X$ such that $z_n \to \xi$. Then for any fixed $x \in X$,

$$\begin{aligned} |\alpha(\xi, y, x) - \alpha(\xi, y', x)| &= \lim_{n\to\infty}|d(x, z_n) - d(y, z_n) - d(x, z_n) + d(y', z_n)| \\ &= \lim_{n\to\infty}|d(y', z_n) - d(y, z_n)| \leq d(y, y'). \end{aligned}$$

For $\xi \in X$ the calculation is analogous. Similarly, α can be shown to be Lipschitz with respect to the third variable.

Definition 2.1. We say that a geodesic $h \in GX$ belongs to the stable set $W^s(g)$ of a geodesic g if g, h are asymptotic. Two points $x, x' \in X$ are said to be equidistant from a point $\xi \in \partial X$ if $\alpha(\xi, x, x') = 0$.

We say that a geodesic $h \in GX$ belongs to the strong stable set $W^{ss}(g)$ of a geodesic g if $h \in W^s(g)$ and $g(0), h(0)$ are equidistant from $g(\infty) = h(\infty)$.

Similarly, if $h, g \in GY$, we say that $h \in W^{ss}(g)$ $\big($respectively $W^{s}(g)\big)$ if there exist lifts $\overline{h}, \overline{g} \in GX$ of h, g such that $\overline{h} \in W^{ss}(\overline{g})$ $\big($respectively $W^{s}(\overline{g})\big)$.

The following proposition is a consequence of condition (C) and of the properties of the α function.

Proposition 2.2. *Let $f, g \in GX$ with $f \in W^{ss}(g)$. Then*

$$\lim_{t\to\infty} d\big(f(t), g(t)\big) = 0.$$

Proof. We first show that if $\alpha\big(\xi, f(0), g(0)\big) = 0$, where $\xi = f(+\infty) = g(+\infty)$, then

$$\alpha\big(\xi, f(T), g(T)\big) = 0 \text{ for all } T \in \mathbb{R}. \tag{2.2}$$

Fix $T > 0$ (we work similarly for $T < 0$). Choose a sequence $\{t_n\} \subset \mathbb{R}$, $t_n > T$ converging to $+\infty$. Then

$$\begin{aligned}
\alpha\big(f(t_n), f(0), g(0)\big) &= d\big(f(t_n), g(0)\big) - d\big(f(t_n), f(0)\big) \\
&= d\big(f(t_n), g(0)\big) - d\big(f(t_n), f(T)\big) - d\big(f(T), f(0)\big) \\
&= \alpha\big(f(t_n), f(T), g(0)\big) - T.
\end{aligned}$$

By taking the limits as $t_n \to \infty$ we have, by continuity of a, that

$$\alpha\big(\xi, f(T), g(0)\big) = T.$$

A similar calculation shows that

$$\alpha\big(g(t_n), f(T), g(0)\big) = \alpha\big(f(t_n), f(T), g(T)\big) + T$$

which implies that $\alpha\big(\xi, f(T), g(T)\big) = \alpha\big(\xi, f(T), g(0)\big) - T = 0$. This completes the proof of equation (2.2). A repetition of the argument above asserts that for any $s \in \mathbb{R}$,

$$\alpha\big(\xi, f(T), g(T+s)\big) = s \text{ for all } T \in \mathbb{R}. \tag{2.3}$$

Now let c be the real number posited by Condition (C) making

$$\lim_{t\to\infty} d\big(f(t), g(t+c)\big) = 0.$$

We show that $c = 0$ concluding the proof of the proposition. Assume on the contrary that $d \neq 0$. Let T_0 be large enough so that

$$\big|d\big(f(T_0), g(T_0+c)\big)\big| < |c|/2.$$

Choose a sequence $\{t_n\} \subset \mathbb{R}$ converging to $+\infty$ with $t_n > T_0$. Then

$$\begin{aligned}
\big|\alpha\big(f(t_n), f(T_0), g(T_0+c)\big)\big| &= \big|d\big(f(t_n), g(T_0+c)\big) - d\big(f(t_n), f(T_0)\big)\big| \\
&\leq \big|d\big(f(T_0), g(T_0+c)\big)\big| < |c|/2
\end{aligned}$$

and by taking the limit as $t_n \to \infty$ we have, by continuity of a, that

$$\big|\alpha\big(\xi, f(T_0), g(T_0+c)\big)\big| \leq |c|/2,$$

a contradiction, by equation (2.3). $\square$

We will need the following two lemmata concerning Busemann functions and strong stable sets.

Lemma 2.3. *Let β a geodesic of X and $x \in X$ arbitrary. Then:*

(a) The function $\alpha\big(\beta(+\infty), x, \cdot\big) : \mathrm{Im}\beta \to \mathbb{R}$ is an isometry.

(b) If γ is any geodesic asymptotic with β, then there exists a unique re-parametrization β' of β such that $\alpha\big(\beta(+\infty), \gamma(0), \beta'(0)\big) = 0$, i.e. $\gamma \in W^{ss}(\beta')$.

(c) Let γ be a geodesic ray in X such that $\beta(-\infty) = \gamma(+\infty)$. Then,

$$\alpha\big(\gamma(+\infty), \gamma(0), \beta(t)\big) = t + \alpha\big(\gamma(+\infty), \gamma(0), \beta(0)\big).$$

In other words, the Busemann function b_γ associated to γ is linear when restricted to $\mathrm{Im}\beta$.

Proof. (a) Fix $t, t' \in \mathbb{R}$. Let $\{x_n\} \subset \mathrm{Im}\beta$ be a sequence converging to $\beta(+\infty)$. It is easily shown that for any $x \in X$ and for all n large enough $\big($namely, $\forall\ n$ for which $x_n > \max\{t, t'\}\big)$

$$\begin{aligned} |\alpha(x_n, x, \beta(t)) - \alpha(x_n, x, \beta(t'))| &= |d(x_n, \beta(t)) - d(x_n, \beta(t'))| \\ &= |\beta(t) - \beta(t')|. \end{aligned}$$

Using the continuity of the α function and the fact that $x_n \to \beta(+\infty)$ we obtain that $\alpha\big(\beta(+\infty), x, \cdot\big)$ is an isometry on $\mathrm{Im}\beta$.

Part (b) follows from (a) by choosing $x = \gamma(0)$ and then defining $\beta'(t) = \beta(t+T)$ where T is the unique real number such that $\beta(T)$ is the inverse image of 0 via the isometry $\alpha\big(\beta(+\infty), x, \cdot\big)$, i.e. $\alpha\big(\beta(+\infty), x, \beta(T)\big) = 0$.

(c) Set $\xi = \beta(-\infty) = \gamma(+\infty)$. Using a sequence $\{x_n\}$ converging to ξ and the continuity of the α function it is easily shown that

$$\alpha\big(\xi, \gamma(0), x\big) - \alpha\big(\xi, \beta(0), x\big) = \alpha\big(\xi, \gamma(0), \beta(0)\big),\ \forall\ x \in X.$$

Hence, for arbitrary $t \in \mathbb{R}$ we have

$$\alpha\big(\xi, \gamma(0), \beta(t)\big) = \alpha\big(\xi, \beta(0), \beta(t)\big) + \alpha\big(\xi, \gamma(0), \beta(0)\big).$$

Pick $\{t_n\} \subset \mathbb{R}$, with $t_n \to -\infty$. Then,

$$\begin{aligned} \alpha\big(\xi, \beta(0), \beta(t)\big) &= \lim_{n\to\infty} \alpha\big(\beta(t_n), \beta(0), \beta(t)\big) \\ &= \lim_{n\to\infty} \big(t + |t_n| - |t_n|\big) = t. \end{aligned}$$

This completes the proof of the lemma. □

Lemma 2.4. *(a) For any $g \in GY$ and $c \in \mathbb{R}$, $\overline{W^{ss}(c \cdot g)} = c \cdot \left(\overline{W^{ss}(g)}\right)$.*

(b) Let $h_1, g_1 \in GY$ with $h_1 \in W^{ss}(g_1)$ and $\mathcal{O}_1 \subset GY$ an open set containing h_1. Then there exists an open set $\mathcal{A}_1$ containing g_1 such that for any $g \in \mathcal{A}_1$, $W^{ss}(g) \cap \mathcal{O}_1 \neq \emptyset$.

(c) If $h \in \overline{W^{ss}(g)}$, then $\overline{W^{ss}(h)} \subset \overline{W^{ss}(g)}$.

Proof. (a) If $h \in \overline{W^{ss}(c \cdot g)}$, there exist a sequence $\{h_n\}_{n\in\mathbb{N}} \subset W^{ss}(c \cdot g)$ with $h_n \to h$. It is clear from the definitions that $(-c)\cdot h_n \to (-c)\cdot h$ and $\{(-c)\cdot h_n\}_{n\in\mathbb{N}} \subset W^{ss}(g)$. This shows that $(-c)\cdot h \in \overline{W^{ss}(g)}$ and, hence, $h = c \cdot \big((-c)\cdot h\big) \in c \cdot \left(\overline{W^{ss}(g)}\right)$. Similarly, we show the converse inclusion.

(b) The trivialization $H : GX \to \partial^2 X \times \mathbb{R}$ described in section 1.1 above $\big($see equation (1.7)$\big)$ maps a geodesic $f \in GX$ to a triple where the third coordinate is a real number. We will be denoting this real number by s_f, i.e.,

$$H(f) = \big(f(-\infty), f(+\infty), s_f\big). \tag{2.4}$$

Lift g_1, h_1 to geodesics $\overline{g_1}, \overline{h_1} \in GX$ and consider an open neighborhood $\overline{\mathcal{O}_1}$ of $\overline{h_1}$ of the form

$$\overline{\mathcal{O}_1} := H^{-1}\big(O_1 \times O_1' \times \left(s_{\overline{h}_1} - \varepsilon_1', s_{\overline{h}_1} + \varepsilon_1'\right)\big)$$

where O_1, O_1' are open neighborhoods of $\overline{h_1}(+\infty)$, $\overline{h_1}(-\infty)$ (respectively) in ∂X with $O_1 \cap O_1' = \emptyset$ and ε_1' positive real, all chosen so that

$$p\left(\overline{\mathcal{O}_1}\right) \subseteq \mathcal{O}_1.$$

Claim. We may choose $\varepsilon_1 > 0$ and distinct open neighborhoods $A_1, A_1' \subset \partial X$ of $\overline{g_1}(-\infty), \overline{g_1}(+\infty)$ (respectively) such that the neighborhood

$$\overline{\mathcal{A}_1} := H^{-1}\big(A_1 \times A_1' \times \left(s_{\overline{g}_1} - \varepsilon_1, s_{\overline{g}_1} + \varepsilon_1\right)\big)$$

satisfies the following

$$\forall\, \overline{g} \in \overline{\mathcal{A}_1}\ \exists\, \overline{h} \in \overline{\mathcal{O}_1} \text{ such that } \overline{h} \in W^{ss}(\overline{g}).$$

Then, by taking $\mathcal{A}_1 := p\left(\overline{\mathcal{A}_1}\right)$ the proof of the lemma is complete: for, if $g \in \mathcal{A}_1$, there exists $\overline{g} \in \overline{\mathcal{A}_1}$ with $p(\overline{g}) = g$ and, by the claim, there exists $\overline{h} \in \overline{\mathcal{O}_1}$ such that $\overline{h} \in W^{ss}(\overline{g})$. As $p\left(\overline{\mathcal{O}_1}\right) \subseteq \mathcal{O}_1$, the geodesic $h = p\left(\overline{h}\right)$ belongs to $\mathcal{O}_1$ and satisfies $h \in W^{ss}(g)$.

Proof of Claim. Choose closed balls $B\big(\overline{h_1}(0)\big)$, $B\big(\overline{g_1}(0)\big)$ around $\overline{h_1}(0)$, $\overline{g_1}(0)$ respectively, both with radius ε_1'. As X is proper, closed balls are compact sets and so is ∂X. Thus the (continuous) generalized Busemann function α restricted to $\partial X \times B\big(\overline{h_1}(0)\big) \times B\big(\overline{g_1}(0)\big)$ is uniformly continuous. This implies that for the number $\varepsilon_1'/2 > 0$, there exists a compact subset ∂B of ∂X containing $\overline{h_1}(+\infty)$ and a number $\lambda > 0$ such that for all $(\xi, x, y)\,(\xi', x', y') \in \partial B \times B\big(\overline{h_1}(0)\big) \times B\big(\overline{g_1}(0)\big)$ satisfying $d(x, x') < \lambda$ and $d(y, y') < \lambda$ the following inequality holds:

$$|\alpha(\xi, x, y) - \alpha(\xi', x', y')| < \varepsilon_1'/2. \tag{2.5}$$

Fix $\varepsilon_1 < \min\{\lambda/2, \varepsilon_1'/2\}$. We may choose small enough neighborhoods O_2, O_2' containing $\overline{h_1}(-\infty), \overline{h_1}(+\infty)$ respectively, so that if $\overline{h}$ is a geodesic with $\overline{h}(+\infty) \in O_2'$ and $\overline{h}(-\infty) \in O_2$, then a suitable re-parametrization of $\overline{h}$ (called again $\overline{h}$) satisfies

$$d\big(\overline{h}(0), \overline{h_1}(0)\big) < \varepsilon_1.$$

We may assume that these neighborhoods O_2, O_2' satisfy the inclusions $O_2 \subset O_1$ and $O_2' \subset O_1' \cap \partial B$. Set

$$\overline{\mathcal{O}_2} = H^{-1}\big(O_2 \times O_2' \times \left(s_{\overline{h}_1} - \varepsilon_1, s_{\overline{h}_1} + \varepsilon_1\right)\big).$$

Then we have

$$\forall\, \overline{h} \in \overline{\mathcal{O}_2} \Longrightarrow d\big(\overline{h}(0), \overline{h_1}(0)\big) < \varepsilon_1 + \varepsilon_1 = 2\varepsilon_1. \tag{2.6}$$

Moreover, using equation (1.8) and the fact that $\varepsilon_1 < \varepsilon_1'/2$ we have

$$\forall\, s \in (-\varepsilon_1'/2, \varepsilon_1'/2) \ \text{ and } \forall\, \overline{h} \in \overline{\mathcal{O}_2} \Longrightarrow s \cdot \overline{h} \in \overline{\mathcal{O}_1}. \tag{2.7}$$

In a similar fashion, we may choose neighborhoods $A_1 \subset \partial X$ containing $\overline{g_1}(-\infty)$ and $A_1' \subset O_2'$ containing $\overline{g_1}(+\infty) = \overline{h_1}(+\infty)$ such that for the neighborhood

$$\overline{\mathcal{A}_1} = H^{-1}\big(A_1 \times A_1' \times \left(s_{\overline{g}_1} - \varepsilon_1, s_{\overline{g}_1} + \varepsilon_1\right)\big)$$

we have

$$\forall\ \overline{g} \in \overline{\mathcal{A}_1} \Longrightarrow d\big(\overline{g}\,(0)\,, \overline{g_1}\,(0)\big) < 2\varepsilon_1. \tag{2.8}$$

Let $\overline{g} \in \overline{\mathcal{A}_1}$ be arbitrary. Since $A_1' \subset O_2'$, choose $\overline{h} \in \overline{\mathcal{O}_2}$ with $\overline{h}\,(+\infty) = \overline{g}\,(+\infty)$. Then

$$\begin{array}{ll} d\big(\overline{h}\,(0)\,, \overline{h_1}\,(0)\big) < \lambda & \text{by (2.6) and the fact that } \varepsilon_1 < \lambda/2, \\ d\big(\overline{g}\,(0)\,, \overline{g_1}\,(0)\big) < \lambda & \text{by (2.8) and the fact that } \varepsilon_1 < \lambda/2, \\ \overline{h}\,(+\infty)\,, \overline{h_1}\,(+\infty) \in \partial B & \text{by construction.} \end{array}$$

The above three equations combined with (2.5) imply that

$$\left|\alpha\big(\overline{h}\,(+\infty)\,, \overline{h}\,(0)\,, \overline{g}\,(0)\big) - \alpha\big(\overline{h_1}\,(+\infty)\,, \overline{h_1}\,(0)\,, \overline{g_1}\,(0)\big)\right| < \varepsilon_1'/2.$$

As $\alpha\big(\overline{h}\,(+\infty)\,, \overline{h_1}\,(0)\,, \overline{g_1}\,(0)\big) = 0$, we have that

$$-\varepsilon_1'/2 < \alpha\big(\overline{h}\,(+\infty)\,, \overline{h}\,(0)\,, \overline{g}\,(0)\big) < \varepsilon_1'/2.$$

By lemma 2.3, there exists a real $s \in (-\varepsilon_1'/2, \varepsilon_1'/2)$ such that

$$\alpha\big(\overline{g}\,(+\infty)\,, s \cdot \overline{h}\,(0)\,, \overline{g}\,(0)\big) = 0.$$

Moreover, by equation (2.7), $s \cdot \overline{h} \in \overline{\mathcal{O}_1}$. Therefore, $s \cdot \overline{h} \in W^{ss}\,(\overline{g})$ which completes the proof of the claim.

Part (c) follows immediately from part (b). □

3. Topological transitivity

The geodesic flow $\mathbb{R} \times GY \to GY$ is said to be topologically transitive if given any open sets $\mathcal{O}$ and $\mathcal{U}$ in GY, there exists a sequence $\{t_n\} \subset \mathbb{R}$, $t_n \to +\infty$ such that $t_n \cdot \mathcal{O} \cap \mathcal{U} \neq \emptyset$ for all $n \in \mathbb{N}$. It is apparent that topological mixing implies topological transitivity. However, in the proof of topological mixing in section 4 below we will need a property equivalent to topological transitivity, namely, that $\overline{W^s\,(f)} = GY$ for any $f \in GY$. In this section we will establish this property without dealing with its equivalence to topological transitivity.

Proposition 3.1. *Let X be a $CAT\,(-1)$-space, Γ a non-elementary discrete group of isometries of X and Y the quotient space $Y = X/\Gamma$. Assume that the non-wandering set Ω of the geodesic flow on Y equals GY. Then for any $f \in GY$, $\overline{W^s\,(f)} = GY$.*

For the proof of the above proposition we will need the following result:

Proposition 3.2. *Let X, Γ and Y be as above and $\Lambda\,(\Gamma)$ the limit set of the action of Γ on X. Then, $\Omega = GY$ if and only if $\Lambda\,(\Gamma) = \partial X$.*

The proof of this proposition is given in [10] for ideal polyhedra but it applies verbatim to our context.

Proof of Proposition 3.1. We first claim that

$$\text{for any } \xi \in \partial X,\ \overline{\Gamma\xi} = \partial X. \tag{3.1}$$

Let Fix_h be the set of points in ∂X fixed by hyperbolic elements of Γ, i.e. $Fix_h = \big\{\phi\,(+\infty) \bigm| \phi \in \Gamma, \phi \textit{ hyperbolic}\big\}$. As Fix_h is dense in $\Lambda\,(\Gamma)$ (see [14, Ch. II, §4]) and $\Lambda\,(\Gamma) = \partial X$ (cf. proposition 3.2) it suffices to show that $Fix_h \subseteq \overline{\Gamma\xi}$. Let $\eta \in Fix_h$ be arbitrary. If $\eta = \xi$, we have nothing to show. If $\eta \neq \xi$, then $\eta = \phi\,(+\infty)$ for some hyperbolic $\phi \in \Gamma$. By equation (1.9) it follows that $\phi^n(\xi) \to \eta$, hence, $\eta \in \overline{\Gamma\xi}$. This completes the proof of equation (3.1).

Now let $f, g \in GY$ be arbitrary. We proceed to find a sequence $f_n \in W^s(f)$ such that $f_n \to g$. Lift f, g to geodesics $\overline{f}, \overline{g}$ in GX and let $s_{\overline{g}}$ be the unique real number so that $\left(\overline{g}(-\infty), \overline{g}(+\infty), s_{\overline{g}}\right) = H(\overline{g})$ (cf. equation 2.4). By equation (3.1) there exists a sequence $\{\phi_n\} \subset \Gamma$ such that $\phi_n\left(\overline{f}(+\infty)\right) \to \overline{g}(+\infty)$. Define a sequence $\{\overline{g_n}\} \subset GX$ where each $\overline{g_n}$ is determined by the following three conditions (cf. equation 1.7):

(i) $\overline{g_n}(+\infty) = \phi_n\left(\overline{f}(+\infty)\right)$,
(ii) $\overline{g_n}(-\infty)$ is any sequence : $\overline{g_n}(-\infty) \to \overline{g}(-\infty)$,
(iii) the parametrization is chosen by requiring $s_{\overline{g_n}} = s_{\overline{g}}\ \forall\ n$.

In other words, $\overline{g_n} := H^{-1}\left(\overline{g}(-\infty), \phi_n\left(\overline{f}(+\infty)\right), s_{\overline{g}}\right)$. It is apparent that $\overline{g_n} \to \overline{g}$. Define $\overline{f_n} := \phi_n^{-1}(\overline{g_n})$ and set $f_n = p\left(\overline{f_n}\right)$. As $p\left(\overline{f_n}\right) = p(\overline{g_n})$ and $\overline{g_n} \to \overline{g}$, it follows that $f_n \to p(\overline{g}) = g$. Moreover,

$$\overline{f_n}(+\infty) = \phi_n^{-1}\left(\overline{g}(+\infty)\right) = \phi_n^{-1}\left(\phi_n\left(\overline{f}(+\infty)\right)\right) = \overline{f}(+\infty)$$

thus, $f_n \in W^s(f)$ as required. □

Corollary 3.3. *Under the assumptions of proposition 3.1 above, if there exists a geodesic g whose strong stable set $W^{ss}(g)$ satisfies $\overline{W^{ss}(g)} = GY$, then $\overline{W^{ss}(f)} = GY$ for any closed geodesic $f \in GY$.*

Proof. Let g be a geodesic in GY satisfying $\overline{W^{ss}(g)} = GY$ and let f be a closed geodesic in GY with period, say, ω. By proposition 3.1, $\overline{W^s(f)} = GY$ hence, $g \in \overline{W^s(f)}$. This means that there exists a sequence $\{g_n\} \subset W^s(f)$ such that $g_n \to g$. For each $n \in \mathbb{N}$, consider lifts $\overline{g_n}, \overline{f}$ of g_n, f respectively, satisfying $\overline{g_n} \in W^{ss}\left(\overline{f}\right)$ and use lemma 2.3(b) to obtain a real number t_n such that $t_n \cdot g_n \in W^{ss}(f)$. Each t_n may be expressed by

$$t_n = k\omega + c_n$$

where $k \in \mathbb{Z}$ and $c_n \in [0, \omega)$. By choosing, if necessary, a subsequence we have that $c_n \to c$ for some $c \in [0, \omega]$. Then $c_n \cdot g_n \to c \cdot g$ with $c_n \cdot g_n \in W^{ss}(f)$. This means that for some $c \in [0, \omega]$, $c \cdot g \in \overline{W^{ss}(f)}$. By lemma 2.4(a), we have $c \cdot \overline{W^{ss}(g)} = \overline{W^{ss}(c \cdot g)}$ and, by lemma 2.4(c), $\overline{W^{ss}(c \cdot g)} \subset \overline{W^{ss}(f)}$. Thus, $GY = c \cdot GY = c \cdot \overline{W^{ss}(g)} \subset \overline{W^{ss}(f)}$. □

4. Proof of topological mixing

For the proof of theorem 1.2 we follow closely the idea used by Eberlein in [16] in the proof of topological mixing of the geodesic flow on Riemannian manifolds of non-positive curvature. However, since we deal with a more general class of spaces, the difficulties which arise here are of a different nature. We first establish the following:

Proposition 4.1. *Let X, Y and Γ be as in theorem 1.2 above. If $\Omega = GY$, then there exists a geodesic g whose strong stable set $W^{ss}(g)$ satisfies $\overline{W^{ss}(g)} = GY$.*

Proof. We first show that

$$\forall\ \mathcal{O},\mathcal{U} \subseteq GY \ open,\ \exists\ g \in \mathcal{O} : W^{ss}(g) \cap \mathcal{U} \neq \emptyset. \tag{4.1}$$

We will use the letter p to denote both projections $X \to Y$ and $GX \to GY$. Let $\mathcal{O},\mathcal{U} \subseteq GY$ be arbitrary open sets. Using equation (1.10) and the fact that $\partial X = \Lambda(\Gamma)$ (which follows from the assumption $\Omega = GY$ and proposition 3.2 above), we may choose $f \in p^{-1}(\mathcal{O})$ and $h \in p^{-1}(\mathcal{U})$ such that

$$\big(f(+\infty), h(+\infty)\big) = \left(\phi(+\infty), \phi(-\infty)\right) : \phi \in \Gamma \ \ is \ \ hyperbolic.$$

For each n, let ξ_n be in ∂X such that $\alpha\big(\xi_n, f(0), \phi^n(h(0))\big) = 0$. We claim that

$$\xi_n \to f(+\infty) \ \ as \ n \to \infty. \tag{4.2}$$

To see this assume, on the contrary, that $\{\xi_n\}$ (or, a subsequence of it) converges to $\xi \in \partial X$ with $\xi \neq f(+\infty)$. Let β be a geodesic in X such that $\beta(+\infty) = f(+\infty)$ and $\beta(-\infty) = \xi$. Similarly, let $\beta' \in GX$ such that $\beta'(+\infty) = f(+\infty)$ and $\beta'(-\infty) = h(+\infty)$. Since ϕ translates β', i.e., $\phi\big(\mathrm{Im}(\beta')\big) = \mathrm{Im}(\beta')$, it follows that for any $n \in \mathbb{N}$

$$\mathrm{dist}\big(\phi^n(h(0)), \mathrm{Im}\beta'\big) \leq \mathrm{dist}\big(h(0), \mathrm{Im}\beta'\big).$$

As β and β' are asymptotic and $\phi^n(h(0)) \to \beta(+\infty) = \beta'(+\infty)$, a similar statement holds true for β by using condition (C), namely,

$$\exists\, M \in \mathbb{R} : \mathrm{dist}\big(\phi^n(h(0)), \mathrm{Im}\beta\big) \leq M, \ \ for \ all \ n \in \mathbb{N}.$$

For each $n \in \mathbb{N}$, let t_n be the real number realizing the distance in the left-hand side of the above equation (cf. lemma 1.5). By equation (1.9), is clear that $t_n \to +\infty$. As the generalized Busemann function α is Lipschitz (with Lipschitz constant 1) with respect to the third variable, it follows that for all $n \in \mathbb{N}$,

$$\left|\alpha\big(\xi, f(0), \beta(t_n)\big) - \alpha\big(\xi, f(0), \phi^n(h(0))\big)\right| \leq M. \tag{4.3}$$

ξ_n is chosen so that $\phi^n(h(0))$ and $f(0)$ are equidistant from ξ_n, thus,

$$\lim_{n\to\infty}\alpha\big(\xi, f(0), \phi^n(h(0))\big) = \lim_{n\to\infty}\alpha\big(\xi_n, f(0), \phi^n(h(0))\big) = 0.$$

The latter combined with equation (4.3) implies that

$$\left|\lim_{n\to\infty}\alpha\big(\xi, f(0), \beta(t_n)\big)\right| \leq M.$$

This is impossible by lemma 2.3(c) and the fact that $t_n \to +\infty$. Thus equation (4.2) is proved. We next show that

$$\phi^{-n}(\xi_n) \to h(+\infty) \ \ as \ n \to \infty. \tag{4.4}$$

Assume, on the contrary, that $\phi^{-n}(\xi_n)$ (or, a subsequence of it) converges to $\zeta \in \partial X$ with $\zeta \neq h(+\infty)$. We choose a geodesic $\beta \in GX$ with $\beta(+\infty) = \zeta$ and $\beta(-\infty) = h(+\infty)$ and proceed with the proof exactly as in the previous argument by using the facts that

$$\lim_{n\to\infty}\phi^{-n}(f(0)) = h(+\infty)$$

and

$$\alpha\big(\phi^{-n}(\xi_n), \phi^{-n}(f(0)), h(0)\big) = 0.$$

Choose now geodesics $f_n \in GX$, $n \in \mathbb{N}$ such that $f_n(+\infty) = \xi_n$ and $f_n(-\infty) = f(-\infty)$. We may parametrize f_n so that $f_n(0) \to f(0)$. This can be done by requiring $s_{\overline{f_n}} = s_{\overline{f}}$ for all $n \in \mathbb{N}$. $\big($cf. equation (2.4)$\big)$. Similarly, choose $h_n \in GX$

such that $h_n(+\infty) = \xi_n$ and $h_n(-\infty) = \phi^n\big(h(-\infty)\big)$ and parametrize them so that

$$\alpha\big(\xi_n, f_n(0), h_n(0)\big) = 0. \tag{4.5}$$

It is apparent that for n large enough, $f_n \in p^{-1}(\mathcal{O})$ and $h_n \in W^{ss}(f_n)$. If we show that $\phi^{-n}(h_n) \in p^{-1}(\mathcal{U})$ for n large enough, then we would have

$$\begin{aligned}
&p(f_n) \in \mathcal{O},\\
&p(h_n) = p\big(\phi^{-n}(h_n)\big) \in \mathcal{U},\\
&p(h_n) \in W^{ss}\big(p(f_n)\big).
\end{aligned}$$

The above three properties imply that for n large enough, $W^{ss}\big(p(f_n)\big) \cap \mathcal{U} \neq \emptyset$, as required in equation (4.1). We conclude the proof of the proposition by showing that $\phi^{-n}(h_n) \in p^{-1}(\mathcal{U})$. Using equation (4.4) above, it is clear that

$$\begin{aligned}
\big(\phi^{-n}(h_n)\big)(+\infty) &= \phi^{-n}\big(h_n(\infty)\big)\\
&= \phi^{-n}(\xi_n) \to h(+\infty).
\end{aligned} \tag{4.6}$$

Similarly,

$$\begin{aligned}
\big(\phi^{-n}(h_n)\big)(-\infty) &= \phi^{-n}\big(h_n(-\infty)\big)\\
&= \phi^{-n}\Big(\phi^n\big(h(-\infty)\big)\Big) = h(-\infty) \;\; as\ n \to \infty.
\end{aligned} \tag{4.7}$$

By condition (U) and equations (4.6), (4.7), we have

$$d\Big(h(0), \operatorname{Im}\phi^{-n}(h_n)\Big) \to 0$$

as $n \to +\infty$, and, therefore,

$$d\Big(\phi^n\big(h(0)\big), \operatorname{Im} h_n\Big) \to 0 \; as\ n \to \infty. \tag{4.8}$$

Let $h_n(t_n), t_n \in \mathbb{R}$ be the point on $\operatorname{Im} h_n$ which realizes the distance in equation (4.8) above (cf. lemma 1.5). As the function α is Lipschitz with respect to the third variable (with Lipschitz constant 1), we have

$$\Big|\alpha\Big(\xi_n, f(0), \phi^n\big(h(0)\big)\Big) - \alpha\big(\xi_n, f(0), h_n(t_n)\big)\Big| \leq d\Big(\phi^n\big(h(0)\big), h_n(t_n)\Big).$$

Using the defining property of ξ_n, i.e., $\alpha\big(\xi_n, f(0), \phi^n(h(0))\big) = 0$, it follows that

$$\alpha\big(\xi_n, f(0), h_n(t_n)\big) \to 0 \; as\ n \to \infty.$$

Similarly, using the fact that $f_n(0) \to f(0)$ as $n \to \infty$ and the Lipschitz property of α with respect to the second variable, we have

$$\alpha\big(\xi_n, f_n(0), h_n(t_n)\big) \to 0 \; as\ n \to \infty.$$

Since, by lemma 2.3(b), there is a unique point on each $\operatorname{Im} h_n$ which is equidistant from $f_n(0)$ with respect to ξ_n, namely, $h_n(0)$ $\big($cf. equation (4.5)$\big)$, it follows that $t_n \to 0$ which, combined with equation (4.8) implies that

$$d\Big(\phi^n\big(h(0)\big), h_n(0)\Big) \to 0 \; as\ n \to \infty.$$

Therefore, $\phi^{-n}\big(h_n(0)\big) \to h(0)$ as $n \to \infty$. The latter combined with the facts

$$\begin{aligned}
&\big(\phi^{-n}(h_n)\big)(+\infty) \to h(+\infty) \;\; as\ n \to \infty,\\
&\big(\phi^{-n}(h_n)\big)(-\infty) \to h(-\infty) \;\; as\ n \to \infty
\end{aligned}$$

implies that $\phi^{-n}(h_n) \in p^{-1}(\mathcal{U})$ which concludes the proof of equation (4.1).

Using now a countable basis $\{\mathcal{O}_n\}_{n\in\mathbb{N}}$ for the topology of GY, which exists by equation (1.13), the proof is completed by a standard topological argument (cf. [16, Th. 5.2]) which we include here for the readers convenience: If $\mathcal{O}$ is an arbitrary open set in GY, then by equation (4.1) above, there exists $g_1 \in \mathcal{O}$ such that $W^{ss}(g_1) \cap \mathcal{O}_1 \neq \emptyset$. Let $h_1 \in W^{ss}(g_1) \cap \mathcal{O}_1$. By lemma 2.4, there exists an open set $\mathcal{A}_1$ containing g_1 satisfying

$$W^{ss}(h) \cap \mathcal{O}_1 \neq \emptyset$$

for every $h \in \mathcal{A}_1$. Moreover, we may assume that the closure $\overline{\mathcal{A}_1}$ of $\mathcal{A}_1$ lies in $\mathcal{O}_1$ and is compact. Inductively, a sequence of open sets $\mathcal{A}_i$ and a sequence of geodesics g_i are constructed such that

$$\begin{aligned}
&\overline{\mathcal{A}_i} \subset \mathcal{A}_{i-1},\\
&g_i \in \mathcal{A}_i,\\
&for\ every\ h \in \mathcal{A}_i, W^{ss}(h) \cap \mathcal{O}_i \neq \emptyset.
\end{aligned}$$

By the finite intersection property of the compact sets $\overline{\mathcal{A}_i}$ it follows that there exists a $g \in \bigcap_{i=1}^{\infty} \overline{\mathcal{A}_i}$. As $W^{ss}(g) \cap \mathcal{O}_i \neq \emptyset$ for all i, $\overline{W^{ss}(g)} = GY$. □

We will need a pointwise version of topological mixing and a criterion for such property.

Definition 4.2. Let h, f be in GY and let $\{s_n\}_{n\in\mathbb{N}}$ be a sequence converging to $+\infty$ or $-\infty$. We say that h is s_n-mixing with f (notation, $h \sim_{s_n} f$) if for every neighborhood $\mathcal{O}$,$\mathcal{U}$ in GY of h, f respectively, $s_n \cdot \mathcal{O} \cap \mathcal{U} \neq \emptyset$ for all n sufficiently large.

If $h \sim_{s_n} f$ for some $h, f \in GY$, then using decreasing sequences of open neighborhoods of h and f it is easily shown that for each subsequence $\{s'_n\}$ of $\{s_n\}$ there exists a subsequence $\{r_n\}$ of $\{s'_n\}$ and a sequence $\{h_n\} \subset GY$ such that $h_n \to h$ and $r_n \cdot h_n \to f$. The proof of the converse statement is elementary, hence, the following criterion for the s_n-mixing of h, f holds.

Criterion 4.3. If $h, f \in GY$, then $h \sim_{s_n} f$ if and only if for each subsequence $\{s'_n\}$ of $\{s_n\}$ there exists a subsequence $\{r_n\}$ of $\{s'_n\}$ and a sequence $\{h_n\} \subset GY$ such that $h_n \to h$ and $r_n \cdot h_n \to f$.

The following lemma asserts that pointwise topological mixing is transferred via the strong stable relation of geodesics.

Lemma 4.4. *If $f, g, g' \in GY$ so that $f \in \overline{W^{ss}(g)}$ and $g \sim_{s_n} g'$ for some sequence $s_n \to \infty$, then $f \sim_{s_n} g'$.*

Proof. Fix a sequence $\{s_n\}$ with $s_n \to \infty$. It is easy to verify that the set

$$\left\{f \in GY \;\middle|\; f \sim_{s_n} g'\right\}$$

is closed; for, if $\{h_k\}_{k\in\mathbb{N}} \subset GY$ with $h_k \to h$ and each h_k is s_n-mixing with g', let $\mathcal{O}$,$\mathcal{U}$ be neighborhoods of h, g' respectively. Since $h_k \to h$, $\mathcal{O}$ is also a neighborhood of h_{k_0}, for some k_0 large enough. As $h_{k_0} \sim_{s_n} g'$, it follows that $s_n \cdot \mathcal{O} \cap \mathcal{U} \neq \emptyset$ for n sufficiently large which implies that $\left\{f \in GY \;\middle|\; f \sim_{s_n} g'\right\}$ is closed. Therefore, it suffices to prove the assertion of the lemma for $f \in W^{ss}(g)$.

In order to use criterion 4.3 above to show that $f \sim_{s_n} g'$, let $\{t_n\}$ be an arbitrary subsequence of $\{s_n\}$. As $g \sim_{s_n} g'$, there exists (again by criterion 4.3) a subsequence $\{r_n\}$ of $\{t_n\}$ and a sequence $\{g_n\}$ converging to g such that $r_n \cdot g_n \to g'$. Lift g and f to geodesics $\overline{g}$ and $\overline{f}$ in GX such that $\overline{f}(+\infty) = \overline{g}(+\infty)$ and $\alpha\left(\overline{f}(+\infty), \overline{f}(0), \overline{g}(0)\right) = 0$. Lift each g_n to a geodesic $\overline{g_n}$ such that $\overline{g_n}(+\infty) \to \overline{g}(+\infty)$, $\overline{g_n}(-\infty) \to \overline{g}(-\infty)$ and $\overline{g_n}(0) \to \overline{g}(0)$. Define a sequence of geodesics $\left\{\overline{f_n}\right\}_{n\in\mathbb{N}}$ such that $\overline{f_n} \to \overline{f}$ with $\overline{f_n}(+\infty) = \overline{g_n}(+\infty)$ and $\overline{f_n}(-\infty) = \overline{f}(-\infty)$. By the continuity of the α function we have that

$$\lim_{n\to\infty} \alpha\big(\xi_n, \overline{f_n}(0), \overline{g_n}(0)\big) = \alpha\big(\xi, \overline{f}(0), \overline{g}(0)\big) = 0$$

hence, by passing if necessary to a subsequence of $\left\{\overline{f_n}\right\}_{n\in\mathbb{N}}$, we may assume that

$$\alpha\big(\xi_n, \overline{f_n}(0), \overline{g_n}(0)\big) < 1/n, \quad \textit{for all } n \in \mathbb{N}.$$

By lemma 2.3(b) we may choose the parametrization of each $\overline{f_n}$ so that

$$\alpha\big(\xi_n, \overline{f_n}(0), \overline{g_n}(0)\big) = 0, \quad \textit{for all } n \in \mathbb{N}. \tag{4.9}$$

As the change of parametrization tends to 0 as $n \to \infty$, we may assume that the sequence $\left\{\overline{f_n}\right\}_{n\in\mathbb{N}}$ satisfies equation (4.9) and $\overline{f_n} \to \overline{f}$. Moreover, if we set $f_n := p\left(\overline{f_n}\right)$, then $f_n \to f$.

We proceed now to show that $r_n \cdot f_n \to g'$. Let K be an arbitrary compact subset of $\mathbb{R}$ and ε arbitrary positive. By construction, $\overline{f_n} \in W^{ss}(\overline{g_n})$ for all $n \in \mathbb{N}$ and $\overline{f} \in W^{ss}(\overline{g})$. Moreover, by proposition 2.2,

$$\begin{aligned} &\lim_{t\to\infty} d\big(\overline{f_n}(t), \overline{g_n}(t)\big) = 0, \\ &\lim_{t\to\infty} d\big(\overline{f}(t), \overline{g}(t)\big) = 0. \end{aligned} \tag{4.10}$$

Choose a positive real T such that

$$d\big(\overline{f}(T), \overline{g}(T)\big) < \varepsilon/6.$$

The above equation holds for all $t > T$. This follows by convexity of the distance function (see [3, Ch. 2]) and equation (4.10). As $\overline{f_n} \to \overline{f}$ and $\overline{g_n} \to \overline{g}$ we may choose $N \in \mathbb{N}$ such that

$$\begin{aligned} d\big(\overline{f_n}(T), \overline{f}(T)\big) &< \varepsilon/6, \\ d\big(\overline{g_n}(T), \overline{g}(T)\big) &< \varepsilon/6. \end{aligned}$$

Thus, $d\big(\overline{f_n}(T), \overline{g_n}(T)\big) < \varepsilon/2$ and as before, it follows that

$$d\big(\overline{f_n}(t), \overline{g_n}(t)\big) < \varepsilon/2 \text{ for all } t > T.$$

As $r_n \to +\infty$, there exists n_0 such that $r_n \geq T + diam\, K$ for all $n \geq n_0$. Now for all n sufficiently large, namely, $n \geq \max\{N, n_0\}$, we have

$$d\big(\overline{f_n}(r_n + t), \overline{g_n}(r_n + t)\big) < \varepsilon/2, \ \forall\, t \in K$$

which implies that

$$d\big(r_n \cdot f_n(t), r_n \cdot g_n(t)\big) < \varepsilon/2, \ \forall\, t \in K.$$

As $r_n \cdot g_n \to g'$, we have that for all n sufficiently large

$$d\big(r_n \cdot g_n(t), g'(t)\big) < \varepsilon/2, \ \forall\, t \in K.$$

Combining the last two inequalities we obtain that

$$d\big(r_n \cdot f_n(t), g'(t)\big) < \varepsilon, \ \forall\, t \in K.$$

As K, ε were arbitrary, we have shown that for all n sufficiently large, $r_n \cdot f_n$ lies in any neighborhood of g'. Therefore, $r_n \cdot f_n \to g'$ as required. □

Proof of Theorem 1.2. We will show the following property:

$$\forall\, \mathcal{O},\mathcal{U} \subseteq GY \text{ open},\ \exists\, t_0' > 0 : t\cdot\mathcal{O}\cap\mathcal{U} \neq \emptyset\ \forall\, t \geq t_0'. \tag{4.11}$$

Then applying this property to the open sets $-\mathcal{O}, -\mathcal{U}$ we obtain a number

$$t_0'' > 0 : \big(t\cdot(-\mathcal{O})\big)\cap(-\mathcal{U}) \neq \emptyset\ \forall\, t \geq t_0''. \tag{4.12}$$

Setting $t_0 = \max\left\{t_0^{'}, t_0^{''}\right\}$ we have that

$$-\big((-t)\cdot\mathcal{O}\cap\mathcal{U}\big) = \big(-((-t)\cdot\mathcal{O})\big)\cap(-\mathcal{U}) = \big(t\cdot(-\mathcal{O})\big)\cap(-\mathcal{U}).$$

By (4.12) it follows that $-\big((-t)\cdot\mathcal{O}\cap\mathcal{U}\big)\neq\emptyset$, hence $t\cdot\mathcal{O}\cap\mathcal{U}\neq\emptyset$ for all $t\leq -t_0$. This combined with (4.11) completes the proof of theorem 1.2. We proceed now to show equation (4.11). For this it suffices to show that

$$\begin{array}{l}\forall\, h,f\in GY \text{ and } \forall\,\{t_n\} \text{ with } t_n\to\infty, \exists \text{ sub-}\\ \text{sequence } \{s_n\}\subset\{t_n\} \text{ such that } h\sim_{s_n} f.\end{array} \tag{4.13}$$

Let f,h and $\{t_n\}$ be given and g be the geodesic provided by proposition 4.1. By corollary 3.3 we may assume that g is closed. Moreover, by lemma 2.4(a), the conclusion of proposition 4.1 is satisfied by any translate $c\cdot g$ of g, where $c\in\mathbb{R}$. Choose a subsequence $\{s_n\}$ of $\{t_n\}$ such that $s_n\cdot g\to c\cdot g$ for some $c\in[0,\text{period}\,(g)]$. It is apparent that $g\sim_{s_n} c\cdot g$. Since $\overline{W^{ss}(g)} = GY$, $f\in\overline{W^{ss}(g)}$ and hence, by lemma 4.4, $f\sim_{s_n} c\cdot g$. This implies that $-c\cdot g\sim_{s_n} -f$. Applying lemma 4.4 again and using the fact that $-h\in\overline{W^{ss}(-c\cdot g)}$ it follows that $-h\sim_{s_n} -f$, thus $f\sim_{s_n} h$ as required. ☐

5. Applications

In this section we provide classes of spaces, much wider than Riemannian manifolds, satisfying all assumptions posited in theorem 1.2 above. Recall that a metric space is *geodesically complete* if each geodesic segment is the restriction of a geodesic defined on the whole real line. An immediate application is the following

Corollary 5.1. *Let X be a proper geodesically complete $CAT(-1)$-space and Γ a discrete one-ended group of isometries of X with compact quotient $Y = X/\Gamma$. Then the geodesic flow on Y is topologically mixing.*

Proof. X is the universal cover of Y and a hyperbolic space in the sense of Gromov. Since Y is compact, $\Gamma\approx\pi_1(Y)$ is a hyperbolic group whose boundary is isomorphic with the boundary of X (see [15, Ch. 4, Theorem 4.1]). In particular, Γ is non-elementary. Since Γ is one-ended, it follows that the boundary ∂X of X is connected (see for example Bowditch [5, 6]). It is easy now to deduce that for any $x,x'\in X$ there exists a point $\xi\in\partial X$ such that $\alpha(\xi,x,x') = 0$: if $x = x'$, the result is trivial. If $x\neq x'$, extend the geodesic segment joining x with x' to a geodesic, say, g. We may assume that $x = g(s)$, $x' = g(s')$ for some $s,s'\in\mathbb{R}$ with $s < s'$. The continuous function $\alpha(\cdot,x,x')$ restricted to ∂X attains the negative value $\alpha\big(g(+\infty),x,x'\big) = -d(x,x')$ and the positive value $\alpha\big(g(-\infty),x,x'\big) = d(x,x')$. By connectivity of ∂X, $\exists$ a point $\xi\in\partial X$ such that $\alpha(\xi,x,x') = 0$.

In order to apply theorem 1.2, we need to show that $\Omega = GY$. For this it suffices to show that the limit set $\Lambda_X\big(\Gamma\big)$ of the action of Γ on X equals ∂X (cf. proposition 3.2). Γ acts on itself and the limit set $\Lambda_\Gamma\big(\Gamma\big)$ of this action is equal to $\partial\Gamma$. Consider the map $\Gamma\to X$ given by $\gamma\to\gamma(p)$ for some $p\in X$ fixed. This map is a quasi-isometry, hence induces a homeomorphism $\partial\Gamma \to \partial X$ which takes $\Lambda_\Gamma(\Gamma)$ into $\Lambda_X(\Gamma)$. It follows that $\Lambda_X(\Gamma) = \partial X$. □

We proceed now to apply theorem 1.2 to negatively curved polyhedra and to n-dimensional complete ideal polyhedra with curvature less than or equal to -1.

5.1. **Negatively curved polyhedra.** A hyperbolic n-simplex is the convex hull, in hyperbolic n-space $\mathbb{H}^n$, of $n+1$ points in general position. Let Y be a locally finite union of hyperbolic simplices glued together isometrically along faces of the same dimension such that for every simplex σ, each $(k-1)$-face of σ is glued isometrically with some face of some simplex of Y.

Such a space Y is naturally a complete geodesic metric space (by results of Bridson [7] and Moussong [20]) with distance function given as follows: a broken geodesic from a point x to a point y is a map $f : [a, b] \to Y$ with $f(a) = x$, $f(b) = y$ for which there exists a subdivision $a = t_0 < t_1 < \cdots < t_{k+1} = b$ of $[a, b]$ such that for all $i = 0, 1, \ldots, k$ the restriction $f|_{[t_i,t_{i+1}]}$ is a geodesic whose image lies in a single simplex. The length of a broken geodesic f is defined to be

$$\sum_{i=0}^{k} \ell\left(f|_{[t_i,t_{i+1}]}\right) = \sum_{i=0}^{k} |f(t_i) - f(t_{i+1})|$$

where the length inside a simplex is measured with respect to the hyperbolic metric $|\ |$. The distance $d(x, y)$ from x to y is then defined to be the lower bound of the lengths of broken geodesics from x to y.

Since Y is assumed to be locally finite, hence, locally compact, Y becomes a proper geodesic metric space.

Definition 5.2. Such a space Y is called a negatively curved polyhedron if Y with the induced length metric has curvature ≤ -1.

Recall that a geodesic metric space is said to have curvature less than or equal to χ if each $y \in Y$ has a neighborhood V_y such that every geodesic triangle of perimeter strictly less than $\frac{2\pi}{\sqrt{\chi}}$ ($=+\infty$ when $\chi \leq 0$) contained in V_y satisfies $CAT(\chi)$.

Let $\tilde{Y}$ be the universal cover of Y. Then $\tilde{Y}$ is a $CAT(-1)$-space and Y is the quotient $\tilde{Y}/\Gamma$ where Γ is a discrete group of isometries of $\tilde{Y}$ isomorphic to $\pi_1(Y)$.

Notation 5.3. It is explicit in the above definition that a negatively curved polyhedron can be made up using simplices of various dimensions. Let σ be a 1-dimensional simplex in Y such that σ is not the face of any k-simplex, $k \geq 2$, in Y. Such a simplex will be called a *free* 1-simplex. We will use in the sequel a subspace of the 1-skeleton of Y which consists of all free 1-simplices σ in Y. This subspace will be denoted by $Y^{[1]}$ and is not to be confused with the 1-skeleton of Y. Observe that $Y^{[1]}$ may be empty. The (topological) boundary $Y^{[1]} \setminus \operatorname{Int} Y^{[1]}$ of $Y^{[1]}$, denoted by $bd\left(Y^{[1]}\right)$, is a discrete set of points in Y each of which is the 0-face of some simplex of dimension $k \geq 2$.

For the universal cover $\tilde{Y}$ of Y, the same notation (i.e., $\tilde{Y}^{[1]}$) will be used.

It is purely for convenience that we consider simplices of constant curvature -1 instead of simplices of constant curvature $\chi, \chi < 0$. Moreover, we may define our spaces to be negatively curved cell complexes. As any cell complex can be made simplicial by subdivision, this involves no loss of generality. For detailed definitions and properties of negatively curved polyhedra we refer the reader to the treatments of Ballman [3], Bridson [7] and Paulin [21].

Theorem 5.4. *Let Y be a negatively curved polyhedron which is not a graph. Then the geodesic flow on Y is topologically mixing, provided that the non-wandering set Ω equals GY and $\pi_1(Y)$ is non-elementary. In particular, the geodesic flow on any compact negatively curved polyhedron which is not a graph is topologically mixing.*

As explained in section 1.3, 1-dimensional simplicial complexes are of a special nature as far as topological mixing is concerned. This continues to be the case with the zeros on the boundary of the generalized Busemann function α. The next proposition, which asserts the existence of such zeros, is false if the negatively curved polyhedron contains even a single free 1-simplex. This failure calls for specific treatment which is given in the proof of theorem 5.4.

Proposition 5.5. *Let Y be a negatively curved polyhedron and $\tilde{Y}$ its universal cover. Then, $\forall x, y \in \tilde{Y}$ there exists $\xi \in \partial\tilde{Y}$ such that $\alpha(\xi, x, y) = 0$ provided that the midpoint of the geodesic segment $[x, y]$ is not contained in the interior of $\tilde{Y}^{[1]}$ (i.e., is not contained in the interior of a 1-simplex which is not the face of a k-simplex, $k \geq 2$, of $\tilde{Y}$).*

In the proof of the above proposition we will use the notion of the space of directions: if y is a point in Y (or $\tilde{Y}$) we consider the space of directions D_y at the point y. A point in D_y is an equivalence class of geodesic segments emanating from y and angle measurement induces a metric on D_y. For details concerning angles in an arbitrary $CAT(\chi)$-space we refer the reader to [2, Ch. I.3]. In fact, the simplicial structure of Y induces a simplicial structure on D_y so that D_y is locally a $CAT(1)$-space. If x (resp. σ) is a point (resp. a path) in Y with $x \neq y$ (resp. $y \notin \mathrm{Im}\sigma$), then we denote by $d(x)$ $\big($resp. $d(\sigma)\big)$ the direction at y pointing to x (resp. the path $d \circ \sigma$ in D_y). For details concerning the space of directions as well as for the following two facts needed in the sequel we refer the reader to [3, Chapter 10]. Around any point y there exists a neighborhood U_y such that

(5.1) If σ is a geodesic segment in U_y and $y \notin \mathrm{Im}\sigma$, then $d(\sigma)$ has length $< \pi$.

(5.2) If σ is a geodesic segment in U_y, then $d(\sigma)$ is a geodesic segment in D_y.

Proof of Proposition 5.5. Let $x, y \in \tilde{Y}$ with $x \neq y$ and denote by m the midpoint of the geodesic segment $[x, y]$. Consider the set

$$Z = \left\{ z \in \tilde{Y} \mid \alpha(z, x, y) = 0 \right\}.$$

Observe that $Z \neq \emptyset$, as $m \in Z$. It suffices to show that Z is not bounded in $\tilde{Y}$. For, if $\{z_n\}_{n\in\mathbb{N}}$ is a sequence in Z with $d(z_n, m) \to +\infty$ as $n \to \infty$, then by choosing, if necessary, a subsequence we have that $\{z_n\}_{n\in\mathbb{N}}$ converges to some point $\xi \in \partial\tilde{Y}$ and by continuity of the α function,

$$\alpha(\xi, x, y) = \lim_{t\to\infty} \alpha\big(z_n, x, y\big) = 0.$$

We proceed to show that Z is not bounded in $\tilde{Y}$. Assume, on the contrary, that there exists a point $z_0 \in Z$ of maximal distance from x (and, hence, from y). We have two cases:

Case A: $z_0 = m$.

Consider the space of directions D_m at the point m. If D_m is connected, then for points m_1, m_2 on the geodesic segments $[m, x]$ and $[m, y]$, respectively, which

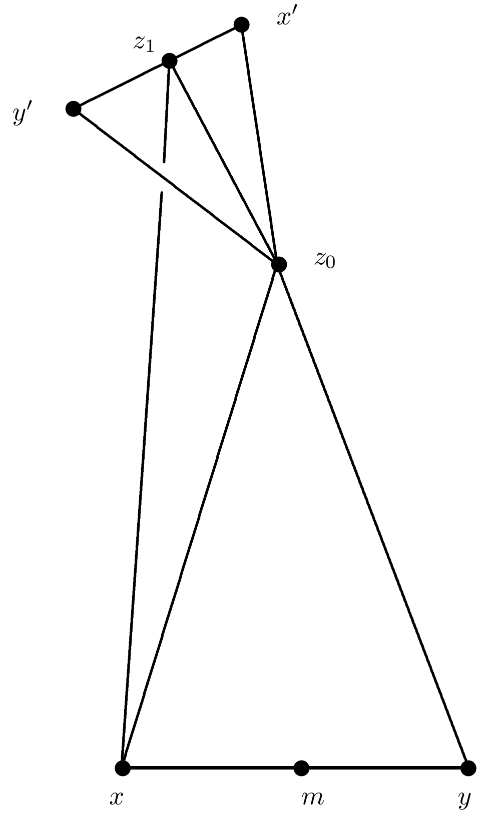

FIGURE 3.

are arbitrarily close to m, there exists a path τ in D_m with endpoints $d(m_1)$ and $d(m_2)$. This path τ determines a path σ in $\tilde{Y}$ with endpoints m_1 and m_2 so that $\tau = d(\sigma)$ and $\mathrm{Im}\sigma$ does not contain m. The generalized Busemann function α attains negative and positive values on the endpoints of σ. Thus, there exists a point z_1 on $\mathrm{Im}\sigma$ with $\alpha(z_1, x, y) = 0$. The piecewise geodesic $[x, z_1] \cup [z_1, y]$ has length, by uniqueness of (length minimizing) geodesics in $\tilde{Y}$, strictly bigger than $d(x, y) = 2d(x, m) = 2d(y, m)$. Thus,

$$d(z_1, x) = d(z_1, y) \gneqq d(x, m)$$

which contradicts the fact that $m = z_0$ is a point in Z of maximal distance from x. Assume now that D_m is not connected. We only have to deal with the case in which the geodesic segments $[m, x]$ and $[m, y]$ determine points $d(x)$ and $d(y)$ in distinct components, say C_x and C_y, of D_m. If there exists a connected component C_0 of D_m distinct from C_x and C_y, then pick a point z_1 so that $d(z_1) \in C_0$. Then the geodesic segments $[x, z_1]$ and $[y, z_1]$ necessarily contain z_0. It follows that $\alpha(z_1, x, y) = 0$ and

$$d(z_1, x) = d(z_1, m) + d(m, x) > d(m, x)$$

which contradicts the fact that $m = z_0$ is a point in Z of maximal distance from x. Assume now that C_x and C_y are the only components of D_m. By the midpoint assumption on $[x, y]$, the sets C_x and C_y cannot be both singletons. Assume C_y

is not a singleton. C_y satisfies $CAT(1)$-inequality and, therefore, the *systole* of $Link(m, \widetilde{Y})$ is greater than or equal to 2π (see [21, theorem 3.15]). Hence, we can extend $[y, m]$ to a geodesic segment $[y, z_1]$ which contains m in its interior so that $d(z_1) \in C_y$. Then $[x, m] \cup [m, z_1]$ and $[y, m] \cup [m, z_1]$ are both geodesic segments. It now follows that

$$d(z_1, x) = d(z_1, y) \gneqq d(m, x)$$

which completes the proof in this case.

Case B: $z_0 \neq m$.

Choose a neighborhood U around z_0 so that statements (5.1) and (5.2) hold. Extend the geodesic segments $[x, z_0]$ and $[y, z_0]$, i.e., choose points $x', y' \in U \setminus \{z_0\}$ so that $[x, z_0] \cup [z_0, x']$ and $[y, z_0] \cup [z_0, y']$ are both geodesic segments. Amongst all possible choices for the pair x' and y' pick one so that

$$\measuredangle_{z_0}(x', y') < \pi.$$

If $d(x') = d(y')$, then for some point z_1 on $[z_0, x'] \cap [z_0, y'] \setminus \{z_0\}$ we have

$$d(z_1, x) = d(z_1, y) \gneqq d(z_0, x)$$

which implies that z_0 is not of maximal distance from x. Hence, we may assume that $d(x') \neq d(y')$.

It is clear that $\alpha(x', x, y) \leq 0$ and $\alpha(y', x, y) \geq 0$. If either $\alpha(x', x, y) = 0$ or $\alpha(y', x, y) = 0$, then, again, the point z_0 is not of maximal distance from x. Thus we may assume that $\alpha(x', x, y) < 0$ and $\alpha(y', x, y) > 0$ which implies that

$$\exists\, z_1 \in [x', y'] : \alpha(z_1, x, y) = 0.$$

We proceed to show that $d(z_1, x) > d(z_0, x)$. Denote by σ, σ_1 and σ_2 the geodesic segments $[x', y'], [x', z_1]$ and $[z_1, y']$ respectively. Observe that $z_0 \notin \mathrm{Im}\sigma$ because x', y' are chosen so that $\measuredangle_{z_0}(x', y') < \pi$. By (5.2) the projection $d(\sigma)$ of σ in the space of directions D_{z_0} is a geodesic segment, hence,

$$d(\sigma_1) + d(\sigma_2) = d(\sigma).$$

As the length of $d(\sigma)$ is smaller, then π, either $d(\sigma_1)$ or, $d(\sigma_2)$ has length $< \pi/2$. In other words, either $\measuredangle_{z_0}(z_1, x') < \pi/2$ or, $\measuredangle_{z_0}(z_1, y') < \pi/2$. We may assume that

$$\measuredangle_{z_0}(z_1, x') < \pi/2 \tag{5.3}$$

(if $\measuredangle_{z_0}(z_1, y') < \pi/2$ we proceed in an identical way). As $[x, z_0] \cup [z_0, x']$ is a geodesic segment, $\measuredangle_{z_0}(x, x') \geq \pi$. Since

$$\measuredangle_{z_0}(x, z_1) + \measuredangle_{z_0}(z_1, x') \geq \measuredangle_{z_0}(x, x')$$

we have, by (5.3), that

$$\measuredangle_{z_0}(x, z_1) > \pi/2.$$

The latter inequality implies that $d(z_1, x) > d(z_0, x)$ which completes the proof of the proposition. □

In the proof of proposition 5.4 we will need the following construction:

Construction. Let Y be a negatively curved polyhedron. Consider the subspace $Y^{[1]}$ of Y (explained in 5.3 above) and let $Y_i, i \in \mathbb{N}$ be the connected components of $Y \setminus \operatorname{Int} Y^{[1]}$. For each i, let $\{\sigma_{i_j} \mid j = 1, 2, \dots\}$ be an enumeration (possibly infinite) of the set $Y_i \cap Y^{[1]}$. We glue the components $Y_i, i \in \mathbb{N}$ together according to the following rule:

> The 0-face σ_{i_j} of Y_i is identified with the 0-face $\sigma_{i'_{j'}}$ of $Y_{i'}$ if there exists a path lying entirely in $Y^{[1]}$ with endpoints σ_{i_j} and $\sigma_{i'_{j'}}$.

Note that the equality $i = i'$ is allowed in the above rule. In this way we obtain a negatively curved polyhedron denoted by Y_{-1} which does not contain any 1-dimensional simplices. The image of a geodesic g in GY determines a unique geodesic line in Y_{-1}. By employing base points on Y and Y_{-1} each geodesic g in GY determines a unique geodesic g_{-1} in GY_{-1}. This map

$$GY \to GY_{-1} \text{ is surjective.} \tag{5.4}$$

Given a subset $\mathcal{V}$ (resp. a point f) in GY we will be denoting by $\mathcal{V}_{-1}$ (resp. f_{-1}) the corresponding subset (resp. point) in GY_{-1} under the above map.

Proof of Theorem 5.4. First observe that in the case of a compact negatively curved polyhedron the assumption $\Omega = GY$ follows exactly as in the proof of Corollary 5.1 above. Moreover, as $\pi_1(Y)$ acts co-compactly on the hyperbolic space $\tilde{Y}$ it follows (by a theorem of Gromov) that $\pi_1(Y)$ is a hyperbolic group, hence, non-elementary. Therefore, theorem 5.4 can be stated for compact negatively curved polyhedra without any hypothesis at all.

If Y does not contain 1-dimensional simplices, then the conclusion of the theorem follows from theorem 1.2 and proposition 5.5. In order to deal with the general case we will modify the proof of theorem 1.2 at the point where assumption (1) of theorem 1.2 is used. This modification will assert that the midpoint assumption of proposition 5.5 is fulfilled, hence, the zeros for the α function needed do, in fact, exist.

Recall that assumption (1) of theorem 1.2 is only used in the proof of proposition 4.1 where given arbitrary open sets $\mathcal{O}, \mathcal{U} \subseteq GY$ we choose $f \in p^{-1}(\mathcal{O})$ and $h \in p^{-1}(\mathcal{U})$ such that

$$\exists \text{ } a \text{ } hyperbolic \text{ } \phi \in \Gamma : \big(f(+\infty), h(+\infty)\big) = (\phi(+\infty), \phi(-\infty))$$

and then assumption (1) is employed to obtain, for each n, a point ξ_n in ∂X such that

$$\alpha\big(\xi_n, f(0), \phi^n(h(0))\big) = 0.$$

We only need to show that by an appropriate choice of f, h and ϕ the midpoint of the geodesic segment $[f(0), \phi^n(h(0))]$ lies, for all n large enough, in the interior of a k-simplex of $\tilde{Y}$ with $k \geq 2$. Let $\mathcal{O}, \mathcal{U} \subseteq GY$ be given. We may choose a geodesic $h' \in \mathcal{U}$, a compact set $K_{h'} \subset \mathbb{R}$ and a real $\varepsilon_{h'} > 0$ such that

(a) the neighborhood $\mathcal{U}'$ around h' determined by $K_{h'}$ and $\varepsilon_{h'}$, i.e. $\mathcal{U}' = \{g \in GY \mid d\big(g(t), h'(t)\big) < \varepsilon_{h'} \ \forall\, t \in K_{h'}\}$, is a subset of $\mathcal{U}$.

We may refine the choices of $h', K_{h'}$ and $\varepsilon_{h'}$ so that, in addition, the following property is satisfied:

(b) for all $t > \max K_{h'}$, $h'(t)$ lies in a single component, say Y_0, of $Y \setminus \operatorname{Int} Y^{[1]}$

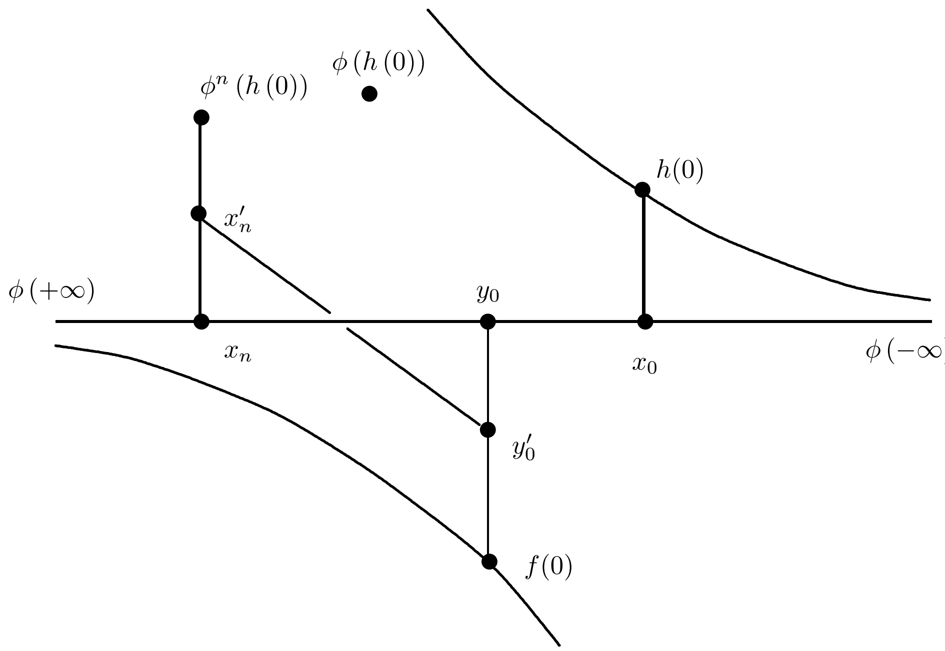

FIGURE 4.

as follows: we need to consider the time $t_{h'} \notin K_{h'}$ at which the geodesic h' first enters the set $Y^{[1]}$, i.e., set

$$t_{h'} = \inf \Big\{ t \in [\max K_{h'}, +\infty) \big| \, h'(t) \in Y^{[1]} \text{ and } h'(t+\varepsilon) \in \operatorname{Int} Y^{[1]} \ \forall \varepsilon > 0 \text{ small} \Big\}.$$

If $t_{h'} = +\infty$, i.e., $h'\big([\max K_{h'}, +\infty)\big)$ does not intersect $\operatorname{Int} Y^{[1]}$, then $h'|_{[\max K_{h'}, +\infty)}$ stays in a single component, say Y_0, of $Y \setminus \operatorname{Int} Y^{[1]}$ and, hence, property **(b)** is satisfied. Suppose now that $t_{h'} \neq +\infty$. Extend (in an arbitrary way) the geodesic ray $h' : (-\infty, t_{h'}] \to Y$ to a geodesic ray $(-\infty, T] \to Y$, for some $T > t_{h'}$, denoted again by h', so that $h'(T) \in bd\big(Y^{[1]}\big)$. Let Y_0 be the component of $Y \setminus \operatorname{Int} Y^{[1]}$ which contains $h'(T)$. We may now extend the geodesic ray $(-\infty, T] \to Y$ to a geodesic line, denoted again by h', so that

$$h'\big([T, +\infty)\big) \subset Y_0.$$

By enlarging, if necessary, the compact set $K_{h'}$ to contain T the choice of $h' \in \mathcal{U}$, $K_{h'} \subset \mathbb{R}$ and $\varepsilon_{h'} > 0$ satisfying **(a)** and **(b)** is complete. We need to do the same thing for $\mathcal{O}$, i.e., to choose a geodesic $f' \in \mathcal{O}$, a compact set $K_{f'} \subset \mathbb{R}$ and a real $\varepsilon_{f'} > 0$ such that

- **(c)** the neighborhood $\mathcal{O}'$ around f' determined by $K_{f'}$ and $\varepsilon_{f'}$ is a subset of $\mathcal{O}$,
- **(d)** for all $t > \max K_{f'}$, $f'(t)$ lies in the same component Y_0 of $Y \setminus \operatorname{Int} Y^{[1]}$.

For this it suffices to find a geodesic f' in $\mathcal{O}$ such that the image of f' intersects Y_0 (we then proceed to alter f' and $K_{f'}$ as we did with properties **(a)** and **(b)** above). Let Y_{-1} be the negatively curved polyhedron constructed above. Since Y_{-1}

does not contain 1-dimensional simplices, the geodesic flow on Y_{-1} is topologically mixing. Hence, the definition 1.1 of topological mixing applied to the neighborhoods $\mathcal{O}_{-1}$ and $\mathcal{U}'_{-1}$ (which are the images of the given neighborhood $\mathcal{O}$ and the above chosen neighborhood $\mathcal{U}'$ under the map (5.4)) implies the existence of a geodesic $f'_{-1} \in \mathcal{O}_{-1}$ so that the image of f'_{-1} intersects the subset of Y_{-1} which corresponds to the component Y_0. By property (5.4) above, choose a pre-image $f' \in \mathcal{O}$ of f'_{-1}. Then, by construction of Y_{-1}, the image of f' intersects Y_0.

Thus, given arbitrary neighborhoods $\mathcal{O}, \mathcal{U} \subseteq GY$ we may choose geodesics $f' \in \mathcal{O}$ and $h' \in \mathcal{U}$ so that for some $t_0 \in \mathbb{R}$, $f'(t)$ and $h'(t) \in Y_0$ for all times $t \geq t_0$. Let $\widetilde{Y_0}$ be the universal cover of Y_0. Since $\widetilde{Y_0}$ embeds isometrically in $\tilde{Y}$ we have, by [15, page 35], that $\partial\widetilde{Y_0}$ injects in $\partial\tilde{Y}$. Since $\Omega = GY$ (equivalently, $\Lambda\big(\pi_1(Y)\big) = \partial\tilde{Y}$) it follows that $\Lambda\big(\pi_1(Y_0)\big) = \partial\widetilde{Y_0}$. As $\partial\widetilde{Y_0}$ is an infinite set, $\pi_1(Y_0)$ is non-elementary which, together with property (1.10), implies that there exists a hyperbolic isometry $\phi \in \pi_1(Y_0) \hookrightarrow \pi_1(Y)$ such that for some lifts $f \in p^{-1}(\mathcal{O})$ and $h \in p^{-1}(\mathcal{U})$ of f' and h', respectively, we have

- $\big(f(+\infty), h(+\infty)\big) = (\phi(+\infty), \phi(-\infty))$,
- the image of the corresponding closed geodesic c_ϕ in Y lies entirely in Y_0.

We proceed now to show that for these choices of f, h and ϕ the midpoint of the geodesic segment $[f(0), \phi^n(h(0))]$ lies, for all n large enough, in the interior of a k-simplex of $\tilde{Y}$ with $k \geq 2$. Then, proposition 5.5 applies to assert the existence of ξ_n in ∂X such that $\alpha\big(\xi_n, f(0), \phi^n(h(0))\big) = 0$ completing the proof of 5.4.

For each $n = 0, 1, 2, \dots$, set x_n to be the projection of $\phi^n(h(0))$ on the geodesic line $(\phi(+\infty), \phi(-\infty))$ (by notation, $\phi^0(h(0)) = h(0)$). Similarly, let y_0 be the projection of $f(0)$ on the same geodesic line. For the reader's convenience, we have gathered all the above notation in figure 4. Let ℓ denote the translation length of the hyperbolic isometry ϕ (i.e., ℓ equals the length of the closed geodesic c_ϕ in Y corresponding to ϕ) and set $\ell_f = d(f(0), y_0)$ and $\ell_h = d(h(0), x_0)$. Observe that, since ϕ is an isometry, $d\big(\phi^n(h(0)), x_n\big) = \ell_h$ for all n. Choose $N \in \mathbb{N}$ such that

$$n\ell > \ell_f + \ell_h + d(x_0, y_0) \;\; \forall \; n \geq N. \tag{5.5}$$

In addition, we may assume that

$$x_n \neq y_0 \; \forall \; n \geq N. \tag{5.6}$$

The latter can be done because the sequence $\{\phi^n(h(0))\}_{n\in\mathbb{N}}$ converges to $\phi(+\infty)$ and $d\big(\phi^n(h(0)), x_n\big) = \ell_h$ for all n.

Fix any $n \geq N$ and consider the piecewise geodesic segment

$$\sigma := [f(0), y_0] \cup [y_0, x_n] \cup [x_n, \phi^n(h(0))].$$

Consider the sub-path σ' of σ which lies entirely in $\widetilde{Y_0}$ and is of largest length with respect to this property. Since $[y_0, x_n] \subset \widetilde{Y_0}$, such a sub-path exists and is of the form

$$\sigma' = [y'_0, y_0] \cup [y_0, x_n] \cup [x_n, x'_n]$$

for some $y'_0 \in [f(0), y_0]$ and $x'_n \in [x_n, \phi^n(h(0))]$.

If $y'_0 = f(0)$ and $x'_n = \phi^n(h(0))$, it is clear that $\sigma(= \sigma')$ lies entirely in $\widetilde{Y_0}$ and so does the geodesic segment $[f(0), \phi^n(h(0))]$. In other words, $[f(0), \phi^n(h(0))]$ does not intersect $\operatorname{Int} Y^{[1]}$, thus, its midpoint lies in the interior of a k-simplex of $\tilde{Y}$ with $k \geq 2$ as required.

If $y_0' \neq f(0)$ and $x_n' \neq \phi^n(h(0))$, then y_0' and x_n' are the 0-faces of two distinct 1-dimensional free simplices attached to $\widetilde{Y_0}$ along y_0' and x_n'. The fact that x_n' and y_0' (as well as the corresponding 1-dimensional simplices) are distinct follows from (5.6). It is easy to see that the piecewise geodesic

$$[f(0), y_0'] \cup [y_0', x_n'] \cup [x_n', \phi^n(h(0))]$$

is, in fact, a geodesic, namely, is equal to the geodesic segment $[f(0), \phi^n(h(0))]$. Now property (5.5) asserts that the midpoint of $[f(0), \phi^n(h(0))]$ lies in the subsegment $[y_0', x_n']$ which is a subset of $\widetilde{Y_0}$.

The cases $y_0' = f(0), x_n' \neq \phi^n(h(0))$ and $y_0' \neq f(0), x_n' = \phi^n(h(0))$ are treated in an identical way. □

It is clear in the above proof that topological mixing in a polyhedron Y which contains free 1-simplices (i.e., $Y^{[1]} \neq \emptyset$) is, in fact, deduced from topological mixing in the space Y_{-1} which does not contain free 1-simplices. There is an alternative approach for this deduction which uses (along with the space Y_{-1}) the topological mixing in graphs as described in remark 1.9. This alternative approach exhibits the important role of graphs in topological mixing of spaces of higher dimension. Since the details are quite technical, we will only give a description of this approach by presenting it in the frame of the following:

Example 5.6. Let Y_A, Y_B be two copies of a (triangulated) torus of genus 2 equipped with a metric of curvature -1. Let $A \in Y_A$, $B \in Y_B$ be 0-simplices of Y_A and Y_B respectively. Let Y be the negatively curved polyherdron consisting of Y_A, Y_B and a single 1-simplex σ (of length 1) whose 0-faces are A and B, i.e.,

$$Y = Y_A \vee_A \sigma \vee_B Y_B.$$

In this example the space Y_{-1} (defined in 5.3) is the one point union of Y_A and Y_B along A and B, i.e.,

$$Y_{-1} = Y_A \vee_{A=B} Y_B.$$

There are countably many geodesic segments in Y_A starting and ending at A. Let Δ_A be the infinite graph consisting of

- a single vertex (denoted again by A),
- for each geodesic segment in Y_A starting and ending at $A \in Y$, there is one loop (of equal length) in Δ_A based at $A \in \Delta_A$.

Thus, Δ_A is an infinite graph (in fact, an infinite rose) whose metric is determined by the geometry of Y_A. Similarly, Δ_B is defined. Let Δ be the disjoint union of Δ_A and Δ_B with one edge with endpoints A and B and of length 1 attached. Moreover, set

$$\Delta_{-1} = \Delta_A \vee_{A=B} \Delta_B.$$

It is clear from the above construction that there is a bijection between $G\Delta$ and the subset of GY consisting of all geodesics $f \in GY$ such that both $\mathrm{Im}\left(f|_{[0,+\infty)}\right)$ and $\mathrm{Im}\left(f|_{(-\infty,0]}\right)$ intersect σ infinitely many times. By proposition 5.5 and theorem 1.2 the geodesic flow on Y_{-1} is topologically mixing. It follows that the lengths of all closed loops in Δ_{-1} generate a dense subset of $\mathbb{R}$, otherwise, in a fashion similar to example 1.8 and using the above-mentioned bijection, it would be possible to construct neighborhoods in GY_{-1} not satisfying definition 1.1. Since closed loops in Δ_{-1} generate a dense subset of $\mathbb{R}$, the same is true for the closed loops in Δ, hence, topological mixing holds for Δ (cf. remark 1.9). On the other hand, it can be shown that topological mixing on Δ implies (in fact, is equivalent with) topological mixing on Y.

In conclusion, topological mixing in Y can proved through graphs once we know that it holds for Y_{-1}. This alternative approach described in this example can be defined and proved in detail for an arbitrary negatively curved polyhedron Y.

Remark 5.7. Let Y be a negatively curved polyhedron; then Y is a graph precisely when $\partial\tilde{Y}$ is totally disconnected. Thus, theorem 5.4 says that the geodesic flow on a negatively curved polyhedron is topologically mixing if the boundary of its universal cover is not totally disconnected. It is plausible to expect that this is the case in any geodesically complete $CAT(-1)$-space, namely, that condition (1) in theorem 1.2 can be replaced by the assumption that $\partial\tilde{Y}$ is not totally disconnected.

5.2. **Ideal polyhedra.** We apply theorem 1.2 to a non-compact class of spaces, namely, to *n-dimensional complete ideal polyhedra*. Important examples of ideal polyhedra have appeared in Thurston's work (see [23], [22, Sec. 10.3]), where 3-manifolds, which are complements of links and knots in $\mathbb{S}^3$, are constructed by gluing together finitely many ideal tetrahedra. In consequence, these finite volume 3-manifolds are equipped by a complete hyperbolic structure. Moreover, the 2-skeleton of these 3-manifolds are examples of 2-dimensional ideal polyhedra.

Definition 5.8. An n-dimensional ideal polyhedron is a locally finite union of ideal hyperbolic n-polytopes glued together isometrically along their $(n-1)$-faces with at least two germs of polytopes along each $(n-1)$-face. The distance function is defined exactly as described in section 5.1 for negatively curved polyhedra, i.e., the distance $d(x,y)$ from x to y is defined to be the lower bound of the lengths of broken geodesics from x to y. With the induced metric, an n-dimensional ideal polyhedron is required to be complete. Since it is locally compact, it is proper and geodesic. Moreover, it is required to have curvature less than or equal to -1.

We note here that in the case $n = 2$ the curvature condition in the above definition can be proved, hence, is redundant (see [11, Prop. 1]). An ideal polyhedron Y is called *finite* if finitely many polytopes are glued together to form Y.

Certain properties of this class of spaces, including transitivity of the geodesic flow, have been studied in [9], [10], [11] and [12].

If Y is an ideal polyhedron of dimension n, then Y is naturally a proper geodesic metric space.

The universal covering $\widetilde{Y}$ of Y is a complete ideal polyhedron of dimension n satisfying $CAT(-1)$ inequality (see [21, Cor. 2.11]). If Y is a finite polyhedron, the non-wandering set Ω of the geodesic flow on Y is equal to GY (see Cor. 10 in [10]) and $\pi_1(Y)$ is a non-elementary group of isometries acting properly discontinuously on $\widetilde{Y}$ (see Cor. 12 in [10]). Moreover, proposition 5.5 applies verbatim to ideal polyhedra. Hence, we obtain the following application of theorem 1.2.

Corollary 5.9. *Let Y be an n-dimensional finite ideal polyhedron. Then the geodesic flow on Y is topologically mixing.*

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