TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 354, Number 3, Pages 1137–1151
S 0002-9947(01)02901-4
Article electronically published on September 21, 2001

RECURRENT DIMENSIONS OF QUASI-PERIODIC SOLUTIONS FOR NONLINEAR EVOLUTION EQUATIONS

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Abstract. In this paper we introduce recurrent dimensions of discrete dynamical systems and we give upper and lower bounds of the recurrent dimensions of the quasi-periodic orbits. We show that these bounds have different values according to the algebraic properties of the frequency and we investigate these dimensions of quasi-periodic trajectories given by solutions of a nonlinear PDE.

1. Introduction

In our previous papers ([10], [11]) we proved that lower bounds of correlation dimensions of quasi-periodic orbits have different values according to algebraic properties, rational (badly) approximable properties, of the irrational frequencies. We introduced a class of irrational numbers, α-order Roth numbers, which contains the class of Roth numbers. These irrational numbers are classified according to badness levels of approximation by rational numbers. On the contrary, Liouville numbers are well known as the irrational numbers which have extremely good approximation by rational numbers. In this paper we introduce a new class of irrational numbers which contains the class of Liouville numbers and we call them α-order Liouville numbers, where α specifies the goodness of approximation by rationals.

The most important and essential properties of a quasi-periodic orbit as a typical one of almost periodic orbits are recursive properties. In this paper first we introduce definitions of recurrent or periodically recurrent dimensions and we give the relations between correlation dimensions and recurrent dimensions. Then we estimate lower and upper bounds of these dimensions for quasi-periodic orbits of a nonlinear discrete dynamical system by using the goodness levels of rational approximations for the irrational frequencies which are α-order Liouville numbers. Next we apply these results to an example of PDE, a nonlinear reaction diffusion equation with two periodic terms, the periods of which are rationally independent.

Our plan of this paper is as follows: In section 2 we introduce definitions of recurrent dimensions and give inequality relations to correlation dimensions. In section 3 we estimate these dimensions, from below and above, for quasi-periodic orbits with frequencies given by α-order Roth or α-order Liouville numbers. In sections 4 and 5 we study a reaction diffusion equation with periodic terms. In section 4 we consider an abstract formulation of the PDE by using the monotone

Received by the editors October 29, 2000 and, in revised form, May 9, 2001.
2000 *Mathematics Subject Classification.* Primary 11K60, 28A80, 35B15.
Key words and phrases. Diophantine approximation, quasi-periodic solutions, fractal dimension.

operator theory and in section 5 we estimate the dimensions of discrete orbits given by the quasi-periodic solutions.

2. Recurrent dimension

Let T be a nonlinear operator on a Banach space X. For an element $x \in X$ we consider a discrete dynamical system given by

$$x_n = T^n x, \quad n \in \mathbf{N}_0 := \mathbf{N} \cup \{0\},$$

and its orbit is denoted by

$$\Sigma_x = \{T^n x : n \in \mathbf{N}_0\}.$$

For a small $\varepsilon > 0$, define upper and lower first ε-recurrent times by

$$\overline{M}_\varepsilon = \sup_{n \in \mathbf{N}_0} \min\{m : T^{m+n}x \in V_\varepsilon(T^n x), \quad m \in \mathbf{N}\},$$
$$\underline{M}_\varepsilon = \inf_{n \in \mathbf{N}_0} \min\{m : T^{m+n}x \in V_\varepsilon(T^n x), \quad m \in \mathbf{N}\},$$

respectively, where $V_\varepsilon(z) = \{y \in X : \|y - z\| < \varepsilon\}$. Then upper and lower recurrent dimensions are defined as follows:

$$\overline{D}_r(\Sigma_x) = \limsup_{\varepsilon \to 0} \frac{\log \overline{M}_\varepsilon}{-\log \varepsilon},$$
$$\underline{D}_r(\Sigma_x) = \liminf_{\varepsilon \to 0} \frac{\log \underline{M}_\varepsilon}{-\log \varepsilon}.$$

If $\overline{M}_\varepsilon = \underline{M}_\varepsilon$ and the limit exits as $\varepsilon \to 0$, we denote $D_r(\Sigma_x) = \overline{D}_r(\Sigma_x) = \underline{D}_r(\Sigma_x)$.

The recurrent properties are essential for almost periodic dynamical systems. Next we define periodically recurrent dimensions of almost periodic orbits. Let the operator T be invertible and consider the almost periodic orbit $\tilde{\Sigma}_x = \{T^m x : m \in \mathbf{Z}\}$. We recall that an orbit is almost periodic if for each $\varepsilon > 0$ there exists a number $l_\varepsilon > 0$ such that for every $m \in \mathbf{Z}$ there exists an integer $\mu \in [m, m + l_\varepsilon] \cap \mathbf{Z}$ with the property

$$\|T^{(\mu+n)}x - T^n x\| \le \varepsilon \quad \text{for all} \quad n \in \mathbf{Z}. \tag{2.1}$$

Here the number μ is called an ε-almost period and l_ε is called an inclusion length for ε-almost period.

By using the inclusion length we can define periodically recurrent dimensions as follows:

$$\overline{D}_p(\tilde{\Sigma}_x) = \limsup_{\varepsilon \to 0} \frac{\log l_\varepsilon}{-\log \varepsilon},$$
$$\underline{D}_p(\tilde{\Sigma}_x) = \liminf_{\varepsilon \to 0} \frac{\log l_\varepsilon}{-\log \varepsilon}.$$

If the limit exists as $\varepsilon \to 0$, we put $D_p(\tilde{\Sigma}_x) = \overline{D}_p(\tilde{\Sigma}_x) = \underline{D}_p(\tilde{\Sigma}_x)$.

Correlation dimensions are very popular and studied in various dynamical systems or in fractal geometry (cf. [2] or [13]). Let $S = \{x_1, x_2, ..., x_n, ...\}$ be an infinite sequence of elements in X and, for a small number $\varepsilon > 0$, define

$$N(\varepsilon) = \lim_{n \to \infty} \frac{1}{n^2} \sum_{i,j=1}^{n} H(\varepsilon - \|x_i - x_j\|)$$

where $H(\cdot)$ is a Heaviside function:

$$H(a) = \begin{cases} 1 & \text{if } a \geq 0, \\ 0 & \text{if } a < 0. \end{cases}$$

The upper and lower correlation dimension of S, $\overline{D}_c(S), \underline{D}_c(S)$, are defined as follows:

$$\begin{aligned} \overline{D}_c(S) &= \limsup_{\varepsilon \downarrow 0} \frac{\log N(\varepsilon)}{\log \varepsilon}, \\ \underline{D}_c(S) &= \liminf_{\varepsilon \downarrow 0} \frac{\log N(\varepsilon)}{\log \varepsilon}. \end{aligned}$$

If $\overline{D}_c = \underline{D}_c$, we say that S has the correlation dimension $D_c(S) = \overline{D}_c = \underline{D}_c$.

Theorem 2.1. *Let X be a Banach space and consider a nonlinear operator T on X and its orbits $\Sigma_x = \{T^n x : n \in \mathbf{N}_0\}$ for some $x \in X$. Then*

$$\underline{D}_c(\Sigma_x) \geq \underline{D}_r(\Sigma_x). \tag{2.2}$$

Proof. From the definition, for every $\delta > 0$, there exists a constant $\varepsilon_0 > 0$ such that, if $0 < \varepsilon < \varepsilon_0$,

$$\underline{D}_r \leq \frac{\log \underline{M}_\varepsilon}{-\log \varepsilon} + \delta.$$

It follows that

$$\varepsilon^{-\underline{D}_r + \delta} \leq \underline{M}_\varepsilon.$$

For a large integer $n \in \mathbf{N}$, let $1 \leq l, m \leq n$. If $0 \leq m - l < \varepsilon^{-\underline{D}_r + \delta}$, we have

$$\|T^m x - T^l x\| = \|T^{m-l} T^l x - T^l x\| \geq \varepsilon,$$

and also, if $0 \leq l - m < \varepsilon^{-\underline{D}_r + \delta}$, we have

$$\|T^l x - T^m x\| = \|T^{l-m} T^m x - T^m x\| \geq \varepsilon.$$

Let $M_n(\varepsilon)$ be a number of elements $T^m x,\ 1 \leq m \leq n$, in the ε-neighborhood of $T^l x,\ 1 \leq l \leq n$;

$$M_n(\varepsilon) = \#\{m \in \mathbf{N} : T^m x \in V_\varepsilon(T^l x), \quad 1 \leq m \leq n\}.$$

Then we have

$$M_n(\varepsilon/2) \leq n \varepsilon^{\underline{D}_r - \delta}$$

and it follows that

$$\frac{1}{n^2} \sum_{l,m=1}^{n} H(\frac{\varepsilon}{2} - \|T^l x - T^m x\|) \leq \frac{1}{n^2} n \cdot n \varepsilon^{\underline{D}_r - \delta} = \varepsilon^{\underline{D}_r - \delta}.$$

Thus we obtain

$$\underline{D}_c = \liminf_{\varepsilon \to 0} \frac{\log N(\frac{\varepsilon}{2})}{\log \frac{\varepsilon}{2}} \geq \liminf_{\varepsilon \to 0} \frac{\log \varepsilon^{\underline{D}_r - \delta}}{\log \frac{\varepsilon}{2}} = \underline{D}_r - \delta$$

for every $\delta > 0$, which yields (2.2). □

On the other hand, we can estimate the upper bound of the correlation dimensions by using the periodically recurrent dimension.

Theorem 2.2. *Let $T^m x, m \in \mathbf{Z}$, be almost periodic and denote*

$$\tilde{\Sigma}_x = \{T^m x : \quad m \in \mathbf{Z}\}.$$

Then we have

$$\overline{D}_p(\tilde{\Sigma}_x) \geq \overline{D}_c(\tilde{\Sigma}_x). \tag{2.3}$$

Proof. From the definition, for every $\delta > 0$, there exists a constant $\varepsilon_0 > 0$ such that, if $0 < \varepsilon < \varepsilon_0$,

$$\overline{D}_p + \delta > \frac{\log l_\varepsilon}{-\log \varepsilon}.$$

It follows that

$$\varepsilon^{-(\overline{D}_p + \delta)} > l_\varepsilon.$$

For a large integer $n \in \mathbf{N}$, let $M_n(\varepsilon)$ be a number of elements $T^m x,\ -n \leq m \leq n$, in the ε-neighborhood of $T^l x,\ -n \leq l \leq n$. Then we have

$$M_n(\varepsilon) \geq \frac{2n}{l_\varepsilon} \geq 2n\varepsilon^{\overline{D}_p + \delta},$$

since for each interval $[a, a + l_\varepsilon),\ a \in \mathbf{Z}$, there exists an integer $m' \in [a, a + l_\varepsilon)$, which satisfies

$$\|T^{m'} T^l x - T^l x\| \leq \varepsilon$$

or

$$\|T^l x - T^{l-m'} x\| = \|T^{m'} T^{l-m'} x - T^{l-m'} x\| \leq \varepsilon.$$

It follows that

$$\begin{aligned} \frac{1}{(2n+1)^2} \sum_{-n \leq l, m \leq n} H(\varepsilon - \|T^l x - T^m x\|) \quad &\geq \quad \frac{1}{(2n+1)^2}(2n+1) \cdot 2n\varepsilon^{\overline{D}_p + \delta} \\ &= \quad \frac{2n}{2n+1}\varepsilon^{\overline{D}_p + \delta}. \end{aligned}$$

Thus we obtain

$$\overline{D}_c \leq \limsup_{\varepsilon \to 0} \frac{\log \varepsilon^{\overline{D}_p + \delta}}{\log \varepsilon} = \overline{D}_p + \delta$$

for every $\delta > 0$, which completes the proof. □

3. Dimensions of quasi-periodic orbits

Let $S(t),\ t \geq 0$, be a semigroup of continous (generally nonlinear) operators on a Banach space X. For each $x \in X$, assume that $S(t+1)x = S(t)x,\ t \geq 0$ and consider the following Hölder conditions:

(G1) There exists a constants $\delta_1 :\ 0 < \delta_1 \leq 1$ and a monotone increasing function $k_1 : \mathbf{R}^+ \to \mathbf{R}^+$, which satisfies

$$\|S(t)x - S(s)x\| \leq k_1(\|x\|)|t - s|^{\delta_1}, \quad t, s \geq 0, |t - s| \leq \varepsilon_0,$$

for a small constant $\varepsilon_0 > 0$.

(G2) There exists a constant $\delta_2 :\ 0 < \delta_2 \leq 1$ and a monotone increasing function $k_2 : \mathbf{R}^+ \to \mathbf{R}^+$, which satisfies

$$\|S(t)x - S(s)x\| \geq k_2(\|x\|)|t - s|^{\delta_2}, \quad t, s \geq 0, |t - s| \leq \frac{1}{2}.$$

For an irrational number $\tau : 0 < \tau < 1$, define a quasi-periodic dynamical system by

$$T^n x = S(\tau n)x, \quad n \in \mathbf{N}_0,$$

then our purpose is to estimate the recurrent dimension under the following algebraic conditions on the frequency τ.

(i) Constant type: there exists a constant $c_0 > 0$ such that

$$|\tau - \frac{r}{q}| \geq \frac{c_0}{q^2} \tag{3.1}$$

for all positive integers r, q.

(ii) Roth number type: for every $\varepsilon > 0$, there exists a constant $c_\varepsilon > 0$ which satisfies

$$|\tau - \frac{r}{q}| \geq \frac{c_\varepsilon}{q^{2+\varepsilon}} \tag{3.2}$$

for all positive integers r, q.

(iii) α_0-order Roth number type: there exist a constant $\alpha_0 > 0$ such that for every $\alpha \geq \alpha_0$ there exists a constant $c_\alpha > 0$ which satisfies

$$|\tau - \frac{r}{q}| \geq \frac{c_\alpha}{q^{2+\alpha}} \tag{3.3}$$

for all positive integers r, q.

These conditions are classified by the rational approximable properties of the irrational number τ.

Consider the following continued fraction of the number τ:

$$\tau = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}} \qquad (a_i \in \mathbf{N}) \tag{3.4}$$

and take the rational approximation as follows. Let $m_0 = 1, n_0 = 0, m_{-1} = 0, n_{-1} = 1$ and define the pair of sequences of natural numbers

$$m_i = a_i m_{i-1} + m_{i-2}, \tag{3.5}$$

$$n_i = a_i n_{i-1} + n_{i-2}, \quad i \geq 1, \tag{3.6}$$

then the elementary number theory gives the Diophantine approximation

$$\frac{1}{m_i(m_{i+1} + m_i)} < |\tau - \frac{n_i}{m_i}| < \frac{1}{m_i m_{i+1}} < \frac{1}{m_i^2} \tag{3.7}$$

where the sequence $\{n_i/m_i\}$ is the best approximation in the sense that

$$|\tau - \frac{n_i}{m_i}| \leq |\tau - \frac{r}{l}|$$

holds for every rational $r/l : \ l \leq m_i$.

An irrational number τ, which has an extremely good approximable property by rational numbers, is called a Liouville number if

$$|\tau - \frac{n_i}{m_i}| \leq \frac{1}{m_i^i}, \quad \forall i.$$

Here we introduce a class of irrational numbers which contains Liouville numbers as follows. We state that an irrational number τ is an α_1-order Liouville number if (iv) there exist constants $c, \alpha_1 > 0$ such that

$$|\tau - \frac{n_i}{m_i}| \leq \frac{c}{m_i^{2+\alpha_1}}, \quad \forall i. \tag{3.8}$$

For the case of the constant type (i), it is well known (cf. [16]) that the uniform boundedness of the sequence $\{a_j\}$ is equivalent to the property (1.1). For the α-order Roth numbers and the α-order Liouville numbers we can show the equivalent conditions to the approximable properties of these numbers by using the growth rate of $\{m_j\}$.

(B1) There exist constants $\beta, K > 0$:

$$m_{j+1} \leq K m_j^{1+\beta}, \quad \forall j. \tag{3.9}$$

We have proved the following two lemmas in [11].

Lemma 3.1. *If Hypothesis* **(B1)** *is satisfied for an irrational number τ, then τ is a Roth number with its order*

$$\alpha_0 = \beta(\beta + 3). \tag{3.10}$$

Lemma 3.2. *If τ is a Roth number with its order α_0, then for every $\beta \geq \alpha_0$, there exists $K_\beta > 0$ which satisfies* **(B1)**:

$$m_{j+1} \leq K_\beta m_j^{1+\beta}, \quad \forall j. \tag{3.11}$$

For the α_1-order Liouville numbers we can show the equivalent condition:

(B2) There exist constants $\alpha_1, L > 0$:

$$m_{j+1} \geq L m_j^{1+\alpha_1}, \quad \forall j. \tag{3.12}$$

Lemma 3.3. *τ is a Liouville number with its order α_1 if and only if τ satisfies the condition* **(B2)**.

Proof. If τ satisfies **(B2)**, then it follows from (3.7) that

$$|\tau - \frac{n_j}{m_j}| < \frac{1}{m_{j+1} m_j} \leq \frac{1}{L m_j^{2+\alpha_1}}.$$

On the other hand, if τ is a Liouville number with the order α_1, then also (3.7) yields

$$\begin{aligned} \frac{c}{m_j^{2+\alpha_1}} \geq |\tau - \frac{n_j}{m_j}| &> \frac{1}{m_j(m_{j+1} + m_j)} \\ &> \frac{1}{2 m_{j+1} m_j}. \end{aligned}$$

Thus we have

$$m_{j+1} > \frac{1}{2c} m_j^{1+\alpha_1}.$$

□

Obviously, **(B2)** is equivalent to the following condition on the partial quotients in the continued fraction expansion of τ.

(B3) There exist constants $\alpha_1, L' > 0$:

$$a_{j+1} \geq L' m_j^{\alpha_1}, \quad \forall j. \tag{3.13}$$

In [11] we have given a sufficient condition for an α-order Roth number, using the partial quotients of the continued fraction expansion.

Lemma 3.4. *Let $\{a_j\}$ be the partial quotients in the continued fraction expansion of τ. Assume that, for a given constant $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$, which satisfies*

$$a_{j+1}a_j^2 \le C_\varepsilon (a_{j-1}a_{j-2}\cdots a_1)^\varepsilon, \quad \forall j.$$

Then we have

$$|\tau - \frac{r}{q}| \ge \frac{c_\varepsilon}{q^{2+\varepsilon}}, \quad \forall q, r \in N$$

where $c_\varepsilon = 1/(16C_\varepsilon)$.

For an α-order Liouville number we can show the following lemma.

Lemma 3.5. *If the partial quotients in the continued fraction expansion of τ satisfies*

$$a_{j+1} \ge L_0 a_j^{\beta+1}, \quad \forall j$$

for some $\beta > 0$ and $L_0 \ge 2^{\beta+1}$, then τ is a Liouville number with its order β.

Proof. By induction we show that the condition **(B2)** is satisfied. Let $1 \le K \le a_1$, then we have

$$m_1 = a_1 m_0 + m_{-1} = a_1 m_0 \ge K m_0 = K m_0^{\beta+1}.$$

Next we assume that $m_j \ge K m_{j-1}^{\beta+1}$. Since

$$\frac{m_{j+1}}{m_j} \ge a_{j+1}, \quad a_j \ge \frac{m_j}{2m_{j-1}},$$

it follows by hypothesis that

$$\begin{aligned}\frac{m_{j+1}}{m_j} &\ge a_{j+1} \ge L_0 a_j^{\beta+1} \\ &\ge L_0(\frac{1}{2})^{\beta+1}(\frac{m_j}{m_{j-1}})^{\beta+1}.\end{aligned}$$

Thus we obtain

$$\begin{aligned}m_{j+1} &\ge L_0(\frac{1}{2})^{\beta+1} m_j^{\beta+1}\frac{m_j}{m_{j-1}^{\beta+1}} \\ &\ge K m_j^{\beta+1}.\end{aligned}$$

□

Example 3.6. For some positive numbers $\kappa, M > 1$, let

$$a_j \sim M^{\kappa^j},$$

that is, there exist constants $d_1 \ge d_2 > 0$:

$$d_1 M^{\kappa^j} \ge a_j \ge d_2 M^{\kappa^j}. \tag{3.14}$$

Assume that

$$M^{\kappa^2-\kappa} \ge \frac{2d_1}{d_2}, \tag{3.15}$$

then τ is a Liouville number with its order β:

$$\beta \leq \frac{\log d_2 + \kappa^2 \log M}{\log 2d_1 + \kappa \log M} - 1. \tag{3.16}$$

In fact, (3.15) yields

$$\log d_2 + \kappa^2 \log M \geq \log 2d_1 + \kappa \log M$$

and (3.16) gives

$$\beta + 1 \leq \frac{\log d_2 + \kappa^{j+1} \log M}{\log 2d_1 + \kappa^j \log M} \tag{3.17}$$

for $j \geq 1$. It follows that

$$a_{j+1} \geq d_2 M^{\kappa^{j+1}} \geq (2d_1)^{\beta+1} \cdot (M^{\kappa^j})^{\beta+1} \geq 2^{\beta+1} a_j^{\beta+1},$$

which satisfies the hypothesis of Lemma 3.5.

The number, which satisfies (3.14), is also a Roth number with its order $(\kappa - 1)(\kappa + 1)$. (For further details and examples, see [11].)

Since the correlation dimensions are estimated by the recurrent dimensions, here we give a lower bound of the recurrent dimension of the quasi-periodic orbit

$$\Sigma_x = \{S(\tau n)x : n \in \mathbf{N}_0\}, \quad x \in X,$$

when the frequency τ is an α-order Roth number and we also give an upper bound of the recurrent dimension when τ is an α-order Liouville number.

Theorem 3.7. *Under the assumption* **(G2)***, assume that there exists a constant* $K_2 > 0$ *such that*

$$\inf_{n \in \mathbf{N}_0} k_2(\|S(\tau n)x\|) \geq K_2$$

and assume that the frequency τ *is a Roth number with its order* α_0*. Then the recurrent dimension of the quasi-periodic orbit* Σ_x *satisfies*

$$\underline{D}_r(\Sigma_x) \geq \frac{1}{\delta_2(1 + \alpha_0)}. \tag{3.18}$$

Proof. Put

$$\varphi(m) = S(\tau m)x, \quad m \in \mathbf{N}_0,$$

then, since we can find an integer n':

$$|m\tau - n'| < \frac{1}{2},$$

it follows from (iii), **(G2)** and the hypothesis that

$$\begin{aligned}
\|\varphi(m+n) - \varphi(n)\| &= \|S(\tau(m+n))x - S(\tau n)x\| \\
&= \|S(\tau(m+n))x - S(\tau n + n')x\| \\
&\geq k_2(\|S(\tau n)x\|)|\tau m - n'|^{\delta_2} \\
&\geq K_2(\frac{c_\alpha}{m^{1+\alpha}})^{\delta_2}, \quad \forall \alpha \geq \alpha_0,
\end{aligned}$$

for all $m \in \mathbf{N}$ and for all $n \in \mathbf{N}_0$. For every $\varepsilon > 0$, there exists $m \in \mathbf{N}$ such that

$$K_2(\frac{c_\alpha}{(m+1)^{1+\alpha}})^{\delta_2} \leq \varepsilon < K_2(\frac{c_\alpha}{m^{1+\alpha}})^{\delta_2}$$

and as $\varepsilon \to 0$, $m \to +\infty$. Thus we can obtain

$$\begin{aligned}\underline{D}_r &= \liminf_{\varepsilon\to 0} \frac{\log \underline{M}(\varepsilon)}{-\log\varepsilon} \\ &\geq \liminf_{\varepsilon\to 0} \frac{\log m}{-\log\varepsilon} \\ &\geq \lim_{m\to\infty} \frac{\log m}{\delta_2(1+\alpha)\log(m+1) - \log K_2 c_\alpha^{\delta_2}} \\ &= \frac{1}{\delta_2(1+\alpha)}\end{aligned}$$

for all $\alpha \geq \alpha_0$. □

Theorem 3.8. *Under the assumption* **(G1)***, assume that the frequency τ is a Liouville number with its order α_1. Then the recurrent dimension of the quasi-periodic orbit Σ_x satisfies*

$$\underline{D}_r(\Sigma_x) \leq \frac{1}{\delta_1(1+\alpha_1)}. \tag{3.19}$$

Proof. Put

$$\varepsilon_k = \frac{k_1(\|S(\tau n_0)x\|)c^{\delta_1}}{m_{k+1}^{\delta_1(1+\alpha_1)}}$$

for some positive integer n_0 and $x \in X$. It follows from the hypothesis that we have

$$|m_{k+1}\tau - n_{k+1}| < \frac{c}{m_{k+1}^{\alpha_1+1}}. \tag{3.20}$$

By the above estimate and **(G1)** we have

$$\begin{aligned}\|\varphi(m_{k+1}+n_0) - \varphi(n_0)\| &= \|S(\tau(m_{k+1}+n_0))x - S(\tau n_0)x\| \\ &= \|S(\tau(m_{k+1}+n_0))x - S(\tau n_0 + n_{k+1})x\| \\ &\leq k_1(\|S(\tau n_0)x\|)|m_{k+1}\tau - n_{k+1}|^{\delta_1} \\ &\leq \frac{k_1(\|S(\tau n_0)x\|)c^{\delta_1}}{m_{k+1}^{\delta_1(1+\alpha_1)}} = \varepsilon_k.\end{aligned}$$

Thus we can obtain

$$\begin{aligned}\underline{D}_r &= \lim_{\varepsilon_0\to 0} \inf_{0<\varepsilon<\varepsilon_0} \frac{\log \underline{M}_\varepsilon}{-\log\varepsilon} \\ &\leq \lim_{k\to\infty} \frac{\log \underline{M}_{\varepsilon_k}}{-\log\varepsilon_k} \\ &\leq \lim_{k\to\infty} \frac{\log m_{k+1}}{\delta_1(1+\alpha_1)\log m_{k+1} - \log k_1(\|S(\tau n_0)x\|)c^{\delta_1}} \\ &= \frac{1}{\delta_1(1+\alpha_1)}.\end{aligned}$$

□

If the partial quotient a_j in the continued fraction expansion of τ satisfies $a_{j+1} \sim m_j^\beta$, that is, there exist constants $c_1, c_2 > 0$ such that

$$c_1 m_j^\beta < a_{j+1} < c_2 m_j^\beta,$$

then it follows from the lemmas that the frequency τ is the Roth number with its order $\beta(\beta+3)$ and the Liouville number with its order β. Thus we have the following estimate.

Corollary 3.9. *Under the assumptions* **(G1)** *and* **(G2)** *with the same hypothesis as that in Theorem 3.7, assume that* $a_{j+1} \sim m_j^{\beta}$. *Then we have*

$$\frac{1}{(1+\beta(\beta+3))\delta_2} \le \underline{D}_r(\Sigma_x) \le \frac{1}{(1+\beta)\delta_1}.$$

4. Example of PDE and abstract formulation

Let Ω be a bounded domain in R^N with its smooth boundary $\partial\Omega$ and let $H^m(\Omega)$, $H_0^m(\Omega)$ be the usual Sobolev spaces. We consider the following initial value problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + g(t)u \ni f(t,x), & (t,x) \in (s,+\infty)\times\Omega, \\ u(t,x) = 0, & x \in \partial\Omega,\ t \ge s, \\ u(s,x) = \varphi(x), & x \in \Omega, \end{cases} \tag{4.1}$$

where we assume the periodicity of the perturbation term $f : R\times\Omega \to R$ and the nonlinear monotone operator (possibly multi-valued) $g(t) \subset R\times R$:

$$f(t+1,x) = f(t,x), \quad g(t+\tau) = g(t), \quad t \in R, \quad x \in \Omega,$$

for an irrational number $0 < \tau < 1$.

To investigate (4.1), we consider an abstract evolution equation. Hereafter, let X be a Hilbert space with its norm and the inner product denoted by $|\cdot|, (\cdot,\cdot)$, respectively, and consider a reflexive Banach space V and its dual V^* with their norms and the dual pair denoted by $\|\cdot\|, \|\cdot\|_*, \langle\cdot,\cdot\rangle$, respectively. Assume that

$$V \subset X \subset V^*$$

where V is densely and continuously embedded in X.

In view of (4.1) we consider an operator $\mathbf{A}(t)$ which has the form

$$\mathbf{A}(t)u = A_0u + G(t)u + f(t)$$

for a single-valued maximal monotone operator A_0 from V to V^*, a multi-valued maximal monotone operator $G(t)$ on X and a perturbation $f : R \to X$. We assume the periodicity

$$f(t+1) = f(t), \quad G(t+\tau) = G(t) \tag{4.2}$$

for every $t \in R$. For simplicity we assume that the domain of $G(t)$ is time-independent and put

$$D := D(G(t)) \subset V, \quad t \in R.$$

We can consider an abstract evolution equation on X.

$$\begin{cases} \dfrac{du}{dt} + \mathbf{A}(t)u \ni 0, & s < t, \\ u(s) = x. \end{cases} \tag{4.3}$$

We assume that the domain of the nonlinear multi-valued operator $\mathbf{A}(t)$ on X is time-independent:

(A1) $D(\mathbf{A}(t)) = D \subset V, \quad t \in R.$

For each $t \in R, \mathbf{A}(t)$ is a maximal monotone operator on X (cf. [6]):

(A2) There exits a constant $\omega > 0$ such that

$$(\mathbf{A}(t)u - \mathbf{A}(t)v, u - v) \geq \omega \|u - v\|^2, \quad u, v \in D.$$

(A3) $\mathbb{R}(I + \lambda \mathbf{A}(t)) = X, \quad t \in R, \quad \lambda > 0$, where $\mathbb{R}(\cdot)$ denotes the range of the operator.

Since the Hölder continuity is essential to determine the dimensions of trajectories, we need the following condition on the resolvent $J_\lambda(t) := (I+\lambda\mathbf{A}(t))^{-1}, \ \lambda > 0$.

(J) There exist a small constant $\varepsilon_0 > 0$, an exponent $0 < \delta \leq 1$ and a monotone increasing function $l : R^+ \to R^+$ such that the resolvent satisfies

$$|J_\lambda(t)x - J_\lambda(\tau)x| \leq \lambda l(|x|)|t - \tau|^\delta$$

for $t, \tau \in R : |t - \tau| \leq \varepsilon_0, \quad x \in X$.

The conditions above are sufficient to construct the associated two-parameter family of operators $U(s, \tau) : X \to X, (s, \tau) \in R \times R^+$, which satisfies the following properties from **(u-i)** to **(u-v)** (see Theorem 3.2, Proposition 2.1 in [4]):

(u-i) $U(s, 0) = I$ (the identity operator), $\quad s \in R$.

(u-ii) $U(s, \sigma + \tau) = U(s + \tau, \sigma)U(s, \tau), \quad s \in R, \quad \sigma, \tau \in R^+$.

(u-iii) For any fixed $\tau \in R^+$, the one-parameter family of maps $U(s, \tau) : X \to X$ with the parameters $s \in R$ is equicontinuous.

A two-parameter family of the operators on X, which satisfies **(u-i)**, **(u-ii)** and **(u-iii)**, is called a process or an evolutionary operator (cf. [4] or [5]).

(u-iv) There exists a monotone increasing function $k : R^+ \to R^+$ which satisfies

$$|U(s, \tau)x - U(s, \sigma)x| \leq k(|x|)|\tau - \sigma|^\delta, \quad \tau, \sigma \in R : \ |\tau - \sigma| \leq \varepsilon_0.$$

(u-v) Given $s \in R, x \in X$, define a local Hölder continuous function $u : [s, +\infty) \to X$ by

$$u(s + \sigma) = U(s, \sigma)u(s), \quad u(s) = x,$$

and let $y_0 \in \mathbf{A}(s)x_0$ and $y(t) \in \mathbf{A}(t)x_0, t \geq s$, such that $y(s) = y_0$ and $y(t)$ is continuous, then $u(t)$ satisfies

$$\begin{aligned} &\int_0^\sigma (y(s + \tau) - \omega(x_0 - u(s + \tau)), x_0 - u(s + \tau))d\tau \\ &\qquad \geq \frac{1}{2}(|x_0 - u(s + \sigma)|^2 - |x_0 - u(s)|^2) \\ &\qquad \geq (u(s) - u(s + \sigma), x_0 - u(s)) \end{aligned} \tag{4.4}$$

where we use **(A2)** and (4.3) in the first inequality and the second inequality is obvious.

Here we introduce some definitions of trajectories associated with the process $U(s, \tau)$.

Definitions. The positive trajectory through $(s, x) \in R \times X$ is the map $U(s, \cdot)x : R^+ \to X$. A complete trajectory through (s, x) is a function $u(\cdot) : R \to X$ such that $u(s) = x$ and $u(t + \tau) = U(t, \tau)u(t)$ for all $(t, \tau) \in R \times R^+$.

Define the σ-translate U_σ by $U_\sigma(s, \tau) = U(s + \sigma, \tau)$, then a process U on X is called almost periodic if for any sequence $\{\sigma_n\}$ of R, there exists a subsequence $\{\sigma'_n\}$ of $\{\sigma_n\}$ such that the sequence $\{U_{\sigma'_n}(s, \tau)x\}$ converges to some $V(s, \tau)x$ in X uniformly in $s \in R$ and pointwise $(\tau, x) \in R^+ \times X$.

(A2) yields relative compactness of each positive trajectory and then our purpose is to estimate the recurrent dimension of the positive trajectory.

Note that for each $s \in R, x \in X$, $u(t)$ in **(u-v)**, which is called an integral solution, is weakly differentiable (cf. [3]). Then, dividing (4.4) by σ, taking the limit $\sigma \downarrow 0$, we have

$$(y_0 - \omega(x_0 - u(s)), x_0 - u(s)) \geq -(u', x_0 - u(s))$$

for every $[x_0, y_0] \in \mathbf{A}(s)$ where u' is the weak derivative of $u(t)$ at $t = s$. Thus the maximal monotonicity yields $-u'(s) \in \mathbf{A}(s)u(s)$.

In view of **(J)**, we assume local Hölder continuity on $f(t), G(t)$ with exponents $\sigma_1, \sigma_2 : 0 < \sigma_1, \sigma_2 \leq 1$, respectively. Let $\varepsilon_0 > 0$ be a sufficiently small constant.

(A4-i) There exists a positive constant L_1 such that

$$|f(t) - f(\tau)| \leq L_1 |t - \tau|^{\sigma_1}, \quad t, \tau \in R : |t - \tau| \leq \varepsilon_0, \quad u \in V.$$

(A4-ii) There exists a monotone increasing function $L_2 : R^+ \to R^+$ such that, if $t, \tau \in R : |t - \tau| \leq \varepsilon_0, x \in D$ and $y \in G(\tau)x$, then there exists $w \in G(t)x$ which satisfies

$$|w - y| \leq L_2(|x|)|t - \tau|^{\sigma_2}.$$

(A5) $A_0 0 = 0, \quad G(t)0 \ni 0$ for every $t \in R$.

Under the above conditions we can show the local Hölder continuity of the resolvents.

Lemma 4.1 ([7]). *Assume from* **(A1)** *to* **(A5)**. *Then* **(J)** *holds in the following sense. For $x \in X$ and for the constant $\delta = \min\{\sigma_1, \sigma_2\}$, the estimate*

$$|J_\lambda(t)x - J_\lambda(\tau)x| \leq \lambda l(|x|)|t - \tau|^\delta, \quad t, \tau \in R : |t - \tau| \leq \varepsilon_0,$$

holds where $J_\lambda(t)x = (I + \lambda \mathbf{A}(t))^{-1}x$.

We can estimate the upper bound of the recurrent dimension of the discrete positive trajectory $u(s + \tau n) = U(s, \tau n)x$ and $u(s + n) = U(s, n)x$.

Theorem 4.2. *Under the assumptions* **(A1)-(A5)**, *assume that τ is a Liouville number with its order α_1 and the unique complete trajectory $u(t)$ satisfies*

$$\sup_{t \in R} \|u(t)\| < \infty. \tag{4.5}$$

Then the recurrent dimension of the discrete positive trajectories

$$\Sigma_x^1 = \bigcup_{n \in \mathbf{N}_0} U(s, \tau n)x \quad \textit{and} \quad \Sigma_x^2 = \bigcup_{n \in \mathbf{N}_0} U(s, n)x$$

satisfies

$$\underline{D}_r(\Sigma_x^1) \leq \frac{1}{\sigma_1(1 + \alpha_1)}, \quad \underline{D}_r(\Sigma_x^2) \leq \frac{1}{\sigma_2(1 + \alpha_1)}.$$

Proof. We have already shown in [9] that $u(t)$ satisfies

$$|u(y + \mu) - u(y)| \leq c \sup_{t \in R}(|f(t + \mu) - f(t)| + \|G(t + \mu)u(t) - G(t)u(t)\|_*),$$

for every $y, \mu \in R$. Thus, for every $m, n, l \in \mathbf{N}$, it follows from the periodicity and **(A4-i)** that we have

$$\begin{aligned} |u(s + \tau(n + m)) - u(s + \tau n)| &\leq c \sup_{t \in R} |f(t + \tau m) - f(t + l)| \\ &\leq cL_1 |\tau m - l|^{\sigma_1} \end{aligned}$$

and it follows from the periodicity of $G(t)$ and **(A4-ii)** that we have

$$\begin{aligned} |u(s+n+l)-u(s+n)| &\leq c\sup_{t\in R}\|G(t+l)u(t)-G(t+\tau m)u(t)\|_* \\ &\leq cL_2(\sup_t\|u(t)\|)|\tau m-l|^{\sigma_2}. \end{aligned}$$

In view of the proof of Theorem 3.8, since we can put

$$\varphi(n)x = U(s,\tau n)x = u(s+\tau n)$$

and

$$\varphi(n)x = U(s,n)x = u(s+n)$$

for every $n \in \mathbf{N}$, following the argument in the proof of Theorem 3.8 with $n = n_0,\ m = m_{k+1}$ and $l = n_{k+1}$ yields the conclusions. □

5. Semilinear parabolic equations

Let $g(t) \subset R\times R$ be a maximal monotone set in $R\times R$ for each $t\in R$. Assume the following conditions.

(g-i) The domain of g is independent of t, $D(g(t)) = D_g$.

(g-ii) $0 \in g(t)0$.

(g-iii) There exist a constant $k_1 > 0$ and a constant $\sigma_2, 0<\sigma_2\leq 1$, such that, if $t,\tau \in R : |t-\tau| \leq \varepsilon_0,\ \ x\in D_g$ and $y\in g(t)x$, then there exists an element $w\in g(\tau)x$:

$$|y-w| \leq k_1(1+|x|)|t-\tau|^{\sigma_2}.$$

(g-iv) There exists a constant $k_0 > 0$ such that

$$\sup\{|y| : y\in g(t)u,\ t\in R\} \leq k_0(1+|u|).$$

Put $X := L^2(\Omega), V := H_0^1(\Omega)$ and $V^* := H^{-1}(\Omega)$, then, applying routine methods in the theory of maximal monotone operators (see Proposition 2.7 in Chapter 2 of [1]), we can define a maximal monotone operator $\bar{g}(t)\subset X\times X$ by

$$\bar{g}(t) = \{[u,v] : u,v\in X \text{ and } v(x)\in g(t)u(x) \text{ a.e.}\}.$$

(See [14], [15] for further details.)

Next we consider the conditions on $f : R\times\Omega \to R$.

(f) For each $t\in R$, $f(t,\cdot)$ is measurable and there exist constants $L_1 > 0$ and $0<\sigma_1\leq 1$ such that

$$|f(t,x)-f(\tau,x)| \leq L_1|t-\tau|^{\sigma_1},\quad t,\tau\in R : |t-\tau|\leq\varepsilon_0,\quad x\in\Omega.$$

Lemma 5.1 ([9]). *Under the assumptions above, the weak solution $u(t)$ of (4.1) satisfies*

$$\sup_{t\in R}\|u(t)\| < \infty.$$

Define the operator A_0 on X by

$$A_0u = -\Delta u,\quad D(A_0) = H^2(\Omega)\cap H_0^1(\Omega),$$

then, the operator A_0 satisfies **(A4-i)**. Furthermore we can show that the maximal monotone operator $\mathbf{A}(t)$ on X defined by

$$\begin{aligned} &\mathbf{A}(t)u = A_0u + \bar{g}(t)u + f(t), \\ &\qquad u\in D(\mathbf{A}(t)) := D = H^2(\Omega)\cap H_0^1(\Omega)\cap D(\bar{g}) \end{aligned}$$

satisfies the conditions from **(A1)** to **(A5)**.

Theorem 5.2. *Under the conditions from* **(g-i)** *to* **(g-iv)**, *and* **(f)**, *assume that* τ *is a Liouville number with its order* α_1. *Then the recurrent dimensions of the discrete trajectories* $\Sigma_x^1 = \bigcup_{n\in\mathbf{N}_0} u(s+\tau n)$ *and* $\Sigma_x^2 = \bigcup_{n\in\mathbf{N}_0} u(s+n)$ *satisfy*

$$\underline{D}_r(\Sigma_x^1) \le \frac{1}{\sigma_1(1+\alpha_1)}, \qquad \underline{D}_r(\Sigma_x^2) \le \frac{1}{\sigma_2(1+\alpha_1)}.$$

Remark 5.3. Here we give a simple example of $g(t) \subset R\times R$ satisfying our assumptions.

$$g(t)x = \begin{cases} [-\varphi(t), \varphi(t)] & \text{if } x = 0, \\ \varphi(t)(x+1) & \text{if } x > 0, \\ \varphi(t)(x-1) & \text{if } x < 0, \end{cases}$$

where $\varphi(t) : R \to R^+$ is Hölder continuous with its exponent $\sigma_2 : 0 < \sigma_2 < 1$ and uniformly bounded: $\sup_{t\in R} \varphi(t) < +\infty$.

Remark 5.4. Following the proof in [9], we can prove that the discrete complete trajectories

$$\Sigma_1 = \{u(\tau n),\ \ n \in \mathbf{Z}\}, \quad \Sigma_2 = \{u(n),\ \ n\in\mathbf{Z}\},$$

are almost periodic and their inclusion lengths satisfy

$$l_\varepsilon^1 \le c_1 \varepsilon^{-\frac{1}{\sigma_1}}, \quad l_\varepsilon^2 \le c_2 \varepsilon^{-\frac{1}{\sigma_2}},$$

respectively. For an arbitrarily given irrational number τ, it follows that

$$\overline{D}_c(\Sigma_1) \le \overline{D}_p(\Sigma_1) \le \frac{1}{\sigma_1},$$
$$\overline{D}_c(\Sigma_2) \le \overline{D}_p(\Sigma_2) \le \frac{1}{\sigma_2}.$$

Remark 5.5. As an example of an evolution equation, its solution of which satisfies the hypotheses of Theorem 3.7 and the estimate

$$\underline{D}_c(\Sigma_x) \ge \underline{D}_r(\Sigma_x) \ge \frac{1}{\delta_2(1+\alpha_0)},$$

we have studied the following linear abstract equation on a Hilbert space (cf. [10]);

$$\frac{du}{dt} + Au = f^*(t), \quad t > 0,$$
$$u(0) = u_0$$

where the perturbation term $f^*(t)$ is quasi-periodic and given as a Weierstrass type function (see also [12]). Under suitable conditions we can prove that the solution satisfies the Hölder condition **(G2)** with the hypothesis in Theorem 3.7.

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