DEGENERATE FIBRES
IN THE STONE-ČECH COMPACTIFICATION
OF THE UNIVERSAL BUNDLE OF A FINITE GROUP

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ABSTRACT. Applied to a continuous surjection $\pi : E \to B$ of completely regular Hausdorff spaces E and B, the Stone-Čech compactification functor β yields a surjection $\beta\pi : \beta E \to \beta B$. For an n-fold covering map π, we show that the fibres of $\beta\pi$, while never containing more than n points, may degenerate to sets of cardinality properly dividing n. In the special case of the universal bundle $\pi : EG \to BG$ of a p-group G, we show more precisely that every possible type of G-orbit occurs among the fibres of $\beta\pi$. To prove this, we use a weak form of the so-called generalized Sullivan conjecture.

1. INTRODUCTION

This paper concerns the interaction of two fundamental, but at first glance unrelated, universal topological constructions: the classifying space, BG, of a group G and the Stone-Čech compactification, βX, of a completely regular topological space X. The former belongs to algebraic topology, the latter to general topology. Why consider them together? We were initially motivated by the simple question of whether an n-to-1 covering map always admits an n-to-1 compactification. As it turns out, *any* compactified universal G-bundle $\beta EG \to \beta BG$ for a non-trivial finite group G gives a counterexample.

Applied to a continuous surjection $\pi : E \to B$ of completely regular Hausdorff spaces, the Stone-Čech compactification functor β yields a continuous surjection $\beta\pi : \beta E \to \beta B$. In the simple case where $E = B \times F$, F a finite set, and π is projection on the first factor, elementary considerations show that $\beta E = \beta B \times F$ and $\beta\pi$ is again projection on the first factor. However, as we shall show, nontrivial phenomena can arise in the slightly more general situation of an n-fold covering map π (a local homeomorphism $\pi : E \to B$ with $|\pi^{-1}(b)| = n$ for any $b \in B$). Indeed, the fibres of $\beta\pi$, while never exceeding n points, can degenerate to sets whose cardinality properly divides n. (Contrast this with the more usual, explosive sort of Stone-Čech "pathology".)

This phenomenon depends on a homotopy invariant, the sectional category, of the map π. Indeed, we show that for any G-bundle $\pi : E \to B$, G a finite group, $\beta\pi$ has degenerate fibres exactly when π has infinite sectional category.

In the special case of a universal G-bundle $\pi : EG \to BG$ for a finite p-group G, we obtain a more precise result: every possible type of G-orbit occurs among the fibres of $\beta\pi$. To prove this, we use a weak form of the so-called "generalized Sullivan

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conjecture" [5], [1], now a theorem of H. Miller. A G-orbit, up to isomorphism, has the form G/H for some subgroup H of G. Thus, at least for p-groups, the lattice of subgroups of the structure group manifests itself in the geometry of the compactified universal bundle — this in spite of the fact that action of the group does not belong to the data fed to the Stone-Čech functor. In effect, the maximal compactification process automatically detects the symmetry of the bundle.

This work lies at the junction of algebraic and general topology, two very nearly autonomous mathematical disciplines. Accordingly, we have included enough detail throughout to make the paper essentially self-contained. In particular, we provide the necessary background on the Stone-Čech functor and homotopy theory (specifically, sectional category and homotopy fixed points). Readers unfamiliar with universal G-bundles should bear in mind the simplest non-trivial example, $G = \mathbb{Z}/2\mathbb{Z}$. The double cover of the infinite real projective space $\mathbb{RP}^\infty$ forms a universal $\mathbb{Z}/2\mathbb{Z}$-bundle. No other finite group has a universal bundle so easily pictured; this case also motivates some of our terminology.

2. Background on β

We record here some basic lore about Stone-Čech compactifications. For further information, see [8].

Let $C(X)$ denote the ring of real-valued continuous functions on a topological space X. The space X is said to be *completely regular* if $C(X)$ separates points from closed sets, [1] and *Tychonoff* if it is both completely regular and Hausdorff. We shall call a space *locally Tychonoff* if every point has a Tychonoff neighborhood. A locally Tychonoff Hausdorff space need not be Tychonoff (an example is the half-disc topology on the upper half-plane [7]); however, it is easy to see that a locally Tychonoff *regular* space is Tychonoff.

For a function $f \in C(X)$, $\mathbf{Z}(f)$ denotes $f^{-1}(0)$, the *zero-set* of f, and $\mathbf{Cz}(f)$ denotes $X \setminus \mathbf{Z}(f)$, the *cozero-set* of f. We write $\mathbf{Z}[X]$ for the family of all zero-sets in X. Finite unions and intersections of zero-sets are again zero-sets, since $\mathbf{Z}(f) \cup \mathbf{Z}(g) = \mathbf{Z}(fg)$ and $\mathbf{Z}(f) \cap \mathbf{Z}(g) = \mathbf{Z}(|f| + |g|)$. (Notice also that $\{x \mid f(x) \geq 0\} = \mathbf{Z}(\min\{f, 0\})$.)

By a *z-filter* on X we mean a non-empty, proper subfamily $\mathfrak{F}$ of $\mathbf{Z}[X]$ closed under finite intersection and enlargement (meaning that $B \in \mathfrak{F}$ whenever $B \in \mathbf{Z}[X]$, $A \in \mathfrak{F}$ and $A \subseteq B$). By a *z-ultrafilter* we mean a maximal z-filter. Customarily, one denotes ultrafilters by small letters u, v, etc. Note that for any finite family $\{u_i\}$ of distinct z-ultrafilters, one can find a pairwise disjoint family $\{A_i\}$ of zero-sets such that $A_i \in u_i$.

Denote by βX the collection of all z-ultrafilters on X. For a zero-set F of X, let $\hat{F} = \{u \in \beta X \mid F \in u\}$. Topologize βX by taking as a basis of closed sets the collection $\{\hat{F} \mid F \in Z[X]\}$. Let $i : X \to \beta X$ denote the natural map taking $x \in X$ to the *principal z*-ultrafilter $i(x) = \{F \in \mathbf{Z}[X] \mid x \in F\}$.

When X is Tychonoff, the map i is a homeomorphism onto its image; accordingly, we identify X with βX. In this case, βX compactifies X — i.e., βX is a compact Hausdorff space and X is dense in βX. In fact, βX provides a universal compactification of X, in the sense that any continuous mapping $X \to Y$, where Y is compact Hausdorff, extends uniquely to a continuous mapping $\beta X \to Y$.

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[1]This means that, for a closed set $A \subset X$ and a point $x \in X \setminus A$, there exists $f \in C(X)$ such that $f(x) = 0$ and $f(A) = 1$.

Given a continuous map $f : X \to Y$, we define $\beta f : \beta X \to \beta Y$ by setting

$$(\beta f)(u) = \{ F \in \mathbf{Z}(Y) \mid f^{-1}(F) \in u \}.$$

Clearly $\beta(f \circ g) = \beta f \circ \beta g$. Thus, β is a functor from completely regular spaces to compact Hausdorff spaces.

Lemma 2.1. *If $\pi : E \to B$ is a surjection between Tychonoff spaces E and B, the map $\beta\pi : \beta E \to \beta B$ is also surjective.*

Proof. Consider a surjection of Tychonoff spaces, $\pi : E \to B$. From the compactness of βE, we have the image of $\beta\pi$ compact, hence closed. But the image of $\beta\pi$ contains B, a dense subset of βB; so the image must coincide with βB. □

3. Stone-Čech Compactification of G-Bundles

For the remainder of this paper G denotes a finite group of order $n > 1$ with identity element e. By a *G-bundle*, we always mean a covering map $\pi : E \to B$ with E a G-space, $B = E/G$ and π the canonical surjection taking each point $x \in E$ to its orbit.

We shall apply the Stone-Čech functor β to such a G-bundle. The action of G extends naturally to an action on βE. As we shall see, G continues to act transitively on the fibres of $\beta\pi : \beta E \to \beta B$. A fibre will therefore have fewer than n points iff it contains a z-ultrafilter fixed by some non-trivial element of G. The main results of this section (Theorems 3.4 and 3.7) give conditions under which a non-trivial subgroup of G fixes a z-ultrafilter in βE.

Henceforth we assume that both the spaces E and B are Tychonoff, so that applications of the Stone-Čech functor will yield compactifications. Actually, in the context of Hausdorff spaces, it suffices to have B Tychonoff:

Lemma 3.1. *Suppose that B is Tychonoff, E is Hausdorff and $\pi : E \to B$ is a local homeomorphism. Then E Tychonoff.* [2]

Proof. It suffices to show that E is regular, since the local homeomorphism π guarantees E locally Tychonoff. We apply the characterization of regular spaces as spaces in which all points have neighborhood bases consisting of closed sets. Fix a point x of E and an open neighborhood U of x small enough to make $\pi|_U$ a homeomorphism. A neighborhood basis of closed sets $\{N_i\}_{i \in I}$ for the point $\pi(x)$ in $\pi(U)$ yields a neighborhood basis $\{U \cap \pi^{-1}(N_i)\}_{i \in I}$ for x. The regularity of E will follow if we show that each set $\widetilde{N}_i = U \cap \pi^{-1}(N_i)$ is closed, not just in U, but in E.

Observe that for a set N closed in U, $N \subset \pi^{-1}(\pi(N))$, a set closed in E, and therefore $y \in \mathrm{cl}(N)$ implies $y \in \pi^{-1}(\pi(N))$, or equivalently, $\pi(y) \in \pi(N)$.

Applying this observation with $N = \widetilde{N}_i$ shows that $y \in \mathrm{cl}(\widetilde{N}_i)$ implies $\pi(y) \in N_i$. Thus there exists $y' \in \widetilde{N}_i$ such that $\pi(y') = \pi(y)$. We can now show that $\widetilde{N}_i$ is closed in E by showing $y' = y$.

Assume not. Then there exist disjoint open sets V and V' in the Hausdorff space E with $y \in V$ and $y' \in V'$. Now $y \in \mathrm{cl}(\widetilde{N}_i)$ gives $y \in \mathrm{cl}(\widetilde{N}_i \setminus V') \cup \mathrm{cl}(V')$. But clearly $y \notin \mathrm{cl}(V')$, and applying the observation above with $N = \widetilde{N}_i \setminus V'$ shows also that $y \notin \mathrm{cl}(\widetilde{N}_i \setminus V')$, a contradiction. □

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[2] The real line with the origin doubled shows the necessity of some hypothesis on E.

Lemma 3.2. *Fix a G-bundle $\pi : E \to B$. Then G acts transitively on each fibre of the map $\beta\pi$. In particular, the fibres of $\beta\pi$ form G-orbits, and hence have cardinality at most n.*

Proof. Given z-ultrafilters u, v in βE such that $v \neq gu$ for any $g \in G$, we shall show that $(\beta\pi)(v) \neq (\beta\pi)(u)$.

For each $g \in G$, choose $C_g \in v$ such that $C_g \notin g^{-1}u$. Then $C = \bigcap_{g \in G} C_g \in v$; so $\pi(C) \in (\beta\pi)(v)$. On the other hand, $gC \notin u$ for any $g \in G$ (since $gC \subseteq gC_g$ and $gC_g \notin u$). Since G acts transitively on the fibres of π, $\pi^{-1}(\pi(C)) = \bigcup_{g \in G} gC$. But as the z-ultrafilter u does not contain any of the (finitely many) sets gC, u does not contain this last set either. Thus, $\pi(C) \notin (\beta\pi)(u)$. $\square$

Definition 3.3. Let H be a subgroup of G. We call a cozero-set $U \subset E$ **H-antipolar** if $U \cap hU = \emptyset$ for some $h \in H$. We call a zero-set $F \subset E$ **H-equatorial** if we have $F \cup hF = E$ for some $h \in H$, or equivalently, if F forms the complement of an H-antipolar set.

Theorem 3.4. *For a z-ultrafilter u on E, the following conditions are equivalent:*
  1. Every element of H fixes u.
  2. u contains every H-equatorial set.

Proof. $1 \Rightarrow 2$: Suppose H fixes u. For an H-equatorial set F, choose $h \in H$ such that $F \cup hF = E$. The z-ultrafilter u must then contain either F or hF. But H fixes u; so $hF \in u$ implies $F \in u$. Hence $F \in u$.

$2 \Rightarrow 1$: Suppose that $hu \neq u$ for some $h \in H$. Then there exist disjoint zero-sets $C_e \in u$ and $C_h \in hu$. Then $C := C_e \cap h^{-1}C_h$ forms a zero-set in u disjoint from hC. Fix a non-negative function $f \in C(E)$ with zero-set C. Then we claim that $F = \{x : f(x) \geq f(hx)\} \cup \{x : f(x) \geq f(h^{-1}x)\}$ gives an H-equatorial zero-set not in u. Certainly $F \cup hF = E$, since if $x \in E \setminus F$, then $f(x) < f(h^{-1}x)$, whence $h^{-1}x \in F$, i.e., $x \in hF$. We shall show that $F \notin u$ by demonstrating that $F \cap C = \emptyset$. Observe that the non-negative function $f \circ h^{-1}$ has zero-set hC. Since $C \cap hC = \emptyset$, f and $f \circ h^{-1}$ never vanish simultaneously; after composing with h, we can say the same for $f \circ h$ and f. But for any $x \in F \cap C$, we must have either $0 = f(x) \geq f(hx) \geq 0$, whence $f(hx) = 0$, or $0 = f(x) \geq f(h^{-1}x) \geq 0$, whence $f(h^{-1}x) = 0$. This is a contradiction in either case, and so $F \cap C = \emptyset$. $\square$

Corollary 3.5. *H fixes points of βE if and only if the class of H-equatorial sets has the finite intersection property.*

Proof. If the H-equatorial sets have the finite intersection property, then they generate a z-filter, which we may extend to a z-ultrafilter. $\square$

Definition 3.6. Call a cozero-set $U \subset E$ **H-sectional** if $U \cap hU = \emptyset$ for all $h \in H \setminus \{e\}$. Call a zero-set $F \subset E$ **H-cosectional** if $F \cup hF = E$ for all $h \in H \setminus \{e\}$, or equivalently, if F forms the complement of an H-sectional set.

Remark. For $\pi_H : E \to E/H$ the natural quotient map, we may characterize H-sectional subsets of E as the images of sections of π_H over cozero subsets of E/H.

Theorem 3.7. *For a z-ultrafilter u on E, the following conditions are equivalent:*
  1. Some element $h \in H \setminus \{e\}$ fixes u.
  2. u contains all H-cosectional sets.

Proof. $1 \Rightarrow 2$: Suppose $hu = u$ for some $h \in H \setminus \{e\}$. For an H-cosectional set F, $F \cup hF = E$. The z-ultrafilter u must then contain either F or hF. Since $hu = u$, $hF \in u$ implies $F \in u$; so $F \in u$ either way.

$2 \Rightarrow 1$: Suppose no $h \in H \setminus \{e\}$ fixes u. We shall produce an H-cosectional set $F \notin u$. Our assumption guarantees that the ultrafilters hu, h running over H, are distinct. Hence, there exist pairwise disjoint zero-sets $G_h \in hu, h \in H$. Since $h^{-1}G_h \in u$, the zero-set $C = \bigcap_{h \in H} h^{-1}G_h$ belongs to u, and $C \cap hC = \emptyset$ for every $h \in H \setminus \{e\}$.

Now fix a non-negative function $f \in C(E)$ with zero-set C. Since $C \cap h^{-1}C = \emptyset$,

$$f(x) = 0 \Rightarrow x \in C \Rightarrow x \notin h^{-1}C \Rightarrow hx \notin C \Rightarrow f(hx) > 0$$

for all $h \in H \setminus \{e\}$. Define zero-sets

$$F_h = \{x : f(x) \geq f(hx)\} \text{ and } F = \bigcup_{h \in H \setminus \{e\}} F_h .$$

Then $F \cap C = \emptyset$, since $x \in C \Rightarrow 0 = f(x) < f(hx) \Rightarrow x \notin F_h$. So $F \notin u$. We claim that F is H-cosectional, i.e., $F \cup hF = E$ for all $h \in H \setminus \{e\}$. For if $x \in E$, then either $f(x) \geq f(h^{-1}x)$, in which case $x \in F_{h^{-1}} \subseteq F$, or $f(h^{-1}x) \geq f(x) = f(h(h^{-1}x))$, in which case $h^{-1}x \in F_h \subseteq F$, whence $x \in hF_h \subset hF$.   □

Corollary 3.8. *H fails to act freely on βH if and only if the class of H-cosectional sets has the finite intersection property.*

4. Fibre Degeneration and Sectional Category

One calls a G-bundle $\pi : E \to B$ *principal* if G acts freely on E. One calls a locally trivial bundle $\pi : E \to B$ *numerable* when one can trivialize it by an open cover of B that supports a locally finite partition of unity; see [3]. The class of numerable bundles includes all locally trivial bundles over a paracompact base.

One calls a numerable principal G-bundle $\pi : EG \to BG$ *universal* if for any numerable principal G-bundle $q : E \to B$ there exists a map $\phi : B \to BG$, unique up to homotopy, such that $q = \phi^*p$. Universal G-bundles have contractible total spaces EG — indeed, this characterizes universal G-bundles among G-bundles.

In this section, we prove the following.

Theorem 4.1. *For any universal principal G-bundle $\pi_G : EG \to BG$ with BG paracompact and for any subgroup $H \neq \{e\}$ of G, βEG contains points fixed by a non-trivial subgroup of H.*

The proof exploits the notion of the *sectional category* of a bundle $\pi : E \to B$. This is the minimum cardinality of an open cover of B with π admitting a section over each set in the cover; see the standard reference [6], which discusses the identical concept under the name *genus*.[3] The term *sectional category* originated with James' survey of the Lusternik-Schnirelmann category [4].

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[3]Many arguments about fibre bundles hinge at some point on the availability of a partition of unity (or something similar) and therefore require suitable hypotheses. Authors deal with this technicality in different ways. The approach of Švarc in [6] employs a nonstandard definition of *open covering* more restrictive than the usual one. Sectional category in our sense thus generally only bounds sectional category in his sense from below. The notion of a numerable bundle serves a similar purpose. Commonly one just assumes all base spaces to be paracompact.

Lemma 4.2. *Given a G-bundle $\pi : E \to B$, write $\pi_H : E \to E/H$ for the canonical quotient map. Then the H-cosectional sets generate a non-trivial filter iff π_H has infinite sectional category.*

Proof. Corollary 3.8 says that βE has points fixed by a non-trivial element of H iff the collection of H-cosectional sets has the finite intersection property, or equivalently, iff E admits no covering by finitely many H-sectional sets. But as remarked in the previous section, a set $U \subseteq E$ is H-sectional iff the canonical quotient map $\pi_H : E \to E/H$, itself an H-bundle, has a section over $\pi_H(U) = U/H$. ☐

We now collect some facts concerning sectional category that we shall use in the proof of Theorem 4.1 and the sequel.

The *G-category* of a G-bundle $\pi : E \to B$ is the minimum cardinality of an open cover of B by sets over which π is trivial as a G-bundle. For principal G-bundles, sectional category and G-category coincide (since precisely trivial principal G-bundles admit sections).

The *Lusternik-Schnirelmann category* of a space X (L-S category for short) equals the minimum cardinality of an open cover $\{U_i\}$ with each U_i contractible in X. For the homotopy invariance of L-S category, see Prop. (1.1) of [4].

Lemma 4.3. *The G-category of any universal principal G-bundle $\pi : E \to B$ coincides with the L-S category of B.*

Proof. The bundle π admits a section over any contractible open set U in B; so the L-S category of B provides an upper bound for the G-category of π. For the opposite bound, note that the existence of a section of π over U implies that U is contractible in B, since the uniqueness aspect of the universal property of the bundle makes the inclusion map from U to B homotopic to a constant map. (The restriction of the numerable bundle π to U certainly yields a numerable bundle, as would any pullback of any numerable bundle.) ☐

Thus, all universal principal G-bundles over B have the same sectional category.

Given a bundle $\pi : E \to B$, we now describe its n-fold *fibre join* $\pi^{(n)} : E^{(n)} \to B$. A point x in $E^{(n)}$ with $\pi^{(n)}(x) = b$ is represented by an n-tuple $(t_1 e_1, \dots, t_n e_n)$, where the non-negative real numbers $t_1, \dots, t_n$ (called *weights*) satisfy $t_1 + \cdots + t_n = 1$ and the points $e_i \in E$ satisfy $\pi(e_1) = \dots = \pi(e_n) = b$; and where it is understood that $t_i e_i$ is independent of e_i when $t_i = 0$ ([3] gives further details).

Fact 4.4 (Prop. (8.1) in [4]; see also [6])**.** Suppose B is paracompact. Then $\pi : E \to B$ has sectional category less than or equal to n if and only if the n-fold fibre join $E^{(n)}$ admits a section.

For the reader's convenience, we recall the short proof. Given an open cover $U_1, \dots, U_n$ of B and a section σ_i of π over each set U_i, any partition of unity $\{t_i\}$ subordinate to $\{U_i\}$ yields a section of $\pi^{(n)}$, namely $b \mapsto (t_1(b)\sigma_1(b), \dots, t_n(b)\sigma_n(b))$ for $b \in B$. Conversely, given a section σ of $\pi^{(n)}$, say $\sigma(b) = (t_1(b)\sigma_1(b), \dots, t_n(b)\sigma_n(b))$ for $b \in B$, we have a cover of B by the open sets $U_i = \{b | t_i(b) \neq 0\}$, and, over each of these, a well-defined section of π determined by σ_i.

Fact 4.5 (Prop. 50 in [6])**.** If a discrete group G contains elements of finite order, then the bundle $\pi : EG \to BG$ has infinite G-category.[4]

Proof of Theorem 4.1 . The natural map $\pi_H : EG \to EG/H$ forms a universal principal H-bundle, since H acts freely on the contractible space EG. (Recall here that the contractibility of their total spaces characterizes universal bundles.) By Lemma 4.3 and Fact 4.5, the finiteness of H implies that $\pi_H : EG \to EG/H$ has infinite sectional category. By Lemma 4.2, the H-cosectional sets generate a non-trivial filter. By Corollary 3.8, βEG contains points fixed by some non-trivial subgroup of H. □

In particular, for H prime cyclic, H itself must fix some points of βEG. Thus for each such H, $\beta\pi$ has degenerate fibres containing no more than $[G : H]$ points.

5. H-Fixed Points and an Auxiliary Bundle

So far we have seen that the infinite sectional category of universal H-bundles $\pi : EH \to BH$ entails that the H-cosectional subsets of EG have the finite intersection property, and this in turn guarantees the existence of points in βEG fixed by some subgroup of H, by Corollary 3.8.

We could prove the existence of points in βEG fixed by H itself, using Corollary 3.5, if we knew that the family of H-equatorial sets in EG had the finite intersection property, or equivalently, that EG had no cover by finitely many H-antipolar sets. In this section, we formulate an equivalent condition in terms of a certain auxiliary bundle's having infinite sectional category.

Write ΔF for the simplex generated by a finite set F.[5] If F carries an action by H, so does ΔF, in which case we may speak of H-antipolar subsets of ΔF. We denote by λF the subcomplex of ΔF consisting of the union of all H-antipolar faces. Given an H-bundle $\pi : E \to B$, we may perform this construction on each fibre to obtain an H-bundle, which we denote by $\lambda\pi : \lambda E \to B$.[6] This is the auxiliary bundle alluded to above. In order to obtain the advertised result, we shall need to assume B normal. We shall also need the following two technical lemmas.

Lemma 5.1. *For the product $B \times F$ of a normal space B and a discrete finite set F, write π_B and π_F for the projections from $B \times F$ onto B and F. Given an open set $V \subseteq B \times F$ and a point $x \in B$, write $Q(x)$ for the set $\pi_F(\pi_B{}^{-1}(x) \cap V)$. Then there exists a map $t : \pi_B(V) \to \Delta F$ such that $t(x) \in \Delta Q(x)$ for all $x \in \pi_B(V)$.*

Proof. Set $U_j = \{x \in \pi(V) : |Q(x)| \leq j\}$. For $x \in U_1$, set $t(x)$ equal to the unique element of $Q(x)$. We proceed by induction on j. Suppose t is already defined on U_j in such a way that for every $A \subset F$ with $|A| \leq j$, $t(x) \in \Delta A$ for all $x \in \mathrm{cl}(Q^{-1}(A))$. We must extend t continuously to U_{j+1} so that for every $A \subset F$ with $|A| = j + 1$ we have $t(x) \in \Delta A$ for all $x \in \mathrm{cl}(Q^{-1}(A))$.

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[4] The proof of Prop. 50 in [6] contains a misprint: The reference to Theorem 10 should instead point to Theorem 17.

[5]Thus, ΔF consists of functions $w \in [0, 1]^F$ with $\sum_{x \in F} w(x) = 1$. We identify elements of F with the vertices of ΔF, i.e., $x \in F$ is identified with the function that is 1 at x and identically zero on $F \setminus \{x\}$. For any $w \in \Delta F$, we speak of $w(x)$ as the *barycentric coordinates* of w.

[6]The simplest way to do this is to begin with the H-bundle $p' : E' \to B$ whose fibre above $b \in B$ is $[0, 1]^{p^{-1}(b)}$, topologized so that its continuous sections correspond to continuous mappings $f : E \to [0, 1]$; the bundles $\Delta p : \Delta E \to B$ and $\lambda p : \lambda E \to B$ can be obtained as sub-bundles of this.

Continuity of a function on the closed set U_j and on each of the finitely many closed sets $\mathrm{cl}(Q^{-1}(A)) \cap U_{j+1}$ ($A \subset F$ with $|A| = j + 1$) guarantees continuity on all of U_{j+1}. Thus we need only extend t continuously from $\mathrm{cl}(Q^{-1}(A)) \cap U_j$ to $\mathrm{cl}(Q^{-1}(A))$ with values in ΔA for each $A \subset F$ with $|A| = j + 1$. The coordinate functions of t associated with $v_i \in F \setminus A$ equal zero identically on $\mathrm{cl}(Q^{-1}(A)) \cap U_j$; so these we simply extend by zero. For the remaining coordinate functions, those associated to vertices in A, we apply the Tietze extension theorem. This yields a function $s : \mathrm{cl}(Q^{-1}(A)) \to [0,1]^A$ agreeing with t on $\mathrm{cl}(Q^{-1}(A)) \cap U_j$. Finally, to extend t to $\mathrm{cl}(Q^{-1}(A))$, we compose s with a retraction from $[0,1]^A$ onto ΔA. $\square$

Lemma 5.2. *Given $\pi : E \to B$, an H-bundle over a normal space B with H acting transitively on fibres, and V an H-antipolar subset of E, there exists a partial section $t : \pi(V) \to \lambda E$ of the bundle $\lambda\pi : \lambda E \to B$ such that $t(x) \in \Delta(\pi^{-1}(x) \cap V)$ for all $x \in \pi(V)$.*

Proof. The special case where E has the form $B \times F$ for a transitive H-set F follows immediately from Lemma 5.1. We now reduce the general case to this special case.

Denote by E' another copy of E. Consider the pullback

$$
\begin{array}{ccc}
E' \times_B E & \xrightarrow{\;q_2\;} & E \\
\downarrow{q_1} & & \downarrow{\pi} \\
E' & \xrightarrow{\;\pi'\;} & B.
\end{array}
$$

We regard the map $q_1 : E' \times_B E \to E'$ as an H-bundle by letting H act in the usual way on E, but trivially on E'. Taking the diagonal map as a section trivializes the bundle $q_1 : E' \times_B E \to E'$. Observe that $q_2^{-1}(V)$ forms an H-antipolar subset of $E' \times_B E$. From Lemma 5.1, $\lambda q_1 : \lambda(E' \times_B E) \to E'$ has a partial section $s : \pi'^{-1}(\pi(V)) \to \lambda(E' \times_B E)$ such that $s(x) \in \Delta(q_1^{-1}(x) \cap q_2^{-1}(V))$ for all $x \in \pi'^{-1}(\pi(V))$.

If we now let H act in the usual way on E', but trivially on E, we may view the map $q_1 : E' \times_B E \to E'$ as H-equivariant. There exists a canonical isomorphism between the quotient of q_1 by this action and the bundle $\pi : E \to B$. Thus, giving a partial section $t : \pi(V) \to \lambda E$ amounts to giving an H-equivariant partial section from $\pi'^{-1}(\pi(V))$ to $\lambda(E' \times_B E)$.

While H-equivariance may fail for the partial section

$$
s : \pi'^{-1}(\pi(V)) \to \lambda(E' \times_B E),
$$

we do get an H-equivariant map when we pass to the fibrewise barycenter $\overline{s} : \pi'^{-1}(\pi(V)) \to \lambda(E' \times_B E)$ of all the translates hsh^{-1} of s, h running over H. Clearly $\overline{s}(x) \in \Delta(q_1^{-1}(x) \cap q_2^{-1}(V))$. The quotient of $\overline{s}$ by H gives us a mapping $t : \pi(V) \to \lambda E$ with $t(x) \in \Delta(\pi^{-1}(x) \cap V)$ for all $x \in \pi(V)$, as desired. $\square$

Theorem 5.3. *Given $\pi : E \to B$, an H-bundle over a normal space B with H acting transitively on the fibres, the following conditions are equivalent:*

   1. $\lambda\pi : \lambda E \to B$ has infinite sectional category.

   2. E admits no cover by finitely many H-antipolar sets.

Proof. $1 \Rightarrow 2$: Suppose that E has a cover by finitely many H-antipolar sets V_i, in which case the open sets $U_i = \pi(V_i)$[7] cover B. Then Lemma 5.2 allows us to find a partial section t_i of $\lambda\pi : \lambda E \to B$ over each U_i.

---

[7] Recall here that π is a covering map, hence, open.

$2 \Rightarrow 1$: Suppose $\lambda\pi : \lambda E \to B$ has finite sectional category. Then B has a finite open cover $\{U_1, \dots, U_n\}$ such that $\lambda\pi : \lambda E \to B$ admits a partial section s_i over each U_i. Each s_i determines an H-antipolar open subset V_i of E with $V_i \cap \pi^{-1}(x)$ equal to the set of vertices of the smallest face containing $s_i(x)$. Then the V_i and all their H-translates cover E.

Note that only the first implication requires assuming B normal. □

Remark 5.4. We might bypass Theorm 5.3 by working instead with a different auxiliary bundle, one having finite non-Hausdorff spaces as fibres. The fibre above x consists of points v_i to represent each of the H-antipolar subsets $V_i \subset \pi^{-1}(x)$; the closure of a particular v_i consists of all the points representing subsets of the associated V_i.

6. p-GROUPS AND THE SULLIVAN CONJECTURE

Applying the results of the previous section to a universal bundle $\pi : EH \to BH$, we see that βEH (and hence, βEG) has points fixed by H iff $\lambda\pi$ has infinite sectional category. Unfortunately, computing the sectional category of the bundle $\lambda\pi : \lambda EH \to BH$ seems difficult due to the combinatorial complexity of the bundle itself; at present we do not know whether these bundles have infinite sectional category for every H. On the other hand, showing that a larger, but structurally simpler, bundle has infinite sectional category suffices for our purposes. This we can do when H is a p-group.

First, for a finite H-space F, write ΛF for the H-invariant subcomplex of ΔF consisting of all cells except the single cell of highest dimension corresponding to the set F itself. Observe that the H-space ΛF contains λF as a subcomplex, since we certainly never have F itself H-antipolar in F. Now write $\Lambda\pi : \Lambda EH \to BH$ for the sub-bundle of $\Delta\pi : \Delta EH \to BH$ obtained by performing this construction on each fibre.

Henceforth assume BH to be paracompact. We claim that $\Lambda\pi$ has infinite sectional category. In light of Fact 4.4, we just need to show that the n–fold fibre join $(\Lambda\pi)^{(n)} : (\Lambda EH)^{(n)} \to (BH)^{(n)}$ has no section for any n.

Fix a particular fibre F of $\pi : EH \to BH$, enumerating the points of F from 1 to $|F|$. We can then identify points of ΛF with n-tuples $(s_1, ..., s_{|F|})$, where $s_i \in [0, 1]$ and $\sum_i s_i = 1$, and where $s_i = 0$ for at least one i. Accordingly, we can represent points of $\Lambda F^{(n)}$, the n–fold join of ΛF, by weighted n–tuples of the form

$$(t_1(s_{1,1}, \dots, s_{|F|,1}), \dots, t_n(s_{1,n}, \dots, s_{|F|,n}))$$

satisfying
(i) $t_j, s_{i,j} \in [0, 1]$ for $j = 1, \dots, n$, $i = 1, \dots, |F|$;
(ii) $\sum_{j=1}^{n} t_j = 1$;
(iii) $\sum_{i=1}^{|F|} s_{i,j} = 1$ for $j = 1, \dots, n$; and
(iv) for each j there exists an i such that $s_{i,j} = 0$.

The representation is unique provided that we identify two such expressions precisely when they differ only at coordinates with weight $t_j = 0$.

Now to each such weighted n–tuple associate the $n \times |F|$ matrix $|a_{i,j}|$, where $a_{i,j} = t_j s_{i,j}$. Observe that this matrix depends only on the point of $\Lambda F^{(n)}$ itself, not on the particular representation of the point by a weighted n–tuple, and that the matrix determines the point. Precisely those $n \times |F|$ matrices arise in this fashion that satisfy

(i) each $a_{i,j} \in [0,1]$;

(ii) each row contains at least one entry equal to 0;

(iii) $\sum_{j=1}^{n} \sum_{i=1}^{|F|} a_{i,j} = 1$.

Of course H acts on the set of such matrices by permuting columns. This action can have no fixed points. Indeed, with the action of H on F transitive and a 0 in every row, a fixed matrix would have to vanish identically, contrary to (iii).

Pull $(\Lambda\pi)^{(n)} : (\Lambda EH)^{(n)} \to BH$ back along $\pi : EH \to BH$ to obtain the bundle

$$\pi \times_{BH} (\Lambda\pi)^{(n)} : EH \times_{BH} (\Lambda EH)^n \to EH.$$

The contractibility of EH makes this bundle trivial, hence isomorphic (as an H-bundle) to the first factor projection

$$\pi_1 : EH \times (\Lambda H)^n \to EH.$$

In other words, we have $(\Lambda\pi)^{(n)} : (\Lambda EH)^{(n)} \to BH$ isomorphic to

$$\pi_1/H : (EH \times (\Lambda H)^n)/H \to EH/H.$$

Thus a section of $(\Lambda\pi)^{(n)} : (\Lambda EH)^{(n)} \to BH$ amounts to an H-equivariant map $EH \to (\Lambda H)^{(n)}$.

On the other hand, an H-equivariant map from a point to $(\Lambda H)^{(n)}$ amounts to an H-fixed point. We just saw that the H-action on $(\Lambda H)^{(n)}$ has no fixed points. For a p-group H, Miller's version of the Sullivan conjecture [5] provides a homotopy equivalence between the homotopy fixed point space of $(\Lambda H)^{(n)}$, namely, $\mathrm{Hom}_H(EH, (\Lambda H)^{(n)})$, and the fixed point space of $(\Lambda H)^{(n)}$. (See [1], Theorem A, for a result weaker than Miller's, easier to prove, and still sufficient for the present purpose.) Therefore $\mathrm{Hom}_H(EH, (\Lambda H)^{(n)}) = \emptyset$, or equivalently, there exists no H-equivariant map $EH \to (\Lambda H)^{(n)}$. We conclude that the bundle $\Lambda\pi : \Lambda EH \to BH$ has infinite sectional category, so we have proved

Theorem 6.1. *For any universal G-bundle $EG \to BG$ with BG normal and paracompact, and for any p-subgroup H of G, βEG has points fixed by H.*

Searching for a weaker hypothesis on the subgroup H leads, by previous considerations, to the following problem:

Problem 6.2. What groups H, not p-groups, have the homotopy fixed point space $\mathrm{Hom}_H(EH, (\lambda H)^{(n)})$ empty for all n?

7. Remarks on More General Groups

Asking, as in the last section, for a larger bundle to have infinite sectional category amounts to asking for the finite intersection property for a larger family of zero-sets. Let us call a cozero-set $U \subset EH$ H-**incomplete** if

$$\bigcap_{h \in H} hU = \emptyset$$

(meaning U misses at least one element in each fibre over BH), and a zero-set H-**coincomplete** if it forms the complement of an H-incomplete set. Theorem 6.1 shows that for a p-group H and a normal, paracompact classifying space BH, the H-incomplete sets generate a non-trivial filter in BH. Since the H-incomplete filter contains the H-equatorial filter, conceivably we might have the former trivial and the latter not. The following proposition gives a sufficient condition for these two filters to coincide.

Proposition 7.1. *For a finite group H, fix k such that $k^2 - k + 1 < |H|$. Suppose an H-incomplete set U in a principal H-bundle misses no more than k points from any fibre that it meets. Then U admits a finite covering by H-antipolar sets.*

Proof. Let

$$\mathfrak{S} = \{S \subset H \mid |S| > |H| - k\} .$$

For each $S \in \mathfrak{S}$, set $U_S = \bigcap_{h \in S} hU$.

Note that the cozero-sets U_S cover U, since $p \in U$ implies $p \in U_S$ for $S = \{h \mid p \in hU\}$.

Now we shall demonstrate the H-antipolarity of each U_S, $S \in \mathfrak{S}$; this means finding a $j \in H$ such that $U_S \cap jU_S = \emptyset$. Actually, any j such that $S \cup jS = H$ suffices, since then $U_S \cap jU_S = U_S \cap U_{jS} = \bigcap_{h \in H} hU = \emptyset$, by the H-incompleteness of U.

To see why we have a $j \in H$ with $S \cup jS = H$, observe that $S \cup jS \neq H$ implies that $H \backslash S$ must contain an x with $j^{-1}x \notin S$, so that $j = x(x^{-1}j) \in (H \backslash S)(H \backslash S)^{-1}$. But $|S| > |H| - k$ implies $|(H \setminus S)(H \setminus S)^{-1}| \leq k^2 - k + 1 < |H|$.   □

On passing to complements, the proposition says that the equality of the H-coincomplete and the H-equatorial filters follows from the assumption that minimal H-coincomplete sets contain no more than k points of those fibres they do not contain entirely.

Remark. One might consider properties intermediate between H-antipolarity and H-incompleteness in hopes of extending Theorem 6.1 to a larger class of groups. Some candidates:

   a) $\bigcap_{h \in K} hU = \emptyset$ for some cyclic proper subgroup K;
   b) $\bigcap_{h \in K} hU = \emptyset$ for some proper subgroup K;
   c) U meets no more than half the points in any orbit.

The last condition at least has combinatorial simplicity to recommend it.

8. Loci of Degeneration

The homotopy equivalence of all universal G-bundles notwithstanding, we can expect sensitivity of the Stone-Čech functor to topological differences between such bundles. We now obtain especially precise results by concentrating specifically on Milnor's well-known models of universal G-bundles. Given a group G, Milnor sets EG equal to the infinite fibre join $G * G * G * \cdots$ (i.e., the direct limit of the fibre joins $G^{(n)}$); the quotient by the natural left action of G then gives BG. For details, see [3].

Henceforth, we deal specifically with Milnor's model of the universal G-bundle. This offers two advantages. First, Milnor's EG and BG are metrizable. Secondly, for each subgroup H of G, Milnor's EG contains Milnor's $EH = H * H * H * \cdots$ as a closed subspace and, correspondingly, Milnor's BG contains a closed subspace canonically homeomorphic to Milnor's BH, namely the quotient of this EH by H. (Indeed, BH actually has $|G/H|$ disjoint copies of EH sitting over it: for each left coset gH the infinite join $gH * gH * gH * \cdots$ gives one such.)

We measure the degeneration of a fibre of $\pi_G : \beta EG \to \beta BG$ by the conjugacy class of the stabilizer of any point in that fibre. Under the hypotheses of Theorem 8.1 (e.g., for G a p-group), each conjugacy class of subgroups of G has an associated non-empty locus in BG where the fibres display just the corresponding

sort of degeneration. We wish to emphasize again that the action of G on EG has no direct role in the data fed to the Stone-Čech functor; rather the compactification process automatically detects the symmetry of the bundle.

Theorem 8.1. *Fix a finite group G. Suppose that for each subgroup H of G and each universal H-bundle $\pi_H : EH \to BH$ with BH normal, $\beta\pi_H : \beta EH \to \beta BH$ has at least one fibre consisting of a single point. Then for each such H, βEG has points with stabilizer exactly H.*

We shall need a preliminary lemma. Given a family $\mathfrak{F}$ of zero-sets in a space X, let $\mathfrak{F}'$ denote the collection of zero-sets having a non-empty intersection with every member of $\mathfrak{F}$. For $\mathfrak{F}$ a z-filter, $\mathfrak{F}'$ just equals the union of the z-ultrafilters extending $\mathfrak{F}$. Note that if $\mathfrak{F} \subseteq \mathfrak{G}$, then $\mathfrak{G}' \subseteq \mathfrak{F}'$, and also that $(\bigcup_i \mathfrak{F}_i)' = \bigcap_i \mathfrak{F}_i'$ for any family $\mathfrak{F}_i$.

A *perfectly normal* space is one in which every closed set is a z-set. Every metrizable space — in particular, Milnor's EG — is perfectly normal.

Lemma 8.2. *For $\mathfrak{F}$ a z-filter on a perfectly normal space X, if a zero-set C belongs to $\mathfrak{F}''$, then every closed neighborhood of C belongs to $\mathfrak{F}$.*

Proof. Fix $C \in \mathfrak{F}''$ and suppose $C \subseteq U \subseteq N$, with U open and N closed. Since $X \setminus N \subseteq X \setminus U$, $C \cap \mathrm{cl}(X \setminus N) = \emptyset$. Since X is perfectly normal, $\mathrm{cl}(X \setminus N) \in \mathbf{Z}[X]$; so $\mathrm{cl}(X \setminus N) \notin \mathfrak{F}'$. Thus $\mathfrak{F}$ must contain a set E disjoint from $\mathrm{cl}(X \setminus N)$. Then from the fact that $E \subseteq N$ and the fact that $\mathfrak{F}$ is a filter we conclude that $N \in \mathfrak{F}$. $\square$

Remark 8.3. The converse of the lemma (of which we shall make no use) actually holds with no restriction on X: for a z-filter $\mathfrak{F}$ on *any* space X, if every closed neighborhood of a zero-set C belongs to $\mathfrak{F}$, then C belongs to $\mathfrak{F}''$. Indeed, given a zero-set $C \notin \mathfrak{F}''$, there exists a zero-set $D \in \mathfrak{F}'$ disjoint from C. Since the complete separation of disjoint zero-sets holds in *any* space, C must have a closed neighborhood N disjoint from D. But N disjoint from $D \in \mathfrak{F}'$ implies $N \notin \mathfrak{F}$, as desired.

Proof of Theorem 8.1. Suppose the theorem is false. This means that G has a subgroup H which does not occur as the stabilizer (in G) of any point of βEG. Write $\mathfrak{H}$ for the H-equatorial filter. Fix an enumeration $K_1, \ldots, K_n$ of the subgroups of G that properly contain H, and write $\mathfrak{K}_i$ for the K_i-equatorial filter.

The free action of H on EG makes the bundle $EG \to EG/H$ a universal H-bundle with a normal base. By hypothesis, the bundle $\beta EG \to \beta(EG/H) = (\beta EG)/H$ then has at least one fibre that consists of just a single point. Thus βEG does possess H-stable points, which guarantees the non-triviality of $\mathfrak{H}$.

By the choice of H, the stabilizer of an H-stable point always contains H properly; so the stabilizer of each H-stable point of βEG lies among the K_i. Thus a z-ultrafilter extending $\mathfrak{H}$ must also extend some $\mathfrak{K}_i$. This means that $\mathfrak{H}' \subseteq \bigcup_i \mathfrak{K}_i'$, and thus $\bigcap_i \mathfrak{K}_i'' \subseteq \mathfrak{H}''$.

Since $\mathfrak{K}_i \subset \mathfrak{K}_i''$, we can obtain a contradiction by showing that $\bigcap_i \mathfrak{K}_i \not\subseteq \mathfrak{H}''$.

As we noted, EG contains a canonical copy of EH. This EH and all its distinct G-translates $g_k EH$ lie over the canonical copy of BH in BG, disjointly (assume here that $g_1 = e$, the identity). By normality, the $g_k EH$ have disjoint open neighborhoods U_k. If necessary, replace $U = U_1$ by $\bigcap g_k^{-1} U_k$, making U disjoint from gU for any $g \notin H$. This makes U K_i-antipolar for each i. Then $EG \setminus U \in \mathfrak{K}_i$ for each

i, whence $EG \setminus U \in \bigcap_i \mathfrak{K}_i$. To obtain the contradiction we seek, it now suffices to show that $EG \setminus U \notin \mathfrak{H}''$.

Since Lemma 8.2 says that a closed neighborhood of an element of $\mathfrak{H}''$ belongs to $\mathfrak{H}$, showing $EG \setminus U \notin \mathfrak{H}''$ reduces to finding a closed neighborhood of $EG \setminus U$ that does not belong to $\mathfrak{H}$. The normality of EG yields disjoint open sets W and W' with $EH \subseteq W$ and $EG \setminus U \subseteq W'$. Certainly $EG \setminus W$ (which contains the closure of W') constitutes a closed neighborhood of $EG \setminus U$. We can finish now by showing that $EG \setminus W \notin \mathfrak{H}$, or equivalently, by showing that W admits no covering by finitely many H-antipolar sets.

Observe that $EH \to EH/H = BH$ also constitutes a universal H-bundle. Reapplying the hypothesis, $\beta EH \to \beta BH$ has fibres consisting of a single point. This guarantees the non-triviality of the H-equatorial filter on EH. Thus we cannot cover EH by finitely many H-antipolar sets. *A fortiori*, neither can we cover W by finitely many H-antipolar sets. □

Acknowledgments

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