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# ON CERTAIN CO–H SPACES RELATED TO MOORE SPACES

MANFRED STELZER

Abstract. We show that certain co–$H$ spaces, constructed by Anick and Gray, carry a homotopy co–associative and co–commutative co–$H$ structure.

## 1. Introduction and statement of results

In a series of papers, Anick and Gray constructed and studied a family of co–$H$ spaces $G_k^{2n}(p^r)$, $p \geq 5$, with some remarkable properties [1], [2], [7], [8]. The limit space, or rather its loop space $\Omega G_\infty^{2n}(p)$, shows up in a secondary version of the EHP sequence. Furthermore, Gray used the $G_k^{2n}(p^r)$ to construct $v_2$–periodic families in the homotopy groups of Moore spaces [8], and Anick used them to decompose loop spaces of finite complexes [1]. It is the aim of this paper to prove the following theorem, which was conjectured in part in [2]:

**Theorem 1.** *The spaces $G_k^{2n}(p^r)$ carry a co–associative and co–commutative co–$H$ structure.*

In section 2 we define the notions of co–$A$ and co–$C$ deviation. Some properties are established, which are used in section 3 to prove Theorem 1.

After the results in this paper were obtained, I learned of Stephen Theriault's thesis, in which he proved the homotopy co–associtivity of $G_k$ (and much more) by using Ganea's characterisation of co–associtivity. But the argument given there has a gap. It is assumed during the proof that, given a homotopy commutative square of co–$H$ spaces and co–$H$ maps, the map induced on the cofibers is also a co–$H$ map. But this is clearly false (see for example the discussion of Zabrodsky's formula in [10], p. 228).

## 2. The co–$A$ and co–$C$ deviation

The co–$A$ deviation was first defined by Harper [10]. It is an obstruction for the existence of a co–associative co–$H$ structure on the mapping cone of a co–$H$ map between co–associative co–$H$ spaces. The dual $A$–deviation and $C$–deviation were studied by Zabrodsky in [13].

Let $f : X, \tau \to Y, \sigma$ be a co–$H$ map between the co–$H$ spaces $X, Y$ with structure maps $\tau, \sigma$. So there are based homotopies

$$\begin{aligned} M_x &: & j \circ \tau &\to \triangle_x \ , \\ M_y &: & j \circ \sigma &\to \triangle_y \ , \\ F &: & (f \vee f) \circ \tau &\to \sigma \circ f \ . \end{aligned}$$

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Here $\triangle$ is the diagonal, and $j$ the inclusion of the wedge into the product.

Suppose $X, \tau$ and $Y, \sigma$ are co–associative resp. co–commutative up to homotopy. In this case we have based homotopies:

$$L_x : (1 \vee \tau) \circ \tau \longrightarrow (\tau \vee 1) \circ \tau , \quad L_y : (1 \vee \sigma) \circ \sigma \longrightarrow (\sigma \vee 1) \circ \sigma$$

and

$$G_x : \tau \longrightarrow T \circ \tau , \quad G_y : \sigma \longrightarrow T \circ \sigma ,$$

where $T$ denotes the switch map.

We will use the following notation for stringing together a list of homotopies. Given $H_1, \dots, H_n : Z \times I \to W$ such that $H_i(Z, 1) = H_{i+1}(Z, 0)$, define

$$\{H_1, \dots, H_n\}(z, t) = H_i(z, nt - i + 1), \quad \text{if } (i-1) \le nt \le i.$$

We also write $H^{-1}$ for the homotopy with $H^{-1}(z, t) = H(z, 1 - t)$.

Suppose $X, \tau$ and $Y, \sigma$ are homotopy co–associative, and fix $L_x, L_y$. Following Harper we define the co–$A$ deviation with respect to $F$ as follows: let

$$A(f, F) = \{(f \vee F) \circ \tau,\ (1 \vee \sigma) \circ F,\ L_y \circ f,\ (\sigma \vee 1) \circ F^{-1},\ (F^{-1} \vee f) \circ \tau,\ f_{(3)} \circ L_x^{-1}\} .$$

This defines a map

$$A(f, F) : X, * \longrightarrow \Lambda Y_{(3)}, c ,$$

where $\Lambda$ denotes the free loop space, $c$ is the map sending the whole circle to the base point, and $X_{(k)}$ and $f_{(k)}$ denote the $k$–fold wedge of the space and the map, respectively.

Let $u : X, * \to \Lambda Y_{(3)},\ c$ be the map which sends $x$ to the constant map

$$f_{(3)} \circ (1 \vee \tau) \circ \tau(x).$$

Then $A(f, F) - u$ factors uniquely over a map from $X, *$ to $\Omega Y_{(3)}, *$, and the homotopy class of the adjoint of this map

$$A_*(f, F) \in [\Sigma X, Y_{(3)}]$$

is the co–$A$ deviation.

The homotopy $F$ is called primitive, if

$$j \circ F = \{(f \times f) \circ M_x,\ M_y^{-1} \circ f\}$$

as tracks, i.e., as homotopy classes relative to $X \times \partial I$. Berstein and Harper proved in [5] that one can choose $F$ to be primitive.

The map $f$ is called a co–$A$ map, if there is a primitive homotopy $F$ such that $A_*(f; F)$ is the class of the constant map.

The definition of the co–$C$ deviation is similar. Suppose $X, \tau$ and $Y, \sigma$ are homotopy co–commutative, and fix $G_x, G_y$. Define

$$C(f, F) = \{F, G_y \circ f, T \circ F^{-1}, (f \vee f) \circ G_x^{-1}\}.$$

This gives us a map $C(f, F) : X, * \to \Lambda Y_{(2)}, c$.

Again there is a map unique up to homotopy from $X$ to $\Omega Y_{(2)}$ such that the composite with the inclusion $\Omega Y_{(2)} \hookrightarrow \Lambda Y_{(2)}$ is $C(f, F) - v$ with $v(x)$ constant of value $(f \vee f) \circ \tau(x)$. The homotopy class of the adjoint $C_*(f, F) \in [\Sigma X, Y_{(2)}]$ is called the co–$C$ deviation of $f, F$.

The map $f$ is called a co–$C$ map, if there is a primitive homotopy $F$ such that $C_*(f, F)$ is the class of the constant map.

A result from [13], which we wish to dualize, is that the $A$–deviation, which is a map from $X \times X \times X$ to $\Omega Y$, can be chosen to be constant on the fat wedge in $X \times X \times X$. The proof used the existence of a strict unit for $H$–spaces. Since there is never a strict counit for a co–$H$ space unless the space is a point, we have not been able to prove the strictly dual theorem, i.e., that $A_*(f, F)$ lifts to the homotopy fiber of $Y_{(3)} \longrightarrow R$, where $R$ is the homotopy limit of the diagram

$$\begin{array}{ccccc} Y_1 \vee Y_2 & \longrightarrow & Y_1 & \longleftarrow & Y_1 \vee Y_3 \\ \downarrow & & & & \downarrow \\ Y_2 & \longleftarrow & Y_2 \vee Y_3 & \longrightarrow & Y_3 \end{array}$$

with the obvious maps. However, the following weak version will suffice for the applications which we have in mind.

**Theorem 2.1.** *The homotopies $L_x, L_y$ and $G_x, G_y$ can be chosen such that*

a) $j \circ A_*(f, F) \simeq *$, *where* $j : Y_{(3)} \to Y \times Y \times Y$, *and*

b) $j \circ C_*(f, F) \simeq *$, *where* $j : Y_{(2)} \to Y \times Y$.

*For a map $g : X \to Y_{(k)}$ we denote the projection onto the $i$–th component by $g_i$.*

The proof of Theorem 2.1 will need

**Lemma 2.2.** *One can choose $L_x$ and $L_y$ such that*

$$\begin{array}{lclllcl} L_{y,1} & = & \sigma_1 \circ M_{y,1}^{-1}, & & L_{x,1} & = & M_{x,1}^{-1} \circ \tau_1, \\ L_{y,2} & = & \{M_{y,1} \circ \sigma_2,\, \sigma_2 \circ M_{y,1}^{-1}\}, & & L_{x,2} & = & \{M_{x,1} \circ \tau_2, \tau_2 \circ M_{x,1}^{-1}\}, \\ L_{y,3} & = & \sigma_2 \circ M_{y,2}, & & L_{x,3} & = & M_{x,2} \circ \tau_2 \end{array}$$

*as tracks, i.e., as homotopy classes relative to $Y \times \partial I$ and $X \times \partial I$.*

*Proof.* Since the loop of the homotopy fiber sequence $F \to Y_{(3)} \to Y \times Y \times Y$ splits, the lemma follows from [5, 1.8]. □

**Lemma 2.3.** *One can choose $G_x$ and $G_y$ such that*

$$G_{x,1} = \{M_{x,1}, M_{x,2}^{-1}\},\ G_{y,1} = \{M_{y,1}, M_{y,2}^{-1}\},$$
$$G_{x,2} = \{M_{x,2}, M_{x,1}^{-1}\},\ G_{y,2} = \{M_{y,2}, M_{y,1}^{-1}\}.$$

*Proof.* Same argument as above. □

*Proof of Theorem* 2.1. a) We choose $F$ primitive and $L_x, L_y$ as in Lemma 2.2. We show that in this case $A(f, F)_i = u_i$, $i \in \{1, 2, 3\}$. (By a slight abuse of notation we also denote by $g_i$ the composition of a map to $\Lambda(Y_{(k)})$ with the $i$-th projection to $\Lambda(Y)$.)

We first have a hard look at $A(f, F)_1$:

$$\begin{array}{rcl} A(f, F)_1 & = & \{f \circ M_{x,1},\, M_{y,1}^{-1} \circ f,\, \sigma_1 \circ M_{y,1}^{-1} \circ f,\, \sigma_1 \circ M_{y,1} \circ f,\, \sigma_1 \circ f \circ M_{x,1}^{-1}, \\ & & M_{y,1} \circ f \circ \tau_1,\, f \circ M_{x,1}^{-1} \circ \tau_1,\, f \circ M_{x,1} \circ \tau_1\} \\ & = & \{f \circ M_{x,1},\, M_{y,1}^{-1} \circ f,\, \sigma_1 \circ f \circ M_{x,1}^{-1},\, M_{y,1} \circ f \circ \tau_1\}. \end{array}$$

*Claim.*

$$\{f\circ M_{x,1},\, M_{y,1}^{-1}\circ f\} = M_{y,1}^{-1}\circ f\circ M_{x,1},$$
$$\{\sigma_1\circ f\circ M_{x,1}^{-1},\, M_{y,1}\circ f\circ \tau_1\} = M_{y,1}\circ f\circ M_{x,1}^{-1}.$$

Here $M_{y,1}^{-1}\circ f\circ M_{x,1}$ is shorthand for $M_{y,1}^{-1}(f\circ M_{x,1},\ \mathrm{id}_I)$, and similarly for $M_{y,1}\circ f\circ M_{x,1}^{-1}$.

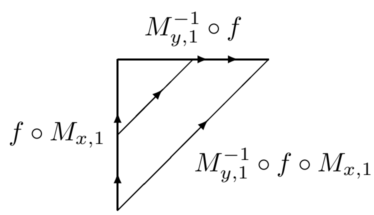
Moreover, for compositions, the first argument may be replaced by – if the expression is complicated and forced by contest.

To see this, consider the homotopies of tracks

$$K_1(x,t,s) := \begin{cases} f\circ M_{x,1}(x,2t) & \text{if } 0\le t\le \frac{s}{2},\\ M_{y,1}^{-1}(f\circ M_{x,1}(x,t+\frac{s}{2}),\, t-\frac{s}{2}) & \text{if } \frac{s}{2}\le t\le 1-\frac{s}{2},\\ M_{y,1}^{-1}(f(x),2t-1) & \text{if } 1-\frac{s}{2}\le t\le 1\,, \end{cases}$$

$$K_2(x,t,s) := \begin{cases} \sigma_1\circ f\circ M_{x,1}^{-1}(x,2t) & \text{if } 0\le t\le \frac{s}{2},\\ M_{y,1}(f\circ M_{x,1}^{-1}(x,t+\frac{s}{2}),\, t-\frac{s}{2}) & \text{if } \frac{s}{2}\le t\le 1-\frac{s}{2},\\ M_{y,1}(f(x),\, 2t-1) & \text{if } 1-\frac{s}{2}\le t\le 1\,. \end{cases}$$

The picture is:




where the path in the middle is $K_1(\dots,s)$. In conclusion, we have shown that $A(f,F)_1 \simeq A(f,F)_1(0) = u_1$. $A(f,F)_2$ is a little more complex:

$$\begin{aligned} A(f,F)_2 &= \{F_1\circ\tau_2,\, \sigma_1\circ F_2,\, L_{y,2}\circ f,\, \sigma_2\circ F_1^{-1},\, F_2^{-1}\circ\tau_1,\, f\circ L_{x,2}^{-1}\}\\ &= \{f\circ M_{x,1}\circ\tau_2,\, M_{y,1}^{-1}\circ f\circ\tau_2,\, \sigma_1\circ f\circ M_{x,2},\, \sigma_1\circ M_{y,2}^{-1}\circ f,\\ &\quad M_{y,1}\circ\sigma_2\circ f,\ \sigma_2\circ M_{y,1}^{-1}\circ f,\, \sigma_2\circ M_{y,1}\circ f,\, \sigma_2\circ f\circ M_{x,1}^{-1},\\ &\quad M_{y,2}\circ f\circ\tau_1,\ f\circ M_{x,2}^{-1}\circ\tau_1,\, f\circ\tau_2\circ M_{x,1},\ f\circ M_{x,1}^{-1}\circ\tau_2\}\\ &= \{\{M_{y,1}^{-1}\circ f\circ\tau_2,\, \sigma_1\circ f\circ M_{x,2},\, \sigma_1\circ M_{y,2}^{-1}\circ f,\, M_{y,1}\circ\sigma_2\circ f\}\,,\\ &\quad \{\sigma_2\circ f\circ M_{x,1}^{-1},\, M_{y,2}\circ f\circ\tau_1,\, f\circ M_{x,2}^{-1}\circ\tau_1,\, f\circ\tau_2\circ M_{x,1}\}\}\,. \end{aligned}$$

As in the case above, one sees that

$$\begin{aligned} \{\sigma_1\circ f\circ M_{x,2},\, \sigma_1\circ M_{y,2}^{-1}\circ f\} &= \sigma_1\circ M_{y,2}^{-1}\circ f\circ M_{x,2},\\ \{M_{y,2}\circ f\circ\tau_1,\, f\circ M_{x,2}^{-1}\circ\tau_1\} &= M_{y,2}\circ f\circ M_{x,2}^{-1}\circ\tau_1. \end{aligned}$$

So it remains to show that

$$\begin{aligned} &\{M_{y,1}^{-1}\circ f\circ\tau_2,\, \sigma_1\circ M_{y,2}^{-1}\circ f\circ M_{x,2},\, M_{y,1}\circ\sigma_2\circ f\}\\ &\quad = \{\sigma_2\circ f\circ M_{x,1}^{-1},\, M_{y,2}\circ f\circ M_{x,2}^{-1}\circ\tau_1,\, f\circ\tau_2\circ M_{x,1}\}^{-1}\,. \end{aligned}$$

To see this, consider the homotopy between tracks:

$$H_1(x,t,s) := \begin{cases} M_{y,1}^{-1}(f\circ\tau_2(x),3t) \quad \text{if } 0\le t\le \frac{s}{3}, \\ M_{y,1}^{-1}(-,s)\circ M_{y,2}^{-1}\left(-,\dfrac{t-\frac{s}{3}}{1-2\frac{s}{3}}\right)\circ f\circ M_{x,2}\left(-,\dfrac{t-\frac{s}{3}}{1-2\frac{s}{3}}\right) \\ \qquad\qquad \text{if } \frac{s}{3}\le t\le 1-\frac{s}{3}, \\ \sigma_2\circ M_{y,1}(f(x),3t-2) \quad \text{if } 1-\frac{s}{3}\le t\le 1, \end{cases}$$

$$H_2(x,t,s) := \begin{cases} \sigma_2\circ f\circ M_{x,1}^{-1}(x,3t) \quad \text{if } 0\le t\le \frac{s}{3}, \\ M_{y,2}\left(-,\dfrac{t-\frac{s}{3}}{1-2\frac{s}{3}}\right)\circ f\circ M_{x,2}^{-1}\left(-,\dfrac{t-\frac{s}{3}}{1-2\frac{s}{3}}\right)\circ M_{x,1}^{-1}(x,s) \\ \qquad \text{if } \frac{s}{3}\le t\le 1-\frac{s}{3}, \\ f\circ\tau_2\circ M_{x,1}(x,3t-2) \quad \text{if } 1-\frac{s}{3}\le t\le 1. \end{cases}$$

Since $H_1(x,t,0) = H_2(x,1-t,0) = M_{y,2}^{-1}(-,t)\circ f\circ M_{x,2}(x,t)$, the identity above is proved.

The case $A(f,F)_3$ is like the first one.

Next we prove b), which is easier. Using the primitivity of $F$ and Lemma 2.3, we find that

$$\begin{aligned}&\{F, G_y\circ f, T\circ F^{-1}, f\vee f\circ G_x^{-1}\}_{,1}\\ &\quad = \{f\circ M_{x,1}, M_{y,1}^{-1}\circ f, M_{y,1}\circ f, M_{y,2}^{-1}\circ f, M_{y,2}\circ f, f\circ M_{x,2}^{-1}, f\circ M_{x,2}, f\circ M_{x,1}^{-1}\},\end{aligned}$$

which is the constant track.

A similar argument works for the second coordinate. □

The following two lemmas were proved in [5].

**Lemma 2.4.** *Let $(f,F):\Sigma X,\tau\to Y,\sigma$ be a co–H map with primitive homotopy $F$. Suppose $Y,\sigma$ is homotopy co–associative and $\Sigma X,\tau$ is the suspension co–H structure. Then for each suspension $\Sigma g:\Sigma Z\to\Sigma X$ the co–A deviation satisfies*

$$A_*(f\circ\Sigma g, F\circ\Sigma g) = A_*(f,F)\circ\Sigma^2 g\ .$$

□

Recall from [5] that for a co–$H$ map $f: X,\tau\to Y,\sigma$ with primitive homotopy $F$, the mapping cone $C_f$ carries a co–$H$ structure $\overline{\sigma}$ defined by

$$\begin{aligned}\overline{\sigma}(x,t) &= \begin{cases} F^{-1}(x,2t) & \text{for} \quad 0\le 2t\le 1, \\ \tau(x), 2t-1 & \text{for} \quad 0\le 2t\le 1, \end{cases}\\ \overline{\sigma}(y) &= \sigma(y)\ .\end{aligned}$$

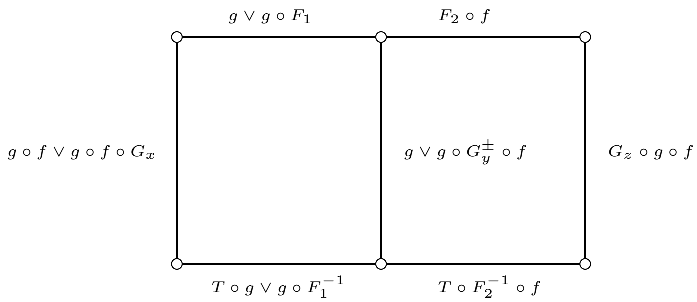
**Lemma 2.5.** *Let $(f,F)$ be a co–A map. Then $\overline{\sigma}$ is homotopy co–associative.* □

There are similar results for the co–$C$ deviations:

**Lemma 2.6.** *Let $(f,F_1): X,\tau\longrightarrow Y,\sigma$ and $(g,F_2): Y,\sigma\longrightarrow Z,\eta$ be co–H maps and primitive homotopies, and suppose all spaces are homotopy co–commutative. Then*

$$C_*(f\circ g,\ \{g\vee g\circ F_1, F_2\circ f\}) = (g\vee g)\circ C_*(f,F_1) + C_*(g,F_2)\circ\Sigma f.$$

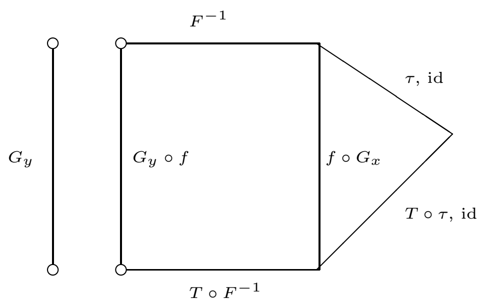
*Proof.* The proof is given by looking at the diagram




□

**Lemma 2.7.** *Let $(f, F)$ be a co–$C$ map and a primitive homotopy. Then the co–$H$ structure on $C_f$ defined by $F$ is homotopy co–commutative.*

*Proof.* Consider the diagram




The square in the middle can be filled since $f$ is a co–$C$ map, and the triangle is extendable by the cone on $G_*$. □

## 3. Review on $G_k^{2n}(p^r)$ and proof of theorem 1

Recall from [2] that the space $G_k^{2n}(p^r)$ has a $CW$ structure as follows:

$$G_k^{2n}(p^r) = S^{2n} \cup_{p^r} e^{2n+1} \cup_{g_1} \ \ldots \ \cup_{g_k} S^{2np^k} \cup_{p^{r+k}} e^{2np^k+1};$$

the attaching maps $g_i$ are divisible by $p^{r+i-1}$ in homotopy.

One also has $G_{k-1}^{2n}(p^r) \cup_{g_k} S^{2np^k} \cup e^{2np^k+1} = G_k^{2n}(p^r)$, and there is a compatible co–$H$ structure on $G_k^{2n}(p^r)$. Denote by $W_k^{r+k}$ the class of spaces that are of finite type and of the homotopy type of bouquets of Moore spaces of type $p^{r+s}$ with $s \in \{0, \ldots, k\}$.

**Lemma 3.1.** *The following statements hold:*

a) $\Sigma^2 \Omega G_k^{2n}(p^r) \in W_r^{r+k}$.
b) $\Sigma \Omega G_k^{2n}(p^r) \wedge \Omega G_k^{2n}(p^r) \in W_r^{r+k}$.

c) *Let $F$ be the homotopy fiber of*

$$j : \bigvee_{i=1}^{3} G_k^{2n}(p^r) \longrightarrow \prod_{i=1}^{3} G_k^{2n}(p^r) \ .$$

*Then $\Omega F$ is homotopy equivalent to a weak product of spaces of the form $\Omega W_j$, $W_j \in W_r^{r+k}$.*

*Proof.* Parts a) and b) are in [2]. For c), recall the two homotopy equivalences

i) $\Sigma(A \times B) \simeq \Sigma A \vee \Sigma B \vee \Sigma A \wedge B$,

ii) $\Omega(X \vee Y) \simeq \Omega X \times \Omega Y \times \Omega(\Sigma\Omega X \wedge \Omega Y)$ .

Apply i) to $\Omega(Y \vee Y \vee Y)$ to find that

$$\Omega F \simeq \Omega\Sigma(\Omega Y)^{\wedge 2} \times \Omega[\Sigma\Omega Y \wedge (\Omega Y \times \Omega Y \times \Omega\Sigma(\Omega Y)^{\wedge 2}] \ ,$$

and ii) to see that

$$\begin{aligned}&\Sigma(\Omega Y \times \Omega Y \times \Omega\Sigma(\Omega Y)^{\wedge 2}) \\ &\quad \simeq \Sigma\Omega Y \vee \Sigma\Omega Y \vee \ \Sigma(\Omega Y)^{\wedge 2} \ \vee \ \Sigma\Omega\Sigma(\Omega Y)^{\wedge 2} \vee \Sigma\Omega Y \\ &\qquad \wedge \ \Omega\Sigma(\Omega Y)^{\wedge 2} \vee \Sigma\Omega Y \wedge \Omega\Sigma(\Omega Y)^{\wedge 2} \vee \Sigma(\Omega Y)^{\wedge 2} \wedge \Omega\Sigma(\Omega Y)^{\wedge 2}.\end{aligned}$$

By Hilton–Milnor, and the fact that $W_r^{r+k}$ is closed under smash products [12], it is enough to show that each wedge factor is in $W_r^{r+k}$ for $Y = G_k^{2n}(p^r)$. By b) $\Sigma(\Omega G_k^{2n}(p^r))^{\wedge 2} \in W_r^{r+k}$, and since this space is 2–connected it is a double suspension. By a) the smash with $\Omega G_k^{2n}(p^r)$ is in $W_r^{r+k}$. The splitting of $\Sigma\Omega\Sigma X$ shows that also $\Sigma\Omega\Sigma\Omega G_k^{2n}(p^r)^{\wedge 2} \in W_r^{r+k}$. □

*Proof of Theorem* 1. The proof is by induction on $k$. For $k = 0$, $G_k^{2n}(p^r)$ is a simply connected Moore space of type $p^r$, $p$ odd. Hence it is a suspension, and so homotopy co–associative. If $n \geq 2$ it is a double suspension, and the homotopy co–commutativity for $n = 1$ was proved in [4]. So suppose the assertion holds for $k-1, k \geq 1$. We show that the co–$A$ and the co–$C$ deviations of $g_k$ vanish. This suffices by Lemmas 2.5 and 2.7. Since $g_k$ is divisible by $p^{r+k-1}$, it follows from Lemmas 2.4 and 2.6 that $A_*(g_k, F)$ and $C_*(g_k, F)$ are also divisible by $p^{r+k-1}$.

By Theorem 2.1 the maps $A_*(g_k, F)$ and $C_*(g_k, F)$ lift, uniquely up to homotopy, to the homotopy fibers $F_1$ and $F_2$, respectively, of

$$j : \bigvee_{i=1}^{s} G_k^{2n}(p^r) \longrightarrow \prod_{i=1}^{s} G_k^{2n}(p^r), \qquad s \in \{2, 3\} \ .$$

Moreover, the lift is also divisible by $p^{r+k-1}$, since the loop of this fibration splits.

We claim that in fact

$$p^{r+k-1}[P^{2np^k}(p^r), \Omega F_i] = 0.$$

The homotopy type of $F_2$ is well known to be $\Sigma\Omega G_{k-1}^{2n}(p^r) \wedge \Omega G_{k-1}^{2n}(p^r)$, and the type of $\Omega F_1$ was determined during the proof of Lemma 3.1.

By [11] the homotopy exponent of $P^\ell(p^s)$ is $p^{s+1}$ for $p$ odd. It follows that the only factors in the product decomposition of $\Omega F_i$ which could contribute a class of order $p^{r+k}$ are of the form $\Omega P^\ell(p^{r+k-1})$. So we have to determine the least $\ell$ for which such a factor occurs. The first class in $H_*(\Omega G_k^{2n}(p^r); \mathbb{Z})$ of order $p^{r+k-1}$ shows up in dimension $2np^{k-1} - 1$ by [2, p. 864].

Consequently the first class of order $p^{r+k-1}$ in $H_*(\Omega\Sigma\Omega G_k^{2n}(p^r) \wedge \Omega G_k^{2n}(p^r); \mathbb{Z})$ shows up in dimension $4np^{k-1} - 2$. Inspection of the proof of Lemma 3.1 shows that in the splitting of $\Omega F_1$ each factor is of the form loop of $\Sigma(\Omega G_k^{2n}(p^r))^{\wedge 2} \wedge Z$. So also in $H_*(\Omega F_1, \mathbb{Z})$ and $H_*(\Omega F_2, \mathbb{Z})$ the first class of order $p^{r+k-1}$ is in dimension $\geq 4np^{k-1} - 2$. This class comes from a factor $\Omega P^{4np^{k-1}\geq}(p^{r+k-1})$.

By [6], the first element of order $p^{r+k}$ in $\pi_*(P^{2m+1}(p^{r+k-1}))$, respectively $\pi_*(P^{2m}(p^{r+k-1}))$ is in dimension $2mp-1$, resp. $4mp-2p-1$.

The universal coefficient sequence for homotopy groups splits at an odd prime. So it is enough to show that $\pi_*(P^\ell(p^{r+k-1}))$ does not contain a class of order $p^{r+k}$ for $* \leq 2np^k + 1$ and $\ell \geq 4np^{k-1} = 2m$. But this follows from what was said above and the trivial estimates

$$\begin{aligned} 2np^k + 1 &< 4mp - 2p - 1 = 8np^k - 2p - 1, \\ 2np^k + 1 &< 2mp - 1 = 4np^k - 1\ . \end{aligned}$$

□

The next two corollaries are just mild strengthenings of two results from [2].

**Corollary 3.2.** *Suppose $X$ is an $H$ space, and $\varphi_{k-1} : G_{k-1}^{2n}(p^r) \to \Sigma X$ is a map such that $\varphi_{k-1}\big|_K$, where $K$ is a skeleton, is a co–$H$ map. Suppose also that $p^{r+k-1}\pi_{2np^k-1}(X; \mathbb{Z}/p^{r+k}) = 0$. Then $\varphi_{k-1}$ has an extension $\varphi_k : G_k^{2n}(p^r) \to \Sigma X$.*

*Proof.* It was shown in [2, 4.1] that the corollary would follow if one could choose the coretraction $\sigma : G_k^{2n}(p^r) \to \Sigma\Omega G_k^{2n}(p^r)$ corresponding to the co–$H$ structure to be a co–$H$ map. Since every 1–connected co–associative co–$H$ space is a cogroup [3], this follows from [9, 4.2] and Theorem 1. □

**Corollary 3.3.** *Let $0 \leq s \leq m \leq \infty$, and suppose that $X$ is an $H$–space such that $p^{r+k-1}\pi_{2np^k-1}(X; \mathbb{Z}/p^{r+k}) = 0$ for $s < k \leq m$. Let $\varphi : P^{2mp^s}(p^{r+s}) \to X$ be a map. Then there is a map $\varphi_m : G_m^{2n}(p^s) \to \Sigma X$ which extends*

$$G_s^{2n}(p^r) \xrightarrow{pinch} G_s^{2n}(p^r)/G_{s-1}^{2n}(p^r) = P^{2np^s+1}(p^{r+s}) \xrightarrow{\Sigma\varphi} \Sigma X\ .$$

*Proof.* Since $G_{s-1}^{2n}(p^r)$ is a sub-co–$H$ space of $G_s^{2n}(p^r)$ the pinch map is a co–$H$ map for the induced co–$H$ structure on the quotient space. This co–$H$ structure is unique [4]. Thus the composite with $\Sigma\varphi$ is a co–$H$ map, and the assertion follows from Corollary 3.2. □

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Sesenheimerstrasse 20, 10627 Berlin, Germany