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ALGEBRAIC STRUCTURE IN THE LOOP SPACE HOMOLOGY BOCKSTEIN SPECTRAL SEQUENCE

JONATHAN A. SCOTT

ABSTRACT. Let X be a finite, n-dimensional, r-connected CW complex. We prove the following theorem:

If $p \geq n/r$ is an odd prime, then the loop space homology Bockstein spectral sequence modulo p is a spectral sequence of universal enveloping algebras over differential graded Lie algebras.

INTRODUCTION

Let ΩX be the Moore loop space on a pointed topological space X. If R is a subring of $\mathbf{Q}$, then $H_*(\Omega X; R)$ has a natural Hopf algebra structure via composition of loops, as long as there is no torsion. The submodule $P \subset H_*(\Omega X; R)$ of primitive elements is a graded Lie subalgebra; in [6], Milnor and Moore showed that if $R = \mathbf{Q}$ and X is simply connected, then $H_*(\Omega X; \mathbf{Q})$ is the universal enveloping algebra of P. In [5], Halperin established the same conclusion for $R \subset \mathbf{Q}$ when X is a finite, simply-connected CW complex, provided that $H_*(\Omega X; R)$ is torsion-free and the least non-invertible prime in R is sufficiently large.

In the presence of torsion, the loop space homology algebra with coefficients in R does not have a natural Hopf algebra structure. However, in [3] Browder showed that the Bockstein spectral sequence $H_*(\Omega X; \mathbf{F}_p) \Rightarrow (H_*(\Omega X; \mathbf{Z})/\text{torsion}) \otimes \mathbf{F}_p$ is a spectral sequence of Hopf algebras. Halperin also proved in [5] that for large enough primes, $H_*(\Omega X; \mathbf{F}_p)$ is the universal enveloping algebra of a graded Lie algebra. The present article establishes this for every term in the Bockstein spectral sequence.

Theorem 1. *Let X be a finite, n-dimensional, q-connected CW complex ($q \geq 1$). If p is an odd prime and $p \geq n/q$, then each term in the mod p homology Bockstein spectral sequence for ΩX is the universal enveloping algebra of a differential graded Lie algebra (L^r, β^r). In addition, the E^∞ term is the universal enveloping algebra of a graded Lie algebra.*

Note that the association $X \rightsquigarrow \{(L^r, \beta^r)\}$ is not functorial; see Example 1.

Remark 1. The hypothesis $p \geq n/q$ cannot be removed. Indeed, $X = \Sigma \mathbf{C}P^p$ is 2-connected, but $(2p+1)$-dimensional. There is a non-vanishing pth power in $H^*(\Omega X; \mathbf{F}_p)$ and so $H_*(\Omega X; \mathbf{F}_p)$ is not primitively generated (see [6]). In particular, it is not a universal enveloping algebra. The Bockstein spectral sequence is constant, so none of the terms is a universal enveloping algebra.

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Remark 2. Theorem 1 neglects the case $p = 2$. If X is n-dimensional and q-connected with $n \leq 2q$, then X is in the stable range. Indeed, if $q = 1$, then X is a finite wedge of 2-spheres. If $q \geq 2$, then $X = \Sigma^2 Y$, for some connected space Y, and the adjunction map $\Sigma Y \to \Omega X$ determines Hopf algebra isomorphisms $TE^r_+(\Sigma Y) \xrightarrow{\cong} E^r(\Omega X)$ for $r \geq 1$. In either case, the Bockstein spectral sequence is a free associative algebra generated by primitive elements, and so is naturally the universal enveloping algebra of a free adjusted Lie algebra in the sense of Sjödin [7].

In [2], under the hypotheses of Theorem 1, Anick associates to X a differential graded Lie algebra L_X over $\mathbf{Z}_{(p)}$ and a natural quasi-isomorphism $UL_X \xrightarrow{\simeq} C_*(\Omega X; \mathbf{Z}_{(p)})$ of Hopf algebras up to homotopy. It follows that there is an isomorphism of mod p Bockstein spectral sequences $E^r(UL_X) \cong E^r(\Omega X)$. The inclusion $\iota_X : L_X \hookrightarrow UL_X$ therefore induces a transformation of Bockstein spectral sequences $E^r(\iota_X) : E^r(L_X) \to E^r(\Omega X)$.

Theorem 2. *The image of each $E^r(\iota_X)$ is contained in L^r.*

Given the results of [2], Theorems 1 and 2 follow immediately from the following purely algebraic result:

Theorem 3. *Let (L, ∂) be a differential graded Lie algebra over $\mathbf{Z}_{(p)}$ that is connected, free as a graded module, and of finite type. The mod p homology Bockstein spectral sequence for $U(L, \partial)$ is a sequence of universal enveloping algebras, $E^r(UL) = U(L^r, \beta^r)$, and converges to the universal enveloping algebra of a graded Lie algebra L^∞. Furthermore, if $\iota : L \hookrightarrow UL$ is the inclusion, then the image of $E^r(\iota)$ is contained in L^r.*

The proof of Theorem 3 depends in an essential way on the work of André [1] and Sjödin [7], which characterizes the cocommutative Hopf algebras of finite type over a field $\boldsymbol{k}$ which can be written as universal enveloping algebras. Namely, such a Hopf algebra A can be written as UL if and only if the dual $A^\sharp$ can be given the structure of a Hopf algebra with divided powers (see Theorem 4).

The structure of the article is as follows.

Section 1. Notation and review of graded Lie algebras, divided powers algebras, Bockstein spectral sequences, acyclic closures and minimal models.

Section 2. In [5], Halperin showed that for a differential graded Lie algebra (L, ∂) over $\mathbf{F}_p$, $H(UL) = UE$ for a graded Lie algebra E. We show that the inclusion $\iota : (L, \partial) \hookrightarrow U(L, \partial)$ satisfies $\operatorname{im} H(\iota) \subset E$.

Section 3. We extend André-Sjödin duality [1, 7], between graded Lie algebras and graded Hopf algebras with divided powers, to the respective differential categories.

Section 4. Proof of Theorem 3.

Section 5. An example is given, demonstrating that the sequence of Lie algebras given by Theorem 3 is not natural.

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1. Preliminaries

Let R be a commutative ring containing $1/2$. All objects are graded by the integers unless otherwise stated. Fix an odd prime p. The ring of integers localized at p is denoted $\mathbf{Z}_{(p)}$ while the prime field is denoted $\mathbf{F}_p$. Differential graded modules, algebras, coalgebras, and Hopf algebras are shortened to DGM, DGA, DGC, and DGH, respectively; a comprehensive treatment of these objects is given in [4].

1.1. **Graded modules.** Let M be a graded module over R. If $x \in M_k$ then we say that x has degree k, and write $|x| = k$. A free graded module M is of *finite type* if each M_k is of finite rank. We raise and lower degrees by the convention $M^k = M_{-k}$. We denote by sM the suspension of M: $(sM)_i = M_{i-1}$. The dual of M is the graded module $M^\sharp = \mathrm{Hom}(M, R)$. If M is finite type and $N = (sM)^\sharp$, then $M = (sN)^\sharp$ via $x(sf) = -f(sx)$, for $x \in M$, $f \in N$.

If V is a graded module over R, then we denote by TV and ΛV the tensor algebra and free commutative algebra on V, respectively. The tensor coalgebra on V is denoted by $T_C V$. The shuffle product ([5], Appendix) makes $T_C V$ into a graded commutative (not cocommutative) Hopf algebra. Note that $TV = \bigoplus_{k\geq 0} T^k V$, $\Lambda V = \bigoplus_{k\geq 0} \Lambda^k V$ and $T_C V = \bigoplus_{k\geq 0} T_C^k V$, with $T^k V$, $\Lambda^k V$, and $T_C^k V$ consisting of words in V of length k. Elements of $T_C^k V$ are denoted $[v_1|\cdots|v_k]$.

The symmetric group S_k acts on $T^k V$ via $\sigma\cdot(x_1\otimes\cdots\otimes x_k) = \pm\, x_{\sigma(1)}\otimes\cdots\otimes x_{\sigma(k)}$, where the sign is determined by the rule $x\otimes y \mapsto (-1)^{|x||y|} y\otimes x$.

1.2. **Graded Lie algebras.** A *graded Lie algebra* is a graded R-module $L = \bigoplus_{k\geq 0} L_k$ along with a degree-zero linear map $[\,,] : L\otimes L \to L$, called the *Lie bracket*, satisfying graded anti-symmetry, the graded Jacobi identity, and the further condition $[x,[x,x]] = 0$ if $x \in L_{\mathrm{odd}}$; see [5] for details.

For example, any non-negatively graded associative algebra A is a graded Lie algebra via the graded commutator bracket $[a,b] = ab - (-1)^{|a||b|} ba$, for $a, b \in A$.

A graded Lie algebra is *connected* if it is concentrated in strictly positive degrees.

The *graded abelian Lie algebra* on the graded set $\{x_j\}$, denoted $L_{\mathrm{ab}}(x_j)$, is the free graded module on the basis $\{x_j\}$, given the trivial Lie bracket.

Let L be a graded Lie algebra, and denote by $L^\flat$ the underlying graded module. The *universal enveloping algebra* of L is the associative algebra $UL = (TL^\flat)/I$, where I is the ideal generated by all elements of the form $x\otimes y - (-1)^{|x||y|} y\otimes x - [x,y]$, for $x, y \in L$. UL has the natural structure of a graded Hopf algebra; the comultiplication is defined by declaring the elements of L to be primitive and then using the universal property.

A *Lie derivation* on a graded Lie algebra L is a linear operator θ on L of degree k such that for $x, y \in L$, $\theta([x,y]) = [\theta(x), y] + (-1)^{k|x|}[x, \theta(y)]$. A *differential graded Lie algebra* (DGL for short) is a pair (L, ∂), where L is a graded Lie algebra, and ∂ is a Lie derivation on L of degree -1 satisfying $\partial\partial = 0$. If (L,∂) is a DGL, then ∂ extends to a derivation on UL, making $U(L,\partial)$ into a DGA.

1.3. **Divided powers algebras.** Divided powers algebras arise as the duals of universal enveloping algebras, in the sense of Theorem 4, below.

Definition 1. A *divided powers algebra*, or Γ-algebra, is a commutative graded algebra A, satisfying either $A = A^{\geq 0}$ or $A = A^{\leq 0}$, equipped with set maps $\gamma^k : A^{2n} \to A^{2nk}$ for $k \geq 0$ and $n \neq 0$ satisfying the following list of conditions.

(1) $\gamma^0(a) = 1; \gamma^1(a) = a$ for $a \in A$;
(2) $\gamma^k(a+b) = \sum_{j=0}^k \gamma^j(a)\gamma^{k-j}(b)$ for $a, b \in A^{2n}$;
(3) $\gamma^j(a)\gamma^k(a) = \binom{j+k}{j}\gamma^{j+k}(a)$ for $a \in A^{2n}$;
(4) $\gamma^j(\gamma^k(a)) = \dfrac{(jk)!}{k!^j j!}\gamma^{jk}(a)$ for $a \in A^{2n}$;
(5) $\gamma^k(ab) = \begin{cases} a^k\gamma^k(b) & \text{if } |a| \text{ and } |b| \text{ even, } |b| \neq 0, \\ 0 & \text{if } |a| \text{ and } |b| \text{ odd.} \end{cases}$

A Γ-*morphism* is an algebra morphism which respects the divided powers operations. A *Hopf* Γ-*algebra* is a Hopf algebra, along with a system of divided powers, such that the coproduct is a Γ-morphism. A Γ-*derivation* on a Γ-algebra A is a derivation θ on A satisfying $\theta(\gamma^k(a)) = \theta(a)\gamma^{k-1}(a)$ for $a \in A^{2n}$, $k \geq 1$. A *differential graded* Γ-*algebra*, or Γ-DGA, is a pair (A, ∂), where A is a Γ-algebra, and ∂ is a Γ-derivation of degree $+1$ satisfying $\partial\partial = 0$. If A is furthermore a Hopf Γ-algebra, and ∂ is also a coderivation, then (A, ∂) is called a *differential graded Hopf* Γ-*algebra*, or Γ-DGH.

Let V be a free graded R-module. Let $\Gamma^k(V)$ be the graded submodule of $T_C^k V$ of elements fixed by the action of the symmetric group S_k. Then $\Gamma(V) = \bigoplus_k \Gamma^k(V)$ is a Hopf subalgebra of $T_C(V)$, called the *free* Γ-*algebra* on V. Divided powers are defined on $\Gamma(V)$ by

(1) $\gamma^0(v) = 1$, $\gamma^1(v) = v$ for $v \in V$,
(2) $\gamma^k(v) = \underbrace{[v|\cdots|v]}_{k \text{ times}}$ for $v \in V^{2n}$

and then extending via conditions (4) and (5) of Definition 1. If $f : V \to A$ is any linear map of degree zero from V into a Γ-algebra A, then f extends to a unique Γ-morphism $\bar{f} : \Gamma(V) \to A$. If V is R-free on a countable, well-ordered basis $\{v_i\}$, then $\Gamma(V)$ is R-free, with basis consisting of elements $\gamma^{k_1}(v_1)\cdots\gamma^{k_s}(v_s)$ where $k_j \geq 0$ and $k_j = 0$ or 1 if $|v_j|$ is odd.

Abusing notation, we will call a Γ-DGH *free* if it is free as a Γ-algebra.

Hopf Γ-algebras and universal enveloping algebras are related in the following sense.

Theorem 4 (André–Sjödin)**.** *Let A be a connected Hopf algebra of finite type over a field $\boldsymbol{k}$, where* char $\boldsymbol{k} \neq 2$. *Then its dual Hopf algebra $A^\sharp$ is isomorphic to the universal enveloping algebra of a graded Lie algebra over $\boldsymbol{k}$ if and only if A is a Hopf* Γ-*algebra.* ☐

Remark 3. Theorem 4 was proved first by André in [1] in dual form. Sjödin proved the result directly in [7] and, using the notion of "adjusted" graded Lie algebras for characteristic 2, extended the result to arbitrary characteristic. Sjödin also proved for the "if" direction that A is free as a Γ-algebra.

If $V \otimes W \xrightarrow{\langle,\rangle} R$ is a pairing, then there is an induced pairing

$$TV \otimes T_C W \to R \tag{1}$$

given by $\langle T^j V, T_C^k W\rangle = 0$ if $j \neq k$, and

$$\langle v_1 \otimes \cdots \otimes v_k, [w_1|\cdots|w_k]\rangle = \pm\langle v_1, w_1\rangle \cdots \langle v_k, w_k\rangle$$

where $\pm$ is the sign of the permutation $v_1, \dots, v_k, w_1, \dots, w_k \mapsto v_1, w_1, \dots, v_k, w_k$. The pairing (1) in turn induces a pairing

$$\Lambda V \otimes \Gamma W \to R. \tag{2}$$

Suppose that V is R-free of finite type, $V = V_{<0}$ or $V = V_{>0}$, and $W = V^\sharp$. Then (1) and (2) induce Hopf algebra isomorphisms $T_C(V^\sharp) \cong (TV)^\sharp$ and $\Gamma(V^\sharp) \cong (\Lambda V)^\sharp$.

1.4. **The Cartan–Chevalley–Eilenberg–Cartan complex.** Denote by $B(A)$ the bar construction on the augmented DGA (A, ∂) ([5], Section 1); recall that the underlying coalgebra of $B(A)$ is $T_C(s\bar{A})$, where $\bar{A}$ is the augmentation ideal. Let (L, ∂) be a DGL. Then $\Gamma(sL) \subset \Gamma(s\overline{UL}) \subset B(UL)$ and $(\Gamma(sL), \partial_0 + \partial_1)$ is a sub-DGC of $B(UL)$, denoted by $C_*(L, \partial)$, called the *chains on* (L, ∂).

The *Cartan–Chevalley–Eilenberg–Cartan complex* on (L, ∂) is the commutative cochain algebra $C^*(L, \partial) = (\Lambda V, d)$, dual to $C_*(L, \partial)$ by the pairing (2), where $V = (sL)^\sharp$, and the differential d is the sum of derivations d_0 and d_1. The *linear part* d_0 preserves word length and is dual to ∂ in that $\langle d_0 v, sx\rangle = (-1)^{|v|}\langle v, s\partial x\rangle$ for $v \in V$, $x \in L$. The *quadratic part* d_1 increases word length by one and is dual to the Lie bracket in L:

$$\langle d_1 v, sx \cdot sy\rangle = (-1)^{|sy|}\langle v, s[x, y]\rangle \tag{3}$$

where the pairing is (2) above with $W = sL = V^\sharp$. We will usually refer to the Cartan–Chevalley–Eilenberg–Cartan complex as the *cochains on* (L, ∂).

1.5. **Bockstein spectral sequences.** Fix a prime p. Let C be a free chain complex over $\mathbf{Z}_{(p)}$. Applying $C \otimes -$ to the short exact sequence of coefficient modules $0 \to \mathbf{Z}_{(p)} \xrightarrow{\times p} \mathbf{Z}_{(p)} \to \mathbf{F}_p \to 0$ leads to a long exact sequence in homology which may be wrapped into the exact couple

$$\begin{array}{ccc} H_*(C) & \longrightarrow & H_*(C) \\ & \nwarrow \quad \swarrow & \\ & H_*(C; \mathbf{F}_p) & \end{array}$$

from which we get the *homology Bockstein spectral sequence modulo* p of C, $(E^r(C), \beta^r)$, mod p BSS for short [3]. If $C = C_*(X)$ is the normalized singular chain complex of a space X, then we refer to the homology BSS mod p of $C_*(X)$ as the mod p homology BSS of X, denoted $(E^r(X), \beta^r)$.

If $H(C)$ is of finite type, then the spectral sequence collapses degreewise to a graded vector space E^∞, where $E^\infty \cong (H(C)/\text{torsion}) \otimes \mathbf{F}_p$.

There is the corresponding notion of *cohomology Bockstein spectral sequence* defined in the obvious manner, using the functor $\mathrm{Hom}(C, -)$ rather than $C \otimes -$.

The mod p BSS of C measures p-torsion in $H_*(C)$: if $x, y \in E^r$, $x \neq 0$, satisfy $\beta^r(y) = x$, then x represents a torsion element of order p^r in $H_*(C)$.

Notation. If $c \in C$ is such that $[\bar{c}] \in E^1$ lives until the E^r term, then we will denote the corresponding element of E^r by $[c]_r$.

1.6. **Acyclic closures and minimal models.** (Reference: [5, Sections 2 and 7]) Consider the graded algebra $\Lambda V \otimes \Gamma(sV)$ over R. Extend the divided powers operations on $\Gamma(sV)$ to $R \oplus \Lambda V \otimes \Gamma^+(sV)$ via rule 5 of Definition 1.

Definition 2 ([5, Section 2]). An *acyclic closure* of the DGA $(\Lambda V, d)$ is a DGA of the form $C = (\Lambda V \otimes \Gamma(sV), D)$ in which D is a Γ-derivation restricting to d in ΛV and $H(C) = H^0(C) = R$.

Remark 4. Let (L, ∂) be a connected DGL over R which is R-free of finite type. Then $C^*(L) = (\Lambda V, d)$ where $V = (sL)^\sharp$. Let C be an acyclic closure for $C^*(L)$, and set $(\Gamma(sV), \bar{D}) = R \otimes_{C^*(L)} C$. By the work of Halperin in [5], we identify $H(UL) = H([\Gamma(sV), \bar{D}]^\sharp)$ and $U(L, \partial) = (\Gamma(sV), \bar{D})^\sharp$.

Let $R = \mathbf{Z}_{(p)}$ or $R = \mathbf{F}_p$, and consider a commutative algebra of the form $(\Lambda W, d)$ over R, where $W = W^{\geq 2}$ is R-free and of finite type. We may write the differential as a sum $d = \sum_{j \geq 0} d_j$ where d_j raises wordlength by j.

Definition 3. If $R = \mathbf{Z}_{(p)}$, the DGA $(\Lambda W, d)$ above is $\mathbf{Z}_{(p)}$-*minimal* if $d_0 : W \to pW$. If $R = \mathbf{F}_p$, $(\Lambda W, d)$ is $\mathbf{F}_p$-*minimal* if $d_0 = 0$.

Suppose (A, ∂) is a cochain algebra satisfying $H^0(A) = R$, $H^1(A) = 0$, $H^2(A)$ is R-free, and $H^*(A)$ is of finite type. Then by [5], Theorem 7.1, there exists a quasi-isomorphism $m : (\Lambda W, d) \xrightarrow{\simeq} (A, \partial)$ from an R-minimal algebra. This quasi-isomorphism is called a *minimal model*.

Associated to an $\mathbf{F}_p$-minimal model $m : (\Lambda W, d) \xrightarrow{\simeq} (A, \partial)$ is its *homotopy Lie algebra*, E. As a graded vector space, $E = (sW)^\sharp$; the bracket is defined by the relation

$$\langle w, s[x, y] \rangle = (-1)^{|sy|} \langle d_1 w, sx \cdot sy \rangle$$

for $w \in W$, $x, y \in E$.

2. The image of $H(L) \to H(UL)$

Let (L, ∂) be a connected DGL over $\mathbf{F}_p$ of finite type. By [5], the choice of minimal model $m : (\Lambda W, d) \xrightarrow{\simeq} C^*(L)$ determines an isomorphism of graded Hopf algebras, $H(UL) \cong UE$, where E is the homotopy Lie algebra of m.

Proposition 5. *With the notation above, the image of $H(\iota) : H(L) \to H(UL)$ lies in E.*

Proof. It suffices to construct the following commutative diagram.

(4)
$$\begin{array}{ccc} H(L, \partial) & \xrightarrow{H(\iota_L)} & H(UL) \\ \theta \downarrow & & \downarrow \cong \\ E & \xrightarrow{\iota_E} & UE \end{array}$$

Recall that $C^*(L, \partial) = (\Lambda V, d)$, where $V = (sL)^\sharp$ and $d = d_0 + d_1$. Recall further that the minimality condition on $(\Lambda W, d)$ implies that the linear part of its differential vanishes. The *linear part* of m is the linear map $m_0 : (W, 0) \to (V, d_0)$ defined by the condition $m - m_0 : W \to \Lambda^{\geq 2} V$. Recall that $E = (sW)^\sharp$ and $UE = \Gamma(sW)^\sharp$ ([5], Theorem 6.2).

The model m extends to a morphism of constructible acyclic closures ([5], Section 2) $\hat{m} : (\Lambda W \otimes \Gamma(sW), D) \to (\Lambda V \otimes \Gamma(sV), D)$ by Proposition 2.7 of [5]. Since $(\Lambda W, d)$ is $\mathbf{F}_p$-minimal, $d_0 = 0$. By Corollary 2.6 of [5], $d_0 = 0$ is equivalent to $\bar{D} = 0$ in $(\Gamma(sW), \bar{D})$. Apply $\mathbf{F}_p \otimes_m -$ to $\hat{m}$ to get a Γ-morphism $\bar{m} : (\Gamma(sW), 0) \to (\Gamma(sV), \bar{D})$.

Let $\pi_L : (\Gamma(sV), \bar{D}) \twoheadrightarrow s(V, d_0)$ and $\pi_E : (\Gamma(sW), 0) \twoheadrightarrow s(W, 0)$ be the projections. The maps π_L and π_E fit into the diagram

$$\begin{array}{ccc} (\Gamma(sW), 0) & \xrightarrow{\pi_E} & s(W, 0) \\ \Big\downarrow{\scriptstyle \overline{m}} & & \Big\downarrow{\scriptstyle sm_0} \\ (\Gamma(sV), \overline{D}) & \xrightarrow{\pi_L} & s(V, d_0) \end{array} \tag{5}$$

For $w \in W$, Proposition 2.7 of [5] states that $\hat{m}(1 \otimes sw) - 1 \otimes sm_0 w$ has total wordlength at least two. It follows that $\bar{m}(sw) - sm_0 w$ has $\Gamma(sV)$-wordlength at least two, so $\pi_L(\bar{m}(sw)) = sm_0 w = sm_0(\pi_E(sw))$, so diagram (5) commutes. Dualize and pass to homology to get (4). $\square$

3. The dual of a Γ-derivation

André [1] and Sjödin [7] proved that the functor $L \rightsquigarrow (UL)^\sharp$ is a natural equivalence from connected graded Lie algebras of finite type over a field $\boldsymbol{k}$ to Hopf Γ-algebras of finite type over $\boldsymbol{k}$. In fact, by [7], $(UL)^\sharp = \Gamma(V)$, where $V^\sharp = L$ as a graded vector space. The same result is proved in [5] over an arbitrary commutative ring containing $1/2$. Let R be a such a ring.

Proposition 6. *Let (UL, ∂) be a* DGH *over R of finite type, where L is a graded Lie algebra over R which is free as an R-module. Then $\partial^\sharp$ is a Γ-derivation if and only if $\partial(L) \subset L$.*

Proof. It suffices to prove the dual statement, namely that $\partial^\sharp : \Gamma V \to \Gamma V$ factors over the surjection $\pi : \Gamma V \to V$ to induce a differential in V. But $\ker(\pi)$ is generated as a module by products and elements of the form $\gamma^k(v)$ for $v \in V$, $k \geq 2$. Since $\partial^\sharp$ is a Γ-derivation, $\partial^\sharp(\gamma^k(v)) = \partial^\sharp(v)\gamma^{k-1}(v)$ is a product. It follows that $\partial^\sharp(\ker(\pi)) \subset \ker(\pi)$, completing the 'only if' portion of the proof.

Conversely, the work of Halperin in [5] allows us to identify $[U(L, \partial)]^\sharp$ with $(\Gamma(L^\sharp), \bar{D})$ as DG Hopf algebras. Since $\bar{D}$ is a Γ-derivation, so too is $\partial^\sharp$. $\square$

We can thus extend the work of André and Sjödin to the differential categories.

Theorem 7. *The functor $L \rightsquigarrow (UL)^\sharp$ is a natural equivalence, from the category of* DGL*'s of finite type over $\boldsymbol{k}$ to the category of Γ-*DGH*'s of finite type over $\boldsymbol{k}$.* $\square$

4. Bockstein spectral sequence of a universal enveloping algebra

In this section, we prove the main algebraic result, Theorem 3, stated in the Introduction.

Let $(\Lambda W, d)$ be a minimal Sullivan algebra over $\mathbf{Z}_{(p)}$. Let $C = (\Lambda W \otimes \Gamma(sW), D)$ be a constructible acyclic closure for $(\Lambda W, d)$ ([5], Section 2). Let $(\Gamma(sW), \bar{D})$ be the quotient $\mathbf{Z}_{(p)} \otimes_{(\Lambda W, d)} C$. $C \otimes \mathbf{F}_p$ is a constructible acyclic closure for $(\Lambda W, d) \otimes \mathbf{F}_p$. Since $(\Lambda W, d)$ is $\mathbf{Z}_{(p)}$-minimal, p divides d_0, so the linear part of the differential vanishes in $(\Lambda W, d) \otimes \mathbf{F}_p$. It follows by Corollary 2.6 of [5] that the differential in $(\Gamma(sW), \bar{D}) \otimes \mathbf{F}_p$ is null, so that p divides $\bar{D}$. Set $E^r = E^r([\Gamma(sW), \bar{D}]^\sharp)$ and $E_r = E_r(\Gamma(sW), \bar{D})$. Let $\rho : \Gamma(sW) \to \Gamma(sW) \otimes \mathbf{F}_p = E_1$ be the reduction homomorphism.

Proposition 8. *With the hypotheses and notation above, for $r \geq 1$, the following statements hold.*

(1) *(E_r, β_r) is isomorphic to a free Γ-DGH $(\Gamma(sW_r), \beta_r)$.*
(2) *There is a Γ-morphism $g_r : E_r \to E_1$ such that if $g_r(z) = \rho(a)$ for some $z \in E_r$, $a \in \Gamma(sW)$, then $z = [a]_r$.*
(3) *There is a graded Lie algebra L^r such that $(E^r, \beta^r) = U(L^r, \beta^r)$ as a DGH.*

Furthermore, there exists a graded Lie algebra L^∞ such that $E^\infty = UL^\infty$ as a Hopf algebra.

Proof. We proceed by induction. For $r = 1$, let $W_1 = W \otimes \mathbf{F}_p$. Since p divides $\bar{D}$, $E_1 = \Gamma(sW_1)$ and $\beta_1 = \bar{D}/p$ (reduced modulo p). Because $\bar{D}$ is a Γ-derivation, so is β_1, establishing the first statement. For the second statement, let g_1 be the identity map on E_1. The third statement follows from the first and Theorem 7. In fact, from the definitions it follows that L^1 is the homotopy Lie algebra of the identity on $(\Lambda W, d) \otimes \mathbf{F}_p$.

Now suppose the three statements are established for $r-1$. We may write $C^*(L^{r-1}, \beta^{r-1}) = (\Lambda W_{r-1}, \delta)$; let $C(r-1) = (\Lambda W_{r-1} \otimes \Gamma(sW_{r-1}), D)$ be a constructible acyclic closure. By Lemma 5.4 of [5], there is a chain isomorphism $\gamma_{r-1} : U(L^{r-1}, \beta^{r-1}) \xrightarrow{\cong} (\Gamma(sW_{r-1}), \bar{D})^\sharp$. It is implicit that γ_{r-1} is a coalgebra isomorphism, which then induces an algebra structure on $(\Gamma(sW_{r-1}), \bar{D})^\sharp$ which makes $(\Gamma(sW_{r-1}), \bar{D})$ into a Γ-DGH. In particular, under γ_{r-1} we identify $\bar{D}$ with β_{r-1}, the differential in E_{r-1}.

Let $m_r : (\Lambda W_r, d) \xrightarrow{\simeq} C^*(L^{r-1}, \beta^{r-1})$ be a minimal model. Let $C'(r)$ be a constructible acyclic closure of $(\Lambda W_r, d)$ ([5], Section 2). Since $(\Lambda W_r, d)$ is $\mathbf{F}_p$-minimal, $d_0 = 0$; so by Corollary 2.6 of [5], $\mathbf{F}_p \otimes_{\Lambda W} C'(r) = (\Gamma(sW_r), 0)$. By [5], Proposition 2.7, m_r induces a Γ-morphism $\bar{m}_r : (\Gamma(sW_r), 0) \to (\Gamma(sW_{r-1}), \bar{D})$. Since $\mathbf{F}_p$ is a field, by Lemma 3.3 of [5], we may identify $H(\bar{m}_r)$ with $\mathrm{Tor}^{m_r}(\mathbf{F}_p, \mathbf{F}_p)$, where Tor is the differential torsion functor [4]. Since m_r is a quasi-isomorphism, $H(\bar{m}_r) : \Gamma(sW_r) \xrightarrow{\cong} H(\Gamma(sW_{r-1}), \beta_{r-1}) = E_r$, so E_r is a free Γ-algebra. Furthermore, by [5], $H(\bar{m}_r^\sharp) : E^r = H(UL^{r-1}) \xrightarrow{\cong} UL^r$ as Hopf algebras, where $L^r = (sW_r)^\sharp$ as graded vector spaces. Therefore $E_r = (E^r)^\sharp$ is a Hopf Γ-algebra.

By the inductive hypothesis, there exists a Γ-morphism $g_{r-1} : E_{r-1} \to E_1$ such that $z = [a]_{r-1}$ whenever $z \in E_{r-1}$, $a \in \Gamma(sW)$ satisfy $g(z) = \rho(a)$. Let $g_r = g_{r-1}\bar{m}_r$. For $u \in E_r$ choose $a \in \Gamma(sW)$ so that $g_{r-1}(\bar{m}_r(u)) = \rho(a)$. Then $m_r(u) = [a]_{r-1}$, hence $\beta_{r-1}[a]_{r-1} = 0$ and $[a]_r \in E_r$ is defined. Since $H(\bar{m}_r)[a]_r = [a]_{r-1}$, $\bar{m}_r([a]_r) = [a]_{r-1} + \beta_{r-1}(v)$ for some $v \in E_{r-1}$. Thus $\bar{m}_r(u - [a]_r) = \beta_{r-1}(v)$, so $u - [a]_r$ is a boundary in $(E_r, 0)$, whence $u = [a]_r$. This establishes the second statement.

Let $u \in E_r$, and suppose for some $a \in \Gamma(sW)$ that $\rho(a) = g_r(u)$. Then $u = [a]_r$, so $\bar{D}a = p^r b$ for some $b \in \Gamma(sW)$. Thus $\beta_r(u) = [b]_r$. Since g_r and ρ are Γ-morphisms, $\rho(\gamma^j(a)) = g_r(\gamma^j(u))$ so $\gamma^j(u) = [\gamma^j(a)]_r$. Furthermore, $\bar{D}(\gamma^k(a)) = p^r b \cdot \gamma^{k-1}(a)$; so

$$\beta_r \gamma^k(u) = \beta_r[\gamma^k(a)]_r = [b \cdot \gamma^{k-1}(a)]_r = [b]_r[\gamma^{k-1}(a)]_r = \beta_r(u) \cdot \gamma^{k-1}(u).$$

Therefore β_r is a Γ-derivation, finally establishing the first statement.

By Proposition 6, we have established the third statement, completing the inductive step.

The E^∞ term. It suffices to show that E_∞ is a Hopf Γ-algebra. From the definitions it follows that E_∞ is a Hopf algebra. We have a sequence of Γ-morphisms

$$\cdots \to E_r \xrightarrow{\bar{m}_r} E_{r-1} \xrightarrow{\bar{m}_{r-1}} E_{r-2} \to \cdots \xrightarrow{\bar{m}_2} E_1.$$

Thus we may identify E_∞ with $\varprojlim E_r$, which is the subset of $\Pi_{r\geq 1}E_r$ consisting of sequences (x_r) that satisfy $x_r = \bar{m}_{r+1}(x_{r+1})$ for all $r \geq 1$. Since each $\bar{m}_r$ is a Γ-morphism, a well-defined system of divided powers on E_∞ is given by $\gamma^k((x_r)) = (\gamma^k(x_r))$. From the definitions, the coproduct is a Γ-morphism. □

Proof of Theorem 3. Let $m : (\Lambda W, d) \xrightarrow{\simeq} C^*(L, \partial)$ be a minimal model. Recall that the underlying algebra of $C^*(L, \partial)$ is ΛV, where $V = (sL)^\sharp$. Let $(\Lambda W \otimes \Gamma(sW), D)$ and $(\Lambda V \otimes \Gamma(sV), D)$ be constructible acyclic closures for $(\Lambda W, d)$ and $C^*(L, \partial)$, respectively. The model m determines a Γ-morphism $\bar{m} : (\Gamma(sW), \bar{D}) \to (\Gamma(sV), \bar{D})$ where $H(\bar{m}^\sharp)$ is an isomorphism. The composition

$$U(L, \partial) \xrightarrow{\cong} (\Gamma(sV), \bar{D})^\sharp \xrightarrow{\simeq} (\Gamma(sW), \bar{D})^\sharp$$

induces an isomorphism of Bockstein spectral sequences, establishing the first statement.

The reduced minimal model $m \otimes \mathbf{F}_p : (\Lambda W, d) \otimes \mathbf{F}_p \xrightarrow{\simeq} C^*(L, \partial) \otimes \mathbf{F}_p$ has homotopy Lie algebra L^1, so by Proposition 5, $\operatorname{im} E^1(\iota) \subset L^1$. Suppose that $\operatorname{im} E^{r-1}(\iota) \subset L^{r-1}$. Let $\iota^{(r-1)} : L^{r-1} \hookrightarrow UL^{r-1}$ be the inclusion. Then $\operatorname{im} E^r(\iota) \subset \operatorname{im} H(\iota^{(r-1)})$. The homotopy Lie algebra of the minimal model $m_r : (\Lambda W_r, d) \xrightarrow{\simeq} C^*(L^{r-1}, \beta^{r-1})$ is L^r, so Proposition 5 states that $\operatorname{im} H(\iota^{(r-1)}) \subset L^r$, completing the induction and the proof. □

5. Example of non-naturality

First we state a proposition, whose proof is straightforward.

Proposition 9. *Define a* DGL *over* $\mathbf{F}_p$ *by* $(L, \partial) = (L_{ab}(e, f), \partial f = e)$*, where* $|f| = 2n$*. Then* $C^*(L, \partial) = (\Lambda(x, y), d)$ *with* $dx = y$ *and* $|x| = 2n$*. A minimal model* $m : (\Lambda(x_1, y_1), 0) \xrightarrow{\simeq} C^*(L, \partial)$*, given by* $x_1 \mapsto x^p$ *and* $y_1 \mapsto x^{p-1}y$*, induces isomorphisms* $\Gamma(sx_1, sy_1) \xrightarrow{\cong} H([UL]^\sharp)$ *and* $H(UL) \xrightarrow{\cong} UL_{ab}(e_1, f_1)$ *with* $|e_1| = |sx_1| = 2np - 1$*,* $|f_1| = |sy_1| = 2np$*.* □

Example 1. Define a DGL (L, ∂) over $\mathbf{Z}_{(p)}$ by $L = L_{\text{ab}}(e, f, g)$, where $|e| = 2n - 1$, $|f| = |g| = 2n$, and $\partial(f) = pe$. Then $L^1 = L_{\text{ab}}(e, f, g)$ (over $\mathbf{F}_p$), with $\beta^1(f) = e$, and $C^*(L^1, \beta^1) = (\Lambda(x, y), dx = y) \otimes (\Lambda(z), 0)$. Recall the model m from Proposition 9. Define DGA morphisms $i, j : (\Lambda(z), 0) \to C^*(L^1, \beta^1)$ by $i(z) = z$, $j(z) = z + y$. Then $\varphi = m \otimes i$ and $\psi = m \otimes j$ are minimal models, both with homotopy Lie algebra $L^2 = L_{\text{ab}}(a, b, c)$, $|a| = 2np - 1$, $|b| = 2np$, and $|c| = 2n$. The two models determine Hopf algebra isomorphisms $\varphi^*, \psi^* : H(UL^1) \to UL^2$, given by $\varphi^*[ef^{p-1}] = \psi^*[ef^{p-1}] = a$, $\varphi^*[g] = \psi^*[g] = c$, $\varphi^*[f^p] = b$, and $\psi^*[f^p] = b + c^p$. The algebra isomorphism $\psi^*(\varphi^*)^{-1} : UL^2 \to UL^2$ is not of the form $U\theta$ for any Lie algebra morphism $\theta : L^2 \to L^2$. Therefore the construction involved in Theorem 3 is not natural.

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Department of Mathematics, University of Toronto, Toronto, Ontario M5S 3G3, Canada

Current address: Aberdeen Topology Centre, Department of Mathematical Sciences, University of Aberdeen, Aberdeen AB24 3UE, United Kingdom

E-mail address: `j.scott@maths.abdn.ac.uk`